Special metric structures and closed forms

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ABSTRACT

In recent work, N. Hitchin described special geometries in terms of a variational problem for closed generic p-forms. In particular, he introduced on 8-manifolds the notion of an integrable $PSU(3)$-structure which is defined by a closed and co-closed 3-form.

In this thesis, we first investigate this $PSU(3)$-geometry further. We give necessary conditions for the existence of a topological $PSU(3)$-structure (that is, a reduction of the structure group to $PSU(3)$ acting through its adjoint representation). We derive various obstructions for the existence of a topological reduction to $PSU(3)$. For compact manifolds, we also find sufficient conditions if the $PSU(3)$-structure lifts to an $SU(3)$-structure. We find non-trivial, (compact) examples of integrable $PSU(3)$-structures. Moreover, we give a Riemannian characterisation of topological $PSU(3)$-structures through an invariant spinor valued 1-form and show that the $PSU(3)$-structure is integrable if and only if the spinor valued 1-form is harmonic with respect to the twisted Dirac operator.

Secondly, we define new generalisations of integrable G_2- and $Spin(7)$-manifolds which can be transformed by the action of both diffeomorphisms and 2-forms. These are defined by special closed even or odd forms. Contraction on the vector bundle $T \oplus T^*$ defines an inner product of signature (n, n), and even or odd forms can then be naturally interpreted as spinors for a spin structure on $T \oplus T^*$. As such, the special forms we consider induce reductions from $Spin(7, 7)$ or $Spin(8, 8)$ to a stabiliser subgroup conjugate to $G_2 \times G_2$ or $Spin(7) \times Spin(7)$. They also induce a natural Riemannian metric for which we can choose a spin structure. Again we state necessary and sufficient conditions for the existence of such a reduction by means of spinors for a spin structure on T. We classify topological $G_2 \times G_2$-structures up to vertical homotopy. Forms stabilised by $G_2 \times G_2$ are generic and an integrable structure arises as the critical point of a generalised variational principle. We prove that the integrability conditions on forms imply the existence of two linear metric connections whose torsion is skew, closed and adds to 0. In particular we show these integrability conditions to be equivalent to the supersymmetry equations on spinors in supergravity theory of type IIA/B with NS-NS background fields. We explicitly determine the Ricci-tensor and show that over compact manifolds, only trivial solutions exist. Using the variational approach we derive weaker integrability conditions analogous to weak holonomy G_2. Examples of generalised G_2- and $Spin(7)$-structures are constructed by the device of T-duality.

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Chapter 0

Introduction

The present thesis is part of a programme initiated by N. Hitchin in a series of papers [28], [29] and [30] which aims to characterise geometries through closed differential forms enjoying special algebraic properties. The algebraic property which plays the prime rôle in this thesis is *stability*. A form $\rho \in \Omega^p(M)$ is said to be *stable* if at any point, its orbit under the natural action of $GL(T_xM)$ is open. Such a stable form induces a reduction of the structure group of M to the stabiliser subgroup of ρ. Analysing this reduction process closely yields in all interesting cases a natural volume form which over a compact manifold can be integrated to give a diffeomorphism-invariant function on the space of stable forms. Since nearby forms are also stable, we can look for critical points for the variation of this function restricted to closed forms in a given de Rham cohomology class, thereby performing a non-linear version of Hodge theory, and adopt the resulting equations as integrability conditions. The appeal of this method is that it unravels in a straightforward way the special geometrical structure of the local moduli space for the resulting geometry.

In this thesis, we will deal with those stable forms whose stabiliser induces a natural Riemannian metric. Now stability is clearly a low-dimensional phenomenon since for large n and $p > 2$,

$$\dim GL(n) = n^2 \ll \dim \Lambda^p T^* = n!/p!(n-p)!$$

(for $p = 2$ and n even, any symplectic form is stable). The complete list of complex representations of reductive Lie groups admitting an open orbit was given by Sato and Kimura [42]. It follows that except for the well-known G_2-geometry, there is only one more case of interest in dimension 8. It is associated with the structure group $PSU(3)$ induced by a stable 3-form, where $PSU(3)$ acts through its adjoint representation $\mathrm{Ad} : PSU(3) \hookrightarrow SO(8)$. The condition to define a critical point for the variational problem is for both G_2- and $PSU(3)$-structures that the defining 3-form is closed and co-closed.

The first task of this research project was to take further the investigation of $PSU(3)$-structures initiated in [29]. First we derive necessary topological conditions for their existence (Section 2.1.1). The embedding of $PSU(3)$ into $SO(8)$ naturally lifts to $Spin(8)$ and restricted to this lift, the vector representation Λ^1 and the two irreducible spin representations Δ_+ and Δ_- of $Spin(8)$ coincide ([29] and Proposition 1.10). As a result, the tangent space is isomorphic to the spinor bundles which imposes quite severe constraints on the topology of the underlying manifold. In particular, we establish the existence of four linearly independent vector fields (Proposition 2.8). With any principal $PSU(3)$-fibre bundle we also associate a characteristic class in $H^2(M, \mathbb{Z}_3)$ which we call the *triality class* (Section 2.1.2). It is the obstruction to lift a $PSU(3)$-structure to an $SU(3)$-structure and Theorem 2.11 completely settles the existence problem for those $PSU(3)$-structures whose triality class vanishes.

Unlike G_2 the group $PSU(3)$ does not act transitively on a sphere. In particular, it follows from the Berger-Simons theorem that $PSU(3)$ does not appear as a possible holonomy group for the Levi-Civita connection except for the symmetric space $SU(3)$ and its non-compact dual. As a consequence, the 3-form of an integrable $PSU(3)$-structure is not parallel with respect to the Levi-Civita connection. Explicit examples of such integrable $PSU(3)$-structures are described in Section 3.2.2. Continuing with the integrable case, we characterise reductions to $PSU(3)$-structures in Riemannian terms through a spinor-valued 1-form which we show to be harmonic with respect to the twisted Dirac operator on $T^*M \otimes (\Delta_+ \oplus \Delta_-)$ if and only if the structure is integrable (Theorem 3.6), thereby completing and correcting an argument given in [29]. In particular, this implies the existence of a Rarita-Schwinger field which is present in various supergravity and string theories. The projection of this spinor-valued 1-form onto the modules $T^*M \otimes \Delta_+$ and $T^*M \otimes \Delta_-$ induces two orientation-*preserving* isometries $TM \to \Delta_\pm$. More generally, we show that the existence of a *supersymmetric map* (Definition 1.1), that is, an isometry $TM \to \Delta_\pm$, induces a reductive Lie algebra structure on $\Delta_\mp$ whose adjoint group preserves the metric structure (Theorem 1.12). As a result, the structure group reduces to the intersection of $SO(\Delta_\pm)$ with the automorphism group of the Lie algebra. By Cartan's classification we obtain, other than the group $PSU(3)$ associated with the Lie algebra $\mathfrak{su}(3)$, the two groups $SO(3) \times SO(3) \times SO(2)$ and $SO(3) \times SO(5)$ which arise as automorphism groups inside $SO(\Delta_\pm)$ of the remaining Lie algebra structures $\mathfrak{su}(2) \oplus \mathbb{R}^5$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$. The corresponding supersymmetric maps are orientation-*reversing*. These groups also arise as stabilisers of certain 3-forms (albeit non-stable, cf. Theorem 1.15). We will briefly analyse the resulting geometries where we impose the same integrability conditions as in the $PSU(3)$-case (see Sections 1.3.3 and 3.2.1). In particular, we shall see that an orientation-reversing isometry $\gamma_\pm : TM \to \Delta_\pm$ associated with $SO(3) \times SO(5)$ induces an almost quaternionic structure on the tangent space. If the $Sp(1) \cdot Sp(2)$-invariant 4-form is closed, then $\gamma_\pm$ is harmonic with respect to the twisted Dirac-operator. Moreover, we will construct an example showing that integrability does not imply the defining 3-form to be covariant constant with respect to the induced Levi-Civita connection.

The aforementioned classification established by Sato and Kimura also indicated the existence of stable spinors in either of the spin representations $\Lambda^{ev,od}$ of $Spin(n,n)$ for $n = 6$ and 7, where the stabiliser subgroup is conjugate to a real form of $SL(6,\mathbb{C})$ or $G_2(\mathbb{C}) \times G_2(\mathbb{C})$ respectively. If we consider the spin structure associated with the orthogonal vector bundle $T \oplus T^*$ together with contraction as inner product of signature (n,n), then spinors can be naturally identified with differential forms. In [30], Hitchin dealt with the 6-dimensional case and introduced the notion of a *generalised Calabi-Yau manifold* which is defined by a stable form with stabiliser $SU(3,3)$. From the algebra of the $Spin(12,\mathbb{C})$-action he constructed a diffeomorphism-invariant function defined on the open orbit and a similar variational principle as for forms of pure degree applies. Moreover, this function is also invariant under the natural action of closed 2-forms. At any point of the manifold, a 2-form (or a *B-field* in accordance with physicists' jargon) can be regarded as an element of $\mathfrak{so}(6,6) = \Lambda^2(T_x \oplus T_x^*)$ and exponentiation to $Spin(6,6)$ yields an action on all geometric objects living in a representation space of this group. In particular, this action preserves the orbit structure of $\Lambda^{ev,od}$ and thus the induced geometry. This observation subsequently led to the broader concept of a *geometry with B-fields* or *generalised geometry* associated with the $SO(n,n)$-structure on $T \oplus T^*$. A "special" generalised geometry designates a G-structure inside this $SO(n,n)$-bundle, a concept which appears all over this thesis.

Proposition 1.19 suggested the existence of stable spinors whose stabiliser subgroup is conjugate to $G_2 \times G_2$ and $G_2(\mathbb{C})$. The first one induces a natural metric giving rise to what we call a *generalised* G_2*-structure* and the investigation of those constitutes the second part of the thesis. There is also an invariant function which was first explicitly described by Gyoja in purely algebraic terms [24]. However, with a view towards setting up the variational principle for these structures, this formulation proved to be rather cumbersome for our purposes. We therefore approached these structures from a group-theoretic point of view and discussed the geometry in terms of G-structures on the manifold. The inclusion $G_2 \times G_2 \hookrightarrow Spin(7) \times Spin(7)$ prescribes a $G_2 \times G_2$-structure of the vector bundle $T \oplus T^*$. First the inclusion $G_2 \times G_2 \hookrightarrow SO(7) \times SO(7)$ induces what we call a generalised metric structure (Section 1.4), i.e. an orthogonal splitting

$$T \oplus T^* = V_+ \oplus V_-$$

into a positive and negative definite subbundle V_+ and V_- with respect to the inner product on $T \oplus T^*$. Equivalently, this splitting can be encoded in a linear map $P : T \to T^*$, and taking the symmetric and skew-symmetric part of its dual $P \in T^* \otimes T^*$ induces an honest Riemannian metric g and a 2-form b on the manifold. The $G_2 \times G_2$-structure then defines a G_2-structure on V_+ and V_- which can be pulled back to yield reductions to two G_2-subbundles inside the $SO(7)$-bundle associated with g. Taking the induced spin structure, this can be rephrased by the existence of two unit spinors Ψ_+ and Ψ_-. Interpreting the square $\Delta \otimes \Delta$ of the irreducible spin representation space Δ of $Spin(7)$ as the module of either even or odd forms (and thus as spinors for $Spin(7,7)$), we show that any stable form whose stabiliser is conjugate to $G_2 \times G_2$ can be written as

$$\rho = e^{-F} \cdot e^{b/2}(\Psi_+ \otimes \Psi_-) \qquad (0.1)$$

(Corollary 1.22). The scalar F is the so–called *dilaton*. Here it appears as a scaling factor but as becomes apparent in Section 3.3, it is assuredly the same object as the well-known scalar field which occurs in various string theories. The identification of the tensor product of spinors with forms is classical, and a special case of the formula (0.1) was already considered in [20]. What is new in our approach is to interpret the tensor product $\Delta \otimes \Delta$ not as a $Spin(7)$-module but as a module for $Spin(7) \times Spin(7)$ which also acts on even or odd forms through the natural map $Spin(7) \times Spin(7) \to Spin(7,7)$.

The same approach also makes sense for $Spin(7)$-structures over 8-manifolds and an analogous formula to (0.1) yields an even or odd form whose stabiliser is conjugate to $Spin(7) \times Spin(7)$ (Section 1.5). However, as in the classical case, such a form is not generic. The parity of ρ depends on the chirality of the spinors Ψ_+ and Ψ_- and in this way, we obtain the natural notion of a *generalised* $Spin(7)$*-structure of even* or *odd type*. More generally it makes sense to consider *generalised* G*-structures* associated with $G \times G \hookrightarrow SO(n,n)$ for any subgroup $G \leqslant SO(n)$. For instance, Gualtieri introduced generalised Kähler structures associated with $U(n) \times U(n)$ [23].

Any classical G_2- and $Spin(7)$-structure induces a canonical generalised G_2- and $Spin(7)$-structure where the two G_2- or $Spin(7)$-bundles inside the principal $SO(7)$- or $SO(8)$-bundle coincide. From this, necessary and sufficient conditions for the existence of reductions to $G_2 \times G_2$ and $Spin(7) \times Spin(7)$ readily follow (Proposition 2.15). We will also prove the existence of *exotic* structures, that is the two reductions to a G_2- and $Spin(7)$-subbundle inside the principal $Spin(7)$- or $Spin(8)$-bundle cannot be transformed into each other by a vertical, that is, fibrewise homotopy. In the generalised G_2-case we will actually classify these structures up to vertical homotopy by an integer invariant which essentially counts (with an appropriate sign convention) the number of points where the two G_2-structures coincide (Theorem 2.18).

We move on to set up the variational formalism to derive various notions of integrability (Sections 3.1.2, 3.1.3 and 3.1.4). The first one is that of *strong* integrability which means that ρ is closed and co-closed in an appropriate sense. Trivial examples are G_2- or $Spin(7)$-structures whose defining form is closed and co-closed. We reformulate this condition in terms of the right-hand side data of (0.1) which leads to the main result of this thesis. Theorem 3.15 characterises integrable generalised G_2- and $Spin(7)$-structures in terms of two linear metric connections whose torsion is skew, closed and adds to 0, and a further equation which links the differential of the dilaton with the torsion 3-form T,

$$(dF \pm \frac{1}{2}T) \cdot \Psi_{\pm} = 0.$$

For the first variational principle we set up in Section 3.1.2, the resulting torsion is just db which we can think of as "internal" or "intrinsic" torsion coming from the omnipresent twist by the B-field b. In presence of an additional closed 3-form H (a bosonic background field in physicists' terminology), we can also consider a twisted version of this principle taking place over a d_H-cohomology class, where d_H is the twisted differential operator on forms defined by $d_H\alpha = d\alpha + H \wedge \alpha$. This 3-form then appears as "external torsion", that is $T = db + H$. Connections with skew-symmetric torsion gained a lot of attention in the recent mathematical literature (see, for instance, [1], [14], [19], [17] and [31]) due to their importance in string and M-theory, and eventually, our reformulation yields the supersymmetry equations in supergravity of type IIA/B with bosonic background fields [20]. Therefore we drop any distinction between "internal" and "external" torsion and work with T rather than with db or H. A further benefit of this approach is an explicit formula for the

Ricci tensor of an integrable generalised G_2- and $Spin(7)$-structure (Theorem 3.21 and 3.28). As a striking consequence of the closedness of the torsion we obtain the following version of a no-go theorem, by using arguments going back to [32] (the result is wrong for non-closed T). Over a compact manifold, any integrable generalised G_2- or $Spin(7)$-structure is induced by two classical G_2- or $Spin(7)$-structures whose defining spinor is parallel with respect to the Levi-Civita connection. In particular, no exotic structure can be strongly integrable (Corollary 3.19 and Corollary 3.27). This provides a full solution to the variational problem which in this sense does not give rise to any truly new geometry, contrasting sharply with the $U(n) \times U(n)$-geometry mentioned above, for which compact examples that are not defined by two integrable Kähler structures do exist [23]. This motivated us to consider a constrained variational problem (as introduced in [29]) which subsequently led to the notion of a *weakly* integrable generalised G_2-structure of *even* or *odd type* (there is no meaningful $Spin(7)$-equivalent). This notion corresponds to weak holonomy G_2, but as the discussion in Section 3.3.3 reveals, it defines a new type of geometry in the sense that a weak holonomy G_2-manifold does not induce a weakly integrable generalised G_2-structure.

Setting up all these notions would, of course, prove fruitless if they were not supplied by non-trivial examples. In the context of generalised geometry, we dispose of the powerful device of T-duality to construct solutions. This duality was known for some time to relate the string theories of type IIA and IIB, but the first rigorous mathematical formulation was given only recently in a paper by Bouwknegt et al. [8] (see also [12]). Starting with a principal S^1-fibre bundle which carries an integrable generalised structure, we define a second topological generalised structure by exchanging the S^1-fibre with another one, but without destroying integrability. With this tool at hand, we can easily construct local examples of strongly integrable structures with closed torsion out of classical G_2- or $Spin(7)$-structures (Section 3.3.3). T-duality also applies to weakly integrable structures. However, the lack of a classical counterpart makes a straightforward application difficult. If examples of weakly integrable structures exist remains a question to be settled.

As we have seen, all the geometries we introduce give rise to a metric and can be defined by closed forms. Consequently, G-structures play a predominant rôle, and this requires a thorough understanding of the representation-theoretic aspects of the groups involved. The recurrent theme of this thesis is the equivalent description of these G-structures by means of spinors. We therefore chose to deal simultaneously with both *classical* structures (which are defined by a form of *pure* degree) and *generalised* structures (which are defined by an even or odd form which we regard as a spinor for $T \oplus T^*$). As a result, we organised the material as follows.

In the first chapter, we discuss the linear algebra of supersymmetric maps which identify vectors isometrically with spinors. Particular emphasis will be given to the *triality principle* which states that in dimension 8, spinors and vectors have the same internal structure. Since the decomposition into irreducible $Spin(n)$-modules of the space of linear maps from the tangent space to the spinor space yields two pieces of "spin 1/2" and "spin 3/2", we can distinguish supersymmetric maps accordingly, each type giving rise to different geometries. From this point of view, we can characterise classical and generalised structures by the existence of one or two supersymmetric maps of according spin.

We are then in a position to deal with global existence issues of topological G-structures. In the second chapter, we first analyse topological reductions to $PSU(3)$ before moving on to generalised G_2- and $Spin(7)$-structures.

The third and final chapter describes several notions of integrability. If the defining form is stable, we can set up various variational principles which provide us with a natural set of integrability conditions. We then adopt these for the non-stable cases, too. The discussion of the representation-theoretic aspects laid down in the first chapter is taken further for both classical and generalised structures in order to derive geometrical properties of those. Finally, we construct concrete examples.

Chapter 1

Linear algebra

In this chapter we deal with the representation-theoretic aspects of some groups which give rise to a natural metric g on a vector space T and also stabilise a form of either pure degree (inducing what we call a *classical* structure for reasons which become apparent in Section 1.5) or an even or odd form (subsequently referred to as a *generalised* structure). The primary aim is first to gain a better grasp of the linear algebra of generic or *stable* forms (Section 1.6) associated with the groups $PSU(3)$ and $G_2 \times G_2$. The latter case provides an example of a generalised structure, where even or odd forms can be naturally identified with spinors for the orthogonal vector space $T \oplus T^*$ with contraction as inner product of signature (n, n) (Section 1.1.3).

In the context of these "metric" structures, spinors associated with the metric g on T naturally appear and motivate an approach in Riemannian terms. Moreover, stability is essentially a low-dimensional phenomenon and as such, special isomorphisms between vectors and spinors exist. This idea will be formalised by the concept of a *supersymmetric* *map* (Definition 1.1). Such a map identifies the tangent space isometrically with the spinor space and completely characterises the geometry. Since the decomposition into irreducible $Spin(n)$-modules of the space of linear maps from the tangent space to the spinor space yields two pieces of "spin 1/2" and "spin 3/2" we can distinguish supersymmetric maps accordingly, each type giving rise to different geometries (Sections 1.2 and 1.3). Particular emphasis will be given to the *triality principle* which states that in dimension 8, spinors and vectors have the same internal structure (Section 1.1.2). Here, a supersymmetric map of spin 1/2 leads to the well-known $Spin(7)$-structures, but for 3/2, we get something new. Theorem 1.12 states that a supersymmetric map $T \to \Delta_\pm$ induces a reductive Lie algebra structure on $\Delta_\mp$ whose adjoint group preserves the metric structure. Other than the group $PSU(3)$ associated with the Lie algebra $\mathfrak{su}(3)$, the groups $SO(3) \times SO(3) \times SO(2)$ and $SO(3) \times SO(5)$ associated with $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$ and $\mathfrak{su}(2) \oplus \mathbb{R}^5$ appear. These groups can be equivalently defined by a form which, however, is not stable. We will investigate these structures in Section 1.3.3.

We then move on to generalised structures. These share the special feature of being invariant not only under linear isomorphisms, but also under a natural action of a 2-form or *B-field* in physicists' language. We spend some time on describing this setup (Sections 1.1.3 and 1.4) before introducing what we call *generalised exceptional structures* (Section 1.5) which are associated with the groups $G_2 \times G_2$ and $Spin(7) \times Spin(7)$. These stabilise a spinor for $Spin(7,7)$ and $Spin(8,8)$ which, as we have just explained above, can be naturally identified with a form. Propositions 1.20 and 1.21 show how these structures fit into the spinorial picture which will be a key theme of this thesis.

For a more detailed discussion and proofs on the general facts we use we suggest Harvey's book [25].

1.1 Clifford algebras and their representations

In this section we recall some fundamental properties of Clifford algebras to fix our notation.

1.1.1 Clifford algebras

For any inner product space (T, g) of dimension n, we can consider the *Clifford algebra* *Cliff*(T, g) which is generated by T and a unit $\mathbf{1}$ subject to the relations

$$v \cdot v = -g(v, v)\mathbf{1}$$

for any $v \in T$. To ease notation, we shall drop the inner product g and write simply *Cliff*(T) whenever the underlying metric is clear from the context. Note that as a vector space, the map

$$J : v_{i_1} \cdot \ldots \cdot v_{i_k} \mapsto v_{i_1} \wedge \ldots \wedge v_{i_k} \tag{1.1}$$

identifies *Cliff*(T) and $\Lambda^* T^*$. We will usually abuse notation and write informally $v_{i_1 \ldots i_k}$ for either $v_{i_1} \cdot \ldots \cdot v_{i_k}$ or $v_{i_1} \wedge \ldots \wedge v_{i_k}$. In the same vein, we speak of the $\mathbb{Z}$- and $\mathbb{Z}_2$-grading on Clifford algebras or exterior forms. For instance, if $J(x) \in \Lambda^p T$ then we shall refer to p as the *degree* of x and denote it by $\deg(x)$. If $x \in$ *Cliff*(T) and $v \in T$ then

$$J(v \cdot x) = v \wedge x - v \llcorner x, \quad J(x \cdot v) = (-1)^{\deg(x)}(v \wedge x + v \llcorner x).$$

A Clifford algebra comes along with several canonical involutions. We denote by σ the most useful for our purposes. It is an anti-automorphism of *Cliff*(T) defined on elements x of degree p by

$$\sigma(x) = \epsilon(p)x \tag{1.2}$$

where $\epsilon(p) = 1$ for $p \equiv 0, 3 \bmod 4$ and $\epsilon(p) = -1$ for $p \equiv 1, 2 \bmod 4$. The Hodge star operator $\star$ on Λ^* then translates into

$$\star J(x) = J(\sigma(x) \cdot vol), \tag{1.3}$$

where vol now denotes the Riemannian volume element seen as an element in *Cliff*(T).

Finally, we briefly recall some aspects of the representation theory of Clifford algebras. These can be identified with (sometimes a direct sum of) endomorphism algebras over the so-called pinor space P which is unique up to isomorphism. We shall usually denote by $\cdot$ the action of the Clifford algebra on P which is also called *Clifford multiplication*. The pinor spaces come equipped with a bilinear form q (this letter being henceforth reserved for it) satisfying

$$q(x \cdot \phi, \psi) = q(\phi, \sigma(x) \cdot \psi). \tag{1.4}$$

This property renders q unique up to a scalar. If the signature of g is $0 \bmod 8$, then q is symmetric if and only if $p = [n/2]$ equals 0 or $3 \bmod 4$ (which, for us, will always be fulfilled), and skew-symmetric otherwise. If we restrict Clifford multiplication to even elements, then the pinor space P can be decomposed into irreducible components referred to as *spin representations*. In particular, we obtain representations for the so–called *spin* group

$$Spin(T, g) = \{v_1 \cdot \ldots \cdot v_{2k} \mid g(v_i, v_i) = \pm 1\}.$$

For $a \in Spin(T, g)$ we define $\pi_0(a) \in SO(T, g)$ by $\pi_0(a)v = a \cdot v \cdot a^{-1}$. The map

$$\pi_0 : Spin(T, g) \to SO(T, g) \tag{1.5}$$

is in fact a double covering. The induced Lie algebra isomorphism is given on basis elements $e_i \cdot e_j$ by

$$\pi_{0*} : e_i \cdot e_j \in \mathfrak{so}(T, g) \leqslant \mathit{Cliff}(T, g) \to 2e_i \wedge e_j \in \mathfrak{so}(T, g) = \Lambda^2 T^*.$$

The construction of explicit representations in the cases we will subsequently consider will occupy us next.

1.1.2 The triality principle

Roughly speaking, the triality principle asserts that in dimension 8, spinors and vectors have the same internal structure. It therefore plays a central rôle if we wish to approach special geometries in dimensions 7 and 8 from a "supersymmetric" point of view.

Recall that the vector representation Λ^1 and the two irreducible spin representations of $Spin(8)$ are all 8-dimensional and real. A convenient way to think of these spaces is to adopt the octonions $\mathbb{O}$ as the underlying vector space. More concretely, let us fix an orthonormal basis $e_1, \ldots, e_8$ in Λ^1 and identify these vectors with the standard basis $1, i, \ldots, e \cdot k$ of $(\mathbb{O}, \|\cdot\|)$ according to the prescription

$$e_0 \equiv 1,\ e_1 \equiv i,\ e_2 \equiv j,\ e_3 \equiv k,\ e_4 \equiv e,\ e_5 \equiv e \cdot i,\ e_6 \equiv e \cdot j,\ e_7 \equiv e \cdot k. \tag{1.6}$$

Here, $\|\cdot\|$ denotes the norm induced by the standard inner product $(x, y) = \mathrm{Re}(x \cdot \bar{y})/2$. If R_u denotes right multiplication by $u \in \mathbb{O}$, then the map

$$u \in \mathbb{O} \mapsto \begin{pmatrix} 0 & R_u \\ -R_{\bar{u}} & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{O} \oplus \mathbb{O}) \tag{1.7}$$

extends to an isomorphism $\mathit{Cliff}(\mathbb{O}) \cong \mathrm{End}(\mathbb{O} \oplus \mathbb{O})$ where $\Delta = \mathbb{O} \oplus \mathbb{O}$ is the (reducible) space of spinors for $Spin(8)$. These two summands can be distinguished after fixing an orientation, since the Clifford volume element acts on those by ±Id. Consequently, we will denote these two spin representation spaces by Δ_+ and Δ_-. Then, if E_{ij} is the basis of skew-symmetric matrices Λ^2 given by

$$E_{ij} = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \begin{matrix} \\ \dots i \\ \\ \dots j \\ \\ \end{matrix}$$

$$\begin{matrix} \vdots & \quad & \vdots \\ i & & j \end{matrix}$$

it is straightforward to check that the resulting matrices of the inclusion $\Lambda^1 \cong \mathbb{O} \hookrightarrow \mathrm{End}(\Delta_+ \oplus \Delta_-)$ are

$$\begin{aligned}
e_0 &\equiv -E_{1,9} - E_{2,10} - E_{3,11} - E_{4,12} - E_{5,13} - E_{6,14} - E_{7,15} - E_{8,16}, \\
e_1 &\equiv E_{1,10} - E_{2,9} - E_{3,12} + E_{4,11} - E_{5,14} + E_{6,13} + E_{7,16} - E_{8,15}, \\
e_2 &\equiv E_{1,11} + E_{2,12} - E_{3,9} - E_{4,10} - E_{5,15} - E_{6,16} + E_{7,13} + E_{8,14}, \\
e_3 &\equiv E_{1,12} - E_{2,11} + E_{3,10} - E_{4,9} - E_{5,16} + E_{6,15} - E_{7,14} + E_{8,13}, \\
e_4 &\equiv E_{1,13} + E_{2,14} + E_{3,15} + E_{4,16} - E_{5,9} - E_{6,10} - E_{7,11} - E_{8,12}, \\
e_5 &\equiv E_{1,14} - E_{2,13} + E_{3,16} - E_{4,15} + E_{5,10} - E_{6,9} + E_{7,12} - E_{8,11}, \\
e_6 &\equiv E_{1,15} - E_{2,16} - E_{3,13} + E_{4,14} + E_{5,11} - E_{6,12} - E_{7,9} + E_{8,10}, \\
e_7 &\equiv E_{1,16} + E_{2,15} - E_{3,14} - E_{4,13} + E_{5,12} + E_{6,11} - E_{7,10} - E_{8,9}.
\end{aligned} \tag{1.8}$$

Moreover, the inner product on $\mathbb{O}$ can be adopted as the $Spin(8)$-invariant inner product on Δ_+ and Δ_-. Consequently the three irreducible representations Λ^1, Δ_+ and Δ_-, albeit non-equivalent as representation spaces of $Spin(8)$, coincide as Euclidean vector spaces.

The representations are distinguished by the action of the volume element as summarised in Figure 1.1. With the three representations π_0, π_+ and π_- at hand, the triality principle

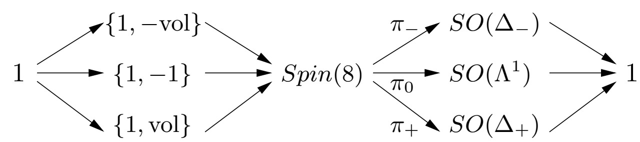

Figure 1.1: The Triality principle

states that

$$\pi_0(g)(x \cdot y) = \pi_-(g)(x) \cdot \pi_+(g)(y) \quad \text{for all } x, y \in \mathbb{O}. \tag{1.9}$$

Another way of formulating triality is this. The group of symmetries of the Dynkin diagram of $Spin(8)$ coincides with the outer isomorphism group. The symmetries are given by a reflection κ which maps the vertex representing the fundamental representation Δ_+ to the vertex of Δ_- and a rotation λ of order 3 (cf. Figure 1.2). Hence $Out(Spin(8)) = \mathfrak{S}_3$ = the permutation group of three elements, and if we regard λ and κ acting as outer automorphisms of $Spin(8)$, we have

$$\pi_0 = \pi_+ \circ \kappa \circ \lambda \text{ and } \pi_- = \pi_+ \circ \lambda^2.$$

Morally this means that we can exchange by an outer automorphism any two of the

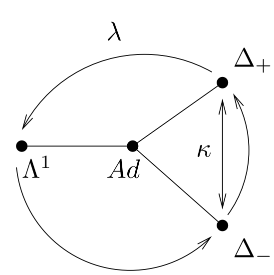

Figure 1.2: The Dynkin diagram of $Spin(8)$

representations Λ^1, Δ_+ and Δ_- while the remaining third one is fixed. We will use these outer automorphisms later on to derive necessary topological conditions for the existence of $PSU(3)$-structures (cf. Section 2.1.1).

1.1.3 The Lie algebra $\mathfrak{so}(n,n)$ and its spin representations

Let T^n be an n-dimensional real vector space and consider the direct sum with its dual $T \oplus T^*$. This direct sum carries a natural orientation by requiring an oriented basis to be of the form $(v_1, \ldots, v_n, \xi_1, \ldots, \xi_n)$ where $\{\xi_j\}$ is the dual basis to $\{v_j\}$. The choice of such a basis identifies the group $SL(T \oplus T^*)$ - the orientation preserving endomorphisms of $T \oplus T^*$ - with $SL(2n, \mathbb{R})$. Furthermore, $T \oplus T^*$ carries the natural inner product

$$(v + \xi, v + \xi) = -\frac{1}{2}\xi(v)$$

of signature (n,n), where $v \in T$ and $\xi \in T^*$. The factor is chosen for convenience and has no geometrical significance. To pick this inner product means singling out a group conjugate to $O(n,n)$ inside $GL(T \oplus T^*) = GL(2n, \mathbb{R})$, and requiring compatibility with the natural orientation brings us down to the group $SO(T \oplus T^*) = SO(n,n)$. We can then consider the Lie algebra $\mathfrak{so}(n,n)$ inside $\mathfrak{gl}(2n)$, the set of endomorphisms of $T \oplus T^*$. Note that $GL(T) \leqslant SO(T \oplus T^*)$ and as a $GL(T)$-space, we have

$$\mathfrak{so}(n,n) = \Lambda^2(T \oplus T^*) = End(T) \oplus \Lambda^2 T^* \oplus \Lambda^2 T.$$

Hence any $X \in \mathfrak{so}(T \oplus T^*)$ can be written in the form

$$X = A \oplus b \oplus \beta = (\sum A^i_j v_i \otimes \xi_j, \sum b_{ij} \xi_i \wedge \xi_j, \sum \beta^{ij} v_i \wedge v_j). \tag{1.10}$$

In particular, any 2-form b defines an element in the Lie algebra $\mathfrak{so}(n,n)$. We will often refer to such a 2-form as a *B-field*. In order to understand its action on $T \oplus T^*$ through exponentiation we represent $\mathfrak{so}(T \oplus T^*)$ as matrices with respect to the splitting $T \oplus T^*$, i.e.

$$\begin{pmatrix} A & \beta \\ B & -A^{tr} \end{pmatrix},$$

where $A \in \mathfrak{gl}(T)$, $b \in \Lambda^2 T^*$ and $\beta \in \Lambda^2 T$. Now the action of $e^b = \exp_{SO(n,n)}(b)$ on $v \oplus \xi \in T \oplus T^*$ is given by

$$e^b(v \oplus \xi) = v \oplus (v \llcorner b + \xi), \tag{1.11}$$

which written in matrix form is

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdot \begin{pmatrix} v \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ b(v) + \xi \end{pmatrix}. \tag{1.12}$$

Next we consider the Clifford algebra $Cliff(T \oplus T^*)$ associated with the inner product $(\cdot\,,\cdot)$ and construct a pinor representation space. We define an action of $T \oplus T^*$ on forms by

$$(v + \xi) \bullet \tau = v \llcorner \tau + \xi \wedge \tau.$$

As this squares to minus the identity it gives rise to an isomorphism

$$Cliff(T \oplus T^*) \cong End(\Lambda^* T^*).$$

We shall reserve the symbol $\bullet$ for this particular Clifford multiplication. The exterior algebra $\Lambda^* T^*$ becomes thus the pinor representation space of $Cliff(T \oplus T^*)$. It splits into the irreducible representation spaces

$$S^{\pm} = \Lambda^{ev,od} T^*$$

of $Spin(T \oplus T^*)$.

Remark: There is an embedding

$$GL_+(T) \hookrightarrow Spin(T \oplus T^*)$$

of the identity component of $GL(T)$ into the spin group of $T \oplus T^*$. From the $GL_+(T)$-representation theoretic point of view we have to twist the spin representations by $(\Lambda^n T)^{1/2}$ so that

$$S^\pm = \Lambda^{ev,od} \otimes (\Lambda^n T)^{1/2}.$$

This is analogous to the complex case

$$U(n) \hookrightarrow Spin^{\mathbb{C}}(2n) = Spin(2n) \times_{\mathbb{Z}_2} S^1,$$

where the even and odd forms get twisted with the square root of the canonical bundle κ,

$$S^\pm = \Lambda^{ev,od}_{\mathbb{C}} T^* \otimes \kappa^{1/2}.$$

As long as we are doing linear algebra this is a mere notational issue but in the global situation we cannot trivialise $\Lambda^n T$ if the manifold is non-orientable. However, this will always be the case for the class of manifolds we consider. Consequently, we omit the twist to ease notation, relying on the context to make it clear whether we consider the standard $GL(T)$-action on exterior forms or the induced action on spinors.

We can define the non-degenerate bilinear form q in (1.4) by taking the top degree of the wedge product $\alpha \wedge \sigma(\beta)$, i.e.

$$q(\alpha, \beta) = (\alpha \wedge \sigma(\beta))_n.$$

It is seen to be invariant under the action of the spin group and from the previous remark we gather that from the $GL(T)$ point of view, q takes values in $\mathbb{R}$ rather than in the volume forms. This form is non-degenerate and symmetric if $n \equiv 0, 3\,(4)$ and skew if $n \equiv 1, 2\,(4)$. Moreover, S^+ and S^- are nondegenerate and orthogonal if n is even and totally isotropic if n is odd.

Finally, we note that the action of a B-field $b \in \mathfrak{so}(n, n)$ on the form $\tau \in S^\pm$ exponentiated to $Spin(T \oplus T^*)$ is given by

$$\exp_{Spin(T\oplus T^*)}(b) \bullet \tau = e^b \bullet \tau = (1 + b + \frac{1}{2} b \wedge b + \ldots) \wedge \tau.$$

We shall therefore also write $\exp(b) \bullet \tau = \exp(b) \wedge \tau$.

1.2 Supersymmetric maps of spin 1/2 and related geometries

As pointed out in the introduction, the purpose of this work is to investigate the Riemannian geometries which arise in Hitchin's variational approach. These geometries fit into a larger picture based on the notion of what we shall call a *supersymmetric map*. In physicists' language a supersymmetry is supposed to exchange bosons (particles that transmit forces, mathematically described as elements lying in a vector representation of the structure group) with fermions (particles that make up matter represented by elements in a spin representation). To formalise this concept, we make the

Definition 1.1. *Let $T = \Lambda^1$ be a Euclidean vector space of dimension 7 or 8 and let Δ denote an irreducible spin representation of the associated spin group $Spin(T, g)$. Then a* supersymmetric map *is an isometry*

$$\gamma : \Lambda^1 \to \Delta$$

onto its image.

A basic instance of this is a unit spinor ψ in the irreducible spin representation Δ of $Spin(7)$ which by Clifford multiplication induces an isometry

$$x \in \Lambda^1 = \mathbb{R}^7 \to x \cdot \psi \in \Delta.$$

Picking a unit spinor is equivalent to the choice of a copy of G_2 inside $Spin(7)$ and it is a general fact that supersymmetric maps are tied to special geometries. Since it is induced by a spinor we say that this isometry is a *supersymmetric map of spin 1/2.* The 1/2 refers to the highest coefficient in the dominant weight of the spin representation. If $x_1, \ldots, x_m$ are the coefficients of a maximal torus of $\mathfrak{so}(2m)$ or $\mathfrak{so}(2m+1)$, then the spin representations $\Delta_\pm$ or Δ are of highest weight

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 + \ldots \pm \frac{1}{2}x_m$$

(where the $-$ occurs for Δ_-). There are also representations of higher spin and in the Section 1.3, we shall also consider *gravitinos*, that is particles of spin 3/2. In this section, however, we will consider geometries which can be characterised by a spinor and give rise to a spin 1/2 supersymmetric map in dimensions 7 or 8. Our treatment is far from being exhaustive. For details and notations, we suggest [13] for $SU(3)$-structures and [10] for G_2- and $Spin(7)$-structures. An exposition which comes particulary close to the spirit of this work and covers several other interesting geometries is [21]. In this article, the authors also deal with structures which are described by two supersymmetric maps. Such geometries will be investigated in Sections 1.4 and 1.5 where we introduce the "generalised" setup.

1.2.1 The group $SU(3)$

In this subsection, we consider $T^6 = \mathbb{R}^6$ together with an $SU(3)$-structure. To see what this means in terms of invariant tensors, we first endow T with an orthogonal complex structure I. Recall that if a real vector space T^{2n} admits an almost complex structure with complex coordinates $z_1, \ldots, z_n$, the complexification of $\Lambda^n T^* \otimes \mathbb{C}$ can be decomposed into the direct sum of complex subspaces $\bigoplus_{p+q=n} \Lambda^{p,q}$ where $\Lambda^{p,q}$ is the space spanned by $dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_q}$. We now introduce an important piece of notation (following [39]) which we shall use throughout this text. Whenever we deal with a complex representation space V, we can also consider its conjugate $\overline{V}$ where a complex scalar z acts by $\bar{z}$. For example, we have $\overline{\wedge^{p,q}} = \wedge^{q,p}$. If V arises as the complexification of a real vector space, we denote that space by $[V]$ which means that

$$[V] \otimes \mathbb{C} = V.$$

In particular, $V \cong \overline{V}$ as *complex* representations and $\dim_{\mathbb{R}}[V] = \dim_{\mathbb{C}} V$. An instance of this are the spaces $\wedge^{p,p}$. If V is not equivalent to $\overline{V}$ then the only way to produce a real vector space is to forget the almost complex structure and to obtain the underlying real vector space $[\![V]\!] = [\![\overline{V}]\!]$. Its complexification is

$$[\![V]\!] \otimes \mathbb{C} = V \oplus \overline{V}$$

and its real dimension is twice the complex dimension of V. For example, we have $T \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$ so that $T = [\![\Lambda^{1,0}]\!]$.

To return to the mainstream of our development we note that the choice of an almost complex structure I brings us down to the structure group $GL(T, I) \cong GL(3, \mathbb{C})$. On the other hand, a non-degenerate 2-form ω yields the structure group $Sp(T, \omega) \cong Sp(6, \mathbb{R})$. If this form lies in $[\Lambda^{1,1}]$, then the structure group reduces to the intersection of $GL(3,\mathbb{C})$ and $Sp(6,\mathbb{R})$ which can be $U(2,1)$ or $U(3)$. The latter occurs if the metric

$$g(x,y) = \omega(Ix,y)$$

is positive definite. This metric also distinguishes a unit circle inside $[\![\Lambda^{3,0}]\!]$. A reduction to $SU(3)$ is then achieved by the choice of a 3-form ψ_+ lying in that circle. This form determines a holomorphic volume form

$$\psi = \psi_+ + i\psi_- \in \Lambda^{3,0},$$

where $\psi_- = I\psi_+$. Choosing a suitable orthonormal basis $e_2, \ldots, e_7$ we can write

$$\begin{aligned} \omega &= e_{32} + e_{54} + e_{67} \\ \Psi &= (e_3 + ie_2) \wedge (e_5 + ie_4) \wedge (e_6 + ie_7) \end{aligned}$$

and decomposing Ψ into the real and imaginary part yields

$$\begin{aligned} \psi_+ &= e_{356} - e_{347} - e_{257} - e_{246} \\ \psi_- &= e_{357} + e_{346} + e_{256} - e_{247} \end{aligned}$$

where as usual we identified e_i with its metric dual e^i and e_{ijk} is shorthand for $e_i \wedge e_j \wedge e_k$. The rather unorthodox choice of the normal form for ω and $\psi_\pm$ in comparison to the standard notation [13] will be justified in Proposition 1.29. Note also the identities

$$\begin{aligned} \omega \wedge \psi_\pm &= 0 \\ \psi_+ \wedge \psi_- &= \frac{2}{3}\omega^3. \end{aligned}$$

Under the action of $SU(3)$, the exterior powers Λ^p decompose as follows.

Proposition 1.1. [13]

$$\begin{aligned} T^* &= [\![\Lambda^1]\!] \\ \Lambda^2 T^* &= [\![\Lambda^{1,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \mathbb{R}\omega \\ \Lambda^3 T^* &= \mathbb{R}\psi_+ \oplus \mathbb{R}\psi_- \oplus [\![\odot^{2,0}]\!] \oplus [\![\Lambda_0^{2,1}]\!] \oplus [\![\Lambda^1]\!] \end{aligned}$$

Note that the induced Hodge star operator determines an isomorphism $\Lambda^p \cong \Lambda^{6-p}$.

Next we want to give a characterisation of $SU(3)$-structures in Riemannian terms, that is we start with a metric on T and ask for the additional invariants to achieve a reduction from $SO(6)$ to $SU(3)$. To this effect, we will discuss $SU(3)$-structures from a spinorial point of view. The pinor space of *Cliff*(T, g) decomposes into two irreducible spin representation spaces S and $\overline{S}$ of $Spin(6)$ which are complex conjugate to each other. In fact there is a special isomorphism $SU(4) = Spin(6)$ and under this identification, the spin representation spaces become the vector representations $S = \mathbb{C}^4$ and $\overline{S} = \overline{\mathbb{C}}^4$. Since

$$SU(4)/SU(3) \cong S^7,$$

we see that $SU(3)$ is the stabiliser of a spinor in S and $\overline{S}$. If we denote these spinors by ψ and $\overline{\psi}$ and choose a representation of the Clifford algebra *Cliff*$(\mathbb{R}^6, g)$, we can reconstruct the forms ω and Ψ by

$$\begin{aligned} \omega(e_i, e_j) &= -iq(e_{ij} \cdot \psi, \psi) \\ \Psi(e_i, e_j, e_k) &= q(e_{ijk} \cdot \psi, \overline{\psi}). \end{aligned}$$

There is clearly some choice involved in this process – we followed the convention in [21].

1.2.2 The group G_2

Next we move up one dimension and consider $T = \mathbb{R}^7$ together with the structure group G_2. It is one of the two so-called exceptional holonomy groups among Berger's list of special holonomy groups. There are several closely interrelated ways of approaching G_2. We can see G_2 inside $GL(8)$ as the automorphism group of the product structure of the octonions. This product structure induces a vector cross product on the imaginary octonions [9] which is also invariant under the action of G_2. From this orthogonal product we can build a 3-form which is generic and stabilised by G_2 and this is the definition of G_2 we shall adopt. More formally, consider the natural action of $GL(7)$ on Λ^3. It has two open orbits one of which contains the aforementioned 3-form and is thus isomorphic to $GL(7)/G_2$. This orbit is commonly denoted by Λ^3_+ and its elements are called *stable*, following the language of [29]. Stable forms will be treated in more detail in Section 1.6. We will usually denote a given G_2-invariant 3-form by φ. Since G_2 sits inside $SO(7)$, the choice of a G_2 structure also induces an orientation and a metric g. The exterior algebra decomposes as follows.

Proposition 1.2. [10]

(i) $\Lambda^1 = \Lambda^1_7 = \mathbb{R}^7$

(ii) $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$ *where*

$$\begin{aligned} \Lambda^2_7 &= \{\alpha \in \Lambda^2 \mid \star(\varphi \wedge \alpha) = 2\alpha\} = \{X \lrcorner \varphi \mid X \in T\} \\ \Lambda^2_{14} &= \{\alpha \in \Lambda^2 \mid \star(\varphi \wedge \alpha) = -\alpha\} = \mathfrak{g}_2 \end{aligned}$$

(iii) $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, *where*

$$\begin{aligned} \Lambda^3_1 &= \{t\varphi \mid t \in \mathbb{R}\} \\ \Lambda^3_7 &= \{\star(\varphi \wedge \alpha) \mid \alpha \in \Lambda^1_7\} \\ \Lambda^3_{27} &= \{\alpha \in \Lambda^3 \mid \alpha \wedge \varphi = 0,\ \alpha \wedge \star\varphi = 0\}. \end{aligned}$$

The notation is standard and the subscript indicates the dimension of the module. The decomposition of the higher order powers follows again from the isomorphism $\Lambda^{7-p} \cong \Lambda^p$ induced by the Hodge $\star$-operator, and modules of the same dimension are equivalent. We will use the notation of Proposition 1.2 throughout this work.

As in the case of $SU(3)$, a spinorial formulation of G_2-structures is available. The group G_2 is simply-connected and the inclusion $G_2 \hookrightarrow SO(7)$ lifts to $Spin(7)$ whose irreducible spin representation Δ is real and 8-dimensional. We have

$$Spin(7)/G_2 \cong S^7,$$

that is, G_2 is the stabiliser of a unit spinor ψ_0 which induces an isometry

$$X \in \Lambda^1 \mapsto X \cdot \psi_0 \in \psi_0^\perp \leqslant \Delta.$$

By fixing a representation of the Clifford algebra $Cliff(\mathbb{R}^7, g)$ we obtain an invariant 3-form by the formula

$$\varphi(e_i, e_j, e_k) = q(e_{ijk} \cdot \psi, \psi). \tag{1.13}$$

The representation we shall use throughout this work is constructed as follows. Identify $T^7 = \mathbb{R}^7$ with the imaginary octonions by $e_1 = i, \ldots, e_7 = e \cdot k$ as in (1.6) and embed T into

$$\mathrm{End}_{\mathbb{R}}(\mathbb{O}) \oplus \mathrm{End}_{\mathbb{R}}(\mathbb{O}) \cong Cliff(T, g)$$

by

$$u \in T \cong \mathrm{Im}\,\mathbb{O} \mapsto \begin{pmatrix} R_u & 0 \\ 0 & -R_u \end{pmatrix},$$

where R_u denotes right multiplication as in (1.7). Projection on the first summand $\mathrm{End}_{\mathbb{R}}(\mathbb{O})$ yields the explicit matrix representation given by

$$\begin{aligned}
e_1 &\equiv E_{1,2} - E_{3,4} - E_{5,6} + E_{7,8},\\
e_2 &\equiv E_{1,3} + E_{2,4} - E_{5,7} - E_{6,8},\\
e_3 &\equiv E_{1,4} - E_{2,3} - E_{5,8} + E_{6,7},\\
e_4 &\equiv E_{1,5} + E_{2,6} + E_{3,7} + E_{4,8},\\
e_5 &\equiv E_{1,6} - E_{2,5} + E_{3,8} - E_{4,7},\\
e_6 &\equiv E_{1,7} - E_{2,8} - E_{3,5} + E_{4,6},\\
e_7 &\equiv E_{1,8} + E_{2,7} - E_{3,6} - E_{4,5}.
\end{aligned}$$

Using this, expression (1.13) becomes

$$\varphi = e_{123} + e_{145} - e_{167} + e_{246} + e_{257} + e_{347} - e_{356} \tag{1.14}$$

which we shall adopt as a normal form of φ. Consequently, we obtain

$$\star\varphi = -e_{1247} + e_{1256} + e_{1346} + e_{1357} - e_{2345} + e_{2367} + e_{4567}. \tag{1.15}$$

Finally, we want to investigate the relationship between $SU(3)$- and G_2-structures. We start with the form point of view. Choose a unit vector $\alpha \in T$ and decompose $T = \mathbb{T} \oplus \mathbb{R}\alpha$ where $\mathbb{T} = \mathbb{R}^6$. Then (1.14) and (1.15) can be written as

$$\varphi = \psi_+ + \omega \wedge \alpha, \quad \star\varphi = \psi_- \wedge \alpha + \frac{1}{2}\omega^2 \tag{1.16}$$

where the forms ω and $\psi_\pm$ are pulled back from $\mathbb{T}$. As usual we identify vectors with 1-forms in the presence of a metric. The data $g_{|\mathbb{T}}$, ω and ψ_+ defines an $SU(3)$-structure on

$\mathbb{T}$. Conversely, take the invariant forms ω and ψ_+ and the metric g_0 of an $SU(3)$-structure and introduce a 1-form α. Define $T = \mathbb{T} \oplus \mathbb{R}\alpha$ with the metric $g = g_0 \oplus \alpha \otimes \alpha$. Then the form φ given as in (1.16) defines a reduction to G_2.

In the spinorial picture, let ψ be the invariant spinor under G_2. The choice of a unit vector α induces an almost complex structure on Δ, since $\alpha \cdot \alpha \cdot \psi = -\psi$. Hence the complexification of Δ becomes

$$\Delta \otimes \mathbb{C} = \Delta^{1,0} \oplus \Delta^{0,1}.$$

In terms of the almost complex structure $I(\psi) = \alpha \cdot \psi$ induced by α, these spaces are

$$\Delta^{1,0} = \{\phi - i\alpha \cdot \phi \mid \phi \in \Delta\} \text{ and } \Delta^{0,1} = \{\phi + i\alpha \cdot \phi \mid \phi \in \Delta\}.$$

Since $\Delta^{1,0}$ and $\Delta^{0,1}$ are 4-dimensional complex vector spaces we have a natural $SU(4) = Spin(6)$ action induced by the cover of $Spin(6) \hookrightarrow Spin(7)$ of the reduction from $SO(7)$ to $SO(6)$ associated with the vector α. Now

$$\Delta = [\![S]\!] = [\![\overline{S}]\!]$$

so that the choice of an inclusion $SU(3) \hookrightarrow SU(4)$ yields a 2-dimensional invariant real subspace spanned by ψ_+ and $\psi_- = \alpha \cdot \psi_+$. The group $SU(3)$ thus sits inside the two copies of $G_{2\pm}$ stabilising the spinors ψ_+ and ψ_- in Δ. The corresponding 3-forms $\varphi_\pm$ are

$$\varphi_\pm = \omega \wedge \alpha \pm \psi_+. \tag{1.17}$$

1.2.3 The group $Spin(7)$

Finally we discuss the geometry associated with $Spin(7)$ over $T = \mathbb{R}^8$. This group is obtained as the reduction from $GL(8)$ to the stabiliser of a certain 4-form which we denote by Ω. Again it can be constructed out of the algebraic data we have on the octonions, but unlike the G_2-invariant 3-form φ it is not generic. However, we still have an inclusion $Spin(7) \hookrightarrow SO(8)$ which induces a metric and an orientation, and Ω is self-dual with respect to the corresponding Hodge $\star$-operator. Since $Spin(7)$ is simply-connected it also lifts to $Spin(8)$.

Proposition 1.3. [10]

(i) $\Lambda^1 = \Lambda^1_8 = \mathbb{R}^8$

(ii) $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21}$ *where*

$$\begin{aligned} \Lambda^2_7 &= \{\alpha \in \Lambda^2 \mid \star(\Omega \wedge \alpha) = 3\alpha\} \\ \Lambda^2_{21} &= \{\alpha \in \Lambda^2 \mid \star(\Omega \wedge \alpha) = -\alpha\} = \mathfrak{spin}(7) \end{aligned}$$

(iii) $\Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48}$, *where*

$$\begin{aligned} \Lambda^3_8 &= \{\star(\Omega \wedge \alpha) \mid \alpha \in T\} \\ \Lambda^3_{48} &= \{\alpha \in \Lambda^3 \mid \alpha \wedge \Omega = 0\}. \end{aligned}$$

(iv) $\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}$ *where*

$$\begin{aligned} \Lambda^4_1 &= \{t\Omega \mid t \in \mathbb{R}\} \\ \Lambda^4_7 &= \{\alpha^*\Omega \mid \alpha \in \Lambda^2_7\} \text{ (\textit{where } } \alpha^* \text{ \textit{denotes the action of} } \Lambda^2 = \mathfrak{so}(8)) \\ \Lambda^4_{27} &= \{\alpha \in \Lambda^4 \mid \Omega \wedge \alpha = 0,\ \star\alpha = \alpha,\ \beta \wedge \alpha = 0 \text{ for all } \beta \in \Lambda^4_7\}, \\ \Lambda^4_{35} &= \{\alpha \in \Lambda^4 \mid \star\alpha = -\alpha\}. \end{aligned}$$

We saw earlier in our discussion of triality that vectors and spinors are equivalent under an outer automorphism. An invariant vector in Λ^1 reduces the structure group from $SO(8)$ to $SO(7)$ and this is covered by an inclusion of $Spin(7)$ to $Spin(8)$. We denote the image by $Spin(7)_0$ to indicate that it fixes an element under the vector representation of $Spin(8)$

induced by π_0 (cf. (1.5)). The triality automorphisms λ and κ permute the representation spaces and consequently, the stabiliser of a unit spinor in $\Delta_\pm$ are two copies of $Spin(7)$ which we denote by $Spin(7)_\pm$. Hence we have

$$Spin(8)/Spin(7)_\pm \cong S^7 \subset \Delta_\pm$$

and $Spin(7)_\pm$ acts irreducibly on $\Delta_\mp$.

As in the previous cases we can compute a normal form of Ω by the formula

$$\Omega(e_i, e_j, e_k, e_l) = q(e_{ijkl} \cdot \psi, \psi).$$

If we fix the representation given by (1.8) we find

$$\begin{aligned}\Omega \;=\; & -e_{0123} - e_{0145} + e_{0167} - e_{0246} - e_{0257} - e_{0347} + e_{0356} + e_{1247} - e_{1256} - e_{1346} \\ & -e_{1357} + e_{2345} - e_{2367} - e_{4567}.\end{aligned}$$

To link G_2- with $Spin(7)$-structures we choose a unit vector $\gamma \in T$ and decompose $T = \mathbb{R}\gamma \oplus \mathbb{T}$ where we endow $\mathbb{T} = \mathbb{R}^7$ with a G_2-structure $\varphi \in \Lambda^3\mathbb{T}^*$. Then the 4-form

$$\Omega = \gamma \wedge \varphi + \star\varphi$$

is stabilised by $Spin(7)$. The spinor description follows from the decomposition

$$\Delta_+ \cong \Delta_- \cong \Lambda^1 \oplus \mathbb{R}\psi_\pm$$

into G_2-modules. In fact, $Spin(8)$ acts transitively on $S^7 \times S^7$ inside $\Delta_+ \times \Delta_-$ with stabiliser G_2,

$$Spin(8)/G_2 = S^7 \times S^7 \subset \Delta_+ \times \Delta_-$$

and consequently, G_2 sits in and actually equals the intersection of $Spin(7)_+$ with $Spin(7)_-$. Moreover, we can show the

Proposition 1.4. *Restricted to $G_2 \subset Spin(8)$, the representations Λ^1, Δ_+ and Δ_- coincide.*

Proof: We denote the four fundamental weights of $Spin(8)$ by ω_1, ω_2, ω_3 and ω_4. The weights of the representation $\Lambda^1 = [1, 0, 0, 0]$ are

$$\pm\omega_1,\ \pm(\omega_1 - \omega_2),\ \pm(\omega_2 - \omega_3 - \omega_4),\ \pm(\omega_3 - \omega_4). \tag{1.18}$$

For the representations $\Delta_+ = [0, 0, 1, 0]$ and $\Delta_- = [0, 0, 0, 1]$ we find the weights

$$\pm(\omega_1 - \omega_4),\ \pm(\omega_1 - \omega_2 + \omega_4),\ \pm(\omega_2 - \omega_3),\ \pm\omega_3 \tag{1.19}$$

and

$$\pm(\omega_1 - \omega_3),\ \pm(\omega_1 - \omega_2 + \omega_3),\ \pm(\omega_2 - \omega_4),\ \pm\omega_4. \tag{1.20}$$

Now $\Delta_+ = \Lambda^1_7 \oplus \mathbb{R}\psi_+$ as a G_2-space where G_2 acts with weights

$$0,\ \pm(\sigma - \tau),\ \pm\sigma,\ \pm(2\sigma - \tau),$$

where σ and τ are the fundamental weights of G_2. Substituting these into 1.18, 1.19 and 1.20 gives in all three cases the same G_2-weights, i.e. they are equivalent. ■

We will also consider the case where we have to orthogonal unit spinors $\psi_\pm \in \Delta_+$. The spinor ψ_+ gives rise to the stabiliser $Spin(7)_+$, so that $\Delta_+ = \mathbb{R}\psi_+ \oplus \mathbb{R}^7$. The group $Spin(7)_+$ acts on $\mathbb{R}^7$ in its vector representation and thus transitively on the sphere S^6 with stabiliser $Spin(6)$. As a $Spin(6)$-space, $\Lambda^1 = \mathbb{R}^8 = [\![\mathbb{C}^4]\!]$, where $\mathbb{C}^4$ denotes the spin representation under the identification $Spin(6) = SU(4)$ as above. Note that $Spin(6)$ stabilises a 2-form in Λ^2, given by

$$\varpi = e_{01} - e_{23} - e_{45} + e_{67}$$

and the three self-dual 4-forms

$$\begin{aligned}
\Omega_1 &= e_{0246} + e_{0257} + e_{0347} - e_{0356} - e_{1247} + e_{1256} + e_{1346} + e_{1357} \\
\Omega_2 &= e_{0247} - e_{0256} - e_{0346} - e_{0357} + e_{1246} + e_{1257} + e_{1347} - e_{1356} \\
\Omega_3 &= -e_{0123} - e_{0145} + e_{0167} + e_{2345} - e_{2367} - e_{4567},
\end{aligned}$$

where $\Omega_3 = \varpi \wedge \varpi/2$.

We summarised the relationship between the groups $SU(3)$, G_2 and $Spin(7)$ in the next table. Note that the decomposition of the vector and spin spaces in the first and second row are taken with respect to the group G_2 and $SU(3)$.

Group	vector space	spinor space
$Spin(8)$	$T = \mathbb{R}\gamma \oplus \mathbb{R}^7$	$\Delta_\pm = \mathbb{R}\psi_\pm \oplus \mathbb{R}^7$
$\cup$		
$G_2 = Spin(7)_+ \cap Spin(7)_-$	$T = \mathbb{R}\alpha \oplus \mathbb{R}^6$	$\Delta = \mathbb{R}\psi_+ \oplus \mathbb{R}\psi_- \oplus \mathbb{R}^6$
$\cup$		
$SU(3) = G_{2+} \cap G_{2-}$	$T = \mathbb{R}^6$	$S = \mathbb{C}^3 \oplus \mathbb{C}\psi,\ \bar{S} = \overline{\mathbb{C}^3} \oplus \overline{\psi}$

Table 1.1: The groups $SU(3)$, G_2 and $Spin(7)$

1.3 Supersymmetric maps of spin $3/2$ and related geometries

In dimension 8, a supersymmetric map sits inside the module

$$\Lambda^1 \otimes \Delta_\pm = \Delta_\mp \oplus \ker \mu_\mp$$

where $\ker \mu_{(\pm)}$ denotes the kernel of the Clifford multiplication

$$\mu_\pm : \Lambda^1 \otimes \Delta_\pm \to \Delta_\mp.$$

As a spin module, $\ker \mu$ is an irreducible space of highest weight

$$\frac{3}{2}x_1 + \frac{1}{2}x_2 + \ldots \pm \frac{1}{2}x_n$$

and consequently, an element of this space will be referred to as a spin 3/2 particle. In the same vein as in the previous section, we try to analyse the geometries associated with a supersymmetric map, this time of spin 3/2. As an example of such a geometry, we first consider the group $PSU(3)$.

1.3.1 The group $PSU(3)$

We shall first adopt a definition of the group $PSU(3)$ which follows closely the approach of [10] for G_2, emphasising the form point of view. Let T be a real vector space of dimension 8. We fix a basis $e_1, \ldots, e_8$ whose dual basis we denote by $e^1, \ldots, e^8$ and choose $e^{12345678}$ as orientation.

Definition 1.2. *Let*

$$\rho = e^{123} + \frac{1}{2}e^1(e^{47} - e^{56}) + \frac{1}{2}e^2(e^{46} + e^{57}) + \frac{1}{2}e^3(e^{45} - e^{67}) + \frac{\sqrt{3}}{2}e^8(e^{45} + e^{67}). \qquad (1.21)$$

Then we define

$$PSU(3) = \{a \in GL_+(T) \mid a^*\rho = \rho\}.$$

The coefficients c_{ijk} of (1.21) are taken from [22], p.50. We shall give a more detailed description of this 3-form in a moment. First we note that as in the case of G_2, we may regard $PSU(3)$ as (the connected component of) an automorphism group.

Proposition 1.5. *The subgroup $PSU(3) \leqslant GL(T) = GL(8)$ is the identity component of the automorphism group of the Lie algebra $\mathfrak{su}(3)$, i.e.*

$$PSU(3) \cong Ad(SU(3)) = SU(3)/Z(SU(3)),$$

where $Z(SU(3))$ denotes the center of $SU(3)$.

Proof: We define a bracket $[\cdot\,,\cdot]$ on $T \times T$ by linearly extending the relations

$$[e_i, e_j] = \sum_k \rho(e_i, e_j, e_k)e_k,$$

that is, the structure constants c_{ij}^k are just the coefficients of ρ and hence are totally skew. We will show that this bracket endows T with a Lie algebra structure isomorphic to $\mathfrak{su}(3)$. It is immediate to verify that we actually have defined a Lie bracket, that is the Jacobi identity holds. Next, consider the Killing form $B(x, y) = \mathrm{Tr}(ad(x) \circ ad(y))$. It is straightforward to check that the basis $e_1, \ldots, e_8$ verifies

$$B(e_i, e_j) = -3\delta_{ij}$$

so that B is negative definite. Hence, $(T, [\cdot\,,\cdot])$ is a compact semi-simple Lie algebra. By Cartan's classification theorem we conclude it must be simple on dimensional grounds and thus isomorphic to $\mathfrak{su}(3)$. Hence $PSU(3)$ is contained in $Aut(\mathfrak{su}(3)) = Ad(SU(3)) \times \mathbb{Z}_2$. Since the orbit of ρ is of dimension less than or equal to 56, the dimension of Λ^3, it follows that $PSU(3)$ is of dimension greater than or equal to 8 and hence equals 8 since this is the dimension of $Ad(SU(3))$. Consequently, the identity component of the automorphism group is contained in $PSU(3)$. Since the other component of $Aut(\mathfrak{su}(3))$ reverses the orientation the assertion follows. ∎

Invariantly formulated, we have

$$\rho(x, y, z) = -\frac{1}{3}B([x, y], z)$$

where B and $[\cdot\,,\cdot]$ denotes the Killing form and the bracket $[\cdot\,,\cdot]$ of $\mathfrak{su}(3)$. The vectors e_1, e_2, e_3, e_8 span a $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathbb{R}e_8$ subalgebra. As an $\mathfrak{su}(2)$-space, we have $\mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathbb{R}e_8 \oplus [\![\mathbb{C}^2]\!]$, where $\mathbb{C}^2$ denotes the vector representation of $\mathfrak{su}(2)$ which we view as a real space $\mathbb{R}^4$ spanned by $e_4, \dots, e_7$. The space of 2-forms $\Lambda^2\mathbb{R}^4$ decomposes into the direct sum of self-dual and anti-self-dual forms (the ± 1-eigenspaces of the Hodge $\star$-operator). By choosing a suitable orientation, the anti-self-dual forms $\omega_{1-} = e^{47} - e^{56}$, $\omega_{2-} = e^{46} + e^{57}$ and $\omega_{3-} = e^{45} - e^{67}$ are acted on trivially by $\mathfrak{su}(2)$, while $\omega_{3+} = e^{45} + e^{67}$ belongs to the space of self-dual forms which is just $\mathfrak{su}(2)$. We can then write

$$\rho = e_{123} + \frac{1}{2}e_1 \wedge \omega_{1-} + \frac{1}{2}e_2 \wedge \omega_{2-} + \frac{1}{2}e_3 \wedge \omega_{3-} + \frac{\sqrt{3}}{2}e_8 \wedge \omega_{3+}. \tag{1.22}$$

and

$$\star\rho = e_{45678} - \frac{1}{2}e_{238} \wedge \omega_{-1} + \frac{1}{2}e_{138} \wedge \omega_{-2} - \frac{1}{2}e_{128} \wedge \omega_{-3} + \frac{\sqrt{3}}{2}e_{123} \wedge \omega_{3+}. \tag{1.23}$$

The next corollary displays yet another feature that $PSU(3)$ has in common with G_2 and which will be crucial in the sequel.

Corollary 1.6. *Under the natural action of $GL_+(8)$, the orbit of the 3-form ρ is diffeomorphic to $GL_+(8)/PSU(3)$ and therefore open.*

Again, we shall refer to a form which lies in this open orbit as *stable*. Expressed in a suitable frame, a stable element has the canonical form (1.21). We refer to such a frame as a $PSU(3)$-*frame*.

As a further corollary of Proposition 1.3.1 we note the

Corollary 1.7. *The Lie group $PSU(3)$ is a compact, connected Lie group of dimension 8 with fundamental group $\pi_1(PSU(3)) = \mathbb{Z}_3$. It acts irreducibly on T. The adjoint representation $Ad : SU(3) \to SO(8)$ descends to an embedding $PSU(3) \hookrightarrow SO(8)$ that lifts to an embedding $PSU(3) \hookrightarrow Spin(8)$.*

Hence the choice of an orientation and a stable 3-form induces a metric g, a Hodge $\star$-operator $\star_g$, a Lie bracket $[\cdot, \cdot]$ and a spin structure. In particular, we will drop any distinction between $\{e_i\}$ and its dual basis $\{e^i\}$.

Next, we will discuss some elements of the representation theory for $PSU(3)$. We will always suppose that we work with a $PSU(3)$-frame $e_1, \ldots, e_8$. We will label irreducible representations by their highest weight expressed in the basis of fundamental weights which in the case of $\mathfrak{su}(3)$ is provided by two elements, σ_1 and σ_2. For example, $T \otimes \mathbb{C}$, the complexification of the adjoint representation of $SU(3)$, is $T \otimes \mathbb{C} = V(\sigma_1 + \sigma_2)$ which we more succinctly write as $(1, 1)$, the i-th component referring to σ_i. This can be seen by fixing the Cartan subalgebra spanned by e_3 and e_8. We obtain the following decomposition of the complexified Lie algebra $\mathfrak{su}(3) \otimes \mathbb{C}$ into root spaces:

$$\mathfrak{su}(3) \otimes \mathbb{C} = \mathfrak{t} \oplus V(\alpha) \oplus V(\bar{\alpha}) \oplus V(\beta) \oplus V(\bar{\beta}) \oplus V(\alpha + \beta) \oplus V(\bar{\alpha} + \bar{\beta}).$$

We use the notation $T = [1, 1]$ for the *real* representation in accordance with the convention introduced on page 30. For subsequent computations we will adopt the following convention. The fundamental roots α and β will be labeled in such a way that the root spaces $V(\alpha)$ and $V(\beta)$ are spanned by

$$x_\alpha = e_4 + ie_5 \text{ and } x_\beta = e_6 - ie_7.$$

This corresponds to the expressions

$$\alpha = -\frac{i}{2}(e^3 + \sqrt{3}e^8) \text{ and } \beta = \frac{i}{2}(-e^3 + \sqrt{3}e^8) \tag{1.24}$$

for the fundamental roots α and β. It follows that $V(\alpha+\beta)$ is generated by $x_{\alpha+\beta} = e_1 + ie_2$. Moreover, the fundamental weights are

$$\begin{aligned} \sigma_1 &= \tfrac{2}{3}\alpha + \tfrac{1}{3}\beta \\ \sigma_2 &= \tfrac{1}{3}\alpha + \tfrac{2}{3}\beta. \end{aligned} \tag{1.25}$$

Note that the Lie structure on $T = \Lambda^1$ induces a $PSU(3)$-invariant operator $d : \Lambda^k \to \Lambda^{k+1}$ which is just the exterior differential restricted to left-invariant forms on the group $SU(3)$,

$$de_i = \sum_{j<k} c_{ijk} e_j \wedge e_k.$$

The following proposition gives the decomposition of the exterior algebra into irreducible $PSU(3)$-modules.

Proposition 1.8.

1. *$\Lambda^1 = T$ is irreducible and of real type.*

2. *$\Lambda^2 = \Lambda^2_8 \oplus \Lambda^2_{20}$, where*

$$\Lambda^2_8 = \{X \lrcorner \rho \mid X \in \Lambda^1\} = \mathfrak{su}(3)$$
$$\Lambda^2_{20} = \{\tau \in \Lambda^2 \mid \tau \wedge \star\rho = 0\}$$

Λ^2_{20} is a representation of complex type and its complexification decomposes as

$$\Lambda^2_{20} \otimes \mathbb{C} = \Lambda^{2\mathbb{C}}_{10+} \oplus \Lambda^{2\mathbb{C}}_{10-}$$

where

$$\Lambda^{2\mathbb{C}}_{10\pm} = \{\alpha \in \Lambda^2 \otimes \mathbb{C} \mid \star(\rho \wedge \alpha) = \pm\sqrt{3}i \cdot \alpha^* \rho\}.$$

Expressed in terms of dominant weights with respect to the basis provided by the fundamental weights σ_1 and σ_2, these spaces are

$$\begin{aligned}\Lambda^2_8 \quad &= [1,1]\\ \Lambda^{2\mathbb{C}}_{10+} &= (0,3)\\ \Lambda^{2\mathbb{C}}_{10-} &= (3,0).\end{aligned}$$

3. $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_8 \oplus \Lambda^3_{20} \oplus \Lambda^3_{27}$, *where*

$$\begin{aligned}&\Lambda^3_1 = \{t\rho \mid t \in \mathbb{R}\}\\ &\Lambda^3_8 = \{\star((X \llcorner \rho) \wedge \rho) \mid X \in \Lambda^1\} \cong \Lambda^2_8\\ &\Lambda^3_{20} = d\Lambda^2_{20}\\ &\Lambda^3_{27} = \{\alpha \in \Lambda^3 \mid \alpha \wedge \rho = 0,\ \alpha \wedge \star\rho = 0\}\end{aligned}$$

Moreover,

$$\Lambda^3_{27} = [2,2].$$

4. $\Lambda^4 = \{\alpha \in \Lambda^4 \mid \star\,\alpha = \alpha\} \oplus \{\alpha \in \Lambda^4 \mid \star\,\alpha = -\alpha\} = \Lambda^{4+}_8 \oplus \Lambda^{4+}_{27} \oplus \star\Lambda^{4-}_8 \oplus \star\Lambda^{4-}_{27}$, *where* $\Lambda^{4\pm}_8$ *and* $\Lambda^{4\pm}_{27}$ *are obtained as projections of the spaces*

$$\begin{aligned}&\{X \llcorner \star\rho \mid X \in \Lambda^1\} \cong \Lambda^1\\ &d\Lambda^3_{27} \cong \Lambda^3_{27}\end{aligned}$$

on the ± 1*-eigenspaces of* $\star$.

The proof of the proposition is a routine application of Schur's lemma. For instance, $\tau = x_\beta \wedge x_{\alpha+\beta}$ is contained in $\Lambda^{2\mathbb{C}}_{10+} = (0,3)$. It is straightforward to check that $\star(\rho\wedge\tau) = \sqrt{3}i\cdot\tau^*\rho$ and thus holds for any form in $\Lambda^{2\mathbb{C}}_{10+}$. Note that the spaces $\Lambda^{2\mathbb{C}}_{10\pm} \cong \Lambda^{3\mathbb{C}}_{10\pm}$ are interchanged if we swap orientations. The forms $\sqrt{3}e_{123} + e_{458} + e_{678}$ and $-\sqrt{3}(e_{145} + e_{167})/2 + e_{238} +$

$e_{478}/2 - e_{568}/2$ lie in Λ^3_8 and Λ^3_{20} and wedged with ρ these get mapped to non-zero elements in Λ^6 etc..

We can approach this decomposition also from a cohomological point of view well suited for our later purposes. The Lie algebra structure on $\Lambda^1 = \mathfrak{su}(3)$ induces a $PSU(3)$-invariant operator $b_k : \Lambda^k \to \Lambda^{k+1}$ by extension of

$$be_i = \sum_{j<k} c_{ijk} e_j \wedge e_k.$$

Since b is built out of the structure constants, it is just the exterior differential operator restricted to the left-invariant differential forms of $SU(3)$ with adjoint $b^* = d^* = - \star\, d \star$. The resulting elliptic complex is isomorphic to the de Rham cohomology $H^*(SU(3), \mathbb{R})$ which is trivial except for the Betti numbers $b_0 = b_3 = 1 = b_5 = b_8$. Hence, $\operatorname{Im} b_k = \ker b_{k+1}$ for $k = 0,\ 1,\ 3,\ 5,\ 6$ and $\operatorname{Im} b_k = \ker b_{k+1} \oplus \mathbb{R}$ for $k = -1,\ 2,\ 4,\ 7$. Schematically, we have

$$\begin{array}{ccccccccccccccccc}
\Lambda^0_1 & & & & & & \Lambda^3_1 & & & & \Lambda^5_1 & & & & & & \Lambda^8_1 \\
 & & \Lambda^1_8 & \xrightarrow{b} & \Lambda^2_8 & & \Lambda^3_8 & \xrightarrow{b} & \Lambda^4_8 & & & & \Lambda^6_8 & \xrightarrow{b} & \Lambda^7_8 & & \\
 & & & & & & & & \Lambda^4_8 & \xrightarrow{b} & \Lambda^5_8 & & & & & & \\
 & & & & \Lambda^2_{20} & \xrightarrow{b} & \Lambda^3_{20} & & & & \Lambda^5_{20} & \xrightarrow{b} & \Lambda^6_{20} & & & & \\
 & & & & & & \Lambda^3_{27} & \xrightarrow{b} & \Lambda^4_{27} & & & & & & & & \\
 & & & & & & & & \Lambda^4_{27} & \xrightarrow{b} & \Lambda^5_{27} & & & & & &
\end{array} \tag{1.26}$$

In particular, we will use the more natural splitting of Λ^4 into $\Lambda^4_o = \ker b_3$ and $\Lambda^4_i = \operatorname{Im} b^*$ instead of the $SO(8)$-equivariant splitting into self- and anti-self-dual forms.

Using the b-operator and its co-differential, we can easily construct the projection operators $\pi^p_q : \Lambda^p \to \Lambda^p_q$. In particular, we find for $p = 2$:

Proposition 1.9. *For any $\alpha \in \Lambda^2$ we have $b(\alpha) = -\alpha^* \rho$. Moreover, $\Lambda^2_8 = \ker b_2$ and the projection operator on the complement is $\pi^2_{20}(\alpha) = \frac{4}{3} b_3^* b_2(\alpha)$. For the complexified modules $\Lambda^2_{10\pm}$, the projection operators are $\pi^2_{10\pm}(\alpha) = \frac{2}{3} b_3^* b_2(\alpha) \mp \frac{8\sqrt{3}}{9} i \star (b_2(\alpha) \wedge \rho)$.*

As in the case of G_2-structures, there is also a half-spin $PSU(3)$-invariant.

First, we can decompose

$$\Delta_+ \otimes \Delta_- = \Lambda^1 \oplus \Lambda^3$$

into $Spin(8)$-spaces and observe that the 3-form stabilised by $PSU(3)$ and divided by its norm induces an isometry $\Delta_+ \to \Delta_-$. The triality principle permutes any two of the representations Λ^1, Δ_+ and Δ_- and since these spaces are isomorphic as Euclidean vector spaces, we conclude that we get two corresponding isometries in $\Lambda^3 \Delta_\pm = \ker \mu_\pm$ which consequently define supersymmetric maps of spin 3/2.

Alternatively, we can establish the existence of such an isomorphism by a weight argument as in Proposition 1.4 or [29]. We restrict the $Spin(8)$-representations Λ^1, Δ_+ and Δ_- to the embedding of $PSU(3)$ which acts on Λ^1 through the adjoint representation of $SU(3)$ with weights

$$0,\ \pm(2\sigma_1 - \sigma_2),\ \pm(-\sigma_1 + 2\sigma_2),\ \pm(\sigma_1 + \sigma_2).$$

Hence, restricting the weights (1.18) to $PSU(3)$ we yield

$$\omega_1 = 0,\ \omega_2 = 2\sigma_1 - \sigma_2,\ \omega_3 = \sigma_1 + \sigma_2,\ \omega_4 = 0.$$

Substituting this into (1.19) and (1.20) we see that $PSU(3)$ acts with equal weights on Λ^1, Δ_+, Δ_-. We showed the

Proposition 1.10. [29] *Restricted to $PSU(3)$, these three representation spaces are equivalent*

$$\Lambda^1 = \Delta_+ = \Delta_- = \mathfrak{su}(3).$$

Remark: In particular Clifford multiplication $\mu : \Lambda^1 \otimes \Delta_\pm \to \Delta_\mp$ induces an orthogonal product

$$\times : \Lambda^1 \otimes \Delta_+ \cong \Lambda^1 \otimes \Lambda^1 \to \Delta_- \cong \Lambda^1$$

in $\mathfrak{su}(3)$. If we think of $\mathfrak{su}(3)$ as the algebra of skew hermitian 3×3 matrices, this product is explicitly given by

$$A \times B = \omega AB - \overline{\omega} BA - \frac{i}{\sqrt{3}} tr(AB) I, \tag{1.27}$$

where $\omega = (1 + i\sqrt{3})/2$ [29].

We conclude that vectors and spinors are essentially the same objects when regarded as being acted on by $PSU(3)$. If we denote the isomorphisms by $\gamma_\pm : \Lambda^1 \to \Delta_\pm$, then these are characterised (up to a scalar) by the equations

$$x \lrcorner \rho(\gamma_\pm) \;=\; \frac{1}{2}\kappa(x \lrcorner \rho) \cdot \gamma_\pm - \gamma_\pm \circ x \lrcorner \rho \;=\; 0$$

valid for all $x \in \Lambda^1$, i.e. $\mathfrak{su}(3)$ acts trivially on $\gamma_\pm$. Their matrices with respect to a

$PSU(3)$-frame and a fixed orthonormal basis of $\Delta_\pm$ are given by

$$\gamma_+ = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}$$

and

$$\gamma_- = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

To what extent we can characterise $PSU(3)$-structures through the existence of a supersymmetric map of spin 3/2 will be the topic of the following section.

We close our discussion of the group $PSU(3)$ with a remark about special $PSU(3)$-orbits in the Grassmannians $\tilde{G}_3(\Lambda^1)$ and $\tilde{G}_5(\Lambda^1)$, the spaces of oriented 3- and 5-dimensional subspaces of Λ^1. This links into another interesting feature of exceptional geometry, namely the existence of special submanifolds.

In general, let (V, g) be an oriented vector space with a metric g and $\tau \in \Lambda^p$ a p-form. We say that τ is a *calibration* [26] if for every oriented k-plane $\xi = e_1 \wedge \ldots \wedge e_k$ in Λ^1 the inequality

$$\tau(e_1, \ldots, e_k) \leq 1$$

holds. In other words any oriented k-plane $i : E \hookrightarrow \Lambda^1$ satisfies $i_E^* \tau \leq vol_E$, where vol_E is the volume element in $\Lambda^p E^*$ induced by the metric g restricted to E. E is *calibrated* by τ if equality actually holds. The *contact set* of a calibration is the set of calibrated planes.

A classical example are the so-called associative and co-associative planes in $\mathbb{R}^7$ which are calibrated by the G_2-invariant form φ and $\star\varphi$. The contact set is parametrised by $G_2/SO(4)$.

Let ρ be a stable form associated with $PSU(3)$.

Proposition 1.11. *We have $\rho(\xi) \leq 1$ with equality if and only if $\xi = Ad(g)\mathfrak{h}$ for $g \in SU(3)$ where $\mathfrak{h}$ is an $\mathfrak{su}(2)$-subalgebra coming from a highest root. Furthermore, $\star\rho(\xi) \leq 1$ with equality if and only if ξ is perpendicular to a 3-plane calibrated by ρ. In particular, $PSU(3)$ acts transitively on either contact set.*

Proof: We adapt the proof from [45]. Let $e_1, \ldots, e_8$ be a $PSU(3)$-frame inducing the Euclidean norm $\|\cdot\|$ and fix the Cartan subalgebra $\mathfrak{t}$ spanned by e_3 and e_8. Define

$$\begin{aligned}
\alpha_1 = \alpha = \tfrac{1}{2}e^3 + \tfrac{\sqrt{3}}{2}e^8, &\quad E_{\alpha_1} = e_5, &\quad F_{\alpha_1} = -e_4\\
\alpha_2 = \beta = \tfrac{1}{2}e^3 - \tfrac{\sqrt{3}}{2}e^8, &\quad E_{\alpha_2} = -e_6, &\quad F_{\alpha_2} = e_7\\
\alpha_3 = \alpha + \beta = e^3, &\quad E_{\alpha_3} = e_1, &\quad F_{\alpha_3} = e_2.
\end{aligned}$$

Then $\|\alpha_i\| \leq 1$ and we immediately verify the relations

$$[H, E_{\alpha_i}] = \alpha_i(H)F_{\alpha_i}, \quad [H, F_{\alpha_i}] = -\alpha_i(H)E_{\alpha_i} \quad \text{and} \quad [E_{\alpha_i}, F_{\alpha_i}] = \alpha_i, \tag{1.28}$$

for $H \in \mathfrak{t}$ and $i = 1, 2, 3$. Let $\xi \in \tilde{G}(3, \mathfrak{su}(3))$. Since $\mathfrak{t}$ is a Cartan subalgebra, $Ad(SU(3))A \cap \mathfrak{t} \neq \emptyset$ for any $0 \neq A \in \mathfrak{su}(3)$. Moreover, ρ is Ad-invariant, so we may assume that $\xi \cap \mathfrak{t} \neq \emptyset$ up to the action of a $g \in SU(3)$. Pick $T \in \xi \cap \mathfrak{t}$ and extend it to a positively oriented basis $\{T, X, Y\}$ of ξ. Then

$$X = T_0 + \sum_{i=1}^{3} s_i E_{\alpha_i} + \sum_{i=3}^{3} t_i F_{\alpha_i},$$

where $T_0 \in \mathfrak{t}$. Hence,

$$\|X\|^2 = \|T_0\|^2 + \sum_{i=1}^{3} s_i^2 + t_i^2$$

which implies

$$\begin{aligned} \|[T, X]\|^2 &= \|\sum_{i=1}^{3} s_i \alpha_i(T) F_{\alpha_i} - t_i \alpha_i(T) E_{\alpha_i}\|^2 \\ &\leq \sum_{i=1}^{3} \|\alpha_i\|^2 \|T\|^2 \, (s_i^2 + t_i^2) \\ &\leq \|T\|^2 \|X\|^2, \end{aligned}$$

and finally

$$\|[T, X]\| \leq \|T\| \|X\| \; .$$

The result follows now by applying the Cauchy-Schwarz inequality

$$|\rho(T, X, Y)| = \|g([T, X], Y)\| \leq \|[T, X]\| \|Y\| \leq \|T\| \|X\| \|Y\| = 1.$$

Furthermore, equality holds if and only if (i) $T \in \mathbb{R}\alpha_i$ and $X \in \mathbb{R}E_{\alpha_i} \oplus \mathbb{R}F_{\alpha_i}$ for an $i \in \{1, 2, 3\}$, (ii) Y is a multiple of $[T, X]$ and (iii) $\rho(T, X, Y) > 0$. Then $Y \in \mathbb{R}E_{\alpha_i} \oplus \mathbb{R}F_{\alpha_i}$ and because of (1.28), ξ is an $\mathfrak{su}(2)$-algebra.

Since $(\star\rho)_{|\xi^\perp} = \star(\rho_{|\xi})$ any calibrated 5-plane is the orthogonal complement of an $\mathfrak{su}(2)$ algebra. Moreover, any two subalgebras of highest root are conjugate. ■

Remark:

By a similar argument, Tasaki showed in [45] that for any simple Lie group G with Killing form B, the 3-form

$$\rho(X,Y,Z) = \frac{1}{\|\delta\|} B(X,[Y,Z])$$

(where δ is the highest root of the Lie algebra) defines a calibration on G and that any calibrated submanifold is a translate of $SU(2)$.

1.3.2 Supersymmetric maps in dimension 8

In dimension 8 triality implies $\ker \mu_\pm = \Lambda^3\Delta_\pm$ so that

$$\begin{aligned}\Lambda^1 \otimes \Delta_+ &= \Delta_- \oplus \Lambda^3\Delta_- \\ \Lambda^1 \otimes \Delta_- &= \Delta_+ \oplus \Lambda^3\Delta_+.\end{aligned}$$

We continue in the vein of Section 1.2 and ask what kind of geometry is induced by a supersymmetric map $\gamma_\pm : \Lambda^1 \to \Delta_\pm$ of spin 3/2, that is $\gamma_\pm \in \Lambda^3\Delta_\mp$. Since this takes only the metric structure into account, by permuting the representations Λ^1, Δ_+ and Δ_- with the triality automorphisms λ and κ (see Page 25), we can rephrase our question as follows. What are the stabilisers of a 3-form that induces an isometry $\Delta_+ \to \Delta_-$?

Remark: Note that there is clearly a choice in this identification of $\Delta_+ \otimes \Delta_-$ with $\Lambda^1 \oplus \Lambda^3$ since the Hodge $\star$-operator induces an isomorphism between Λ^p and Λ^{8-p}.

In what follows, it will be convenient to consider the full module of spinors $\Delta = \Delta_+ \oplus \Delta_-$ endowed with its spin invariant metric $q = q_+ \oplus q_-$, and to think of an odd form as a linear map $\Delta \to \Delta$ whose matrix with respect to this splitting is given by

$$\begin{pmatrix} 0 & A_\rho \\ B_\rho & 0 \end{pmatrix}. \tag{1.29}$$

In the case of ρ being a 3-form, the matrix has to be symmetric as Clifford multiplication is skew for q, hence $B_\rho = A_\rho^{tr}$. Let $\mathfrak{I}_g$ denote the set of 3-forms in $\Delta \otimes \Delta$ that induce an isometry. It is characterised by the next theorem.

Theorem 1.12. *If a 3-form ρ lies in $\mathfrak{I}_g$, then ρ is of unit length and there exists a Lie bracket $[\cdot\,,\cdot]$ on Λ^1 such that*

$$\rho(x, y, z) = g([x, y], z). \tag{1.30}$$

In particular, the adjoint group of the resulting Lie algebra acts as a group of isometries on Λ^1.

Conversely, if there exists a Lie algebra structure on Λ^1 whose adjoint group leaves g invariant, then the 3-form defined by (1.30) and divided by its norm lies in $\mathfrak{I}_g$.

Proof: Inducing an isometry and defining a Lie bracket through (1.30) are both quadratic conditions on the coefficients of ρ which we show to coincide.

To this effect we define the linear map

$$\mathrm{Jac} : \Lambda^3 \otimes \Lambda^3 \to \Lambda^4$$

by skew-symmetrising the contraction to $\Lambda^2 \otimes \Lambda^2$. This is clearly $SO(8)$-equivariant and written in indices with respect to some orthonormal basis $\{e_i\}$, we have

$$\begin{aligned}\mathrm{Jac}(c_{ijk}d_{lmk}) &= c_{[ij}{}^{k}d_{lm]k}\\ &= c_{ij}{}^{k}d_{lmk} + c_{il}{}^{k}d_{mjk} + c_{im}{}^{k}d_{jlk} + c_{jl}{}^{k}d_{imk} + c_{jm}{}^{k}d_{lik} + c_{lm}{}^{k}d_{ijk}\end{aligned}$$

and in particular

$$\mathrm{Jac}(c_{ijk}c_{lmk}) = 2(c_{ij}{}^{k}c_{klm} + c_{li}{}^{k}c_{kjm} + c_{jl}{}^{k}c_{kim}). \tag{1.31}$$

If we consider the skew-symmetric map $[\,\cdot\,,\cdot\,] : \Lambda^2 \to \Lambda^1$ defined through $\rho = \sum_{i<j<k} c_{ijk}e_{ijk}$ by (1.30), then

$$[e_i, e_j] = c_{ij}{}^{k}e_k$$

and (1.31) entails that this defines a Lie bracket if and only if

$$\mathrm{Jac}(\rho \otimes \rho) = 0.$$

Next we analyse the conditions for ρ to induce an isometry. Represent the map $\rho : \Delta \to \Delta$ by a mtarix A_ρ as in (1.29). Then ρ defines an isometry if and only if

$$\begin{aligned}q(\psi,\phi) &= q(\rho\cdot\psi, \rho\cdot\phi)\\ &= q(\rho\cdot\rho\cdot\psi, \phi)\\ &= q_+(A_\rho A_\rho^{tr}\psi_+, \phi_+) + q_-(A_\rho^{tr}A_\rho\psi_-, \phi_-)\\ &= q_+(\psi_+, \phi_+) + q_-(\psi_-, \phi_-).\end{aligned} \tag{1.32}$$

This motivates considering the $Spin(8)$-equivariant linear maps

$$\Gamma_\pm : \rho\otimes\tau \in \Lambda^3\otimes\Lambda^3 \mapsto pr_{\Delta_\pm}(\rho\cdot\tau) \in \Delta_\pm\otimes\Delta_\pm.$$

The condition (1.32) then reads

$$\rho \in \mathfrak{I}_g \text{ if and only if } \Gamma_\pm(\rho\otimes\rho) = \mathrm{Id}_{\Delta_\pm}.$$

If we decompose both the domain and the target space into irreducible components using the algorithm in [39], we find

$$\Lambda^3 \otimes \Lambda^3 \cong \mathbf{1} \oplus 2\Lambda^2 \oplus \Lambda^4_+ \oplus \Lambda^4_- \oplus [0,0,2,2] \oplus [0,1,0,2] \oplus [0,1,2,0] \oplus \oplus [0,2,0,0] \oplus [1,0,1,1] \oplus [2,0,0,0]$$

and

$$\Delta_+ \otimes \Delta_+ = \Lambda^2\Delta_+ \oplus \odot^2\Delta_+ \cong \Lambda^2 \oplus \mathbf{1} \oplus \Lambda^4_+.$$

Note that we labeled some irreducible representations by their highest weight expressed in the basis of fundamental weights of $\mathfrak{so}(8)$ as we did earlier in this section. The modules $\Lambda^4_+ = [0,0,2,0]$ and $\Lambda^4_- = [0,0,0,2]$ are the spaces of self-dual and anti-self-dual 4-forms respectively. Note that

$$\Gamma_+(\rho \otimes \tau)^{tr} = \Gamma_-(\tau \otimes \rho)$$

and consequently, it will suffice to consider the map Γ_+ only. Since the map induced by ρ is symmetric, $\Gamma_+(\rho \otimes \rho) \in \odot^2\Delta_+ = \mathbf{1} \oplus \wedge^4_+$ follows. Moreover Γ_+ maps Λ^4_+ non-trivially into $\Delta_+ \otimes \Delta_+$. To see this, complexify the representations and consider the root vectors x_α, x_β, x_γ and x_δ of $\mathfrak{so}(8) \otimes \mathbb{C}$. Then

$$x_\alpha \wedge x_\beta \wedge x_\gamma \otimes x_\gamma \wedge x_\beta \wedge x_\delta \in \Lambda^4_+ = V(\alpha + 2\beta + 2\gamma + \delta) \leqslant \Lambda^3 \otimes \Lambda^3,$$

where we labeled Λ^4 by its highest weight expressed this time in terms of the fundamental roots α, β, γ, and δ. This vector gets mapped to

$$\Gamma_+(x_\alpha \wedge x_\beta \wedge x_\gamma \otimes x_\gamma \wedge x_\beta \wedge x_\delta) = \|x_\beta\|^2 \|x_\gamma\|^2 \, x_\alpha \cdot x_\delta.$$

Hence a necessary condition for ρ to lie in $\mathfrak{I}_g$ is

$$\Gamma_+(\rho \otimes \rho)_{\odot^2_0 \Delta_+} = 0.$$

This obstruction in Λ^4_+ can be identified with the obstruction $\mathrm{Jac}(\rho\otimes\rho)$ for ρ to induce an adapted Lie algebra as we shall show next. In fact, we claim that

$$\Gamma_+(\rho\otimes\rho) = -\frac{1}{2}\,\mathrm{Jac}(\rho\otimes\rho)_{\Lambda^4_+} + \|\rho\|^2\,\mathrm{Id}\,. \tag{1.33}$$

This yields directly the assertion of the theorem.

In order to derive (1.33), we first note that Clifford multiplication induces a map

$$\rho\otimes\tau\in\Lambda^3\otimes\Lambda^3\mapsto\rho\cdot\tau\in\Lambda^0\oplus\Lambda^2\oplus\Lambda^4\oplus\Lambda^6$$

if we regard the product $\rho\cdot\tau$ as an element of $\mathit{Cliff}(\Lambda^1,g)=\Lambda^*$ rather than as a map $\Delta_+\otimes\Delta_+\oplus\Delta_-\otimes\Delta_-$. The various components of $\rho\cdot\tau$ under this identification come from the number of "coinciding pairs" in the expression e_{ijklmn}. This means that if we have, say, three coinciding pairs, then $i=l$, $j=m$ and $k=n$, hence $e_{ijklmn}=1$. The 2-form component is obtained from two such pairs etc. (e.g. $e_{ijklmn}=e_{im}$ if, say, $j=l$ and $k=n$). Since the map induced by ρ is symmetric, $\Gamma_+(\rho\otimes\rho)$ takes values in the components $\bigodot^2\Delta_+=\Lambda^0\oplus\Lambda^4_+$ only. Hence

$$\rho=\sum_{i<j<k}c_{ijk}e_{ijk}$$

gets mapped to

$$\begin{aligned}\rho\cdot\rho &= \sum_{i<j<k,\,l<m<n} c_{ijk}c_{lmn}e_{lmnijk}\\ &= \sum_{\substack{i<j<k,\,l<m<n\\ \text{3 pairs coincide}}} c_{ijk}c_{lmn}e_{lmnijk} + \sum_{\substack{i<j<k,\,l<m<n\\ \text{1 pair coincides}}} c_{ijk}c_{lmn}e_{lmnijk}.\end{aligned}$$

Now the first sum is just

$$\sum_{i<j<k}c_{ijk}^2\mathbf{1}=\|\rho\|^2\,\mathbf{1}$$

which leaves us with the contribution of the sum with one pair of equal indices. No matter which indices of the two triples $\{i < j < k\}$ and $\{l < m < n\}$ coincide, the skew-symmetry of the c_{ijk} and e_{ijk} allows us to rearrange and rename the indices in such a way that

$$\begin{aligned}
\sum_{\substack{i<j<k,\, l<m<n \\ 1 \text{ pair coincides}}} c_{ijk} c_{lmn} e_{lmnijk} &= \sum_a \sum_{\substack{j<k,\, m<n \\ j,k,m,n \text{ distinct}}} c_{ajk} c_{amn} e_{amnajk} \\
&= -\sum_a \sum_{\substack{j<k,\, m<n \\ j,k,m,n \text{ distinct}}} c_{ajk} c_{amn} e_{mnjk} \\
&= -\frac{1}{2}\,\mathrm{Jac}(\rho \otimes \rho).
\end{aligned}$$

whence (1.33).

Finally note that the metric g is necessarily *ad*-invariant because of the skew-symmetry of ρ. ■

The 3-forms in $\mathfrak{I}_g$ thus encapsulate the data of a Lie algebra structure whose adjoint action preserves the metric on Λ^1. We also say that the Lie structure is *adapted* to the metric g. We shall write $\mathfrak{k}$ if we think of Λ^1 as a Lie algebra. The classification of the resulting structures shall occupy us next.

According to the theorem of Levi-Malcev, we can decompose $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{r}$ into a semi-simple sub-algebra $\mathfrak{m}$ and the so-called *radical* $\mathfrak{r}$, i.e. the maximal solvable ideal of $\mathfrak{k}$. Since a semi-simple Lie algebra is the direct sum of simple ones (which are completely classified), we are left with the solvable part in order to determine the Lie algebra structure of $\mathfrak{k}$ completely. By restricting the metric to $\mathfrak{r}$, we see that $\mathfrak{r}$ is adapted to g.

Proposition 1.13. *Let $\mathfrak{s}$ be a solvable Lie algebra which is adapted to some metric g. Then $\mathfrak{s}$ is abelian.*

Proof: We shall proceed by induction over n, the dimension of $\mathfrak{s}$. If $n = 1$, then $\mathfrak{s}$ is abelian and the assertion is trivial. Now assume that the assumption holds for all $1 \leq m < n$. Let $\mathfrak{a}$ be an *abelian ideal* of $\mathfrak{s}$. Then, for all $A \in \mathfrak{a}$ and X, $Y \in \mathfrak{s}$, *ad*-invariance of g implies

$$g(X, [Y, A]) = g(Y, [A, X]) = 0,$$

for if $Y \in \mathfrak{a}$, then $[Y, A] = 0$ and if $Y \in \mathfrak{a}^{\perp}$, then $[A, X] \in \mathfrak{a} \perp Y$. Hence $\mathfrak{a} \subset \mathfrak{z}$, the center of $\mathfrak{s}$. If $\mathfrak{z}$ were to be trivial, then any abelian ideal of $\mathfrak{s}$ would also be trivial which is equivalent to saying that $\mathfrak{s}$ is semi-simple, contradicting our assumption. Hence we can write $\mathfrak{s} = \mathfrak{z} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is an orthogonal complement to $\mathfrak{z}$. Now for all $X \in \mathfrak{s}$, $Z \in \mathfrak{z}$ and $H \in \mathfrak{h}$ we have

$$g(Z, [X, H]) = g(H, [Z, X]) = 0$$

so that $[X, H] \in \mathfrak{z}^{\perp} = \mathfrak{h}$. Equivalently, $\mathfrak{h}$ is an ideal of $\mathfrak{s}$. As such, it is adapted and solvable since $\mathfrak{s}$ is adapted and solvable. Since the dimension of $\mathfrak{h}$ is strictly less than n, our induction hypothesis applies and we deduce that $\mathfrak{s}$ is abelian. ■

Corollary 1.14. *An adapted Lie algebra $\mathfrak{k}$ is reductive.*

Proof: We need to show that the center $\mathfrak{z} \subset \mathfrak{r}$ actually equals the radical of $\mathfrak{k}$. But for any $X \in \mathfrak{k}$ and R_1, $R_2 \in \mathfrak{r}$, the previous proposition implies that

$$g(R_1, [X, R_2]) = g(X, [R_2, R_1]) = 0.$$

Hence $[X, R_2] \in \mathfrak{r} \cap \mathfrak{r}^{\perp} = \{0\}$ for any $X \in \mathfrak{k}$ and consequently, $\mathfrak{r} = \mathfrak{z}$. ■

As a result, any Lie algebra which is adapted to g is of the form $\mathfrak{m} \oplus \mathfrak{z}$ with $\mathfrak{m}$ semi-simple. Cartan's classification of simple Lie algebras then asserts that Λ^1 is isomorphic to one of the following Lie algebras.

1. $\mathfrak{k}_1 = \mathfrak{su}(3)$
2. $\mathfrak{k}_2 = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus E_2$
3. $\mathfrak{k}_3 = \mathfrak{su}(2) \oplus E_5$.

Here, E_5 denotes the abelian Lie algebra $\mathbb{R}^5$ which we think of as a Euclidean vector space. We refer to a 3-form in $\mathfrak{I}_g$ to be of *type* 1, 2 or 3 according to the Lie algebra it induces. Hence there is a disjoint decomposition of $\mathfrak{I}_g$ into the sets $\mathfrak{I}_{g1}$, $\mathfrak{I}_{g2}$ and $\mathfrak{I}_{g3}$ acted on by $SO(8)$ and pooling the forms of type 1, 2 or 3 respectively. Since the automorphism groups $Aut(\mathfrak{k}_i)$ of $\mathfrak{k}_1$, $\mathfrak{k}_2$ and $\mathfrak{k}_3$ are $PSU(3) \times \mathbb{Z}_2$, $SO(3) \times GL(5)$ and $SO(3) \times SO(3) \times GL(2)$ (recall that $Aut(\mathfrak{su}(2)) = SU(2)/Z_2 = SO(3)$), the stabiliser of ρ in (1.30) inside $SO(8)$ is $PSU(3)$, $SO(3) \times SO(5)$ and $SO(3) \times SO(3) \times SO(2)$ respectively. On the other hand, there is a natural decomposition of $\mathfrak{I}_g$ into the subsets $\mathfrak{I}_{g\pm}$ of forms whose induced map $A_\rho : \Delta_- \to \Delta_+$ has positive or negative determinant. The orbit structure of the $SO(8)$-action on $\mathfrak{I}_g$ and $\mathfrak{I}_{g\pm}$ is given by the next proposition.

Proposition 1.15. *The sets $\mathfrak{I}_{g1}$, $\mathfrak{I}_{g2}$ and $\mathfrak{I}_{g3}$ can be described as follows.*

1. $\mathfrak{I}_{g1} = SO(8)/PSU(3)$
2. $\mathfrak{I}_{g2} = (S^1 - \{pt\}) \times SO(8)/(SO(3) \times SO(3) \times SO(2))$
3. $\mathfrak{I}_{g3} = SO(8)/(SO(3) \times SO(5))$

Furthermore,

$$\mathfrak{I}_{g-} = SO(8)/PSU(3)$$

and $\mathfrak{I}_{g+}$ *foliates over the circle* S^1 *with principal orbits* $SO(8)/(SO(3) \times SO(3) \times SO(2))$ *over* $S^1 - \{pt\}$ *and a degenerate orbit* $SO(8)/(SO(3) \times SO(5))$ *at* $\{pt\}$*.*

Proof: First we note that since the 3-forms under consideration are of the form $g(\cdot, [\cdot\,,\cdot])$, these forms are built out of the structure constants of the corresponding Lie algebra. To begin with, let us show that $\mathfrak{I}_{g1} = SO(8)/PSU(3)$ and that this is contained in $\mathfrak{I}_{g-}$. We will choose a certain $\rho_1 \in \mathfrak{I}_{g1}$ whose orbit under $SO(8)$ is the whole set $\mathfrak{I}_{g1}$. Fix an orthonormal basis $\{e_i\}$ and define

$$\rho_1 = \frac{1}{2}e_{123} + \frac{1}{4}e_{147} - \frac{1}{4}e_{156} + \frac{1}{4}e_{246} + \frac{1}{4}e_{257} + \frac{1}{4}e_{345} - \frac{1}{4}e_{367} + \frac{\sqrt{3}}{4}e_{458} + \frac{\sqrt{3}}{4}e_{678}. \quad (1.34)$$

The form ρ_1 is of unit length and its coefficients are $1/2$ of the structure constants c_{ijk} of $\mathfrak{su}(3)$ as given in Section 1.3.1. Hence $\rho_1 \in \mathfrak{I}_{g1}$ and its stabiliser is $PSU(3)$. To establish transitivity, take any $\rho \in \mathfrak{I}_{g1}$ and endow Λ^1 with the induced $\mathfrak{su}(3)$ structure. We have to find a transformation in $SO(8)$ which puts ρ into the normal form of (1.34). Since g is ad-invariant and positive definite, there exist a strictly negative real number λ such that

$$\lambda g = B_{\mathfrak{su}(3)},$$

where $B_{\mathfrak{k}}(x, y) = \mathrm{Tr}(ad_x \circ ad_y)$ designates the Killing form of the Lie algebra $\mathfrak{k}$. Let $\tilde{f}_1, \ldots, \tilde{f}_8$ be a basis for $\mathfrak{su}(3)$ such that $[\tilde{f}_i, \tilde{f}_j] = c_{ijk}\tilde{f}_k$. Then

$$\lambda g(\tilde{f}_i, \tilde{f}_j) = B_{\mathfrak{su}(3)}(\tilde{f}_i, \tilde{f}_j) = -3\delta_{ij}.$$

Consequently, the basis $f_i = \sqrt{\frac{|\lambda|}{3}}\tilde{f}_i$ is orthonormal and has structure constants $c_{ijk}/\sqrt{3|\lambda|}$. Hence we have shown the existence of an orthonormal basis $f_1, \ldots, f_8$ of g such that

$$\rho(f_i, f_j, f_k) = g([f_i, f_j], f_k) = -\frac{1}{3}\sqrt{\frac{|\lambda|}{3}}B_{\mathfrak{su}(3)}([\tilde{f}_i, \tilde{f}_j], \tilde{f}_k) = \sqrt{\frac{|\lambda|}{3}}c_{ijk}$$

whence

$$\rho = \sqrt{\frac{|\lambda|}{3}} \sum_{i<j<k} c_{ijk} f_{ijk}.$$

Since ρ is of unit norm we have $\lambda = -3/4$. Mapping the basis e_i into f_i yields the desired transformation in $SO(8)$. Next we compute the determinant of A_{ρ_1} by identifying e_i with the octonions as in (1.6) and fixing the representation (1.8). This gives

$$A_{\rho_1} = \frac{1}{4} \begin{pmatrix} \sqrt{3} & 0 & 0 & 3 & -\sqrt{3} & 0 & 0 & 1 \\ 2 & -\sqrt{3} & -1 & 0 & 2 & -\sqrt{3} & -1 & 0 \\ 0 & 3 & -\sqrt{3} & 0 & 0 & -1 & -\sqrt{3} & 0 \\ -1 & 0 & 2 & \sqrt{3} & 1 & 0 & -2 & -\sqrt{3} \\ -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} & 0 & 0 & 3 \\ -2 & -\sqrt{3} & -1 & 0 & -2 & -\sqrt{3} & -1 & 0 \\ 0 & -1 & -\sqrt{3} & 0 & 0 & 3 & -\sqrt{3} & 0 \\ 1 & 0 & 2 & -\sqrt{3} & -1 & 0 & -2 & \sqrt{3} \end{pmatrix}, \qquad (1.35)$$

hence $\det(A_{\rho_0}) = -1$. Moreover, we have $\det \pi_\pm(a) = 1$ for any $a \in Spin(8)$. This follows from the fact that its generators $e_i \cdot e_j$ square to $-Id$, therefore they are of determinant 1. The $Spin(8)$-equivariance of the embedding

$$\rho \in \Lambda^3 \hookrightarrow \begin{pmatrix} 0 & A_\rho \\ A_\rho^{\mathrm{tr}} & 0 \end{pmatrix} \in \Delta \otimes \Delta$$

entails

$$A_{\pi_0(a)^* \rho_1} = \pi_+(a) \circ A_{\rho_1} \circ \pi_-(a)^{-1},$$

whence $\mathfrak{I}_1 \subset \mathfrak{I}_-$.

Next we turn to structures of type 2 and 3 which are characterised by a non-trivial center. We shall see in a moment that these two cases are intimately related as type 3 can be conceived as a limit case of type 2. First we assume that Λ^1 carries a Lie algebra structure isomorphic to $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus E_2$ where for the sake of clarity we labeled the two subalgebras isomorphic to $\mathfrak{su}(2)$ by the subscripts 1 and 2. Hence there exist strictly-negative constants λ_1 and λ_2 such that

$$B_{\mathfrak{k}|\mathfrak{su}(2)_1} = \lambda_1 g, \quad B_{\mathfrak{k}|\mathfrak{su}(2)_2} = \lambda_2 g.$$

Since $\mathfrak{su}(2)_1$ and $\mathfrak{su}(2)_2$ are ideals of $\mathfrak{k}$, the restriction of $B_{\mathfrak{k}}$ to these is just the Killing form of $B_{\mathfrak{su}(2)}$. We claim that the set $\{\lambda_1, \lambda_2\}$ is invariant under the action of $SO(8)$. More concretely, if we transform ρ by $A \in SO(8)$, then $A^*\rho$ gives rise to a new Lie bracket $[\cdot\,,\cdot]_A$ which is just

$$[v, w]_A = A[A^{-1}v, A^{-1}w].$$

We write $A^*\mathfrak{k}$ for the resulting Lie algebra which has the same type as $\mathfrak{k}$. We obtain again a set $\{\lambda_1^A, \lambda_2^A\}$ of negative numbers which we claim to be $\{\lambda_1, \lambda_2\}$. To show this we pick two non-zero vectors v_1 and v_2 in $\mathfrak{su}(2)_1$ and $\mathfrak{su}(2)_2$ respectively so that

$$\lambda_1 = \frac{B_{\mathfrak{su}(2)_1}(v_1, v_1)}{g(v_1, v_1)}, \quad \lambda_2 = \frac{B_{\mathfrak{su}(2)_2}(v_2, v_2)}{g(v_2, v_2)}.$$

The Killing form of $A^*\mathfrak{k}$ can be evaluated by the formula

$$B_{A^*\mathfrak{k}}(Av, Av) = B_{\mathfrak{k}}(v, v),$$

since for an orthonormal basis $\{e_i\}$ of Λ^1 we have

$$\begin{aligned} B_{A^*\mathfrak{k}}(Av, Av) &= \mathrm{Tr}(\mathrm{ad}_{Av}^{A^*\mathfrak{k}} \circ \mathrm{ad}_{Av}^{A^*\mathfrak{k}}) \\ &= \sum_i g([Av, [Av, e_i]_A]_A, e_i) \\ &= \sum_i g(A[v, [v, A^{-1}e_i]], e_i) \\ &= \mathrm{Tr}(\mathrm{ad}_v^{\mathfrak{k}} \circ \mathrm{ad}_v^{\mathfrak{k}}) \\ &= B_{\mathfrak{k}}(v, v). \end{aligned}$$

Hence the set $\{\lambda_1^A, \lambda_2^A\}$ of the transformed Lie algebra $A^*\mathfrak{k}$ is just

$$\lambda_1^A = \frac{B_{A^*\mathfrak{su}(2)_1}(Av, Av)}{g(Av, Av)} = \frac{B_{\mathfrak{su}(2)}(v, v)}{g(v, v)} = \lambda_1$$

and

$$\lambda_2^A = \frac{B_{A^*\mathfrak{su}(2)_2}(Av, Av)}{g(Av, Av)} = \frac{B_{\mathfrak{su}(2)^\perp}(v, v)}{g(v, v)} = \lambda_2$$

which proves our claim.

For $\lambda_1 + \lambda_2 = -2$, an example of a form in $\mathfrak{I}_{g2}$ is given by

$$\rho_2 = \sqrt{\frac{|\lambda_1|}{2}} e_{123} + \sqrt{\frac{|\lambda_2|}{2}} e_{456}.$$

Now pick a $\rho \in \mathfrak{I}_{g2}$. In each ideal $\mathfrak{su}(2)_i$ we can find a basis $\tilde{f}_{i_1}, \tilde{f}_{i_2}, \tilde{f}_{i_3}$ such that

$$[\tilde{f}_{i_l}, \tilde{f}_{i_m}] = \epsilon_{lmn} \tilde{f}_{i_n}$$

(where ϵ_{ijk} is totally skew in its indices with $\epsilon_{123} = 1$) and

$$B_{\mathfrak{su}(2)}(\tilde{f}_{i_l}, \tilde{f}_{i_m}) = -2\delta_{lm}.$$

Hence for $i = 1$, 2 the basis $f_{i_l} := \sqrt{\frac{|\lambda_i|}{2}} \tilde{e}_{i_l}$ is orthonormal for g which we extend to a basis of Λ^1. The only non-trivial coefficients of ρ are

$$\rho(f_{i_1}, f_{i_2}, f_{i_3}) = g([f_{i_1}, f_{i_2}], f_{i_3}) = -\frac{1}{2}\sqrt{\frac{|\lambda_i|}{2}} B([\tilde{f}_{i_1}, \tilde{f}_{i_2}], \tilde{f}_{i_3}) = \sqrt{\frac{|\lambda_i|}{2}}.$$

Any mixed term like $\rho(f_{1_1}, f_{1_2}, f_{2_1})$ etc. has to vanish since the two sub-algebras $\mathfrak{su}(2)$ are ideals in $\mathfrak{k}_1$. Moreover, ρ is of norm 1 which implies $\lambda_1 + \lambda_2 = -2$ so that $\lambda_2 = -2 - \lambda_1 < 0$ for $-2 < \lambda_1 < 0$. If ρ induces the same pair $\{\lambda_1, \lambda_2\}$, then

$$\rho = \sqrt{\frac{|\lambda_1|}{2}} f_{1_1 1_2 1_3} + \sqrt{\frac{|\lambda_2|}{2}} f_{2_1 2_2 2_3} = A^*\rho_2$$

for the transformation $A \in SO(8)$ which maps e_i to f_i (after choosing a suitable labeling for the basis $\{f_i\}$).

The matrix of ρ_2 is given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \sqrt{2+\lambda_1} & \sqrt{-\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{-\lambda_1} & \sqrt{2+\lambda_1} & 0 & 0 & 0 & 0 \\ \sqrt{2+\lambda_1} & \sqrt{-\lambda_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{-\lambda_1} & \sqrt{2+\lambda_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2+\lambda_1} & \sqrt{-\lambda_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-\lambda_1} & \sqrt{2+\lambda_1} \\ 0 & 0 & 0 & 0 & \sqrt{2+\lambda_1} & \sqrt{-\lambda_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{-\lambda_1} & \sqrt{2+\lambda_1} & 0 & 0 \end{pmatrix} \tag{1.36}$$

and therefore $\det(A_{\rho_2}) = 1$, that is $\mathfrak{I}_{g2} \subset \mathfrak{I}_{g+}$.

The limit case $\{-2, 0\}$ corresponds precisely to structures of type 3. Any form in $\mathfrak{I}_{g3}$ can be written as $\rho_3 = e_{123}$ for a suitably chosen orthonormal basis. Its associated matrix is

$$A_{\rho_3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

whose determinant equals 1, hence $\mathfrak{I}_{g3} \subset \mathfrak{I}_{g+}$.

It follows that after rescaling, $\mathfrak{I}_{g2} = (S^1 - \{pt\}) \times SO(8)/(SO(3) \times SO(3) \times SO(2))$, and $\mathfrak{I}_{g+} = \mathfrak{I}_{g2} \cup \mathfrak{I}_{g3}$ foliates over the circle S^1 with principal orbits $SO(8)/(SO(3) \times SO(3) \times SO(2))$ over $S^1 - \{pt\}$ and a degenerate orbit $SO(8)/(SO(3) \times SO(5))$ at pt. ■

Since Theorem 1.12 only uses the Euclidean structure of Λ^1 which by triality coincides with that of Δ_+ and Δ_-, we derive the same conclusion for isometries in $\Lambda^1 \otimes \Delta_+$ and $\Lambda^1 \otimes \Delta_-$. Note however that the characterisation of $\mathfrak{I}_{g\pm}$ does depend on the module under consideration as the outer triality morphisms reverse the orientation. By abuse of language, any element sitting inside Λ^3, $\Lambda^3\Delta_+$ or $\Lambda^3\Delta_-$ will be referred to as a map of spin 3/2. If such an element defines an isometry, it corresponds to the reduction of the structure group $SO(\Lambda^1)$, $SO(\Delta_+)$ or $SO(\Delta_-)$ to the group $K_1 = PSU(3)$, $K_2 = SO(3) \times SO(3) \times SO(5)$ and $K_3 = SO(3) \times SO(5)$. We obtain then a subgroup $\widetilde{K}_i$ of $Spin(8)$ through the lift by π_0, π_+ and π_-. We will also refer to the sub-types i_0, i_+ and i_- to render the provenance of $\widetilde{K}_i$ explicit. Unlike for 3-forms of type 2 and 3, $\widetilde{K}_1$ is not connected. Table 1.2 displays all three possibilities where we have to divide by $\mathbb{Z}_2$ in case of $i = 2$ or 3. Note that $Sp(1) \times_{\mathbb{Z}_2} Sp(2)$ is usually denoted by $Sp(1)Sp(2)$ or $Sp(1) \cdot Sp(2)$, giving rise to what is called an *almost quaternionic* structure.

type i	1	2	3
K_i	$PSU(3)$	$SO(3) \times SO(3) \times SO(2)$	$SO(3) \times SO(5)$
$\widetilde{K}_i$	$PSU(3) \times \mathbb{Z}_2$	$SU(2) \cdot SU(2) \times U(1)$	$Sp(1) \cdot Sp(2)$

Table 1.2: The stabilisers and their covers in $Spin(8)$

1.3.3 The groups $SO(3)\times SO(3)\times SO(2)$ and $SO(3)\times SO(5)$

A group $\widetilde{K}_i$ coming, say, from an i_+-structure might intersect the kernel of π_0 and π_- trivially and therefore induce a geometry on Λ^1 and Δ_- (the two being identified through the isometry $\Lambda^1 \to \Delta_-$) which is different from that of Δ_+ (cf. Figure 1.1). In the case of $PSU(3)$, we have already seen that this is not the case as we get isometries which identify Λ^1, Δ_+ and Δ_- as $PSU(3)$-representations spaces.

Next, we want to carry out a similar analysis for structures of type 2 and 3. Again, we may restrict ourselves to structures of type 2_0 and 3_0 – the remainder is a matter of applying triality.

To this effect, we recall that $\mathfrak{su}(2)$ has one fundamental weight σ and that the irreducible representations are

$$[n] = \bigodot^{n} \mathbb{C}^2$$

of dimension $n+1$. In particular, we get the standard representation $\mathbb{C}^2$ for $n=1$ and the adjoint representation $\mathfrak{su}(2)$ for $n=2$. They are real for n even and quaternionic for n odd. Consequently, the irreducible representations of $\widetilde{K}_2$ can be labeled by $[n_1, n_2, m] = [n_1]\otimes[n_2]\otimes[m]$ where the first two factors correspond to the $SU(2)$-representations $n_1\sigma_1$ and $n_2\sigma_2$. The third factor is the irreducible S^1-representation $S_m : \theta(z) \mapsto e^{im\theta}\cdot z$ which is one dimensional and complex. According to the discussion in the previous section we obtain

$$\Lambda^1 = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus E_2 = [2,0,0]\oplus[0,2,0]\oplus[\![0,0,2]\!].$$

Hence, $\widetilde{K}_2$ acts with weights (cf. (1.18))

$$0,\ 2\sigma_l,\ 2\sigma_r,\ 2m.$$

As an $SO(8)$-space, $\Lambda^1 = [1,0,0,0]$ and the following substitution of weights

$$\omega_1 = 0,\ -\omega_1 + \omega_2 = 2m,\ -\omega_2 + \omega_3 + \omega_4 = 2\sigma_l,\ \omega_3 - \omega_4 = 2\sigma_r$$

yields

$$\omega_1 = 0,\ \omega_2 = 2m,\ \omega_3 = \sigma_l + \sigma_r + m,\ \omega_4 = \sigma_l - \sigma_r + m \tag{1.37}$$

for the restrictions of ω_i to $\tilde{G}_2$. Substituting (1.37) into (1.19) and (1.20) gives

$$\pm(-\sigma_l + \sigma_r - m),\ \pm(\sigma_l - \sigma_r - m),\ \pm(-\sigma_l - \sigma_r + m),\ \pm(\sigma_l + \sigma_r + m).$$

We conclude

$$\Delta_+ = \Delta_- = [\![1,1,1]\!].$$

In particular, the action of K_2 on $\Delta_\pm$ preserves an almost complex structure. Note however that this structure does not reduce to $SU(4)$ as the torus component acts non-trivially on $\lambda^{4,0}\Delta_\pm$.

Permuting with the triality automorphisms yields analogous results for structures of type 2_+ and 2_-. We summarise our results in the Table 1.3.

$\rho \in$	Λ^3	$\Lambda^3\Delta_+$	$\Lambda^3\Delta_-$
type	0	$-$	$+$
Λ^1	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus E_2$	$[\![\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S_1]\!]$	$[\![\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S_1]\!]$
Δ_+	$[\![\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S_1]\!]$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus E_2$	$[\![\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S_1]\!]$
Δ_-	$[\![\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S_1]\!]$	$[\![\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S_1]\!]$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus E_2$

Table 1.3: The action of $\widetilde{K}_2 = SU(2) \cdot SU(2) \times U(1)$ on Λ^1, Δ_+ and Δ_-

Next we turn to structures of type 3. Again we consider $\rho \in \Lambda^3$ and assume that as a $\widetilde{K}_3 = Sp(1) \cdot Sp(2)$ representation space, we have

$$\Lambda^1 = \mathfrak{su}(2) \oplus E_5 = [2,0,0] \oplus [0,1,0].$$

The first index n_1 refers to the $SU(2) = Sp(1)$-representation labeled by $n\sigma$, while the last two indices (m_1, m_2) designate the irreducible $Sp(2) = Spin(5)$-representation with respect to the basis of fundamental weights τ_1 and τ_2. The weights of the action on Λ^1 are

$$0,\ 2\sigma,\ \tau_1,\ \tau_1 - 2\tau_2.$$

Substituting

$$\omega_1 = 0,\ \omega_1 - \omega_2 = 2\sigma,\ \omega_2 - \omega_3 - \omega_4 = \tau_1,\ \omega_3 - \omega_4 = \tau_1 - 2\tau_2$$

we obtain as weights on Δ_+ and Δ_-

$$\pm(-\sigma - \tau_1 + \tau_2),\ \pm(\sigma - \tau_1 + \tau_2),\ \pm(\sigma - \tau_2),\ \pm(\sigma + \tau_2)$$

and thus

$$\Delta_+ = \Delta_- = [1,0,1] = [\mathbb{C}^2 \otimes \mathbb{H}^2].$$

Our conclusions are displayed in the Table 1.4. Choosing the orientation suitably, we see in particular that an almost quaternionic structure on Λ^1 is equivalent to a Lie algebra structure of type 3 on Δ_+.

1.4 Generalised metric structures

So far we have discussed "classical" supersymmetric structures. This means that we are given one supersymmetric map which forces the structure group to reduce. Equivalently,

$\rho \in$	Λ^3	$\Lambda^3\Delta_+$	$\Lambda^3\Delta_-$
type	0	$-$	$+$
Λ^1	$\mathfrak{su}(2) \oplus E_5$	$[\mathbb{C}^2 \otimes \mathbb{H}^2]$	$[\mathbb{C}^2 \otimes \mathbb{H}^2]$
Δ_+	$[\mathbb{C}^2 \otimes \mathbb{H}^2]$	$\mathfrak{su}(2) \oplus E_5$	$[\mathbb{C}^2 \otimes \mathbb{H}^2]$
Δ_-	$[\mathbb{C}^2 \otimes \mathbb{H}^2]$	$[\mathbb{C}^2 \otimes \mathbb{H}^2]$	$\mathfrak{su}(2) \oplus E_5$

Table 1.4: The action of $\widetilde{K}_3 = Sp(1) \cdot Sp(2)$ on Λ^1, Δ_+ and Δ_-

we could consider a *homogeneous* form, i.e. a form of pure degree. Next we want to describe geometries which are induced by the existence of two supersymmetric maps. For this we use the "generalised" setup as introduced in [30] which describes geometrical structures by even or odd forms. The idea is to work with the so-called *Narain* or *generalised T-duality group* $O(n, n)$ (cf. [27] and references therein) which is naturally associated with the vector space $T \oplus T^*$ and contains subgroups of the form $G \times G$. For example, $G_2 \times G_2$ and $Spin(7) \times Spin(7)$ sit naturally inside in $O(7, 7)$ and $O(8, 8)$. In this section, we investigate the geometry associated with $O(n) \times O(n)$ giving rise to what we call a *generalised metric structure.*

Recall that the Euclidean structures on a real vector space $T = \mathbb{R}^n$ are parametrised by the homogeneous space $GL(n)/O(n)$ where the subgroup $O(n)$ determines a certain Euclidean inner product g up to a scale. In the generalised case we are then looking for a tensor which yields a reduction from $O(n, n)$ to $O(n) \times O(n)$, so we are interested in the space

$$O(n, n)/O(n) \times O(n).$$

To this effect let us analyse the invariants attached to $O(n) \times O(n)$. First, picking a particular copy of $O(n) \times O(n)$ inside $O(n,n)$ defines an orthogonal splitting

$$(T \oplus T^*, (\cdot,\cdot)) = (V_+ \oplus V_-, g_+ \oplus g_-)$$

into a positive and a negative definite subspace V_+ and V_-. We refer to such a splitting also as a *metric splitting.* Conversely, any choice of such a metric splitting defines the subgroup

$$O(V_+, g_+) \times O(V_-, g_-) \cong O(n) \times O(n)$$

inside $O(n,n)$.

Definition 1.3. *A* generalised metric structure *is a reduction from $O(n,n)$ to $O(n) \times O(n)$.*

Figure 1.3 suggests how to characterise a metric splitting algebraically. If we think of the

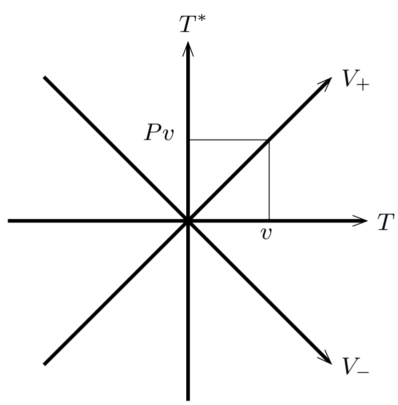

Figure 1.3: Metric splitting of $T \oplus T^*$

coordinate axes T and T^* as a lightcone, choosing a subgroup conjugate to $O(n) \times O(n)$ inside $O(n,n)$ boils down to the choice of a spacelike V_+ with a timelike V_- as orthogonal complement. Interpreting V_+ as the graph of a map $P : T \to T^*$ yields a metric and a 2-form as the symmetric and the skew part of the dual $P \in T^* \otimes T^*$. More formally, we prove the

Proposition 1.16. *The choice of an equivalence class in the space* $O(n,n)/O(n) \times O(n)$ *is equivalent to either set of the following data:*

(i) *A metric splitting*

$$T \oplus T^* = V_+ \oplus V_-$$

into subbundles (V_+, g_+) *and* (V_-, g_-) *with positive and negative definite metrics* $g_\pm = (\cdot\,,\cdot)_{|V_\pm}$ *respectively.*

(ii) *A Riemannian metric* g *and a 2-form* b *on* T*.*

Proof: The equivalence between a choice of a group isomorphic to $O(n) \times O(n)$ and (i) is true almost by definition. Consequently we are left to show that a metric splitting gives rise to the second set of data on T and that conversely, this data allows us to reconstruct the original metric splitting.

Suppose then we are given a metric splitting $T \oplus T^* = V_+ \oplus V_-$. Since the spaces $V_\pm$ are definite and T and T^* are isotropic, $V_\pm$ can be graphed over T, that is there are linear maps

$$P_\pm : T \to T^*$$

such that

$$V_\pm = \{t \oplus P_\pm t \mid t \in T\}.$$

Define the map $\widetilde{P}_+ : T \times T \to \mathbb{R}$ by

$$\widetilde{P}_+(s,t) = (s, P_+(t)).$$

Then the antisymmetric part

$$b(s,t) = \widetilde{P}_+(s,t)/2 - \widetilde{P}_+(t,s)/2$$

gives rise to a 2-form and the symmetric part defines a metric

$$g(s,t) = \widetilde{P}_+(s,t)/2 + \widetilde{P}_+(t,s)/2.$$

Indeed, g is positive definite. If $t \in T$ is non-zero, then

$$g(t,t) = (t, P_+t) = (t \oplus P_+t, t \oplus P_+t)/2 = g_+(t \oplus P_+t, t \oplus P_+t)/2 > 0.$$

Since V_+ and V_- are orthogonal to each other an analogous decomposition of P_- yields a metric g_- and a 2-form b_- that satisfy

$$g_- = -g \quad \text{and} \quad b_- = b.$$

Conversely, assume we are given a metric g and a 2-form b on T. Define the "metric diagonal" by $D_\pm = \{t \oplus \mp t \llcorner g \mid t \in T\}$ (the unnatural sign in the definition of $D_\pm$ stems from our definition of the inner product on $T \oplus T^*$). Then $D_+ \oplus D_-$ is a metric splitting of $T \oplus T^*$ which induces g but yields a vanishing 2-form. Twisting by e^b (cf. (1.11) and (1.12)) gives

$$V_\pm = \{t \oplus t \llcorner (\mp g + b) \mid t \in T\} = e^b(D_\pm),$$

which induces b as well as the metric g. ■

Remark: Since we are working with a fixed inner product $(\cdot\,,\cdot)$ on $T \oplus T^*$, the scale of the induced tensor $\mathcal{G}$ is also fixed, that is, g and b are uniquely determined. Note also that the left and the right hand side $O(n)$ in $O(n) \times O(n)$ are essentially the same in the sense that they induce (up to a sign) the same metric on T and thus the same $O(T)$. The picture to bear in mind is then the following. Endowing T with a metric g reduces the structure group of T to $O(T)$. This allows us to define the "trivial" generalised metric defined by the metric diagonal $D_+ \oplus D_-$. To achieve full generality we have to twist this splitting with a B-field.

As a corollary, we obtain

Corollary 1.17.

$$O(n,n)/O(n) \times O(n) = \{P : T \to T^* \mid \langle Pt, t\rangle > 0 \textit{ for all } t \neq 0\}.$$

In the same vein we can also consider oriented structures. The space $T \oplus T^*$ carries a natural orientation , and an orientation of V_+ fixes thus the orientation on V_- by requiring the orientation of $V_+ \oplus V_-$ to coincide with the natural one on $T \oplus T^*$ and vice versa. This determines then an orientation on T.

Proposition 1.18. *The choice of an equivalence class in the space $SO(n,n)/SO(n)\times SO(n)$ is equivalent to the choice of*

- *an orientation of T*
- *a metric g on T*
- *a B-field $b \in \Lambda^2 T^*$.*

1.5 Generalised exceptional structures

Having introduced the notion of a generalised metric structure we are now in a position to discuss generalised exceptional structures, that is, structures associated with the groups $G_2 \times G_2$ and $Spin(7) \times Spin(7)$. As we saw earlier, the "classical" G_2- and $Spin(7)$-structures are associated with special orbits in an irreducible spin representation space of

$Spin(7)$ and $Spin(8)$. If we want to continue along these lines to obtain a meaningful notion of generalised exceptional structures we have to exhibit a special orbit of the $Spin(7,7)$- and $Spin(8,8)$-action on even and odd forms. The next result shows that at least locally, the groups $G_2 \times G_2$ and $Spin(7) \times Spin(7)$ arise as the stabiliser of a spinor.

Proposition 1.19.

(i) *Assume we are given a G_2-structure on $T = \mathbb{R}^7$ with stable 3-form φ. Then the stabiliser of the even form*

$$\rho_{\pm} = 1 \pm \star\varphi \in S^+ = \Lambda^{ev} T^*$$

under the spin action of $\mathfrak{so}(7,7)$ is isomorphic to $\mathfrak{g}_2 \otimes \mathbb{C}$ and $\mathfrak{g}_2 \oplus \mathfrak{g}_2$ respectively. The same is true for the odd form

$$\hat{\rho}_{\pm} = \pm\varphi + vol_{\varphi} \in S^- = \Lambda^{od} T^*.$$

(ii) *Assume we are given a $Spin(7)$-structure on $T = \mathbb{R}^8$ with invariant 4-form Ω. Then the stabiliser of the even form*

$$\rho_{\pm} = 1 \pm \Omega + vol_{\Omega} \in S^+ = \Lambda^{ev} T^*$$

under the spin action of $\mathfrak{so}(8,8)$ is isomorphic to $\mathfrak{so}(7) \otimes \mathbb{C}$ and $\mathfrak{so}(7) \oplus \mathfrak{so}(7)$ respectively.

Proof: We consider the action of the Lie algebra $\mathfrak{so}(7,7)$ on $S^{\pm}$ and show that the stabiliser in either case is an irreducible symmetric Lie algebra which are dual to each other. We then deduce the result by appealing to Cartan's complete classification of those.

Let T be endowed with a G_2-structure. Then it comes equipped with a Riemannian metric g and we subsequently identify vectors with their duals. We denote the stabiliser of $\rho_{\pm}$ by $\mathfrak{g}_{\pm}$.

First we show that $\mathfrak{g}_\pm = \mathfrak{g}_2 \oplus \mathfrak{g}_2$ as a G_2-module. To this effect, pick a generic element in $A \oplus b \oplus \beta \in \mathfrak{so}(T \oplus T^*)$ (cf. (1.10)) where we now see β as 2-form. By decomposing the equation

$$(A \oplus b \oplus \beta) \bullet \rho_\pm = 0 \tag{1.38}$$

into homogeneous components, (1.38) is equivalent to the conditions

$$\begin{aligned} A^* \star \varphi = 0 \quad &\text{and} \quad \mathrm{Tr}(A) = 0 \\ b \wedge \star\varphi = 0 \quad &\text{and} \quad \beta \llcorner \star\varphi \pm b = 0 \end{aligned} \tag{1.39}$$

The symbol A^* denotes the usual action of $\mathfrak{gl}$ on forms so that the set of equations in the first row simply states that $A \in \mathfrak{g}_2(T)$ – the Lie algebra of the G_2-structure on T. Moreover, $b \wedge \star\varphi = 0$ implies that $b \in \Lambda^2_{14}$ (cf. Proposition 1.3). Here and in the sequel, we will make intensive use of the general formulae

$$\begin{aligned} v \llcorner \star\xi &= (-1)^{\deg(\xi)} \star (v \wedge \xi) \\ v \wedge \star\xi &= (-1)^{\deg(\xi)} \star (v \llcorner \xi) \end{aligned}$$

which hold for an arbitrary form ξ of pure degree. If we decompose β in its components $\beta_7 \oplus \beta_{14}$ with respect to the G_2-decomposition, Proposition 1.2 gives

$$-2\beta_7^\flat + \beta_{14}^\flat = \mp b_{14}$$

and thus $\beta_{14} = \mp b$ and $\beta_7 = 0$. We define

$$\mathfrak{m}_\pm = \{ b \oplus \mp b \in \Lambda^2 \oplus \Lambda^2 \mid b \in \Lambda^2_{14} \}$$

which is isomorphic to $\Lambda^2_{14} = \mathfrak{g}_2$. Hence, as a G_2-module, we get

$$\mathfrak{g}_\pm = \mathfrak{g}_2 \oplus \mathfrak{g}_2.$$

It remains to understand the Lie algebra structure of $\mathfrak{g}_\pm$. We claim that $\mathfrak{g}_\pm$ is an orthogonal symmetric irreducible Lie algebra. If we use the matrix picture of $\mathfrak{so}(T \oplus T^*)$ then the stabiliser $\mathfrak{g}_\pm$ consists of matrices of the form

$$\begin{pmatrix} A & \mp b \\ b & A \end{pmatrix},$$

where $A \in \mathfrak{g}_2(T)$ and $b \in \mathfrak{m}_\pm$ are skew-symmetric matrices. Put $\mathfrak{h} = \mathfrak{g}_2(T)$. The commutator $[C, D] = C \circ D - D \circ C$ is the usual one for matrix algebras and we immediately deduce the relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \ [\mathfrak{h}, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm, \ [\mathfrak{m}_\pm, \mathfrak{m}_\pm] \subset \mathfrak{h}.$$

This amounts to saying that the Lie algebra $\mathfrak{g}_\pm$ is symmetric. In particular, $\mathfrak{m}_\pm$ is an $\mathfrak{h}$-irreducible representation space. We conclude from Cartan's classification theorem that we are left with only two possibilities for the Lie algebra structure of $\mathfrak{g}_\pm$, either the direct sum of Lie algebras $\mathfrak{g}_2 \oplus \mathfrak{g}_2$ or the complex Lie algebra $\mathfrak{g}_2 \otimes \mathbb{C}$. We show that both cases occur and $\mathfrak{g}_+ = \mathfrak{g}_-^* =$ the dual of $\mathfrak{g}_-$. Recall that the dual of $\mathfrak{g}_-$ is defined to be the real subalgebra $\mathfrak{h} \oplus i\mathfrak{m}_-$ inside the complexification $\mathfrak{g}_- \otimes \mathbb{C}$. Hence the dual is represented by matrices of the form

$$\begin{pmatrix} A & ib \\ ib & A \end{pmatrix}.$$

We easily check that

$$\begin{pmatrix} A & ib \\ ib & A \end{pmatrix} \in \mathfrak{g}_-^* \mapsto \begin{pmatrix} A & -b \\ b & A \end{pmatrix} \in \mathfrak{g}_+$$

sets up a (real) Lie algebra isomorphism.

Finally we have to determine which of the two Lie algebras $\mathfrak{g}_\pm$ is compact. To that end we will show that the Killing form $B_{\mathfrak{g}_-}$ restricted to $\mathfrak{m}_-$ is negative definite, hence $\mathfrak{g}_- = \mathfrak{g}_2 \oplus \mathfrak{g}_2$.

First, we introduce a suitable inner product on $\mathfrak{g}_-$ to compute the Killing form. For this we remark that the matrices in $\mathfrak{g}_-$ are skew-symmetric, since the matrices $A \in \mathfrak{h}$ and $b \in \mathfrak{m}_-$ are skew. Therefore, for any non-zero matrix $M \in \mathfrak{g}_-$, we have

$$-\mathrm{Tr}(M \circ M) = -\sum_{i,j} M_{ij} M_{ji} = \sum_{i,j} M_{ij}^2 > 0.$$

It defines therefore a positive definite, ad-invariant inner product on $\mathfrak{g}_-$ which we denote by $\langle \cdot, \cdot \rangle$. Fix an orthonormal basis $\{X_i\}$ for this inner product and pick a nonzero $X \in \mathfrak{m}$. Then

$$\begin{aligned} B_{\mathfrak{g}_-}(X,X) &= \mathrm{Tr}(ad_X \circ ad_X) = \sum_i \langle [X,[X,X_i]], X_i \rangle \\ &= -\sum_i \langle [X,X_i],[X,X_i] \rangle < 0 \end{aligned}$$

The last equality follows from ad-invariance. This completes the proof for $\rho_\pm$ in the first case. For $\hat{\rho}$ the equations (1.39) become

$$\begin{aligned} A^* \varphi = 0 \quad &\text{and} \quad \mathrm{Tr}(A) = 0 \\ \beta \wedge \varphi = 0 \quad &\text{and} \quad b \llcorner \varphi \pm \beta = 0. \end{aligned}$$

Thus we deduce again $\beta_{14} = \mp b_{14}$ and hence the same result holds for $\hat{\rho}_\pm$. Finally in (ii), we obtain the equations

$$\begin{aligned} A^* \Omega = 0 \quad &\text{and} \quad \mathrm{Tr}(A) = 0 \\ b \pm \beta \llcorner \Omega = 0 \quad &\text{and} \quad \pm b \wedge \Omega + \beta \llcorner vol_\Omega = 0. \end{aligned}$$

The first row states that $A \in \mathfrak{spin}(7)$. As $\star\Omega = \Omega$, decomposing b and β into irreducible $Spin(7)$-components (cf. Proposition 1.3) yields from the first equation in the second row

$$b_7 = \mp 3\beta_7 \text{ and } b_{21} = \mp \beta_{21}.$$

The second equation, however, implies

$$b_7 = \mp \frac{1}{3}\beta_7 \text{ and } b_{21} = \mp \beta_{21}.$$

Hence $b_7 = \beta_7 = 0$ and $b_{21} = \mp\beta_{21}$ and the proof of (i) carries over word for word. ■

Remark: Note that $\hat{\rho}_\pm = \star\sigma(\rho_\pm)$ and $\rho_\pm = \star\sigma(\rho_\pm)$, where σ is the involution defined in (1.2).

Motivated by the previous proposition we define

Definition 1.4.

(i) *We call a reduction from the structure group* $SO(7,7)$ *of* $T \oplus T^*$ *to* $G_2 \times G_2$ *a* generalised G_2- structure*.*

(ii) *We call a reduction from the structure group* $SO(8,8)$ *of* $T \oplus T^*$ *to* $Spin(7) \times Spin(7)$ *a* generalised $Spin(7)$-structure*.*

We refer to both of these structures also as generalised exceptional structures*.*

The previous proposition suggests that these structures can be, like in the classical case, defined by a form which, however, is of mixed degree. To get things rolling, we first look at the tensorial invariants on T which are induced by such a reduction along the lines of Proposition 1.16. Since $G_2 \times G_2$ determines some group $SO(V_+) \times SO(V_-)$ conjugate to $SO(7) \times SO(7)$, such a structure induces a generalised oriented metric structure (g, b). We then consider the irreducible spin representation Δ of $Spin(7) = Spin(T, g)$. The group $G_2 \times G_2$ acts on $\Delta \otimes \Delta$ and stabilises a decomposable line element $\Psi_+ \otimes \Psi_-$ which we choose to be of unit norm. This gives rise to two groups $G_{2\pm}$ conjugated to G_2 inside $Spin(7)$ associated with the spinors Ψ_+ and Ψ_-. However, these are only determined up to a scalar since $e^{-F}\Psi_+ \otimes e^F\Psi_- = \Psi_+ \otimes \Psi_-$. To remove this ambiguity up to a simultaneous sign flip in Ψ_+ and Ψ_- we let Ψ_+ and Ψ_- be of unit norm and introduce the additional scalar F. We refer to it as the *dilaton*.

We can carry out the same analysis for the group $Spin(7) \times Spin(7)$ which sits inside some $SO(8) \times SO(8)$. Note that the stabilised spinors can lie in either of the $Spin(T, g)$-representation spaces Δ_+ or Δ_- according to whether or not the specific copy of $Spin(7)$ inside $Spin(T, g)$ contains plus or minus the volume element of $T = \mathbb{R}^8$. By choosing a suitable global orientation on T, we can always assume that the spinor ψ_+ stabilised by the left-hand side factor of $Spin(7) \times Spin(7)$ lives in Δ_+.

We summarise this discussion in the next proposition.

Proposition 1.20.

(i) *A generalised G_2-structure induces the following data on $T = \mathbb{R}^7$:*

- *an orientation*
- *a metric g*
- *a 2-form b*
- *two unit spinors $\psi_\pm \in \Delta$.*
- *a scalar f.*

Here, Δ denotes the irreducible spin representation space of $Spin(T, g) = Spin(7)$.

(ii) *A generalised $Spin(7)$-structure induces the following data on $T = \mathbb{R}^8$:*

- *an orientation*
- *a metric g*
- *a 2-form b*

- *a unit spinor* $\psi_+ \in \Delta_+$ *and a unit spinor* ψ_- *in either* Δ_+ *or* Δ_-.

- *a scalar* f

Here, Δ_+ *and* Δ_- *denote the irreducible spin representation spaces of* $Spin(T, g) = Spin(8)$.

Since for the generalised $Spin(7)$-case we have to take into account the chirality of the spinors, we make the

Definition 1.5. *A generalised* $Spin(7)$*-structure which induces two spinors of the same or opposite chirality is said to be of* even *or* odd *type.*

Remark: The reason for this nomenclature will become apparent in a moment. Note that the subscript $\pm$ of the spinors does not indicate the chirality of the spinor. Its meaning will become clear in Theorem 3.15.

The key idea for establishing a converse of Proposition 1.20 is to interpret the exterior algebra as the spin representation space for $Spin(7) \times Spin(7)$ and $Spin(8) \times Spin(8)$ and to relate it to the tensor product $\Delta \otimes \Delta$ of the "classical" spin representations of $Spin(7)$ and $Spin(8)$. To discuss both cases in parallel we let (T, g) denote an inner product space of dimension 7 or 8 and consider the groups $Spin(7) = Spin(T^7, g)$ or $Spin(8) = Spin(T^8, g)$. Again, let Δ be the spin representation space of $Spin(7)$ or $Spin(8)$ which in the latter case can be decomposed into the irreducible $Spin(8)$-modules Δ_+ and Δ_-. We regard the tensor product $\Delta \otimes \Delta$ as a $Spin(7) \times Spin(7)$- or $Spin(8) \times Spin(8)$-representation space. We can construct a further representation space of these groups out of the following embedding into $\mathit{Cliff}(T \oplus T^*, (\cdot, \cdot))$ defined through a metric splitting of $T \oplus T^* = V_+ \oplus V_-$. Write the associated Clifford algebra as a $\mathbb{Z}_2$-graded tensor product

$$Cliff(V_+, g_+)\hat{\otimes} Cliff(V_-, g_-) \cong Cliff(T \oplus T^*),$$

where the isomorphism is given by extension of

$$v_+ \hat{\otimes} v_- \mapsto v_+ \cdot v_-.$$

In particular, this maps $Spin(V_+, g_+) \times Spin(V_-, g_-)$ into $Spin(T \oplus T^*)$. On the other hand, the maps

$$\pi_\pm : x \in (T, \pm g) \mapsto -x \oplus x \llcorner (\pm g - b)$$

induce isomorphisms

$$Cliff(T, \pm g) \cong Cliff(V_\pm, g_\pm)$$

which map $Spin(T, g) = Spin(T, -g)$ isomorphically onto $Spin(V_\pm, g_\pm)$. The compounded algebra isomorphism

$$\begin{aligned} \iota_b : Cliff(T, g)\hat{\otimes} Cliff(T, -g) &\rightarrow Cliff(T \oplus T^*) \\ x \hat{\otimes} y &\mapsto (-x \oplus x \llcorner (g - b)) \bullet (-y \oplus y \llcorner (-g - b)) \end{aligned}$$

maps

$$Spin(T) \times Spin(T) \rightarrow Spin(V_+) \times Spin(V_-) \leqslant Spin(T \oplus T^*).$$

We will suppress the subscript b and simply write ι if $b \equiv 0$. Recall also our convention to denote by $\bullet$ the product and the Clifford action of $Cliff(T \oplus T^*)$. Now we are in a position to compare the actions on $\Delta \otimes \Delta$ and $\Lambda^{ev,od}$. The so-called *pinor product* $\psi_0 \hat{\circ} \psi_1$ for two spinors ψ_0 and ψ_1 is the endomorphism

$$\psi_0 \hat{\circ} \psi_1(\phi) = q_\Delta(\psi_1, \phi)\psi_0$$

(where q_Δ denotes the spin-invariant inner product on Δ). Clifford multiplication identifies any element in the Clifford algebra with an endomorphism of Δ. It relates to the pinor product by [25]

$$(x \cdot \psi_0)\hat{\circ}\psi_1 = x \circ (\psi_0 \hat{\circ} \psi_1) \text{ and } \psi_0\hat{\circ}(x \cdot \psi_1) = (\psi_0\hat{\circ}\psi_1) \circ \sigma(x), \tag{1.40}$$

where σ is the Clifford involution of (1.2). If we regard $\psi_0\hat{\circ}\psi_1$ as an element in *Cliff*$(T \oplus T^*)$, then

$$\sigma(\psi_0\hat{\circ}\psi_1) = \psi_1\hat{\circ}\psi_0.$$

Let R denote the representations of *Cliff*(T^8) or *Cliff*(T^7) as defined in Sections 1.1.2 and 1.2.2. Recall that J was the vector space isomorphism between *Cliff*(T) and $\Lambda^* T^*$ defined in (1.1). Then we define the following maps

$$L_b^8 : \Delta \otimes \Delta \xrightarrow{\hat{\circ}} \mathrm{End}(\Delta_+ \oplus \Delta_-) \xrightarrow{R^{-1}} \mathit{Cliff}(T^8, g) \xrightarrow{J} \Lambda^* T^* \xrightarrow{e^{b/2}\wedge} \Lambda^* T^*$$

and

$$L_b^7 : \Delta \otimes \Delta \xrightarrow{\hat{\circ}\oplus 0} \mathrm{End}(\Delta) \oplus \mathrm{End}(\Delta) \xrightarrow{R^{-1}} \mathit{Cliff}(T^7, g) \xrightarrow{J} \Lambda^* T^* \xrightarrow{e^{b/2}\wedge} \Lambda^* T^*.$$

As the 2-form b is to play a crucial part in the latter development, it is worthwhile emphasising it in the notation. If $b = 0$ we shall simply write $L_0 = L$. Here and from now on we will drop the superscripts 7 and 8, relying on the context to determine which particular map is under consideration. We denote by $L_b^{ev,od}$ the map L_b followed by the projections on the even or odd part of the exterior algebra, i.e.

$$L_b = L_b^{ev} \oplus L_b^{od}.$$

Note that

$$\begin{aligned} L_b(\Delta_+ \otimes \Delta_+ \oplus \Delta_- \otimes \Delta_-) &= \Lambda^{ev} \\ L_b(\Delta_+ \otimes \Delta_- \oplus \Delta_- \otimes \Delta_+) &= \Lambda^{od}, \end{aligned}$$

which justifies our notion of a generalised "even" or "odd" $Spin(7)$-structure. Whenever it makes sense, we regard the maps $L_b^{ev,od}$ as the restriction of L_b to the spinors of equal or opposite chirality. Moreover, we denote by $L^{ev}_{\pm}$ and $L^{od}_{b,\pm}$ the restrictions to $\Delta_+ \otimes \Delta_+$, $\Delta_- \otimes \Delta_-$, $\Delta_- \otimes \Delta_+$ and $\Delta_+ \otimes \Delta_-$ respectively. The next proposition says that up to a sign twist, the action of T on $\Delta \otimes \Delta$ and $\Lambda^{ev,od}$ commute.

Proposition 1.21. *For any $x \in T$ we have*

$$\begin{aligned} L_b^{ev,od}(x \cdot \varphi \otimes \psi) &= \iota_b(x \hat{\otimes} 1) \bullet L_b^{od,ev}(\varphi \otimes \psi) \\ L_b^{ev,od}(\varphi \otimes x \cdot \psi) &= \pm\iota_b(1 \hat{\otimes} x) \bullet L_b^{od,ev}(\varphi \otimes \psi). \end{aligned}$$

We postpone the proof of Proposition 1.21 to draw several corollaries first.

Corollary 1.22. *For unit vectors $x_1 \ldots, x_k \in (T, g)$ and $y_1, \ldots, y_l \in (T, -g)$ with $k + l$ even we have*

$$L_b^{ev,od}(x_1 \cdot \ldots \cdot x_k \cdot \varphi \otimes y_1 \cdot \ldots \cdot y_l \cdot \psi) = (-1)^l s(l) \iota_b(x_1 \cdot \ldots \cdot x_k \hat{\otimes} y_1 \cdot \ldots \cdot y_l) \bullet L_b^{ev,od}(\varphi \otimes \psi),$$

where $s(l) = 1$ for $l \equiv 0, 1 \bmod 4$ and $s(l) = -1$ for $l \equiv 2, 3 \bmod 4$. For $k + l$ odd we obtain

$$L_b^{ev,od}(x_1 \cdot \ldots \cdot x_k \cdot \varphi \otimes y_1 \cdot \ldots \cdot y_l \cdot \psi) = (-1)^l s(l) \iota_b(x_1 \cdot \ldots \cdot x_k \hat{\otimes} y_1 \cdot \ldots \cdot y_l) \bullet L_b^{od,ev}(\varphi \otimes \psi).$$

Moreover, $L_b^{ev,od}$ preserves any invariant space of a subgroup of $Spin(7) \times Spin(7)$ or $Spin(8) \times Spin(8)$. In particular, we get the following decomposition into irreducible k-dimensional representations V_k of $G_2 \times G_2$ and $Spin(7) \times Spin(7)$.

(i) $\Lambda^{ev,od} T^{7*} = V_1 \oplus V_{l,7} \oplus V_{r,7} \oplus V_{49}$

(ii) $\Lambda^{ev,od} T^{8*} = V_1 \oplus V_{l,7} \oplus V_{r,7} \oplus V_{49} \oplus V_{64}$.

V_1 corresponds to the trivial representation spanned by $\psi_+ \otimes \psi_-$, $V_{l,7}$ to $\psi_+^{\perp} \otimes \psi_-$, $V_{r,7}$ to $\psi_- \otimes \psi_+^{\perp}$ and V_{49} to $\psi_+^{\perp} \otimes \psi_-^{\perp}$.

Proof: If $k+l$ is even, even- or oddness of k implies that of l. We abbreviate $x_1 \cdot \ldots \cdot x_l$ and $y_1 \cdot \ldots \cdot y_k$ by $\underline{x}$ and $\underline{y}$. Now suppose that k is even, then

$$L_b^{ev}(\underline{x} \cdot \varphi \otimes \underline{y} \cdot \psi) = \iota_b(\underline{x} \hat{\otimes} 1) \bullet L_b^{ev}(\varphi \otimes \underline{y} \cdot \psi).$$

Moreover, l is even and because of the sign flip for L_b^{od}, we get an extra minus sign whenever l is of the form $4m+2$ while this sign cancels for $l = 4m$. Hence the sign is given by $s(l)$. The other cases are discussed in the same way where the parity flips if $k+l$ is odd. The remaining claim is straightforward. ■

Corollary 1.23. *Restricted to the action of $a \in Spin(T) \mapsto (a,a) \in Spin(T) \times Spin(T)$, we have*

$$L^{ev,od}(a \cdot \varphi \otimes a \cdot \psi) = \pi_0(a)^* L^{ev,od}(\varphi \otimes \psi)$$

for any $a \in Spin(T,g)$, where $\pi_0(a)^*$ denotes the action on forms induced by the vector representation $\pi_0 : Spin(T,g) \to SO(T,g)$. Hence, $L^{ev,od}$ defines a $Spin(T)$-equivalence between $\Delta \otimes \Delta$ and $\Lambda^{ev,od}$ or Λ^* (which is unique up to a scalar).

Proof: To derive the assertion we look at the action of the Lie algebra, that is we want to show

$$L^{ev,od}(A \cdot \varphi \otimes \psi + \varphi \otimes A \cdot \psi) = \pi_{0*}(A)^* L^{ev,od}(\varphi \otimes \psi)$$

for any $A \in \mathfrak{spin}(T,g)$. We fix an orthonormal basis $\{e_i\}$. Since $e_i \wedge (e_j \llcorner \tau) = -e_j \llcorner (e_i \wedge \tau)$ for any form τ we get for $e_{ij} = e_i \cdot e_j$ with $i \neq j$ that

$$\begin{aligned}
L^{ev,od}(e_{ij}\cdot\varphi\otimes\psi + \varphi\otimes e_{ij}\cdot\psi) &= \iota(e_{ij}\oplus 0)\bullet L^{ev,od}(\varphi\otimes\psi) - \\
&\quad -\iota(0\oplus e_{ij})\bullet L^{ev,od}(\varphi\otimes\psi) \\
&= 2(e_j\llcorner(e_i\wedge L^{ev,od}(\varphi\otimes\psi)) + \\
&\quad + e_j\wedge(e_i\llcorner L^{ev,od}(\varphi\otimes\psi)) \\
&= \pi_{0*}(e_i\cdot e_j)^* L^{ev,od}(\varphi\otimes\psi),
\end{aligned}$$

hence the result. ■

Corollary 1.24.

(i) *Generalised G_2-structures are in 1-1 correspondence with lines of forms ρ in Λ^{ev} or Λ^{od} whose stabiliser under the action of $Spin(7,7)$ is isomorphic to $G_2 \times G_2$. We refer to ρ as the* structure form *of the generalised G_2-structure. This form can be uniquely written (modulo a simultaneous sign change for Ψ_+ and Ψ_-) as*

$$\rho = e^{-f}L^{ev}(\psi_+\otimes\psi_-) \text{ or } \rho = e^{-f}L^{od}(\psi_+\otimes\psi_-).$$

(ii) *Generalised $Spin(7)$-structures are in 1-1 correspondence with lines of forms ρ in Λ^{ev} or Λ^{od} whose stabiliser under the action of $Spin(8,8)$ is isomorphic to $Spin(7)\times Spin(7)$. We refer to ρ as the* structure form *of the generalised $Spin(7)$-structure. We speak of an* even *or* odd *structure according to whether ρ is even or odd. This form can be uniquely written (modulo a simultaneous sign change for Ψ_+ and Ψ_-) as*

$$\rho = e^{-f}L(\psi_+\otimes\psi_-).$$

Now we turn to the proof for Proposition 1.21.

Proof: (of proposition 1.21) First we assume that $b \equiv 0$. Starting with the 8-dimensional case, we see that

$$\begin{aligned} L(x \cdot \varphi \otimes \psi) &= J(R^{-1}(R(x)(\varphi)\hat{\circ}\psi)) \\ &= J(R^{-1}(R(x) \circ (\varphi\hat{\circ}\psi))) \\ &= J(x \cdot R^{-1}(\varphi\hat{\circ}\psi)) \\ &= -x \llcorner L(\varphi \otimes \psi) + x \wedge L(\varphi \otimes \psi) \\ &= \iota(x\hat{\otimes}1) \bullet L(\varphi \otimes \psi), \end{aligned}$$

where we have used (1.40). Hence we get

$$L^{ev}(x \cdot \varphi \otimes \psi) = \iota(x\hat{\otimes}1) \bullet L^{od}(\varphi \otimes \psi), \quad L^{od}(x \cdot \varphi \otimes \psi) = \iota(x\hat{\otimes}1) \bullet L^{ev}(\varphi \otimes \psi).$$

Similarly, we find

$$\begin{aligned} L(\varphi \otimes x \cdot \psi) &= J(R^{-1}(\varphi\hat{\circ}R(x)(\psi))) \\ &= J(R^{-1}(\varphi\hat{\circ}\psi \circ R(\hat{x}))) \\ &= -J(R^{-1}(\varphi\hat{\circ}\psi) \cdot x) \\ &= x \llcorner L^{ev}(\varphi \otimes \psi) - x \wedge L^{ev}(\varphi \otimes \psi) + \\ &\quad + x \llcorner L^{od}(\varphi \otimes \psi) + x \wedge L^{od}(\varphi \otimes \psi) \\ &= -\iota(1\hat{\otimes}x) \bullet L^{ev}(\varphi \otimes \psi) + \iota(1\hat{\otimes}x) \bullet L^{od}(\varphi \otimes \psi), \end{aligned}$$

so that

$$L^{ev}(\varphi \otimes x \cdot \psi) = \iota(1\hat{\otimes}x) \bullet L^{od}(\varphi \otimes \psi), \quad L^{od}(\varphi \otimes x \cdot \psi) = -\iota(1\hat{\otimes}x) \bullet L^{ev}(\varphi \otimes \psi).$$

Next we turn to the case $n = 7$. By convention, we let T^7 act through the inclusion

$$T^7 \hookrightarrow \mathit{Cliff}(T^7, g) \overset{R}{\cong} \mathrm{End}(\Delta) \oplus \mathrm{End}(\Delta)$$

followed by projection on the first summand. Thus

$$\begin{aligned} L(x\cdot\varphi\otimes\psi) &= J(R^{-1}((x\cdot\varphi)\hat{\circ}\psi\oplus 0)) \\ &= J(x\cdot R^{-1}(\varphi\hat{\circ}\psi\oplus 0)) \end{aligned}$$

and we can argue as above. The same applies to $L(\varphi\otimes x\cdot\psi)$ so that the case $b=0$ is shown.

Now let b be an arbitrary B-field. For the sake of clarity we will temporarily denote by $\widetilde{\exp}$ the exponential map from $\mathfrak{so}(7)$ to $Spin(7)$, while the untilded exponential takes values in $SO(7)$. The adjoint representation Ad of the group of units inside a general Clifford algebra *Cliff*(V) restricts to the double cover $Spin(V)\to SO(V)$ still denoted by Ad. As a transformation in $SO(V)$ we then have $Ad\circ\widetilde{\exp}=\exp\circ ad$. Since $ad(v\wedge w)=[v,w]/4$ we obtain

$$e^b = Ad(\widetilde{e}^{\sum b_{ij}e_i\cdot e_j/2})$$

for the B-field $b=\sum_{i<j}b_{ij}e_i\wedge e_j$. Hence

$$\begin{aligned} L_b^{ev,od}(x\cdot\varphi\otimes\psi) &= \widetilde{e}^{b/2}\bullet L^{ev,od}(x\cdot\varphi\otimes\psi) \\ &= (\widetilde{e}^{b/2}\bullet\iota(x\hat{\otimes}1))\bullet L^{od,ev}(\varphi\otimes\psi) \\ &= (\widetilde{e}^{b/2}\bullet\iota(x\hat{\otimes}1)\bullet\widetilde{e}^{-b/2})\bullet L_b^{od,ev}(\varphi\otimes\psi) \\ &= Ad(\widetilde{e}^{b/2})(\iota(x\hat{\otimes}1))\bullet L_b^{od,ev}(\varphi\otimes\psi) \\ &= e^b(\iota(x\hat{\otimes}1))\bullet L_b^{od,ev}(\varphi\otimes\psi) \\ &= \iota_b(x\hat{\otimes}1)\bullet L_b^{od,ev}(\varphi\otimes\psi). \end{aligned}$$

Similarly, the claim is checked for $L_b^{ev,od}(\varphi\otimes x\cdot\psi)$ which completes the proof. ■

The next corollary works out how a 3-form acts by wedging and contraction. This will become important in Section 3.3.1 where 3-forms naturally appear as the torsion form of a linear metric connection.

Corollary 1.25. *Let $\tau \in \Lambda^3$. Then the identities*

$$\begin{aligned}\tau \llcorner L^{ev,od}(\varphi \otimes \psi) &= \frac{1}{8} L^{od,ev}(-\tau \cdot \varphi \otimes \psi \pm \varphi \otimes \tau \cdot \psi \mp \\ &\quad \mp \sum_i (e_i \llcorner \tau) \cdot \varphi \otimes e_i \cdot \psi + \sum_i e_i \cdot \varphi \otimes (e_i \llcorner \tau) \cdot \psi)\end{aligned}$$

and

$$\begin{aligned}\tau \wedge L^{ev,od}(\varphi \otimes \psi) &= \frac{1}{8} L^{od,ev}(\tau \cdot \varphi \otimes \psi \pm \varphi \otimes \tau \cdot \psi \mp \\ &\quad \mp \sum_i (e_i \llcorner \tau) \cdot \varphi \otimes e_i \cdot \psi - \sum_i e_i \cdot \varphi \otimes (e_i \llcorner \tau) \cdot \psi)\end{aligned}$$

hold.

Proof: From Proposition 1.21 we derive

$$\begin{aligned}L^{ev,od}(x \cdot \varphi \otimes \psi) &= x \wedge L^{od,ev}(\varphi \otimes \psi) - x \llcorner L^{od,ev}(\varphi \otimes \psi) \\ L^{ev,od}(\varphi \otimes x \cdot \psi) &= \pm x \wedge L^{od,ev}(\varphi \otimes \psi) \pm x \llcorner L^{od,ev}(\varphi \otimes \psi)\end{aligned}$$

which implies

$$\begin{aligned}x \wedge L^{ev}(\varphi \otimes \psi) &= \frac{1}{2} L^{od}(x \cdot \varphi \otimes \psi - \varphi \otimes x \cdot \psi) \\ x \wedge L^{od}(\varphi \otimes \psi) &= \frac{1}{2} L^{ev}(x \cdot \varphi \otimes \psi + \varphi \otimes x \cdot \psi)\end{aligned}$$

and

$$\begin{aligned}x \llcorner L^{ev}(\varphi \otimes \psi) &= \frac{1}{2} L^{od}(-x \cdot \varphi \otimes \psi - \varphi \otimes x \cdot \psi) \\ x \llcorner L^{od}(\varphi \otimes \psi) &= \frac{1}{2} L^{ev}(-x \cdot \varphi \otimes \psi + \varphi \otimes x \cdot \psi).\end{aligned}$$

Fix an orthonormal basis $e_1, \dots, e_7$. By using the identities above we gather that

$$\begin{aligned} e_{ijk} \llcorner L^{ev,od}(\varphi \otimes \psi) &= \frac{1}{8} L^{od,ev}(-e_{ijk} \cdot \varphi \otimes \psi \pm \varphi \otimes e_{ijk} \cdot \psi \pm \\ &\pm e_{ik} \cdot \varphi \otimes e_j \cdot \psi - e_i \cdot \varphi \otimes e_{kj} \cdot \psi \pm \\ &\pm e_{ji} \cdot \varphi \otimes e_k \cdot \psi - e_j \cdot \varphi \otimes e_{ik} \psi \pm \\ &\pm e_{kj} \cdot \varphi \otimes e_i \cdot \psi - e_k \cdot \varphi \otimes e_{ji} \cdot \psi) \end{aligned}$$

and

$$\begin{aligned} e_{ijk} \wedge L^{ev,od}(\varphi \otimes \psi) &= \frac{1}{8} L^{od,ev}(e_{ijk} \cdot \varphi \otimes \psi \pm \varphi \otimes e_{ijk} \cdot \psi \pm \\ &\pm e_{ik} \cdot \varphi \otimes e_j \cdot \psi + e_i \cdot \varphi \otimes e_{kj} \cdot \psi \pm \\ &\pm e_{ji} \cdot \varphi \otimes e_k \cdot \psi + e_j \cdot \varphi \otimes e_{ik} \psi \pm \\ &\pm e_{kj} \cdot \varphi \otimes e_i \cdot \psi + e_k \cdot \varphi \otimes e_{ji} \cdot \psi). \end{aligned}$$

For instance, we have

$$\begin{aligned} e_{ijk} \wedge L^{ev}(\varphi \otimes \psi) &= \frac{1}{2} e_{ij} \wedge L^{od}(e_k \cdot \varphi \otimes \psi - \varphi \otimes e_k \cdot \psi) \\ &= \frac{1}{4} e_i \wedge L^{ev}(e_{jk} \cdot \varphi \otimes \psi + e_k \cdot \varphi \otimes e_j \cdot \psi - \\ &\quad - e_j \cdot \varphi \otimes e_k \cdot \psi - \varphi \otimes e_{jk} \cdot \psi) \\ &= \frac{1}{8} L^{od}(e_{ijk} \cdot \varphi \otimes \psi - e_{jk} \cdot \varphi \otimes e_i \cdot \psi + e_{ik} \cdot \varphi \otimes e_j \cdot \psi - \\ &\quad - e_k \cdot \varphi \otimes e_{ij} \cdot \psi - e_{ij} \cdot \varphi \otimes e_k \cdot \psi + e_j \cdot \varphi \otimes e_{ik} \cdot \psi - \\ &\quad - e_i \cdot \varphi \otimes e_{jk} \cdot \psi + \varphi \otimes e_{ijk} \cdot \psi). \end{aligned}$$

The remaining assertions are established in the same way. If $\tau = \sum_{i<j<k} \tau_{ijk} e_{ijk}$ is a general 3-form, then

$$
\begin{aligned}
\tau \llcorner L^{ev,od}(\varphi \otimes \psi) &= \frac{1}{6} \sum_{i,j,k} \tau_{ijk} e_{ijk} \llcorner L^{ev,od}(\varphi \otimes \psi) \\
&= \frac{1}{48} L^{od,ev}(-6\tau \cdot \varphi \otimes \psi \pm \varphi \otimes 6\tau \cdot \psi \mp \\
&\quad \mp 3 \sum_{ijk} \tau_{ijk} e_{jk} \cdot \varphi \otimes e_i \cdot \psi + 3 \sum_{ijk} \tau_{ijk} e_i \cdot \varphi \otimes e_{jk} \cdot \tau) \\
&= \frac{1}{8} L^{od,ev}(-\tau \cdot \varphi \otimes \psi \pm \varphi \otimes \tau \cdot \psi \mp \\
&\quad \mp \sum_i (e_i \llcorner \tau) \cdot \varphi \otimes e_i \cdot \psi + \sum_i e_i \cdot \varphi \otimes (e_i \llcorner \tau) \cdot \psi),
\end{aligned}
$$

since $e_i \llcorner \tau = \sum_{j,k} \tau_{ijk} e_{jk}/2$. The second identity is derived in a similar fashion. ■

Choosing a generalised metric structure and an orientation on T also sets up the isomorphism L_b. Since the spin representation Δ of $Spin(7)$ is of real type so is the tensor product. We remark that up to isomorphism, there is only one 49-dimensional irreducible representation space of $Spin(7) \times Spin(7)$, hence Λ^{ev} and Λ^{od} are isomorphic as $Spin(7) \times Spin(7)$-spaces. Consequently, there is (up to a scalar) a unique $Spin(7) \times Spin(7)$-invariant in $\Lambda^{ev,od} \otimes \Lambda^{od,ev}$, that is we obtain an equivariant map $\Lambda^{ev,od} \to \Lambda^{od,ev}$. The description of this invariant will occupy us next. Morally it is the Hodge $\star$-operator twisted with a B-field and the anti-automorphism σ defined in (1.2).

Definition 1.6. *The* box operator *or* generalised Hodge $\star$-operator $\Box_{g,b} : \Lambda^{ev,od}T^* \to \Lambda^{od,ev}T^*$ *(n odd) or* $\Box_{g,b} : \Lambda^{ev,od}T^* \to \Lambda^{ev,od}T^*$ *(n even) associated with g and b is defined by*

$$\Box_{g,b}\rho = e^{b/2} \wedge \star_g \sigma(e^{-b/2} \wedge \rho).$$

We will need the following technical lemma which is an immediate consequence from the definition of σ.

Lemma 1.26. *Let $\rho^{ev,od} \in \Lambda^{ev,od}T^*$ be an even or an odd form and let $x \in T^*$. Then*

1. $$\begin{aligned} \star\sigma(\rho^{ev,od}) &= \sigma(\star\rho^{ev,od}), \quad n = 7 \\ \star\sigma(\rho^{ev,od}) &= \pm\sigma(\star\rho^{ev,od}), \quad n = 8 \end{aligned}$$
2. $\sigma(e^b \wedge \rho^{ev,od}) = e^{-b} \wedge \sigma(\rho^{ev,od})$

Now we can prove the following

Proposition 1.27.

(i) *If $n = 7$, then*

$$\Box_{g,b} L_b = L_b$$

or equivalently,

$$\Box_{g,b} L_b^{ev,od} = L_b^{od,ev}.$$

(ii) *If $n = 8$, then*

$$\Box_{g,b} L_{b,\pm}^{ev,od} = \pm L_{b,\pm}^{ev,od}.$$

In particular, $\Box_{g,b}$ is $Spin(7) \times Spin(7)$- and $Spin(8) \times Spin(8)$-equivariant.

Proof: For any $\varphi, \psi \in \Delta \otimes \Delta$ ($n = 7$ or 8) we have according to (1.3) and Lemma 1.26

$$\begin{aligned} \Box_{g,b} L_b(\varphi \otimes \psi) &= e^{b/2} \wedge \star\sigma(e^{-b/2} \wedge L_b(\varphi \otimes \psi)) \\ &= e^{b/2} \wedge \star\sigma(L(\varphi \otimes \psi)) \\ &= e^{b/2} \wedge J(R^{-1}(\varphi \hat{\circ} \psi) \cdot \lambda). \end{aligned}$$

Now for $n = 7$, $R^{-1}((\varphi\hat{\circ}\psi \oplus 0) \cdot vol)$ is just $R^{-1}(\varphi\hat{\circ}\psi \oplus 0)$ while for $n = 8$ the sign changes according to the chirality of the spinor, since the volume element acts on $\Delta_\pm$ by $\pm\mathrm{id}$. ■

If g and b are induced by ρ we will also use the sloppier notation $\square_\rho$ or drop the subscript altogether. Nevertheless it is important to bear in mind that the $\square$ operator is really induced by the choice of a generalised oriented metric structure. Consequently, if this generalised metric gets conjugated by an element in $A \in O(7,7)$ or $O(8,8)$, then the $\square$-operator will transform naturally for the lift $\widetilde{A} \in Pin(7,7)$ or $Pin(8,8)$ which means that

$$\square_{\widetilde{A}\bullet\rho}\widetilde{A} \bullet \rho = \widetilde{A} \bullet \square_\rho \rho. \tag{1.41}$$

For concrete computations it is useful to have an explicit description of the structure forms. Taking our standard representation R of *Cliff*($\mathbb{R}^7$) or *Cliff*($\mathbb{R}^8$), we can readily compute the coefficients from the description as a tensor product by using the following formula.

Lemma 1.28. *Extend g to an inner product on Λ^* (which we still denote by g) and let I be a multi-index. Then*

$$g(L(\varphi \otimes \psi), e_I) = \frac{1}{16} q(R(e_I) \cdot \psi, \varphi).$$

This lemma is a direct consequence of Theorem 13.73 in [25]. The process of passing from the tensor product to the exterior algebra is also referred to as *fierzing* in the physics literature – we saw an instance of this procedure in Section 1.2 where we related the square of a unit spinor with a form. To ease the notation we assume our map L to be rescaled by $1/16$ so that we can henceforth discard this factor. Now suppose we are given two unit spinors ψ_+ and ψ_-, say in the 7-dimensional case. There is a pair of orthonormal spinors $\widetilde{\psi}_0, \widetilde{\psi}_1$ with $\psi_+ = \widetilde{\psi}_0$ and $\psi_- = c\widetilde{\psi}_0 + s\widetilde{\psi}_1$. As the notation suggests, this is merely the projection onto

$\widetilde{\psi}_0$ and $\widetilde{\psi}_1$ whose coefficients are given by the sine and cosine of the angle $\sphericalangle(\psi_+, \psi_-)$. By Theorem 14.69 in [25], the action of $Spin(7)$ on the Stiefel variety $V_2(\Delta)$, the set of pairs of orthonormal spinors, is transitive. Hence, there is an $w \in Spin(T, g)$ such that $\widetilde{\psi}_0 = w \cdot \psi_0$ and $\widetilde{\psi}_1 = w \cdot \psi_1$, where, $\psi_0 = 1, \psi_1 = i, \ldots, \psi_7 = e \cdot k$ is the standard basis of $\mathbb{O}$ which we use in our explicit matrix representation. Since $L^{ev,od}(w \cdot \psi_0 \otimes w \cdot \psi_1) = \pi_0(w)^* L^{ev,od}(\psi_0 \otimes \psi_1)$, we may assume that $\psi_+ = \psi_0$ and $\psi_- = c\psi_0 + s\psi_1$. In the case $n = 8$, we have to distinguish between the odd and even case. $Spin(8)$ acts transitively on $V_2(\Delta_+)$ and hence for structures of even type we can apply the same argument as before. In the odd case we appeal again to Theorem 14.69 in [25] to see that the action of $Spin(8)$ on $\Delta_+ \times \Delta_-$ is transitive on the product of spheres $S^7 \times S^7$. Hence we may assume that ψ_+ and ψ_- are both unit spinors which are orthogonal to each other. The computation of the normal form is now straightforward.

Proposition 1.29.

(i) *Let $\rho^{ev,od} \in \Lambda^{ev,od} T^{7*}$ be a form stabilised by $G_2 \times G_2$. Then there exists a unique $b \in \Lambda^2$ and $f \in \mathbb{R}$ with*

$$\rho = e^{-f} e^{b/2} \wedge \rho_0^{ev,od}$$

and an orthonormal basis $e_1, \ldots, e_7$ such that

$$\begin{aligned} \rho_0^{ev} \quad = \quad & \cos(a) + \sin(a)(-e_{23} - e_{45} + e_{67}) + \\ & + \cos(a)(-e_{1247} + e_{1256} + e_{1346} + e_{1357} - e_{2345} + e_{2367} + e_{4567}) + \\ & + \sin(a)(e_{1246} + e_{1257} + e_{1347} - e_{1356}) - \sin(a) e_{234567} \end{aligned}$$

and

$$\begin{aligned}\rho_0^{od} &= \sin(a)e_1 + \sin(a)(e_{247} - e_{256} - e_{346} - e_{357}) + \\ &+ \cos(a)(e_{123} + e_{145} - e_{167} + e_{246} + e_{257} + e_{347} - e_{356}) + \\ &+ \sin(a)(-e_{12345} + e_{12367} + e_{14567}) + \cos(\alpha)e_{1234567},\end{aligned}$$

where $a = \sphericalangle(\psi_+, \psi_-)$.

(ii) *Let* $\rho \in \Lambda^{ev}T^{8*}$ *be an even form stabilised by* $Spin(7) \times Spin(7)$. *Then there exists a unique* $b \in \Lambda^2$ *and* $f \in \mathbb{R}$ *with*

$$\rho = e^{-f}e^{b/2} \wedge \rho_0^{ev}$$

and an orthonormal basis $e_0, \ldots, e_7$ *such that*

$$\begin{aligned}\rho_0^{ev} &= \cos(a) + \sin(a)(e_{01} - e_{23} - e_{45} + e_{67}) + \\ &+ \cos(a)(e_{0123} + e_{0145} - e_{0167} + e_{0246} + e_{0257} + e_{0347} - e_{0356} - \\ &- e_{1247} + e_{1256} + e_{1346} + e_{1357} - e_{2345} + e_{2367} + e_{4567}) + \\ &+ \sin(a)(e_{0247} - e_{0256} - e_{0346} - e_{0357} + e_{1246} + e_{1257} + e_{1347} - e_{1356}) + \\ &+ \sin(a)(-e_{012345} + e_{012367} + e_{014567} - e_{234567}) + \cos(a)e_{01234567},\end{aligned}$$

where $a = \sphericalangle(\psi_+, \psi_-)$.

(iii) *Let* $\rho \in \Lambda^{od}T^{8*}$ *be an odd form stabilised by* $Spin(7) \times Spin(7)$. *Then there exists a unique* $b \in \Lambda^2$ *and* $f \in \mathbb{R}$ *with*

$$\rho = e^{-f}e^{b/2} \wedge \rho_0^{od}.$$

There exists an orthonormal basis $e_0, \ldots, e_7$ *such that*

$$\begin{aligned}\rho_0^{od} &= -e_0 + e_{123} + e_{145} - e_{167} + e_{246} + e_{257} + e_{347} - e_{356} + \\ &+ e_{01247} - e_{01256} - e_{01346} - e_{01357} + e_{02345} - e_{02367} - e_{04567} + e_{1234567}.\end{aligned}$$

If the spinors are not parallel, we can express the homogeneous components of ρ in terms of the invariant forms of the intersection $G_{2+} \cap G_{2-} = SU(3)$, $Spin(7, \psi_+)_+ \cap Spin(7, \psi_-)_+ = Spin(6)$ or $Spin(7)_+ \cap Spin(7)_- = G_2$ (cf. Section 1.2).

Corollary 1.30. *In terms of the underlying $SU(3)$- and G_2-invariants (cf. Section 1.2, in particular Table 1.1), we find for these 3 cases (letting s and c be shorthand for the (co-)sine of a)*

(i) *generalised G_2*

$$\rho_0^{ev} = c + s\omega - c(\psi_- \wedge \alpha + \frac{1}{2}\omega^2) + s\psi_+ \wedge \alpha - \frac{1}{6}s\omega^3$$

and

$$\rho_0^{od} = s\alpha - c(\psi_+ + \omega \wedge \alpha) - s\psi_- - \frac{1}{2}s\omega^2 \wedge \alpha + cvol_g.$$

(ii) *generalised $Spin(7)$, even type*

$$\begin{aligned} \rho_0^{ev} &= c + s\varpi + c(\Omega_1 - \frac{1}{2}\varpi^2) + s\Omega_2 - \frac{s}{3}\varpi^3 + cvol_g \\ &= \gamma \wedge \rho_0^{od,G_2} + \rho_0^{ev,G_2}, \end{aligned}$$

where ρ_0^{ev,G_2} and ρ_0^{od,G_2} are the forms defined in (i).

(iii) *generalised $Spin(7)$, odd type*

$$\begin{aligned} \rho_0^{od} &= -\gamma - \varphi + \gamma \wedge \star_7\varphi + \frac{1}{7}\varphi \wedge \star_7\varphi \\ &= \gamma \wedge (-1 + \star_7\varphi) - \varphi + vol_7 \end{aligned}$$

where $\star$ and vol_7 are taken with respect to the metric g restricted to $\gamma^\perp$.

Remark: The 1-form $s\alpha$ of the generalised G_2-structure form is the dual of the vector field induced by ψ_+ and the projection of ψ_- onto the line $\mathbb{R}\widetilde{\psi}_1$. Similarly the 1-form $-\gamma$ in (iii) is the dual of the vector field x such that $x \cdot \psi_+ = \psi_-$.

Since B-fields act as isometries on the spin spaces $\Lambda^{ev,od}$ we obtain the following relation between the dilaton f, the induced Riemannian volume form vol_g and the "norm" $q(\rho, \Box_\rho \rho)$ of the spinor ρ.

Corollary 1.31.

(i) *If* $\rho = e^{-f} L_b^{ev}(\psi_+ \otimes \psi_-)$ *defines a generalised* G_2*-structure, then*

$$q(\rho, \Box_\rho \rho) = 8e^{-2f} vol_g.$$

(ii) *If* $\rho = e^{-f} L_b(\psi_+ \otimes \psi_-)$ *defines a generalised* $Spin(7)$*-structure of even or odd type, then*

$$q(\rho, \Box_\rho \rho) = 16e^{-2f} vol_g.$$

1.6 Stable forms

Let T be an n-dimensional real vector space and consider a form ρ in $\Lambda^p T^*$ or $\Lambda^{ev,od} T^*$. We regard these spaces as $GL(n)$- or $\mathbb{R}^\times \times Spin(n,n)$-modules under their natural action respectively and refer to the $GL(n)$-case as *classical* and to the $\mathbb{R}_{>0} \times Spin(n,n)$-case as *generalised*.

Definition 1.7. ρ *in* $\Lambda^p T^*$ *or* $\Lambda^{ev,od} T^*$ *is said to be* stable *if it lies in an open orbit under the action of* $GL(n)$ *or* $\mathbb{R}_{>0} \times Spin(n,n)$.

We refer to a geometrical structure defined by a stable form also as a *variational structure* for reasons that become apparent in Chapter 3. As pointed out in [29], save for the obvious cases $(n \in \mathbb{N}, p = 1)$ and $(n \in 2\mathbb{N}, p = 2)$ (the orbit of a *symplectic* form), stability is an exceptional phenomenon restricted to low dimensions since $\dim GL(T) = n^2$ or $\dim \mathbb{R}_{>0} \times Spin(n,n) = n(n-1)/2+1$ is usually much smaller than $\dim \Lambda^p T^* = n!/p!(n-p)!$ or $\dim \Lambda^{ev,od} T^* = 2^{n-1}$.

A complete classification of representation spaces that admit an open orbit under the action of a connected reductive complex Lie group was given in [42]. Note also that if there is an open orbit under the $GL(T)$ action in $\Lambda^p T^*$ then there is also one in the dual space $\Lambda^p T \cong \Lambda^{n-p} T^* \otimes \Lambda^n T$ and consequently in $\Lambda^{n-p} T^*$ [29].

Let us consider some examples that are relevant to us.

Example:

(i) Fix a G_2-structure over $T = \mathbb{R}^7$ defined by the 3-form φ. The dimension of its orbit is

$$\dim GL(7) - \dim G_2 = 35 = \dim \Lambda^3 T^*,$$

and hence the form is stable in accordance to our earlier use of this notion.

(ii) Take the case of a spin 3/2 supersymmetric map induced by a $\mathfrak{su}(3)$-structure on $T = \mathbb{R}^8$. The stabiliser of the 3-form ρ inside $GL_+(8)$ is conjugate to $PSU(3)$ and so the orbit is open:

$$\dim GL(8) - \dim PSU(3) = 56 = \dim \Lambda^3 T^*.$$

Note, however, that 3-forms inducing an orientation-preserving isometry are not stable.

(iii) Finally, consider a generalised G_2-form $\rho \in \Lambda^{ev,od} T^*$ over $T = \mathbb{R}^7$. Its stabiliser is conjugate to $G_2 \times G_2$ inside $Spin(7,7)$ and thus

$$\dim \mathbb{R}_{>0} \times Spin(7,7) - 2 \dim G_2 = 64 = \dim \Lambda^{ev} T^*.$$

Hence, if we let the real numbers act by rescaling then we get an open orbit under the action of $\mathbb{R}_{>0} \times Spin(7,7)$.

Remark: Introducing the additional action of $\mathbb{R}_{>0}$ to obtain an open orbit seems *ad hoc* at first glance. It appears, however, already in the classical cases, albeit in disguise. In the classical cases the forms are homogeneous and the scalar matrices in $GL(T)$ act by rescaling.

The important feature all of the previous examples share is that the respective structure groups give rise to a natural metric. There are other examples – symplectic forms, a special 3-form on $\mathbb{R}^6$ with stabiliser $SL(3,\mathbb{C})$ and a generalised version with stabiliser $SU(3,3) \leqslant Spin(6,6)$ which lead to non-metrical structures [28], [30].

Moreover all these structures we have mentioned (metrical or not) give rise to a canonically defined volume form and thus fit into a general formalism set up in [29]. We shall explain the classical case first. Let U denote the open orbit under the $GL(T)$ action. Then we can construct a smooth, $GL(T)$-equivariant map

$$\phi : U \subset \Lambda^p T^* \to \Lambda^n T^*.$$

Taking a scalar matrix in $GL(T)$, the equivariance of ϕ implies that

$$\phi(\lambda^p \rho) = \lambda^n \phi(\rho).$$

Therefore ϕ is necessarily homogeneous of degree n/p.

Since ϕ is smooth we can consider its derivative $D\phi_\rho$ at $\rho \in U$ which is an invariant element in $(\Lambda^p T^*)^* \otimes \Lambda^n T^*$. Now $(\Lambda^p T^*)^* \cong \Lambda^{n-p} T^* \otimes \Lambda^n T$ so that the derivative lives in $\Lambda^{n-p} T^*$. It follows that there exists a unique element $\hat{\rho} \in \Lambda^{n-p} T^*$ for which

$$D\phi_\rho(\dot{\rho}) = \hat{\rho} \wedge \dot{\rho}.$$

We call the form $\hat{\rho}$ the *companion* of ρ. To determine what $\hat{\rho}$ is we can take $\dot{\rho} = \rho$ and apply Euler's formula to get

$$\hat{\rho} \wedge \rho = \frac{n}{p} \phi(\rho).$$

In particular, we see that $\hat{\rho}$ is also invariant under the stabiliser of ρ.

Example:

(i) In the case of a symplectic form ω over $\mathbb{R}^{2m}$ one can directly compute the differential $D\phi$ to find $\hat{\omega} = \omega^{m-1}/(m-1)!$ [29].

(ii) In the G_2- and $PSU(3)$-case we have to look at the elements in Λ^4 and Λ^5 which are stabilised by G_2 and $PSU(3)$ – up to a factor this is just $\star\rho$. As we can always rescale the volume map ϕ, we may assume that in these cases $\hat{\rho} = \star\rho$ [29].

Next we consider the action of $\mathbb{R}_{>0} \times Spin(n,n)$ on the spin modules $\Lambda^{ev,od}$. Here, we are left with the cases $n = 6$ and $n = 7$. The case $n = 6$ with stabiliser isomorphic to $SU(3,3)$ leads to a so-called *generalised Calabi-Yau* structure and was dealt with in [30]. We will focus on generalised G_2-structures. Let U be the open orbit U in Λ^{ev} or Λ^{od}. We define

$$\phi : \rho \in U \mapsto q(\Box_\rho \rho, \rho) \in \Lambda^7 T$$

and since $\Box$ transforms naturally under the action of $Spin(7,7)$ (cf. (1.41)), we immediately conclude

Proposition 1.32. *ϕ is homogeneous of degree 2 and $Spin(7,7)$-invariant.*

Remark: The existence of such invariants for complex Lie groups acting with an open orbit holds in general [42] and follows from purely algebraic considerations. In this context, an explicit formula of the invariant for $Spin(14,\mathbb{C})$ was given in [24]. However, in view of the variational formalism (see Section 3.1), this description proves to be rather cumbersome for our purposes which motivated our approach in terms of G-structures.

The differential of this map is obtained in the same way as for classical G_2-structures. Since the form q is non-degenerate, we can write

$$D\phi_\rho(\dot{\rho}) = q(\hat{\rho}, \dot{\rho}).$$

for a unique $\hat{\rho} \in \Lambda^{od}$. The $Spin(7,7)$-invariance of ϕ then implies that $\hat{\rho}$ is a $G_2 \times G_2$ invariant and by rescaling ϕ appropriately we conclude that $\hat{\rho} = \Box_\rho \rho$.

Chapter 2

Topology

This chapter deals with global issues of classical and generalised supersymmetric structures.

First we consider the case of $PSU(3)$-structures and exhibit obstructions to their existence. $PSU(3)$-manifolds are spinnable and the key feature we use is the natural identification between the tangent and the spinor bundle which imposes quite severe constraints on the topology of the underlying manifold. As a consequence, the topological constraints we derive hold in the more general situation where the tangent bundle coincides with the spinor bundles Δ_+ and Δ_-. We call such a structure *doubly supersymmetric*. The $PSU(3)$-case is special in the sense that it admits an orthogonal product and in particular, we can establish the existence of four linearly independent vector fields (Proposition 2.8). Basic examples of $PSU(3)$-manifolds are obtained from topological $SU(3)$-structures ($SU(3)$ acting in its adjoint representation) through the projection $SU(3) \to PSU(3)$. Conversely, a $PSU(3)$-structure induces such an $SU(3)$-structure if a certain cohomology class – the so-called *triality class* – vanishes (Section 2.1.2). We will give necessary and sufficient conditions for a $PSU(3)$-structure with vanishing triality class to exist over a compact manifold (Theorem 2.11).

In the second part of the chapter we investigate the topology of generalised structures. The two unit spinors of these structures induce reductions to their respective stabilisers, resulting in two subbundles associated with G_2 or $Spin(7)$ inside the orthonormal frame bundle. Necessary and sufficient conditions for their existence easily follow from the classical G_2- or $Spin(7)$-case (Proposition 2.15). Generalised G_2-structures are classified by the top cohomology module $H^7(M,\mathbb{Z})$ (Theorem 2.18). If M is compact, this yields an integer invariant which essentially counts the number of points where the two G_2-subbundles inside the orthonormal frame bundle coincide. We also discuss generalised $Spin(7)$-structures, but these are harder to analyse on dimensional grounds. We conclude with some examples.

2.1 The topology of $PSU(3)$-structures

2.1.1 Obstructions to the existence of $PSU(3)$-structures

Definition 2.1. *A Riemannian 8-dimensional spin manifold is said to be* doubly supersymmetric *if and only if the tangent bundle $T = \Lambda^1$ and the spinor bundles Δ_+ and Δ_- are associated with a principal G-fibre bundle such that there exist G-invariant isomorphisms between any two of the three bundles Λ^1, Δ_+ or Δ_-, i.e. $T = \Delta_+ = \Delta_-$.*

As an example, consider an orientable 8-manifold M which comes equipped with a stable 3-form ρ. This means that at any point $x \in M$ the form $\rho_x \in \Lambda^3 T^*_x$ lies in the open orbit isomorphic to $GL_+(8)/PSU(3)$. The existence of such a stable form requires the structure group of M to reduce to $PSU(3)$. As we have seen in the first chapter, this implies the existence of a (then globally) defined $PSU(3)$-invariant isometry $\gamma_\pm : T \to \Delta_\pm$ which identifies T, Δ_+ and Δ_-. Another example which will be important in the sequel are generalised $Spin(7)$-structures of *odd* type to be defined in Section 2.2. As pointed out in Section 1.4, the existence of unit spinors in Δ_+ and Δ_- implies a reduction to G_2 for which T, Δ_+ and Δ_- are isomorphic as G_2-representation spaces. In this section we are primarily interested in the $PSU(3)$-case, but the results hold in general for any doubly supersymmetric structure.

Definition 2.2. *A* topological $PSU(3)$-structure *on M is a reduction from the frame bundle to a principal $PSU(3)$-fibre bundle via the inclusion $Ad : PSU(3) \hookrightarrow GL(8)$.*

The Lie algebra $\mathfrak{su}(3)$, the symmetric space $SU(3) = SU(3) \times SU(3)/SU(3)$ and its non-compact symmetric dual $SL(3,\mathbb{C})/SU(3)$ provide trivial examples of topological $PSU(3)$-structures defined by the bi-invariant 3-form $\rho(X,Y,Z) = B(X,[Y,Z])$, where B denotes again the Killing form.

Remark: Since $PSU(3)$ is a subgroup of $SO(8)$ and $Spin(8)$ (cf. Corollary 1.7), every topological $PSU(3)$-structure induces

- an orientation
- a metric g
- an associated Hodge $\star$-operator
- a spin structure $P_{Spin(8)} = P_{PSU(3)} \times Spin(8)$
- an orthogonal multiplication $\times : T \otimes T \to T$ (cf. (1.27)).

In this section, we will reserve the letter P for a G-principal fibre bundle which induces a doubly supersymmetric structure. The existence of a reduction to G-principal fibre bundle is topologically obstructed. Generally speaking, if we have a G-structure over M with classifying map $f : M \to BG$, then there exists a reduction to a subgroup $i : H \to G$ if and only if there exists a map $g : M \to BH$ such that the diagram in Figure 2.1 commutes. Now

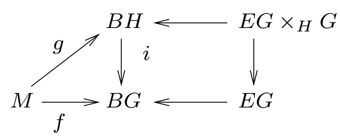

Figure 2.1:

assume that C is a characteristic class in $H^*(BG, R)$ such that $i^*C = 0$ in $H^*(BH, R)$. Then $f^*C = g^*i^*C$ has to vanish in $H^*(M, R)$. A classical example is the reduction of $SO(n)$ to $SO(n-1)$. In this case, the Euler class of M has to be zero.

It is known [37], [47] that the cohomology rings for $BSpin(8)$ are

$$H^*(BSpin(8), \mathbb{Z}_2) \cong \mathbb{Z}_2[w_4(E), w_6(E), w_7(E), w_8(E), \epsilon(E)]$$

and

$$H^*(BSpin(8), \mathbb{Z}) \cong \mathbb{Z}[q_1(E), q_2(E), e(E), \delta w_6(E)]/\langle 2\delta w_6(E)\rangle$$

where ϵ and the so-called *spin characteristic classes* $q_1(E)$ and $q_2(E)$ are defined through the relations

$$p_1(E) = 2q_1(E),\ p_2(E) = q_1^2(E) + 2e(E) + 4q_2(E) \text{ and } \rho_2(q_2(E)) = \epsilon(E).$$

Here, $\rho_2 : H^*(BSpin(8), \mathbb{Z}) \to H^*(BSpin(8), \mathbb{Z}_2)$ is the morphism induced from the reduction mod 2.

In our situation, we also have the outer automorphisms κ and λ to play with (cf. Figure 1.2). Let $E = ESpin(8)$ denote the universal $Spin(8)$-bundle. The automorphisms κ and λ define further $Spin(8)$-bundles over $BSpin(8)$ by $E_\kappa = E \times_\kappa Spin(8)$ and $E_\lambda = E \times_\lambda Spin(8)$. They induce two classifying maps $BSpin(8) \to BSpin(8)$ which we still denote by κ and λ, that is, we have $E_\kappa \cong \kappa^* E$ and $E_\lambda \cong \lambda^* E$. Let $p_i(E)$, $w_i(E)$ and $e(E)$ denote the i-th Pontrjagin, the i-th Stiefel-Whitney and the Euler class of the universal spin bundle E. The corresponding characteristic classes for M will be denoted by p_i, w_i and e which are the pull-back of the universal classes under the classifying map $f : M \to BSpin(8)$. Now $\pi_+ = \pi_0 \circ \lambda^2 \circ \kappa$ and $\pi_- = \pi_0 \circ \lambda^2 \circ \kappa \circ \lambda^2$ by (1.9), so we have

$$C(\Delta_+) = f^* \circ (\lambda^2 \circ \kappa)^* C(E), \quad C(\Delta_-) = f^* \circ (\lambda^2 \circ \kappa \circ \lambda^2)^* C(E),$$

where C denotes any characteristic class.

Lemma 2.1. [48] *For κ, $\lambda : BSpin(8) \to BSpin(8)$, we have*

$$\begin{aligned} \kappa^*(q_2(E)) &= \quad q_2(E) + e(E) & \lambda^*(q_2(E)) &= -e(E) - q_2(E) \\ \kappa^*(e(E)) &= -e(E) & \lambda^*(e(E)) &= \quad q_2(E). \end{aligned}$$

Since the bundles T, Δ_+ and Δ_- coincide, we derive

$$e = f^* e(E) = f^* \circ (\lambda^2 \circ \kappa)^* e(E) = -f^* q_2(E) = -q_2$$

and

$$e = f^* e(E) = f^* \circ (\lambda^2 \circ \kappa \circ \lambda^2)^* e(E) = -f^* e(E) = -e.$$

Since $H^8(M, \mathbb{Z})$ has no torsion, e and thus q_2 must be zero. Furthermore [47], $q_2 = q_2(\Delta_+)$ is uniquely determined by the relation

$$16 q_2(\Delta_+) = 4 p_2(\Delta_+) - p_1^2(\Delta_+) - 8 e(\Delta_+)$$

and therefore we obtain as a

Corollary 2.2. *If M^8 is doubly supersymmetric, then*

$$w_1 = w_2 = 0$$

$$e = 0$$

$$4p_2 = p_1^2.$$

Since the degree of the first non-vanishing Stiefel-Whitney class must always be a power of 2 (cf. 8-B in [36]), we also get that $w_3 = 0$.

Remark: Corollary 2.2 fails to hold if we merely assume a "simply" supersymmetric structure. For example, a $Spin(7)$-structure which stabilises a unit spinor $\Psi_+ \in \Delta_+$ yields a $Spin(7)$-equivariant isometry $X \in T \mapsto X \cdot \Psi_+ \in \Delta_-$, but this does not imply the vanishing of the Euler class (see also [34] Subsection IV.10 or Proposition 2.15).

As an application of Corollary 2.2, we prove the following

Proposition 2.3. *Let G be a simple compact Lie group and $(G/H, g)$ a Riemannian homogeneous space which is doubly supersymmetric. If $M = G/H$ has vanishing Euler class, then G/H is diffeomorphic to $SU(3)$.*

Proof: Since G sits inside the isometry group of (M, g), its dimension is less than or equal to $9 \cdot 8/2 = 36$. If we had equality, then M would be diffeomorphic to a torus or – up to a finite covering – to the 8-sphere ([53], Corollary 11.6.3). While the first case is excluded for G has to be simple, the second case is impossible since $e(S^8) \neq 0$. Hence G must be – up to a covering – a group of type $A_1, \ldots, A_5$, B_2, B_3, C_3, D_4 or G_2. As a closed subgroup of G, H is compact and hence reductive. Therefore H is covered by a direct product of simple Lie groups and a torus, that is the Lie algebra of H is isomorphic to

$$\mathfrak{h} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k \oplus \mathfrak{t}^l.$$

If we denote by rk the rank of a Lie group, we get the following necessary conditions.

$$k \leq rk(G)$$
$$l + \sum rk(\mathfrak{g}_i) \leq rk(G)$$
$$l + \sum dim(\mathfrak{g}_i) = dim(G) - 8,$$

which yields the possibilities displayed in Table 2.1. It follows that H is of maximal rank,

G	H up to a covering	$dim(H)$	$rk(H)$
A_2	$\{1\}$	0	0
A_3	$A_1 \times A_1 \times S^1$	7	3
A_4	$A_3 \times S^1,\ G_2 \times S^1 \times S^1,\ A_2 \times A_2$	16	4
A_5	$A_1 \times A_4,\ A_1 \times A_1 \times B_3,\ A_1 \times A_1 \times C_3$	27	5
B_2	$S^1 \times S^1$	2	2
B_3	$A_1 \times A_2$	13	3
C_3	$A_1 \times A_2$	13	3
D_4	$A_1 \times A_1 \times G_2$	20	4
G_2	$A_1 \times A_1$	6	2

Table 2.1:

that is $rk(H) = rk(G)$, unless $G = SU(3)$ and $H = \{1\}$. But the first case would imply that $e(G/H) \neq 0$ [41] which is impossible by Corollary 2.2. ∎

Since the Euler class is zero, there exists a nowhere vanishing vector field. In fact we are going to see that if M is closed and carries a $PSU(3)$-structure, then there exist four linearly independent vector fields. Let $[M]$ denote the fundamental class of M. We start with the

Lemma 2.4. *If M^8 is a compact spin manifold such that $p_1^2 = 4p_2$, then*

$$sgn(M) = 16\hat{A}[M].$$

In particular, $sgn(M) \equiv 0 \mod 16$.

Proof: We have $sgn(M) = (7p_2 - p_1^2)[M]/45 = p_2[M]/15$. On the other hand,

$$\hat{A}[M] = \frac{1}{2^7 \cdot 45}(7p_1^2 - 4p_2)[M] = \frac{1}{2^4 \cdot 15}p_2[M] = \frac{1}{16}sgn(M).$$

Since M^8 is spin, $\hat{A}[M]$ is the index of the Dirac operator and is therefore an integer. This implies the second assertion. ■

If (b_4^+, b_4^-) denotes the signature of the Poincaré pairing on H^4, then $sgn(M) = b_4^+ - b_4^-$. As a result of this and the previous lemma we obtain

Corollary 2.5. *Let M^8 be a compact simply-connected doubly supersymmetric manifold. If $\hat{A}[M] = 0$ (e.g. if there exists a metric with strictly positive scalar curvature), then*

$$1 + b_2 + b_4^+ = b_3.$$

For example, $H^*(SU(3), \mathbb{R})$ is isomorphic to the space of invariants in $\Lambda^*\mathfrak{su}(3)$ since $SU(3)$ is symmetric. Therefore, $b_2 = 0$, $b_3 = 1$ and $b_4^+ = 0$ in accordance with the corollary.

As $e(M) = 0$ and $sgn(M) \equiv 0 \mod 4$, we can assert the existence of two linearly independent vector fields X and Y ([46]. If the doubly-supersymmetric structure is induced by a $PSU(3)$-structure, then the orthogonal product $\times$ (1.27) produces a third non-vanishing vector field. In particular, this causes the sixth Stiefel-Whitney class of M^8 to be zero. We are then in a position to apply the following Proposition

Proposition 2.6. [50] *Let M be a closed connected smooth spin manifold of dimension 8. If $w_6(M) = 0$, $e(M) = 0$ and $(4p_2(M) - p_1^2(M))[M] \equiv 0 \bmod 128$ and there is a $k \in \mathbb{Z}$ such that $4p_2(M) = (2k-1)^2 p_1^2(M)$ and $k(k+2)p_2(M)[M] \equiv 0 \bmod 3$, then M has 4 linearly independent vector fields.*

Taking $k = 0$ establishes the existence of four linearly independent vector fields. As a consequence, the fifth Stiefel-Whitney class has to vanish.

Proposition 2.7. *We have $w_4^2 = 0$. In particular, all the Stiefel-Whitney numbers have to vanish.*

Proof: Note that by Wu's formula, we get

$$\mathrm{Sq}^k(w_m) = w_k w_m + \binom{k-m}{1} w_{k-1} w_{m+1} + \ldots + \binom{k-m}{k} w_0 w_{m+k},$$

where

$$\binom{x}{i} = \frac{x(x-1) \cdot \ldots \cdot (x-i+1)}{i!}.$$

A further theorem of Wu asserts that

$$w_k = \sum_{i+j=k} \mathrm{Sq}^i(v_j),$$

with elements $v_k \in H^k(M, \mathbb{Z}_2)$ defined through the identity $v_k \cup x[M] = \mathrm{Sq}^k(x)[M]$ which has to hold for any $x \in H^{n-k}(M, \mathbb{Z}_2)$. In particular, we have $v_i = 0$ for $i > 4$. It follows that $v_1 = v_2 = v_3 = 0$, $w_4 = v_4$ and $w_8 = \mathrm{Sq}^4 w_4 = w_4^2 = 0$. ∎

We summarise our results in the following theorem.

Theorem 2.8. *For a topological $PSU(3)$-structure over a closed and oriented 8-manifold M to exist, the following conditions have to hold.*

$$w_i = 0 \text{ except for } i = 4 \text{ and } w_4^2 = 0$$

$$e = 0$$

$$p_1^2 = 4p_2.$$

In particular, there exist four linearly independent vector fields on M and all Stiefel-Whitney numbers vanish.

We close this section by a brief remark about supersymmetric structures of type 3. As follows from the discussion in 1.3.3, their existence is tied to almost quaternionic structures (i.e. a reduction from $SO(8)$ to $Sp(1) \cdot Sp(2)$). This problem was settled in [49]. In particular, the authors proved the following result.

Proposition 2.9. [49] *Let ξ be an oriented 8-dimensional vector bundle over a closed connected smooth spin manifold M. If there is $R \in H^4(M, \mathbb{Z})$ such that the conditions*

- $Sq^2 \rho_2 R = 0$
- $(Rp_1 - 2R^2)[M] \equiv 0 \bmod 16$
- $w_2(\xi) = 0$
- $w_6(\xi) = 0$
- $4p_2(\xi) - p_1^2(\xi) - 8e(\xi) = 0$
- $(p_1^2(\xi) - p_1 p_1(\xi) - 8e(\xi) + 8R^2 + 4Rp_1(\xi) + 4Rp_1)[M] \equiv 0 \bmod 32$

are satisfied, then the structure group of ξ can be reduced to $Sp(1) \cdot Sp(2)$. If $H^2(M, \mathbb{Z}_2) = 0$, then all the previous conditions are also necessary.

If $\xi = T$, the result can be refined to

Proposition 2.10. [49] *Let M be an oriented closed connected smooth manifold of dimension 8. If*

- $w_2 = 0$
- $w_6 = 0$
- $4p_2 - p_1^2 - 8e = 0$

and there exists an $R \in H^4(M, \mathbb{Z})$ such that

- $Sq^2 \rho_2 R = 0$
- $(Rp_1 - 2R^2)[M] \equiv 0 \bmod 16$
- $(R^2 + Rp_1 - e)[M] \equiv 0 \bmod 4,$

then M carries an almost quaternionic Kähler structure. The first and third condition are always necessary while the remaining ones are necessary if $H^2(M, \mathbb{Z}_2) = 0$.

These propositions give similar necessary conditions as Theorem 2.8. Sufficient conditions for $PSU(3)$-structures shall occupy us next.

2.1.2 $PSU(3)$-structures and the triality class

Let M^8 be again a connected, closed and spinnable 8-manifold. Consider a complex rank 3 vector bundle E with vanishing first Chern class, i.e. E is associated with a principal $SU(3)$-fibre bundle. If the adjoint bundle $\mathfrak{su}(E) = P_{SU(3)} \times \mathfrak{su}(3)$ is isomorphic to the tangent bundle, then T is associated with a $PSU(3)$-structure coming from the projection $p : SU(3) \to PSU(3)$. However, not every $PSU(3)$-structure arises in this way. A basic Čech cohomology argument implies that principal G-fibre bundles over M are classified by $H^1(M, G)$ (see, for example, [34] Appendix A). The exact sequence (where $\mathbb{Z}_3$ is central)

$$1 \to \mathbb{Z}_3 \to SU(3) \xrightarrow{p} PSU(3) \to 1$$

gives rise to an exact sequence

$$\ldots \to H^1(M, \mathbb{Z}_3) \to \mathrm{Prin}_{SU(3)}(M) \xrightarrow{p_*} \mathrm{Prin}_{PSU(3)}(M) \xrightarrow{t} H^2(M, \mathbb{Z}_3).$$

(where $\mathrm{Prin}_G(M)$ denotes the set of G-principal bundles over M). Hence, a principal $PSU(3)$-bundle P is induced by an $SU(3)$-bundle if and only if the obstruction class $t(P) \in H^2(M, \mathbb{Z}_3)$ vanishes. Following [4] where the authors consider $PSU(3)$-structures over 4-manifolds, we call this class the *triality class*. By the universal coefficients theorem this obstruction vanishes trivially if $H^2(M, \mathbb{Z}) = 0$ and $H^3(M, \mathbb{Z})$ has no torsion elements of order divisible by three.

If $f : M \to BPSU(3)$ is a classifying map for P, then $t(P) = f^*t$ for the *universal triality class* $t \in H^2(BPSU(3), \mathbb{Z}_3)$. It is induced by $c_1(E_{U(3)})$, the first Chern class of the universal $U(3)$-bundle $E_{U(3)}$ [55]. Let $\overline{p} : U(3) \to PU(3)$ be the natural projection. Note that the inclusion $SU(3) \subset U(3)$ induces an isomorphism between $PSU(3)$ and $PU(3)$ and therefore identifies $BPSU(3)$ with $BPU(3)$. Since $BPU(3)$ is simply connected and $\pi_2(BU(3)) = \mathbb{Z} \to \pi_2(BPU(3)) = \mathbb{Z}_3$ is the reduction mod 3 map $\rho_3 : Z \to \mathbb{Z}_3$, the Hurewicz isomorphism theorem and the universal coefficients theorem imply that $H^2(BPU(3), \mathbb{Z}_3) = \mathbb{Z}_3$ and that

$$B\overline{p}^* : H^2(BPU(3), \mathbb{Z}_3) \to H^2(BU(3), \mathbb{Z}_3)$$

is an isomorphism. Then

$$t = (B\overline{p}^*)^{-1}\rho_{3*}c_1(E_{U(3)}).$$

If the triality class vanishes the problem of finding sufficient conditions for the existence of a $PSU(3)$-structure reduces to the problem for the existence of a complex rank 3 vector bundle E with $\mathfrak{su}(E) = T$. This question is settled in the next theorem.

Theorem 2.11. *Suppose that M is a connected, closed and spinnable 8-manifold. Then the frame bundle reduces to a principal $PSU(3)$-fibre bundle P with $t(P) = 0$ if and only if $e = 0$, $4p_2 = p_1^2$, $w_6 = 0$, p_1 is divisible by 6 and $p_1^2[M] \in 216\mathbb{Z}$.*

Proof: Let us first assume that there exists a $PSU(3)$-fibre bundle P coming from a reduction $Ad : SU(3) \to SO(8)$ with principal bundle $\widetilde{P}$. In virtue of Theorem 2.8 we only need to show the last two conditions. We define the complex vector bundle $E = \widetilde{P} \times_{SU(3)} \mathbb{C}^3$ so that $\mathfrak{su}(E) = T$, and compute the Pontrjagin classes of M. We have $\mathfrak{su}(3) \otimes \mathbb{C} = \mathfrak{sl}(3, \mathbb{C})$, hence $T \otimes \mathbb{C}$ equals $\mathrm{End}_0(E)$, the bundle of trace-free complex endomorphisms. The Chern character of $T \otimes \mathbb{C}$ equals (see for instance [38])

$$ch(T \otimes \mathbb{C}) = 8 + p_1 + \frac{1}{12}(p_1^2 - 2p_2).$$

On the other hand,

$$ch(End(E)) = ch(E \otimes \overline{E}) = ch(End_0(E)) + 1.$$

Now for a complex vector bundle with $c_1(E) = 0$,

$$ch(E) = 3 - c_2(E) + \frac{1}{2}c_3(E) + \frac{1}{12}c_2(E)^2$$

and $c_i(E) = (-1)^i c_i(\overline{E})$ which implies

$$ch(E \otimes \overline{E}) = ch(E) \cup ch(\overline{E}) = 9 - 6c_2(E) + \frac{3}{2}c_2(E)^2.$$

As a consequence

$$p_1 = p_1(\mathfrak{su}(E)) = -6c_2(E), \quad p_2 = p_2(\mathfrak{su}(E)) = 9c_2(E)^2. \tag{2.1}$$

In particular, p_1 is divisible by 6 and we also rederive the relation $4p_2 = p_1^2$. Moreover M is spinnable, hence the spin index $\hat{A} \cup \mathrm{ch}(E)[M]$ is an integer. Since

$$\hat{A} = 1 - p_1/24 + (-4p_2 + 7p_1^2)/5760 = 1 - p_1/24 + p_1^2/960$$

it follows

$$\hat{A} \cup ch(E)[M] = 3\hat{A}[M] + p_1 c_2(E)/24 + c_2(E)^2/12 = 3\hat{A}[M] - p_1^2/216[M] \in \mathbb{Z},$$

which means $p_1^2[M] \in 216\mathbb{Z}$, proving the necessity of the conditions.

For the proof of the converse I am indebted to ideas of M. Crabb [15]. Let $B \subset M$ be an embedded open disc in M and consider the exact sequence

$$K(M, M - B) \to K(M) \to K(M - B).$$

We have $K(M, M - B) = \widetilde{K}(S^8) \cong \mathbb{Z}$ and the sequence is split by the spin index

$$x \in K(M) \mapsto \hat{A} \cup \mathrm{ch}(x)[M] \in \mathbb{Z}$$

which therefore classifies the stable extensions over $M - B$ to M. The first step consists in finding a stable complex vector bundle ξ over $M - B$ such that $c_1(\xi) = 0$ and $\mathfrak{su}(3)$ is stably equivalent to $T_{|M-B}$. To that end, let $[(M - B)_+, BSU(\infty)] \subset K(M - B)$ denote the set of pointed homotopy classes, the subscript $+$ indicating a disjoint basepoint. Let (c_2, c_3) be the map which takes an equivalence class of $[(M - B)_+, BSU(\infty)]$ to the second and third Chern class of the associated bundle.

Lemma 2.12. *The image of the mapping*

$$(c_2, c_3) : [(M - B)_+, BSU(\infty)] \to H^4(M, \mathbb{Z}) \oplus H^6(M, \mathbb{Z})$$

is the set $\{(u, v) \mid Sq^2\rho_2 u = \rho_2 v\}$.

Proof: We first prove that for a complex vector bundle ξ with $c_1(\xi) = 0$, we have

$$Sq^2\rho_2 c_2(\xi) = \rho_2 c_3(\xi). \tag{2.2}$$

Now if W_i denote the Stiefel-Whitney classes of the *real* vector bundle underlying ξ this is equivalent to $Sq^2 W_4 = W_6$. On the other hand, Wu's formula implies

$$Sq^2 W_4 = W_2 W_4 + W_6$$

and thus (2.2) since $W_2 = \rho_2 c_1 = 0$. Next let $i : F \hookrightarrow K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$ denote the homotopy fibre of the induced map

$$Sq^2 \circ \rho_2 + \rho_2 : K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6) \to K(\mathbb{Z}_2, 6).$$

The relation (2.2) implies that the map $(c_2, c_3) : BSU(\infty) \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$ is null-homotopic. Consequently, (c_2, c_3) lifts to a map $k : BSU(\infty) \to F$, thereby inducing an isomorphism of homotopy groups $\pi_i(BSU(\infty)) \to \pi_i(F)$ for $i \leq 7$ and a surjection for $i = 8$. By the exact homotopy sequence for fibrations we conclude that $\pi_4(F) = \mathbb{Z}$, $\pi_6(F) = 2\mathbb{Z}$ and $\pi_i(F) = 0$ for i otherwise. On the other hand, the Chern class $c_2 : \widetilde{K}(S^4) = \mathbb{Z} \to H^4(S^4, \mathbb{Z}) = \mathbb{Z}$ is an isomorphism and $c_3 : \widetilde{K}(S^6) = \mathbb{Z} \to H^6(S^6, \mathbb{Z}) = \mathbb{Z}$ is multiplication by 2. Since $M - B$ is at most 8-dimensional, it follows that the induced map $k_* : [(M - B)_+, BSU(\infty)] \to [(M - B)_+, F]$ is surjective. The horizontal row in the commutative diagram

$$\begin{array}{ccccc} [(M-B)_+, F] & \xrightarrow{i_*} & H^4(M,\mathbb{Z}) \oplus H^6(M,\mathbb{Z}) & \xrightarrow{Sq^2\rho_2 + \rho_2} & H^6(M,\mathbb{Z}_2) \\ & \nwarrow^{k_*} & \uparrow (c_2, c_3) & & \\ & & [(M-B)_+, BSU(\infty)] & & \end{array}$$

is exact, hence $\mathrm{im}\,(c_2, c_3) = \mathrm{im}\, i_* = \ker\,(Sq^2 \circ \rho_2 + \rho_2)$. $\square$

By assumption $p_1 \in H^4(M,\mathbb{Z})$ is divisible by 6 and therefore we can write $p_1 = -6u$ for $u \in H^4(M,\mathbb{Z})$. On the other hand, $p_1 = 2q_1$, where q_1 is the first spin characteristic class and satisfies $\rho_2(q_1) = w_4$. Hence

$$Sq^2\rho_2(u) = Sq^2 w_4 = w_2 w_4 + w_6 = 0,$$

and the previous lemma implies the existence of a stable complex vector bundle ξ such that $c_1(\xi) = 0$, $c_2(\xi) = u$ and $c_3(\xi) = 0$. From (2.1) it follows that $p_1(\mathfrak{su}(\xi)) = p_1$ and since $w_2(\mathfrak{su}(\xi)) = 0$, $\mathfrak{su}(\xi)$ and T are stably equivalent over the 4-skeleton $M^{(4)}$ [54]. Then $\mathfrak{su}(\xi)$ and T are stably equivalent over $M - B$ as the restriction map $KO(M-B) \to KO(M^{(4)})$ is injective. This follows from the exact sequence

$$KO(M^{(i+1)}, M^{(i)}) \to KO(M^{(i+1)}) \to KO(M^{(i)}).$$

By definition $KO(M^{(i+1)}, M^{(i)}) = \widetilde{KO}(M^{(i+1)}/M^{(i)})$ and $M^{(i+1)}/M^{(i)}$ is a disjoint union of spheres S^{i+1}. But $\widetilde{KO}(S^{i+1}) = 0$ for $i = 4$, 5 and 6 and therefore the map $KO(M^{(i+1)}) \to KO(M^{(i)})$ is injective. Since $M = M^{(8)}$ is the disjoint union of $M^{(7)}$ and a finite number of open embedded discs, the assertion follows. Next we extend ξ over B to a stable bundle on M. The condition to be represented by a complex vector bundle E of rank 3 is $c_4(\xi) = 0$. As pointed out above, such a bundle exists if

$$\hat{A} \cup \mathrm{ch}(\xi)[M] = 3\hat{A}[M] + p_1 u/24 + u^2/12$$

is an integer and this holds by assumption. Next $p_2(\mathfrak{su}(\xi)) = 9u^2 = p_2$ and as a consequence, $\mathfrak{su}(\xi)$ is stably isomorphic to T [54]. Finally, two stably isomorphic oriented real vector bundles of rank 8 are isomorphic as $SO(8)$-bundles if they have the same Euler class. Since $e(\mathfrak{su}(E)) = 0$, we have $T = \mathfrak{su}(E)$. ■

Corollary 2.13. *If M is closed and carries a $PSU(3)$-structure with vanishing triality class, then*

$$\hat{A}[M] \in 40\mathbb{Z} \text{ and } sgn(M) \in 640\mathbb{Z}.$$

2.2 The topology of generalised exceptional structures

First we introduce the (rather obvious) global notion of a generalised metric.

Definition 2.3. *A* generalised metric *over a manifold M^n is an orthogonal decomposition*

$$T \oplus T^* = V_+ \oplus V_-$$

into a positive and negative subbundle with respect to the natural inner product $(\cdot\,,\cdot)$.

As we have seen in Section 1.5, a generalised metric structure corresponds to a unique pair (g, b) of a Riemannian metric g and a 2-form b. In particular, any Riemannian manifold defines trivially a generalised metric structure whose existence is therefore unobstructed. The additional choice of an orientation then gives rise to an $SO(n) \times SO(n)$ structure and in particular to a globally defined $\square$-operator.

Next, there is the natural notion of a generalised exceptional structure.

Definition 2.4.

(i) *A* topological generalised G_2-structure *over a 7-manifold M^7 is defined by an even or* *odd form ρ whose stabiliser under the action of $Spin(7,7)$ is conjugate to $G_2 \times G_2$ at any point. We will denote this structure by the pair (M^7, ρ) and call ρ the* structure form.

(ii) *A* topological generalised $Spin(7)$-structure of even or odd type *over an 8-manifold M^8 is defined by an even or odd form ρ whose stabiliser under the action of $Spin(8,8)$ is conjugate to $Spin(7) \times Spin(7)$ at any point. We will denote this structure by the pair (M^8, ρ) and call ρ the* structure form.

We refer to these structures also as topological generalised exceptional structures.

We will usually drop the adjective "topological" and simply refer to a generalised G_2- and $Spin(7)$-structures if there is no risk of confusion. Propositions 1.20 and 1.21 assert in particular the existence of a metric g or equivalently an $SO(7)$- or $SO(8)$-principal fibre bundle which admits a reduction to a G_2- or $Spin(7)$. The inclusions $G_2 \subset Spin(7)$ and $Spin(7) \subset Spin(8)$ determine a unique spin structure for which we can consider the associated spinor bundles Δ (over M^7) or Δ_+ and Δ_- (over M^8). From Propositions 1.20, 1.21 and 1.27 we deduce the following statement.

Theorem 2.14.

(i) *A topological generalised G_2-structure (M^7, ρ) is characterised by the following data,*

- *an orientation,*
- *a metric g,*
- *a 2-form b,*
- *two unit spinors $\Psi_+, \Psi_- \in \Delta$ and a function F such that*

$$e^{-F} L_b(\Psi_+ \otimes \Psi_-) = \rho + \Box_{g,b}\rho.$$

(ii) *An even or odd topological generalised* $Spin(7)$*-structure* (M^8, ρ) *for* $\rho \in \Lambda^{ev}$ *(even type) or* $\rho \in \Lambda^{od}$ *(odd type) is characterised by the following data,*

- *an orientation,*
- *a metric* g*,*
- *a 2-form* b*,*
- *two unit half spinors* $\Psi_+, \Psi_- \in \Delta$ *of either equal (even type) or opposite chirality (odd type) and a function* F *such that*

$$e^{-F} L_b(\Psi_+ \otimes \Psi_-) = \rho.$$

Remark: In the case of a generalised $Spin(7)$-structure we fix the orientation in such a way that Ψ_+ lies in Δ_+ unless otherwise stated.

Thus, a generalised exceptional structure may be regarded as a Riemannian structure whose orthonormal frame bundle admits two G_2- or $Spin(7)$-subbundles associated with the stabilisers of $\Psi_\pm$ in $Spin(7)$ or $Spin(8)$. The B-field b encapsulates the additional information in which copy of $Spin(7) \times Spin(7)$ or $Spin(8) \times Spin(8)$ inside $Spin(7,7)$ or $Spin(8,8)$ the stabiliser of ρ is actually sitting.

The most trivial example of a generalised structure would be a Riemannian spin manifold which admits a nowhere vanishing spinor $\Psi = \Psi_+ = \Psi_-$ and where we put $b = 0$, $F = 0$. A generalised structure defined by a single G_2- or $Spin(7)$-structure, possibly with a non-vanishing B-field b and dilaton F is said to be *straight* (in the generalised $Spin(7)$-case this obviously makes only sense for structures of even type). The existence of a nowhere vanishing spinor field in dimension 7 or 8 is a classical result ([34] Subsection IV.10).

Proposition 2.15.

(i) *A spinnable 7-fold M^7 always carries a nowhere vanishing spinor and hence admits a topological G_2-structure.*

(ii) *A differentiable 8-fold M^8 carries a nowhere vanishing spinor if and only if M^8 is spin and for an appropriate choice of orientation satisfies*

$$p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0.$$

Conversely, any generalised structure gives rise to a Riemannian spin manifold with a nowhere vanishing spinor and consequently, the topological obstructions in Proposition 2.15 are also necessary. Since the spinor Ψ_+ induces a supersymmetric map

$$X \in T \overset{\cong}{\mapsto} X \cdot \Psi_+ \in \Delta_-$$

we see that for a generalised $Spin(7)$-structure of odd type we also need a nowhere vanishing vector field X defined by $X \cdot \Psi_+ = \Psi_-$ which requires the vanishing of the Euler class $\chi(M)$.

Corollary 2.16.

(i) *A 7-fold M carries a topological generalised G_2-structure if and only if M is spin.*

(ii) *An 8-fold M carries an even topological generalised $Spin(7)$-structure if and only if M is spin and*

$$8\chi(M) + p_1(M)^2 - 4p_2(M) = 0. \tag{2.3}$$

(iii) *A differentiable 8-fold M carries an odd topological generalised $Spin(7)$-structure if and only if M is spin and*

$$\chi(M) = 0, \quad p_1(M)^2 - 4p_2(M) = 0. \tag{2.4}$$

Example: The 7-sphere is spinnable and therefore admits a generalised G_2-structure. The tangent bundle of the 8-sphere is stably trivial and therefore all the Pontrjagin classes vanish. Since the Euler class is non-trivial, there exists no generalised $Spin(7)$-structure on an 8-sphere. However, equations (2.3) and (2.4) are automatically satisfied for manifolds of the form $M^8 = S^1 \times N^7$ with N^7 spinnable.

An ordinary G_2- or $Spin(7)$-structure sitting inside the orthonormal frame bundle P induces trivially a generalised one which settles the existence question. But do there exist genuine generalised structures where the two G_2- or $Spin(7)$-subbundles inside P cannot be homotopically transformed into each other? And if yes, can we classify them? In the case of a generalised $Spin(7)$-structure the first question obviously only makes sense for structures of even type and for the remainder of this section, we shall only consider generalised $Spin(7)$-structures where the two induced spinors live in Δ_+.

To render our question more precise, we regard G_2- or (even) $Spin(7)$-structures as being defined by a (continuous) section of the sphere bundle $p_{\mathbb{S}} : \mathbb{S} \to M$ of Δ. On the space of sections $\Gamma(\mathbb{S})$ we then introduce the following equivalence relation. Two spinors Ψ_0 and Ψ_1 in Δ are considered to be equivalent (denoted $\Psi_0 \sim \Psi_1$) if and only if we can deform Ψ_0 into Ψ_1 through sections which means that there exists a continuous map $F : M \times I \to \mathbb{S}$ such that $F(x,0) = \Psi_0(x)$, $F(x,1) = \Psi_1(x)$ and $p_{\mathbb{S}} \circ F(x,t) = x$. An equivalence class will be denoted by $[\Psi]$. If two sections are vertically homotopic, then the two corresponding G_2- or $Spin(7)$-structures are isomorphic as principal G_2- or $Spin(7)$-bundles over M. In particular the generalised structure defined by the pair (Ψ, Ψ_0) is equivalent to a straight structure if and only if $\Psi \sim \Psi_0$. The set of generalised structures with fixed $\Psi_+ = \Psi$ is just $Gen(M) = \Gamma(\mathbb{S})/\sim$ which is what we aim to determine.

The question if whether or not two sections are vertically homotopic can be tackled by using obstruction theory (for a more detailed account of obstruction theory than we need it here, see [43] or [52]). Assume more generally that we are given a fibre bundle over a not necessarily compact n-fold M^n, and two sections s_1 and s_2 which are vertically homotopy equivalent over the q-skeleton $M^{[q]}$ of M. For technical simplicity, we assume the fibre to be connected. The obstruction for extending the vertical homotopy to the $q+1$-skeleton lies in $H^{q+1}(M, \pi_{q+1}(F))$. In particular, there is the first non-trivial obstruction $\delta(s_1, s_2) \in H^m(M, \pi_m(F))$ called the *primary difference* of s_1 and s_2 for the least integer m such that $\pi_m(F) \neq 0$. It is a homotopy invariant of the sections s_1 and s_2 and enjoys the additivity property

$$\delta(s_1, s_2) + \delta(s_2, s_3) = \delta(s_1, s_3). \tag{2.5}$$

Coming back first to the generalised G_2-case we consider the sphere bundle $\mathbb{S}$ over M^7 with fibre S^7. Consequently, the primary difference between two sections lies in $H^7(M, \mathbb{Z})$ and is the only obstruction for two sections to be vertically homotopy equivalent. Moreover, the additivity property implies that $\delta(\Psi, \Psi_1) = \delta(\Psi, \Psi_2)$ if and only if $\delta(\Psi_1, \Psi_2) = 0$, that is, $\Psi_1 \sim \Psi_2$, and for any class $d \in H^7(M, \mathbb{Z})$, there exists a section Ψ_d such that $d = \delta(\Psi, \Psi_d)$ [43], [52]. As a consequence, we obtain the

Proposition 2.17. *The set of generalised G_2-structures can be identified with*

$$Gen(M) = H^7(M, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \textit{if } M \textit{ is compact} \\ 0, & \textit{if } M \textit{ is non-compact} \end{cases}.$$

Generalised G_2-structures are therefore classified by an integer invariant which over a compact M^7 has the natural interpretation as the number of points (counted with an appropriate sign convention) where the two G_2-structures coincide. To see this we associate with every equivalence class $[\Phi]$ the intersection class $\#(\Psi(M), \Phi(M)) \in H_{14}(\mathbb{S}, \mathbb{Z})$ of the 7-dimensional oriented submanifolds $\Psi(M)$ and $\Phi(M)$ inside $\mathbb{S}$. Since the total space of the sphere bundle is 14-dimensional, $\#(\Psi, \Phi)$ counts the number of points in M where the two spinors Ψ and Φ coincide. Taking the cup product of the Poincaré duals of $\Psi(M)$ and $\Phi(M)$ then sets up a map

$$[\Phi] \in Gen(M) \mapsto PD(\Psi(M)) \cup PD(\Phi(M)) \in H^{14}(\mathbb{S}, \mathbb{Z}).$$

On the other hand, the Gysin sequence implies that integration along the fibre defines an isomorphism $\pi_{\mathbb{S}*} : H^{14}(\mathbb{S}, \mathbb{Z}) \to H^7(M, \mathbb{Z})$ and therefore, any generalised G_2-structure induced by the equivalence class $[\Phi]$ over a compact 7-fold M gives rise to a well-defined cohomology class $d(\Psi, \Phi) \in H^7(M, \mathbb{Z})$. The following theorem shows this class to coincide with the primary difference (for the proof I benefited from discussions with W. Sutherland and M. Crabb).

Theorem 2.18. *We have*

$$d(\Psi, \Phi) = \delta(\Psi, \Phi).$$

In particular generalised G_2-structures are classified by the number of points where the two underlying G_2-structures coincide.

Proof: We regard the spinor bundle Δ as an 8-dimensional oriented real vector bundle over M and consider the two sections Ψ and Φ of the sphere bundle. The primary difference $\delta(\Psi, \Phi)$ can be represented geometrically by the zero-set (a finite set with signs) of the section $(m, x) \mapsto (1 - x)\Psi(m) + x\Phi(m)$ of the pullback of Δ to $M \times \mathbb{R}$, and deformed to be transverse to the zero-section. In particular, if Ψ and $-\Phi$ never coincide then the primary difference is 0. Therefore the intersection number, defined geometrically by making $\Psi(M)$

and $\Phi(M)$ transverse and taking the coincidence set, will be $\delta(\Psi,-\Phi)$ (with appropriate sign conventions). By virtue of (2.5), we have $\delta(\Psi,-\Phi)=\delta(\Psi,\Phi)+\delta(\Phi,-\Phi)$. The difference class $\delta(\Phi,-\Phi)$ corresponds to the self-intersection number $\#(\Phi(M),\Phi(M))$ which is 0 since M is 7-dimensional. It follows that we can identify the intersection class $d(\Psi,\Phi)$ with the primary difference $\delta(\Psi,\Phi)$. ■

The index we have just defined is a topological obstruction to closed strong integrability as we will explain in the next chapter (see Definition 3.4 and Corollaries 3.19 and 3.27). Outside the singularity set, the structure group reduces to $SU(3)$, the stabiliser of two orthogonal spinors. There we can express the structure form ρ in terms of $SU(3)$-invariants, but as can be seen from the normal form given in Proposition 1.29, this description breaks down at a singular point where $\sin(a)=0$, that is, both spinors are parallel.

A generalised structure is said to be *exotic* if it is defined by two inequivalent spinors. Here are two examples.

Example:

(i) Choose a G_2-structure over S^7 represented by the spinor Ψ. Since the tangent bundle of S^7 is trivial, so is the sphere bundle of Δ, i.e. $\mathbb{S}=S^7\times S^7$. Consequently,

$$Gen(S^7)=\Gamma(\mathbb{S})/\!\sim\;=[S^7,S^7]=\pi_7(S^7)=\mathbb{Z}$$

and any map $\Phi: S^7\to S^7$ which is not homotopic to a constant gives rise to an exotic generalised G_2-structure for $\Psi\equiv const.$

(ii) The tangent bundle of the manifold $M^8=S^1\times S^7$ is also trivial. As observed above, M carries a unit spinor and we can again trivialise the sphere bundle $\mathbb{S}$, so that $Gen(M)=$

$[S^1 \times S^7, S^7]$, containing the set $[S^7, S^7] = \pi_7(S^7) = \mathbb{Z}$. Choosing a non-trivial homotopy class in $\pi_7(S^7)$ which we extend trivially to $S^1 \times S^7$ defines an exotic generalised $Spin(7)$-structure.

In the $Spin(7)$-case the sphere bundle is 15-dimensional with fibre isomorphic to S^7 and an 8-dimensional base, so that two transverse sections will intersect in a curve. We meet the first obstruction for the existence of a vertical homotopy in $H^7(M, \mathbb{Z})$ which by Poincaré duality trivially vanishes if $H_1(M, \mathbb{Z}) = 0$ (e.g. if M is simply connected). The second obstruction lies in the top cohomology module. Since $\pi_8(S^7) = \mathbb{Z}_2$, we obtain

Proposition 2.19. *Let M^8 be an 8-manifold with $H_1(M, \mathbb{Z}) = 0$ which admits a generalised $Spin(7)$-structure. Then the set of generalised $Spin(7)$-structures with a fixed $\Psi_+ = \Psi$ is given by*

$$Gen(M) = H^8(M, \pi_8(S^7)) \cong \begin{cases} \mathbb{Z}_2, & \text{if } M \text{ compact} \\ 0, & \text{if } M \text{ non-compact} \end{cases}.$$

Again we see that over a non-compact M^8 with $H_1(M, \mathbb{Z}) = 0$ any two sections of the sphere bundle can be vertically transformed into each other while this is clearly not the case for the general compact case as shown by the example above.

Remark: The stable homotopy group $\pi_{n+k}(S^n)$ is isomorphic to the framed cobordism group of k-manifolds [44]. It is conceivable that the $\mathbb{Z}_2$-class is the framed (or spin) cobordism class of the 1-manifold where the two sections coincide.

Chapter 3

Geometry

The motivation for adopting the supersymmetric approach was to provide a framework within which we can understand the algebra of special metrics associated with stable forms. Stability also provides us with a natural set of integrability conditions arising from Hitchin's variational principle. These integrability conditions define the geometry, following the general philosophy of this thesis, through closed forms. Based on the papers [29] and [30], we will formulate in the first section various kinds of this principle. It leads to the notions of integrable $PSU(3)$- and to (closed) strongly or weakly integrable generalised G_2-structures. The variational principle is perfectly general and in particular, no metric is needed to set up these equations.

Starting on a Riemannian manifold however, we can adopt the supersymmetric point of view and reformulate these integrability conditions in terms of the corresponding spinorial invariants, providing thus a *spinorial solution* to the variational problem. As a result, we obtain some well-known equations of mathematical physics, namely the Rarita-Schwinger equation (Theorem 3.6) and the supersymmetry equations of supergravity of type IIA/B with bosonic background fields (Theorem 3.15). We will also analyse related geometries as introduced in Chapter 1 where we impose the integrability conditions from the variational or the supersymmetric ansatz. Moreover, there is also a practical value to the use of spinors as they are easier to manipulate than forms. In particular, we will use these spinor field equations to derive integrability conditions on the Ricci tensor (Sections 3.2.2 and 3.3.2). A further important consequence of the spinorial formulation is a no-go theorem for closed strongly integrable generalised G_2- and $Spin(7)$-structures in the sense that any solution over a compact manifold is induced by two classical G_2- or $Spin(7)$-structures with a parallel spinor (Corollary 3.19). In this sense, the (unconstrained) variational principle leads to classical geometries. However, we shall see that the notion of weak integrability potentially yields an interesting new type of geometry (cf. the discussion in Section 3.3.3) for which non-trivial compact examples could exist. The spinorial picture is less useful for the construction of explicit examples and therefore, we step back to work with forms again. A particularly striking advantage of forms over spinors in this context is the device of T-duality which easily yields non-trivial generalised solutions from classical ones (Section 3.3.3).

3.1 The variational problem

3.1.1 The unconstrained variational problem in the classical case

We recall the set-up of the variational problem as given in [29]. Let (M^n, ρ) be a closed and oriented manifold which is endowed with a *stable* form ρ of pure degree p. This means that ρ is a section of the open subset of $\Omega^p(M)$ whose intersection with any fiber $\Lambda^p T_x^* M$

can be identified with an open orbit U (cf. Section 1.6). This provides us with a natural volume form $\phi(\rho)$ which asks to be integrated

$$V(\rho) = \int_M \phi(\rho).$$

Since stability is a generic condition, we can differentiate the volume functional and we shall consider its variation over a fixed cohomology class in $H^p(M, \mathbb{R})$.

Theorem 3.1. [29] *A closed stable form $\rho \in \Omega^p(M)$ is a critical point in its cohomology class if and only if $d\hat{\rho} = 0$.*

Proof: The first variation of V is

$$\delta V_\rho(\dot{\rho}) = \int_M DV_\rho(\dot{\rho}) = \int_M \hat{\rho} \wedge \dot{\rho}.$$

Since we vary over a fixed cohomology class, $\dot{\rho}$ is exact, i.e. $\dot{\rho} = d\alpha$. As a consequence of Stokes' theorem, this variation vanishes for all $\alpha \in \Omega^{p-1}(M)$ if and only if

$$\int_M \hat{\rho} \wedge d\alpha = \pm \int_M d\hat{\rho} \wedge \alpha = 0$$

which holds precisely when $\hat{\rho}$ is closed. ■

Remark: Note that the condition $d\hat{\rho} = 0$ is non-linear in ρ as the $\wedge$-operation also depends on ρ. In this way, we can see the variational problem as performing a non-linear version of Hodge theory [29].

3.1.2 The unconstrained variational problem in the generalised case

Here we follow the approach given in [30] for generalised Calabi-Yau manifolds. Assume M^7 to be closed, oriented and provided with a stable form ρ giving rise to a generalised

G_2-structure. Since the structure group $GL_+(7)$ lifts to $Spin(7,7)$, ρ is a section of a vector bundle associated with $GL_+(7)$ and with typical fiber isomorphic (as a *vector space*) to $\Lambda^{ev,od}\mathbb{R}^{7*}$. This becomes the spin representation for $GL_+(7) \subset Spin(7,7)$ if we twist with the square root $\Omega^7(M)^{1/2}$ (compare with the remark on Page 27). In this subsection, we shall, for sake of clarity, denote by $\lambda^{ev,od}$ the spin representation of $GL_+(7) \subset Spin(7,7)$, while $GL_+(7)$ acts on $\Lambda^{ev,od}$ as usual by $A^*\rho$.

Let us consider the volume functional ϕ as described in Proposition 1.32. It is homogeneous of degree 2 and can therefore be considered from a representation theoretic point of view as a $GL(7)_+$-equivariant function

$$\phi : U \subset \Lambda^{ev,od} \to \Lambda^7,$$

since for an element $A \in GL_+(7)$ we have

$$\phi(A^*\rho) = (\det A)^{-1} \cdot \phi(\sqrt{\det A} \cdot A^*\rho) = (\det A)^{-1}\phi(A \bullet \rho) = (\det A)^{-1}\phi(\rho).$$

The non-degenerate form q now takes values in Λ^7 so that

$$D\phi_\rho(\dot{\rho}) = q(\hat{\rho}, \dot{\rho})$$

is also a volume form. As in the classical case stable forms are sections of the open subset in $\Omega^{ev,od}(M)$ and ϕ is a well-defined function on this set. We obtain the following analogue of Theorem 3.1.

Theorem 3.2. *A closed stable form $\rho \in \Omega^{ev,od}(M)$ is a critical point in its cohomology class if and only if $d\hat{\rho} = 0$.*

Proof: The argument is a direct translation of the proof of Theorem 3.1. Take the first variation of $V(\rho)$ over $[\rho]$

$$\delta V(d\alpha) = \int_M D\phi(d\alpha) = \int_M q(\hat{\rho}, d\alpha) = \int_M \hat{\rho} \wedge \sigma(d\alpha).$$

Moreover, it is straightforward to check that

$$d\sigma(\tau^{ev,od}) = \mp\sigma(d\tau^{ev,od})$$

for any even or odd form $\tau \in \Omega^{ev,od}(M^7)$. Thus

$$\delta V(\dot{\rho}) = \int_M q(\hat{\rho}, d\alpha) = \pm \int_M q(d\hat{\rho}, \alpha)$$

and the variation vanishes for all α if and only if $\hat{\rho}$ is closed. ■

3.1.3 The constrained variational problem

Next we generalise the constrained variational problem for classical G_2-manifolds (there is no $PSU(3)$-analogue) described in [29] to our setup. On a closed oriented manifold M^7, the non-degenerate pairing q induces a non-degenerate pairing between the spaces of forms $\Omega^{ev}(M)$ and $\Omega^{od}(M)$ by

$$\int_M q(\sigma, \tau).$$

If $\sigma = d\gamma$ is exact, then Stokes' theorem implies that

$$\int_M q(d\gamma, \tau) = \int_M q(\gamma, d\tau),$$

and this vanishes for all γ if and only if τ is closed. Hence we obtain a non-degenerate pairing which allows us to identify

$$\Omega^{ev}_{\text{exact}}(M^7)^* \cong \Omega^{od}(M^7)/\Omega^{od}_{\text{closed}}(M^7).$$

The exterior differential maps the latter space isomorphically onto $\Omega^{ev}_{\text{exact}}(M^7)$ so that

$$\Omega^{ev}_{\text{exact}}(M^7)^* \cong \Omega^{ev}_{\text{exact}}(M^7).$$

Consequently, the non-degenerate pairing becomes a non-degenerate quadratic form on $\Omega^{ev}_{exact}(M^7)$ given by

$$Q(d\gamma) = \int_M q(\gamma, d\gamma).$$

The same conclusion holds for odd instead of even forms.

Theorem 3.3.

A stable form $\rho \in \Omega^{ev,od}_{\text{exact}}(M)$ is a critical point subject to the constraint $Q(\rho) = const$ if and only if there exists a real constant λ with $d\hat{\rho} = \lambda\rho$.

Proof: From the proof of Theorem 3.2, the first variation of V at $\rho = d\gamma$ is

$$(\delta V)_\rho(d\alpha) = \int_M q(\hat{\rho}, d\alpha)$$

whereas the differential of Q at ρ is

$$(\delta Q)_\rho(d\alpha) = 2\int_M q(\alpha, \rho).$$

By Lagrange's theorem, we see that for a critical point we have $d\hat{\rho} = \lambda\rho$. ■

3.1.4 The twisted variational problem

To produce the most general form $e^{b/2}\Psi_+ \otimes \Psi_-$ of a spinor with stabiliser $G_2 \times G_2$, we first considered the "unperturbed" spinor $\rho_0 = \Psi_+ \otimes \Psi_-$ and then hit it with a B-field. The equations for ρ_0 to define a critical point are

$$d\rho_0 = 0, \quad d\hat{\rho}_0 = 0$$

which under the action of a B-field are equivalent to the inhomogeneous set of equations

$$d\rho_0 + \frac{1}{2} db \wedge \rho_0 = 0, \quad d\hat{\rho}_0 + \frac{1}{2} db \wedge \hat{\rho}_0 = 0.$$

Following [30] we can, more generally, consider the twisted differential operator

$$d_H \rho = d\rho + H \wedge \rho,$$

where H is now a closed (but not necessarily exact) 3-form. The closedness guarantees that d_H still defines a differential complex. Hence it makes sense to speak of a d_H-cohomology class and we can set up the variational problem to take place over such a cohomology class. The following theorem is a direct translation from Theorem 13 in [30].

Theorem 3.4. *A d_H-closed stable form $\rho \in \Omega^{ev,od}(M)$ is a critical point of $V(\rho)$ in its d_H-cohomology class if and only if $d_H \hat{\rho} = 0$.*

Proof: This time, the first variation is

$$(\delta V)_\rho(d\alpha) = \int_M q(\hat{\rho}, d\alpha + H \wedge \alpha) = \pm \int_M q(d\hat{\rho}, \alpha) + \hat{\rho} \wedge \sigma(\alpha) \wedge \sigma(H),$$

where ρ and α are both either even or odd. Since

$$\hat{\rho} \wedge \sigma(H \wedge \alpha) = \hat{\rho} \wedge \sigma(\alpha) \wedge \sigma(H) = H \wedge \hat{\rho} \wedge \sigma(\alpha)$$

the first variation is

$$(\delta V)_\rho(d\alpha) = \int_M q(d\hat{\rho} + H \wedge \hat{\rho}, \alpha).$$

This vanishes for all α if and only if $d_H \hat{\rho} = 0$. ■

3.2 Integrable variational structures and related geometries in the classical case

3.2.1 The spinorial solution of the variational problem and related geometries

Next we will consider the geometries introduced in Section 1.3.2 together with the following integrability conditions. For geometries of type 1_0, 2_0 or 3_0, that is, the reduction of the structure group of the tangent bundle to $PSU(3)$, $SO(3)\times SO(3)\times SO(2)$ or $SO(3)\times SO(5)$ (cf. Section 1.3.3) is induced by an honest 3-form $\rho \in \Omega^3(M)$, we impose the integrability condition coming from the variational principle $d\rho = d\star\rho = 0$. If the geometry is defined by a supersymmetric map $\gamma_\pm : \Lambda^1 \to \Delta_\pm$, then this geometry is said to be *integrable* if and only if $\gamma_\pm$ is harmonic with respect to the twisted Dirac operator on $T^*M \otimes \Delta$. In the case of $PSU(3)$, we will show these conditions to coincide. The latter condition therefore provides what we call a spinorial solution to the variational problem. We will analyse this case in some detail before we briefly investigate the related geometries of type 2 and 3.

$PSU(3)$-geometry

The formal solution to the variational problem in the classical case suggests the following definition.

Definition 3.1. *A $PSU(3)$-manifold (M,ρ) is called* integrable *if and only if*

$$d\rho = 0, \quad d\star\rho = 0. \tag{3.1}$$

Recall that in 1.3.2 we proved that as a consequence of the triality principle the $PSU(3)$-

invariant 3-form ρ gives rise to two isometries $\gamma_\pm : \Lambda^1 \to \Delta_\pm$. In this section we want to derive an equivalent formulation of integrability in terms of $\gamma = \gamma_+ \oplus \gamma_- \in \Lambda^1 \otimes (\Delta_+ \oplus \Delta_-)$. This means that we start with a spinnable (not necessarily compact) Riemannian manifold (M^8, g, γ) whose metric is induced by the $PSU(3)$-structure associated with γ, and ask for the conditions on γ which makes this structure integrable.

To this end we have to analyse the action of the covariant derivative on ρ and γ, but first a general outline of the strategy is in order. The following discussion will also prove useful for the generalised case so we start with a generic Riemannian manifold (M^n, g). Consider the Levi-Civita connection form

$$Z : TP_{SO(n)} \to \mathfrak{so}(n),$$

which is an $\mathfrak{so}(n)$-valued 1-form on the orthonormal frame bundle $P_{SO(n)}$. If we are given a reduction from this $SO(n)$-bundle to a subbundle P_G acted on by the subgroup $G \leqslant SO(n)$, then the Lie algebra $\mathfrak{so}(n)$ splits into the subalgebra $\mathfrak{g}$ and an orthogonal complement $\mathfrak{m}$

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}.$$

We can decompose the restriction of the Levi-Civita form Z to P_G accordingly into

$$Z_{|TP_G} = \widetilde{Z} \oplus \Gamma$$

where $\widetilde{Z}$ is a $\mathfrak{g}$-valued connection form on P_G. It defines a connection for the G-fibre bundle and in particular, any G-invariant object will be parallel for the induced affine connection $\widetilde{\nabla}$. Note however that this property is in general not sufficient to guarantee the uniqueness of $\widetilde{\nabla}$ (see, for instance Remark 2 in Section 4.2 of [11]). Γ is a tensorial 1-form of type Ad, i.e. it can be seen as a form on M taking values in the associated bundle

$$P_G \times_{\mathrm{Ad}} \mathfrak{m}.$$

At $x \in M^n$ we can interpret Γ_x as an element of $\mathbb{R}^n \otimes \mathfrak{m}$ and in this sense Γ measures the failure of the Levi-Civita connection to reduce to P_G. It is also referred to as the *torsion* of the structure. The decomposition of the G-module $\mathbb{R}^n \otimes \mathfrak{m}$ into irreducible modules gives rise to the different geometrical *types* of G-structures according to the non-trivial components of the torsion. Knowing the type allows us to extract further geometrical information. For instance, if we consider a G_2-structure on a 7-manifold, then $\mathfrak{m} = \Lambda^2_7$ is isomorphic to the vector representation of G_2. Consequently, $\mathbb{R}^7 \otimes \mathfrak{m}$ decomposes into

$$\mathbb{R}^7 \otimes \mathfrak{m} = \Lambda^0_1 \oplus \Lambda^1_7 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27}.$$

A G_2-structure whose associated Γ takes values in say $\Lambda^0_1 = \mathbb{R}$ is then said to be of *type* Λ^0_1 (these G_2-structures are also called *nearly parallel G_2-manifolds*, see [18]). In particular, the metric structure it defines is Einstein. Generally speaking, the type can be detected by looking at the action of Z on a G-invariant object. For structures of type Λ^0_1 the covariant derivative of the 3-form φ and the corresponding spinor Ψ is given by

$$\nabla_X \varphi = -2\lambda X \lrcorner \star\varphi \text{ and } \nabla_X \Psi = \lambda X \cdot \Psi.$$

Here, as in the sequel, the covariant derivative operator ∇ will *always* denote the Levi-Civita connection. We will revisit this example later on when we consider generalised G_2-structures.

Coming back to the mainstream of the development, we write the covariant derivative of the $PSU(3)$-invariant 3-form as

$$\nabla \rho = T(\rho)$$

with $T \in \Lambda^1 \otimes \Lambda^2$. T acts through its Λ^2 factor which we view as the Lie algebra of $SO(8)$. Since $\mathfrak{su}(3) \leqslant \Lambda^2$ stabilises ρ, we can assume that $T \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp$ so that T really encodes the action of the Lie algebra valued 1-form Γ above. Consequently, we refer to T as the *torsion* of the $PSU(3)$-structure. Unless the space is symmetric or the holonomy reduces to a proper subgroup of $PSU(3)$, inspection of Berger's list [6] implies that the torsion does not vanish. In this case we say that the integrable $PSU(3)$-structure is *non-trivial*. To make contact with the variational principle, we remark that $d\rho$ and $d^{\star}\rho$ are the image of T under the $PSU(3)$-equivariant morphisms

$$\tilde{d} : \Lambda^1 \otimes \mathfrak{su}(3)^{\perp} \to \Lambda^4$$

and

$$\widetilde{d^{\star}} : \Lambda^1 \otimes \mathfrak{su}(3)^{\perp} \to \Lambda^2$$

which map the tensor $A(\rho) \in \Lambda^1 \otimes \Lambda^3$ to Λ^4 and Λ^2 by skew-symmetrisation and contraction respectively. These last two operations also map $\nabla\rho$ to $d\rho$ and $-d^{\star}\rho$ and therefore,

$$\tilde{d}T = d\rho, \quad \widetilde{d^{\star}}T = -d^{\star}\rho.$$

The integrability conditions (3.1) are thus equivalent to

$$T \in \ker \tilde{d} \cap \ker \widetilde{d^{\star}}.$$

The next step consists in decomposing $\Lambda^1 \otimes \mathfrak{su}(3)^{\perp}$ into irreducible modules. Let $\wedge : \Lambda^1 \otimes \mathfrak{su}(3)^{\perp} \to \Lambda^3$ denote the natural skewing map. Then $\Lambda^1 \otimes \mathfrak{su}(3)^{\perp} \cong \ker \wedge \oplus \rho^{\perp}$, where $\rho^{\perp} = [1,1] \oplus [\![3,0]\!] \oplus [2,2]$ is the orthogonal complement of ρ in Λ^3. Moreover, the natural contraction map $\llcorner : \ker \wedge \subset \Lambda^1 \otimes \mathfrak{su}(3)^{\perp} \to \Lambda^1$ splits $\ker \wedge$ into a direct sum isomorphic to $\ker \llcorner \oplus \Lambda^1$ where $\ker \llcorner \cong [2,2] \oplus [\![4,1]\!]$. On the other hand we obtain from Proposition 1.8

$$\Lambda^2 = [1,1] \oplus [\![3,0]\!], \quad \Lambda^4 = 2[1,1] \oplus 2[2,2].$$

Schur's Lemma implies at once

$$[\![4,1]\!] \leqslant \ker \tilde{d} \cap \ker \widetilde{d^\star}.$$

In order to determine the kernel completely, it will be convenient to complexify the modules. We find

$$\Lambda^1 \otimes \mathfrak{su}(3)^\perp \otimes \mathbb{C} = \begin{array}{rcl} \Lambda^1 \otimes \Lambda^2_{10+} & = & (1,1)_+ \oplus (0,3) \oplus (2,2)_+ \oplus (1,4) \\ \oplus & & \\ \Lambda^1 \otimes \Lambda^2_{10-} & = & (1,1)_- \oplus (3,0) \oplus (2,2)_- \oplus (4,1). \end{array}$$

The modules $(1,1)_\pm$ and $(2,2)_\pm$ have non-trivial projections to both $\ker \wedge$ and $\rho^\perp$. In particular, they map non-trivially under $\wedge$.

Proposition 3.5. *The $PSU(3)$-structure defined by ρ is integrable if and only if*

$$T \in [\![4,1]\!].$$

Remark: The main difficulty of the proof resides in the fact that the modules $[1,1]$ and $[2,2]$ appear twice in Λ^4. Even if both components in $\Lambda^1 \otimes \mathfrak{su}(3)^\perp$ map non-trivially under $\widetilde{d}$, we can still have a component of the kernel isomorphic to $[1,1]$ or $[2,2]$ which trivially intersects our fixed copies of $[1,1]$ or $[2,2]$. Therefore, the argument presented in [29] is inconclusive.

Proof: We only have to check that $d\rho = 0$ and $d^*\rho = 0$ implies $T \in [\![4,1]\!]$. Let us determine the kernel of $\widetilde{d^*}$ and recall that $a^*_\pm \rho = \pm\sqrt{3} i \star (a_\pm \wedge \rho)$ for any $a_\pm \in \Lambda^2_{10\pm}$ (Proposition 1.8). By complexifying, it follows that restricted to the $PSU(3)$-invariant modules $\Lambda^1 \otimes \Lambda^2_{10\pm}$,

$$\widetilde{d^*}(\sum e_j \otimes a^\pm_j) = \mp\sqrt{3} i \sum e_j \lrcorner \star (a^\pm_j \wedge \rho) = \pm\sqrt{3} i \sum \star(e_j \wedge a^\pm_j \wedge \rho). \qquad (3.2)$$

In virtue of the remarks preceding the theorem, the kernel of the skewing map $\Lambda^1 \otimes \Lambda^2_\pm$ is isomorphic to $(1,4)$ and $(4,1)$, so this vanishes outside these modules if and only if $\sum e_i \wedge a_i^\pm$ lies in $\widetilde{1} \oplus [2,2]$, the kernel of the map which wedges 3-forms with ρ. Invoking Schur's Lemma, $\ker \widetilde{d}^* \cong [1,1] \oplus 2[2,2] \oplus [\![4,1]\!]$, where the precise embedding of $[1,1]$ will be of no importance to us.

Next we consider the operator $\widetilde{d}$. If we can show that it is surjective, then $\ker \widetilde{d} \cong [\![3,0]\!] \oplus [\![4,1]\!]$ and consequently, the kernels of $\widetilde{d}^*$ and $\widetilde{d}$ intersect in $[\![4,1]\!]$. Let $\iota_{\rho^\perp}$ denote the injection of $\rho^\perp$ into $\Lambda^1 \otimes \Lambda^2_{20}$ obtained by projecting the natural embedding of Λ^3 into $\Lambda^1 \otimes \Lambda^2$. We first prove the relation

$$b_3(\alpha) = \frac{1}{2}\widetilde{d}(\iota_{\rho^\perp}(\alpha)), \quad \alpha \in \rho^\perp \subset \Lambda^3 \tag{3.3}$$

which shows that $\ker b_4 \subset \operatorname{Im} \widetilde{d}$. By (1.26), the kernel of b_3 is isomorphic to $\mathbf{1} \oplus [\![1,2]\!]$, so the claim needs only to be checked for the module $[1,1] \oplus [2,2]$ in Λ^3. A sample vector is obtained by

$$p_3(e_{128}) = \alpha_8 \oplus \alpha_{27} = \tfrac{1}{8}(5e_{128} + \sqrt{3}e_{345} + \sqrt{3}e_{367} - 2e_{458} + 2e_{678}), \tag{3.4}$$

where $p_3 = b_4^* b_3$. That both components α_8 and α_{27} are non-trivial can be seen as follows. Restricting p_3 to Λ^3_8 and Λ^3_{27} is multiplication by real scalars x_1 and x_2 since the modules are representations of real type. If one, say x_1, vanished, then $p_3^2(e_{128}) = p_3(\alpha_{27}) = x_2 \cdot \alpha_{27}$. However

$$p_3^2(e_{128}) = \tfrac{1}{64}(39e_{128} + 7\sqrt{3}e_{345} + 7\sqrt{3}e_{367} - 18e_{458} + 18e_{678})$$

which is not a multiple of (3.4). Moreover, we have indeed

$$\begin{aligned} b_3 p_3(e_{128}) &= \tfrac{1}{32}(7\sqrt{3}e_{1245} + 7\sqrt{3}e_{1267} - 9e_{1468} - 9e_{1578} + 9e_{2478} - 9e_{2568}) \\ &= \tfrac{1}{2}\widetilde{d}(\iota_{\rho^\perp} p_3(e_{128})) \end{aligned}$$

which proves (3.3). For the inclusion $\operatorname{Im} b_5^* \subset \operatorname{Im} \widetilde{d}$ we consider the vector $e_1 \otimes e_{18}$ in $\ker \wedge$. Then $\widetilde{d}(e_1 \otimes e_{18}) = -e_{1238}/2 - e_{1478}/4 + e_{1568}/4$ takes values in both components of $\operatorname{Im} b_5^* \subset \Lambda^4$ since $b_4^* \widetilde{d}(e_1 \otimes e_{18}) = 0$ and otherwise

$$b_5^* b_4 \widetilde{d}(e_1 \otimes e_{18}) = \tfrac{1}{32}(-10e_{1238} - 5e_{1478} + 5e_{1568} + 3e_{2468} + 3e_{2578} + 3e_{3458} - 3e_{3678})$$

would be a multiple of $\widetilde{d}(e_1 \otimes e_{18})$. Hence $\widetilde{d}$ is surjective and the assertion follows. ■

Let $\mathcal{D}$ denote the twisted Dirac operator on $\Lambda^1 \otimes \Delta$, so $\mathcal{D}$ can be locally written as

$$\mathcal{D}(X \otimes \Psi) = \sum_{i=1}^{8} \nabla_{e_i} X \otimes e_i \cdot \Psi + X \otimes e_i \cdot \nabla_{e_i} \Psi.$$

Consider the embedding of Δ into $\Lambda^1 \otimes \Delta = \Delta \oplus \Lambda^3\Delta_+ \oplus \Lambda^3\Delta_-$ given by

$$i(\psi)(X) = -\frac{1}{8} X \cdot \psi$$

and the projection $p : \Lambda^1 \otimes \Delta \to \Delta$

$$p(\gamma) = \sum_{i=1}^{8} e_i \cdot \gamma(e_i).$$

With respect to the decomposition $\operatorname{Im} i \oplus \ker p$, the Dirac operator $\mathcal{D}$ takes the form (see Prop. 2.7 in [51])

$$\begin{pmatrix} -\frac{3}{4} i \circ D \circ i^{-1} & 2i \circ \delta \\ \frac{1}{4} P \circ i^{-1} & Q \end{pmatrix}, \tag{3.5}$$

where $D : \Delta \to \Delta$ is just the usual Dirac-Operator, $\delta : \Lambda^1 \otimes \Delta \to \Delta$ the twisted co-differential, $P : \Delta \to \Lambda^3\Delta_+ \oplus \Lambda^3\Delta_-$ the Twistor operator $P(\sigma) = \nabla\sigma - i \circ p(\nabla\sigma)$, and Q the so-called *Rarita-Schwinger* operator which arises in supergravity and string theories. To introduce more physicist's jargon still, we make the

Definition 3.2. *We call a spin 3/2 field* ψ*, i.e. a section* $\psi \in \Gamma(\Lambda^3\Delta_- \oplus \Lambda^3\Delta_-)$*, a* Rarita-Schwinger field *if it satisfies*

$$Q(\psi) = 0.$$

Theorem 3.6. *The* $PSU(3)$*-structure is integrable if and only if* $\gamma = \gamma_+ \oplus \gamma_- \in \Gamma(\Lambda^1 \otimes \Delta)$ *is harmonic with respect to the twisted Dirac-operator* $\mathcal{D} : \Gamma(\Lambda^1 \otimes \Delta) \to \Gamma(\Lambda^1 \otimes \Delta)$*, that is* $\mathcal{D}\gamma = 0$.

Proof: The proof uses the same idea as Proposition 3.5. Let us consider the $PSU(3)$-equivariant morphism

$$\widetilde{\mathcal{D}} : A \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp \mapsto \mu(A(\gamma)) \in \Lambda^1 \otimes \Delta$$

which maps $A(\gamma) \in \Lambda^1 \otimes \Lambda^1 \otimes \Delta$ through Clifford multiplication to $\mu(A(\gamma))$ in $\Lambda^1 \otimes \Delta$. Since the $PSU(3)$-invariant supersymmetric map $\gamma = \gamma_+ \oplus \gamma_-$ has now components in both $\Delta_- \otimes \Lambda^1$ and $\Delta_+ \otimes \Lambda^1$, we will split the Dirac operator $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_+ \oplus \widetilde{\mathcal{D}}_-$ accordingly, i.e. $\widetilde{\mathcal{D}}_\pm(X \otimes a) = \mu_\pm(X \cdot a(\gamma_\pm)) \in \Delta_\pm \otimes \Lambda^1$. We now have to show that

$$\ker \widetilde{\mathcal{D}}_+ \cap \ker \widetilde{\mathcal{D}}_- = [\![4, 1]\!].$$

The intersection $\ker \widetilde{\mathcal{D}}_+ \cap \ker \widetilde{\mathcal{D}}_-$ contains at least the module $[\![4, 1]\!]$. First we show that $[\![3, 0]\!]$ is not contained in this intersection by taking the vector

$$\begin{aligned}
\tau_{[\![3,0]\!]} &= id \otimes \pi^2_{20}(\iota_{\rho^\perp} b_2(4e_{18})) \\
&= -\sqrt{3}e_1 \otimes e_{45} - \sqrt{3}e_1 \otimes e_{67} + 2e_2 \otimes e_{38} - 2e_3 \otimes e_{28} + \sqrt{3}e_4 \otimes e_{15} + \\
&\quad e_4 \otimes e_{78} - \sqrt{3}e_5 \otimes e_{14} - e_5 \otimes e_{68} + \sqrt{3}e_6 \otimes e_{17} + e_6 \otimes e_{58} - \sqrt{3}e_7 \otimes e_{16} - \\
&\quad e_7 \otimes e_{48} + 2e_8 \otimes e_{23} + e_8 \otimes e_{47} - e_8 \otimes e_{56}.
\end{aligned}$$

A straightforward, if tedious, computation shows $\widetilde{\mathcal{D}}(\tau_{[\![3,0]\!]}) \neq 0$. For the remainder of the proof, it will again be convenient to complexify the torsion module $\Lambda^1 \otimes \mathfrak{su}(3)^\perp$ and to consider $(1,1)_\pm$ and $(2,2)_\pm$. The invariant 3-form ρ induces equivariant maps $\rho_\mp : \Delta_\pm \to \Delta_\mp$ whose matrices with respect to the choices made in (1.8) are given by (1.35) for ρ_+ and by its transpose for ρ_-. By Schur's Lemma, we have

$$\widetilde{\mathcal{D}}_-((2,2)_+) = z \cdot \rho_- \otimes id \circ \widetilde{\mathcal{D}}_+((2,2)_+) \tag{3.6}$$

for a complex scalar z. Since the operators $\widetilde{\mathcal{D}}_\pm$ are real and $(2,2)_-$ is the complex conjugate of $(2,2)_+$, the same relation holds for $(2,2)_-$ with $\bar{z}$. The vector $\tau_0 = 6(e_1 \otimes e_{18} - e_2 \otimes e_{28})$ is clearly in $\ker \llcorner \subset \ker \wedge$ and projecting the second factor to Λ^2_{10+} yields

$$\begin{aligned} id \otimes \pi^2_{10+}(\tau_0) \quad &= \quad e_1 \otimes (3e_{18} + i\sqrt{3}e_{23} - i\sqrt{3}e_{47} + i\sqrt{3}e_{56}) + \\ &\qquad e_2 \otimes (i\sqrt{3}e_{13} - 3e_{28} + i\sqrt{3}e_{46} + i\sqrt{3}e_{57}). \end{aligned}$$

Since any possible component in $(1,4)$ gets killed under $\widetilde{\mathcal{D}}$, we can plug this into (3.6) to find $z = (1 + i\sqrt{3})/8$ which shows that $(2,2)_\pm$ map non-trivially under $\widetilde{\mathcal{D}}$. On dimensional grounds, $\ker \widetilde{\mathcal{D}}_\pm$ therefore contains the module $(2,2)$ with multiplicity one. Their intersection, however, is trivial, for suppose otherwise. Let $(2,2)_0$ denote the corresponding copy in $\ker \widetilde{\mathcal{D}}_+$. It is the graph of an isomorphism $M : (2,2)_+ \to (2,2)_-$ since it intersects $(2,2)_\pm$ trivially. Now if $\tau = \tau_+ \oplus M\tau_+ \in (2,2)_0$ were in $\ker \widetilde{\mathcal{D}}_-$, then

$$\begin{aligned} \widetilde{\mathcal{D}}_-(\tau_+ \oplus M\tau_+) \quad &= \quad z \cdot \rho \otimes id \circ \widetilde{\mathcal{D}}_+(\tau_+) \oplus \bar{z} \cdot \rho \otimes id \circ \widetilde{\mathcal{D}}_+(M\tau_+) \\ &= \quad \rho \otimes id \circ \widetilde{\mathcal{D}}_+(z \cdot \tau_+ \oplus \bar{z} \cdot M\tau_+) \\ &= \quad 0. \end{aligned}$$

Consequently, $z \cdot \tau_+ \oplus \bar{z} \cdot M\tau_+ \in \ker \widetilde{\mathcal{D}}_+$, that is, $\bar{z} \cdot M\tau_+ = Mz \cdot \tau_+$ or $\bar{z} = z$ which is a contradiction. This shows that the kernels of $\widetilde{\mathcal{D}}_\pm$ intersect at most in $2(1,1) \oplus [\![2,3]\!]$ and furthermore, that the condition $\mathcal{D}(\gamma_+) = 0$ or $\mathcal{D}(\gamma_-) = 0$ on its own is not sufficient to guarantee the close- and cocloseness of ρ. This argument also applies to $(1,1)_\pm$. However, since $(1,1)$ appears twice in $\Delta_\pm \otimes \Lambda$, we first need to project onto $\Delta_\mp \cong (1,1)$ via Clifford multiplication before asserting the existence of a complex scalar z such that

$$\mu_+ \circ \widetilde{\mathcal{D}}_-((1,1)_+) = z \cdot \rho_+(\mu_- \circ \widetilde{\mathcal{D}}_+((1,1)_+)).$$

For the computation of z, we can use the vector

$$2\sqrt{3}ie_1 \otimes \pi^2_{10+}(e_{18}) = e_1 \otimes (\sqrt{3}ie_{18} - e_{23} + e_{47} - e_{56}) \in (1,1)_+ \oplus (2,2)_+ \oplus (1,4),$$

as possible non-trivial components in $(2,2)_+ \oplus (1,4)$ get killed under $\mu_\mp$. We find $z = 2(1-\sqrt{3}i)$ which shows that $(1,1)$ occurs with multiplicity at most one in $\ker \widetilde{\mathcal{D}}_\pm$ and that it is not contained in their intersection. Consequently, $\ker \widetilde{\mathcal{D}}_+ \cap \ker \widetilde{\mathcal{D}}_- = [\![4,1]\!]$, which proves the theorem in conjunction with Proposition 3.5. ■

$SO(3)\times SO(3)\times SO(2)$- and $SO(3)\times SO(5)$-geometry

Next we briefly investigate geometries of type 2 and 3. Recall that we divided these structures into the subtypes 0, + and − according to the case where the structure group $SO(\Lambda^1)$, $SO(\Delta_-)$ and $SO(\Delta_+)$ reduces to a group conjugated to $SO(3)\times SO(3)\times SO(2)$ (type 2) and to $SO(3)\times SO(5)$ (type 3) (see 1.3.3). The discussion of integrable $PSU(3)$-structures leaves us with natural integrability conditions for geometries of type 2 and 3.

Definition 3.3.

(i) *We call a structure of type* 2_0 *or* 3_0 integrable *if and only if the associated 3-form* ρ_{2_0} *or* ρ_{3_0} *satisfies*

$$d\rho_{2_0,3_0} = 0, \quad d{\star}\rho_{2_0,3_0} = 0. \tag{3.7}$$

(ii) *We call a structure of type* $2_\pm$ *or* $3_\pm$ integrable *if and only if the spinor-valued 1-form* $\gamma = \gamma_\pm \in \Gamma(T^*M \otimes \Delta_\pm)$ *is harmonic with respect to the twisted Dirac operator on* $\Gamma(T^*M \otimes \Delta)$, *i.e.*

$$\mathcal{D}\gamma = 0. \tag{3.8}$$

The integrability conditions (3.7) and (3.8) can be analysed in terms of the resulting torsion along the lines of the $PSU(3)$-case, although the situation becomes more difficult as the decomposition of type 2-structures into irreducible modules of $\Lambda^1 \otimes \mathfrak{g}^\perp$ breaks up into small pieces of various multiplicities. We shall deal with two natural questions. For integrable structures of subtype 0, it makes sense to ask if the holonomy of the metric reduces to the groups $SO(3) \times SO(3) \times SO(2)$ or $SO(3) \times SO(5)$. We will show that this is not to be expected for a generic metric of type 2_0 or 3_0. An example illustrating this fact will be given in the next section. On the other hand, if the structure is of subtype $2_\pm$ we saw in Section 1.3.3 that we obtain an almost complex structure on the tangent bundle. We will show that under the integrability condition (3.8) not even partial integrability (in the sense that some of the torsion components of an almost complex structure as described on p. 39 in [39] vanish) can be expected. We start with the analysis of integrable structures of subtype 0.

Integrable structures of type 2_0. The structure group reduces from $SO(8) = SO(\Lambda^1)$ to

$SO(3) \times SO(3) \times SO(2)$. We find the decomposition

$$\begin{aligned}\Lambda^1 \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1)^\perp &= ([2,0,0] \oplus [0,0,2] \oplus [\![0,0,2]\!]) \otimes ([2,2,0] \oplus [\![2,0,2]\!] \oplus [\![0,2,2]\!]) \\ &= [\![0,2,4]\!] \oplus 2[\![0,0,2]\!] \oplus [\![0,2,2]\!] \oplus 3[0,2,0] \oplus [\![0,4,2]\!] \oplus \\ &\quad [\![2,0,4]\!] \oplus [\![2,0,2]\!] \oplus 3[2,0,0] \oplus 3[\![2,2,2]\!] \oplus 2[2,2,0] \oplus \\ &\quad [2,4,0] \oplus [\![4,0,2]\!] \oplus [4,2,0].\end{aligned}$$

Moreover, we have

$$\Lambda^2 = \mathbf{1} \oplus [0,2,0] \oplus [\![0,2,2]\!] \oplus [2,0,0] \oplus [\![2,0,2]\!] \oplus [2,2,0]$$

and

$$\Lambda^4 = 2[\![0,0,2]\!] \oplus 2[0,2,0] \oplus 2[2,0,0] \oplus 2[2,2,0] \oplus 2[\![2,2,2]\!].$$

From this we can partially determine the kernel by an application of Schur's lemma. For instance, the kernel of the map $\widetilde{\delta} : \Lambda^1 \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1)^\perp \to \Lambda^2$ restricted to the sub-module $3[\![2,2,2]\!]$ contains on dimensional grounds at least a module isomorphic to $[\![2,2,2]\!]$ etc.. Using also

$$\dim\ker \widetilde{d} \cap \dim\ker \widetilde{\delta} = \dim\ker \widetilde{d} + \dim\ker \widetilde{\delta} - \dim(\ker \widetilde{d} + \ker \widetilde{\delta}),$$

we immediately draw the

Proposition 3.7. *We have*

$$[\![0,2,4]\!] \oplus [\![0,4,2]\!] \oplus [\![2,0,4]\!] \oplus [\![2,2,2]\!] \oplus [\![4,0,2]\!] \oplus [2,4,0] \oplus [4,2,0] \subset \ker \widetilde{d} \cap \ker \widetilde{d^\star}.$$

In particular, the holonomy of an integrable structure of type 2_0 should not reduce to $SO(3) \times SO(3) \times SO(2)$. This is confirmed by the example constructed in Section 3.2.2.

Integrable structures of type 3_0. Next, we assume that the structure group reduces from $SO(8)$ to $SO(3)\times SO(5)$. We obtain

$$\begin{aligned}\Lambda^1\otimes(\mathfrak{so}(3)\oplus\mathfrak{so}(5))^\perp &= ([2,0,0]\oplus[0,1,0])\otimes[2,1,0]\\ &= [0,1,0]\oplus[2,0,0]\oplus[2,0,2]\oplus[2,1,0]\oplus[2,2,0]\oplus[4,1,0]\end{aligned}$$

while

$$\Lambda^2=[0,0,2]\oplus[2,0,0]\oplus[2,1,0]$$

and

$$\Lambda^4=2[0,1,0]+2[2,0,2]$$

which implies the

Proposition 3.8. *We have*

$$[2,2,0]\oplus[4,1,0]\subset\ker\tilde{d}\cap\ker\widetilde{d^\star}.$$

Again we give an explicit example where the holonomy does not reduce to $SO(3)\times SO(5)$ in Section 3.2.2.

Integrable structures of type $2_\pm$. Here, the structure group $SO(8)=SO(\Delta_\pm)$ of the $Spin(8)$-bundle reduces to $SO(3)\times SO(3)\times SO(2)$ (cf. Table 1.3). By choosing the orientation appropriately we may assume that we deal with a structure of type 2_+ which means that $\gamma=\gamma_+\in\Lambda^3\Delta_-$ defines an isometry $\gamma_+:\Lambda^1\to\Delta_+$. As we saw in Section 1.3.3, this forces the structure group of the tangent bundle to reduce to a subgroup of $U(4)$, that is, we obtain an almost complex structure on the tangent bundle. To discuss the integrability of this almost complex structure under the condition (3.8), we decompose

$$\begin{aligned}\Lambda^1 \otimes \mathfrak{u}(4)^\perp &= [\![0,1,1,-3]\!] \oplus [\![1,1,0,-1]\!] \oplus [\![1,0,0,-3]\!] \oplus [\![1,0,0,1]\!] \\ &= [\![V]\!] \oplus [\![\lambda_0^{1,2}]\!] \oplus [\![\lambda^{0,3}]\!] \oplus [\![\lambda^{0,1}]\!], \end{aligned} \tag{3.9}$$

where the labeling of the irreducible $U(4)$-modules was chosen in accordance with [39]. From the point of view of $U(4)$, we have $\Lambda^1 \otimes \mathbb{C} = \mathbb{C}^4 \oplus \overline{\mathbb{C}}^4$ where $U(4)$ acts on its vector representation with weights

$$-\alpha_1 + \alpha_2 + \alpha_4, -\alpha_2 + \alpha_3 + \alpha_4, -\alpha_3 + \alpha_4, \alpha_1 + \alpha_4.$$

This is refined by the actual action of $SU(2) \times SU(2) \times U(1)/\mathbb{Z}_2$ which acts on $\mathbb{C}^4 = (1,1,1)$ with weights

$$-\sigma_1 - \sigma_2 + \tau, -\sigma_1 + \sigma_2 + \tau, \sigma_1 - \sigma_2 + \tau, \sigma_1 + \sigma_2 + \tau.$$

Substituting these weights and decomposing the $U(4)$-modules appearing in (3.9) accordingly, we obtain

$$\begin{aligned}
[\![V]\!] &= [\![0,1,1,-3]\!] &&= [\![1,1,3]\!] \oplus [\![1,3,3]\!] \oplus [\![3,1,3]\!] \\
[\![\lambda_0^{1,2}]\!] &= [\![1,1,0,-1]\!] &&= [\![1,1,-1]\!] \oplus [\![1,3,-1]\!] \oplus [\![3,1,-1]\!] \\
[\![\lambda^{0,3}]\!] &= [\![1,0,0,-3]\!] &&= [\![1,1,-3]\!] \\
[\![\lambda^{0,1}]\!] &= [\![1,0,0,1]\!] &&= [\![1,1,1]\!].
\end{aligned}$$

The actual torsion of the structure lives in

$$\begin{aligned}
\Lambda^1 \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1)^\perp &= [\![1,1,1]\!] \otimes ([2,2,0] \oplus [\![2,0,2]\!] \oplus [\![0,2,2]\!]) \\
&= 2[\![1,1,3]\!] \oplus [\![1,3,3]\!] \oplus 3[\![1,1,1]\!] \oplus 2[\![1,3,1]\!] \oplus [\![3,1,3]\!] \oplus \\
&\quad \oplus 2[\![3,1,1]\!] \oplus [\![3,3,1]\!]
\end{aligned}$$

and is given by the kernel of the map

$$\widetilde{\mathcal{D}} : \Lambda^1 \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1)^\perp \to \Lambda^1 \otimes \Delta_-.$$

Now

$$\Lambda^1 \otimes \Delta_- = [\![1,1,1]\!] \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus E_2) = 3[\![1,1,1]\!] \oplus [\![3,1,1]\!] \oplus [\![1,3,1]\!] \oplus [\![1,1,3]\!]$$

which implies

Proposition 3.9. *We have*

$$[\![1,1,3]\!] \oplus [\![1,3,1]\!] \oplus [\![3,1,1]\!] \oplus [\![1,3,3]\!] \oplus [\![3,1,3]\!] \oplus [\![3,3,1]\!] \subset \ker \tilde{\mathcal{D}}.$$

In particular, a generic integrable structure of type $2_\pm$ is expected to have torsion in the components $[\![V]\!]$, $[\![\lambda_0^{1,2}]\!]$ and $[\![\lambda^{0,3}]\!]$, and the almost complex structure is not integrable.

Integrable structures of type $3_\pm$*.* We finally look at structures of type $3_\pm$ where the structure group $SO(8) = SO(\Delta_\pm)$ of the $Spin(8)$-bundle reduces to $SO(3) \times SO(5)$ (cf. Table 1.4). Again we assume to deal with a structure of subtype $+$. We have

$$\Lambda^1 \otimes (\mathfrak{so}(3) \oplus \mathfrak{so}(5))^\perp = [1,0,1] \otimes [2,1,0] = [1,0,1] \oplus [1,1,1] \oplus [3,0,1] \oplus [3,1,1].$$

On the other hand,

$$\Lambda^1 \otimes \Delta_- = [1,0,1] \otimes ([2,0,0] \oplus [0,1,0]) = 2[1,0,1] \oplus [3,0,1] \oplus [1,1,1]$$

and therefore

Proposition 3.10. *We have*

$$[3,1,1] \subset \ker \tilde{\mathcal{D}}.$$

Recall that an almost quaternionic structure on M^8 is defined by an $Sp(1)\cdot Sp(2)$-invariant 4-form Ω [38], [39]. This defines a quaternionic Kähler structure if and only if

$$\nabla\Omega = T(\Omega) = 0.$$

Since

$$\Lambda^3 \cong \Lambda^5 = [1,0,1]\oplus[1,1,1]\oplus[3,0,1]$$

and $\tilde{d} : \Lambda^1\otimes(\mathfrak{so}(3)\oplus\mathfrak{so}(5))^\perp \to \Lambda^5$ is onto [39], the condition $d\Omega = 0$ implies $T\in[3,1,1]$ and therefore $D\gamma = 0$.

Corollary 3.11. *Any almost quaternionic 8-manifold* (M^8,Ω) *with* $d\Omega = 0$ *defines an integrable structure of type* $3_\pm$*.*

3.2.2 Geometrical properties and examples

As in the case of special holonomy, a solution to the spinor field equation (3.8) puts constraints on the Ricci tensor of the metric. To see this we recall that according to Proposition 2.8 in [51] (using the notation of (3.5)), we have

$$(D\circ\delta - \delta\circ D)(\gamma) = \frac{1}{2}p(\gamma\circ\mathrm{Ric}) \tag{3.10}$$

for any $\gamma\in\Gamma(T^*M\otimes\Delta)$. Now write $\gamma = \sum_i e_i\otimes\gamma_i$ and regard Ric as an endomorphism of T so that

$$\gamma\circ\mathrm{Ric} = \sum_{i,j}\mathrm{Ric}_{ij}e_i\otimes\gamma_j.$$

If the structure is integrable, (3.10) implies

$$p(\gamma\circ\mathrm{Ric}) = \sum_{i,j}\mathrm{Ric}_{ij}e_i\cdot\gamma_j = 0$$

(alternatively, see [29] for a direct derivation of this equation). This means that Ric is in the kernel of the map

$$A \in \bigodot^2 \Lambda^1 \mapsto p(A \circ \gamma) = \sum_{i,j} A_{ij} e_i \gamma_j \in \Delta, \tag{3.11}$$

which is invariant under the stabiliser of γ. In the case of a $PSU(3)$-structure, we have $\odot^2 \wedge^1 = \mathbf{1} \oplus [1,1] \oplus [2,2]$ and $\Delta = 2[1,1]$. Since this map is non-trivial, Ric vanishes on the module $[1,1]$. Similarly, we can decompose the symmetric tensors for structures of type $2_\pm$ and $3_\pm$ into

$$\bigodot^2 \Lambda^1 = \mathbf{1} \oplus [2,0,0] \oplus [0,2,0] \oplus [2,2,0] \oplus [\![0,0,2]\!] \oplus [\![2,2,2]\!]$$

and

$$\bigodot^2 \Lambda^1 = \mathbf{1} + [0,1,0] + [2,0,2]$$

respectively. The map (3.11) takes then values in $\Delta_\mp$ which carries a Lie bracket. It is immediate that for an integrable type 3-structure, Ric must vanish on the 5-dimensional module $[0,1,0]$ while for type 2-structures it vanishes on the component isomorphic to $\Delta_\mp$ (take the matrices (1.36) and check with the base elements $e_i \otimes e_j + e_j \otimes e_i$ to see that the image has maximal rank).

Proposition 3.12.

(i) [29] *If g is an integrable $PSU(3)$-metric, then* Ric *vanishes on the component* $[1,1]$ *inside* $\odot^2 \Lambda^1$.

(ii) *If g is an integrable metric of type $2_\pm$, then* Ric *vanishes on the components* $\Delta_\mp \cong [2,0,0] \oplus [0,2,0] \oplus [\![0,0,2]\!]$ *inside* $\odot^2 \Lambda^1$.

(iii) *If g is an integrable metric of type $3_\pm$, then* Ric *vanishes on the component* $[0,1,0]$ *inside*

$\odot^2\Lambda^1$. *In particular, this holds for any almost quaternionic structure whose* $Sp(1)\cdot Sp(2)$*-invariant 4-form is closed.*

Remark: A quaternionic Kähler manifold is Einstein [39], that is, the Ricci-tensor vanishes on $[0,1,0]\oplus[2,0,2]$ if $\dim = 8$. The weaker condition $d\Omega = 0$ still guarantees the vanishing on the smaller component $[0,1,0]$.

Examples

(i) (*local description of type* 3_0*-geometries*) Consider a 3-form ρ which induces a supersymmetric map $\Delta_+\otimes\Delta_-$ of spin 3/2. Let $\mathrm{Ann}(\rho)\subset\Gamma(T)$ denote the annihilator of ρ, that is the distribution whose sections contracted with ρ yield 0. If the form ρ is closed, then $\mathrm{Ann}(\rho)$ is involutive since

$$[X,Y]\llcorner\rho = \mathcal{L}_x(Y\llcorner\rho) - Y\llcorner\mathcal{L}_X\rho = -Y\llcorner X\llcorner d\rho = 0.$$

If, in addition, the form is co-closed, then $\mathrm{Ann}(\star\rho)$ also defines an involutive distribution. In the case of a $PSU(3)$-structure the annihilator of ρ and $\star\rho$ is $\{0\}$ as a consequence of the genericity of the form, but for forms of type 2 and 3, we get something non-trivial. In particular, we can apply Frobenius' theorem for type 3 structures to assert the existence of local coordinates $(x_1,\ldots,x_8)$ such that $\mathrm{Ann}(\star\rho)=\langle\partial_{x_1},\partial_{x_2},\partial_{x_3}\rangle$ and $\mathrm{Ann}(\rho)=\langle\partial_{x_4},\ldots,\partial_{x_8}\rangle$. Since $\star\rho$ and ρ are the volume forms for $\mathrm{Ann}(\rho)$ and $\mathrm{Ann}(\star\rho)$ respectively, the condition to be of unit norm is just

$$\det_{i,j=1,2,3} g_{ij} = \det_{i,j=1,\ldots,5} g_{ij} = 1.$$

(ii) (*integrable* $PSU(3)$*-structures of positive, negative and zero scalar curvature with vanishing torsion*) As mentioned above, any integrable $PSU(3)$-metric with vanishing torsion

T is either flat or symmetric. For example, consider the 3-form $\rho(X, Y, Z) = B(X, [Y, Z])$ on the Lie algebra $\mathfrak{su}(3)$, the symmetric space $SU(3) = SU(3) \times SU(3)/SU(3)$ or its non-compact dual $SL(3, \mathbb{C})/SU(3)$. Since ρ and $\star\rho$ are Ad-invariant forms, they are closed and hence they define integrable $PSU(3)$-structures. These three cases correspond to zero, positive and negative scalar curvature respectively. In particular, vanishing torsion does not imply Ricci-flatness.

(iii) (*non-trivial local examples of integrable $PSU(3)$-structures, type 2_0- and 3_0-structures which are Ricci-flat*) This example will be built out of a hyperkähler 4-manifold M^4 with a triholomorphic vector field. Let $U \equiv U(x, y, z)$ be a positive harmonic function defined on some domain $D \subset \mathbb{R}^3$ and θ a 1-form on $\mathbb{R}^3$ for which

$$dU = \star d\theta$$

holds. By the Gibbons-Hawking ansatz, the metric on $D \times \mathbb{R}$

$$g = U(dx^2 + dy^2 + dz^2) + \frac{1}{U}(dt + \theta)^2 \tag{3.12}$$

is hyperkähler with associated Kähler forms given by

$$\begin{aligned}
\omega_1 &= U dy \wedge dz + dx \wedge (dt + \theta) \\
\omega_2 &= U dx \wedge dy + dz \wedge (dt + \theta) \\
\omega_3 &= U dx \wedge dz - dy \wedge (dt + \theta).
\end{aligned}$$

The vector field $X = \frac{\partial}{\partial t}$ is triholomorphic, that is it defines an infinitesimal transformation which preserves any of the three complex structures induced by ω_1, ω_2 or ω_3. Conversely, a hyperkähler metric on a 4-dimensional manifold which admits a triholomorphic vector field is locally of the form (3.12).

Following [3], we define the 2-form $\widetilde{\omega}_3$ by changing the sign in ω_3, that is

$$\widetilde{\omega}_3 = U dx \wedge dz + dy \wedge (dt + \theta).$$

This 2-form is closed if and only if

$$U \equiv U(x, z)$$

for $d\omega_3 = 0$ implies

$$d(U dx \wedge dz) = d(dy \wedge (dt + \theta)), \tag{3.13}$$

so that

$$d\widetilde{\omega}_3 = 2d(U dx \wedge dz) = 2\frac{\partial U}{\partial y} dy \wedge dx \wedge dz.$$

Pick such a U and take the standard coordinates $x_1, \ldots, x_4$ of the Euclidean space $(\mathbb{R}^4, g_0)$. Put

$$\begin{array}{llll} e^1 = dx_1, & e^2 = dx_2, & e^3 = dx_3 & e^8 = dx_4 \\ e^4 = \sqrt{U} dy, & e^5 = -\frac{1}{\sqrt{U}}(dt + \theta), & e^6 = -\sqrt{U} dx, & e^7 = \sqrt{U} dz \end{array}$$

which we take as an orthonormal basis on $M^4 \times \mathbb{R}^4$. Endowed with the orientation defined by $(e_4, \ldots, e_7)$, the forms ω_i are anti-self-dual on M^4, while the forms $\widetilde{\omega}_1 = U dy \wedge dz - dx \wedge (dt + \theta)$, $\widetilde{\omega}_2 = U dx \wedge dy - dz \wedge (dt + \theta)$ and $\widetilde{\omega}_3$ are self-dual. An example of an integrable $PSU(3)$-structure is then provided by (cf. 1.22 and 1.23)

$$\begin{aligned} \rho_1 &= dx_1 \wedge dx_2 \wedge dx_3 + \frac{1}{2}(dx_1 \wedge \omega_1 + dx_2 \wedge \omega_2 + dx_3 \wedge \omega_3) + \frac{\sqrt{3}}{2} dx_4 \wedge \widetilde{\omega}_3 \\ &= e_{123} + \frac{1}{2} e_1 \wedge \omega_1 + \frac{1}{2} e_2 \wedge \omega_2 + \frac{1}{2} e_3 \wedge \omega_3 + \frac{\sqrt{3}}{2} e_8 \wedge \widetilde{\omega}_3. \end{aligned}$$

Obviously, the equality $d\rho_1 = 0$ holds. Moreover, we have

$$\begin{aligned} \star\rho_1 = \;& U dx \wedge dz \wedge dy \wedge (dt + \theta) \wedge dx_4 - \frac{1}{2}\omega_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ & + \frac{1}{2}\omega_2 \wedge dx_1 \wedge dx_3 \wedge dx_4 - \frac{1}{2}\omega_3 \wedge dx_1 \wedge dx_2 \wedge dx_4 + \frac{\sqrt{3}}{2}\widetilde{\omega}_3 \wedge dx_1 \wedge dx_2 \wedge dx_3 \\ = \;& e_{45678} - \frac{1}{2}\omega_1 \wedge e_{238} + \frac{1}{2}\omega_2 \wedge e_{138} - \frac{1}{2}\omega_3 \wedge e_{128} + \frac{\sqrt{3}}{2}\widetilde{\omega}_3 \wedge e_{123}, \end{aligned}$$

and one immediately checks that $\star\rho_1$ is also closed.

This ansatz yields then a structure of type 2_0 and 3_0 by defining

$$\rho_2 = \frac{1}{\sqrt{2}} e_{145} + \frac{1}{\sqrt{2}} e_{367} \text{ and } \rho_3 = e_{145}.$$

Then

$$\star\rho_2 = \frac{1}{\sqrt{2}} e_{23678} + \frac{1}{\sqrt{2}} e_{12458} \text{ and } \star\rho_3 = e_{23678},$$

and all these forms are closed since (3.13) implies $de_{45} = de_{67} = 0$.

To check that the holonomy is not contained in the respective stabiliser groups $PSU(3)$, $SO(3) \times SO(3) \times SO(2)$ and $SO(3) \times SO(5)$, we consider the specific example defined by

$$U(x, y, z) = x \text{ on } \{x > 0\} \text{ and } \theta = ydz,$$

and show that $\nabla\rho_i \neq 0$. The metric g on $M^4 \times \mathbb{R}^4$ is given by

$$g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + xdx^2 + xdy^2 + (x + \frac{y^2}{x})dz^2 + \frac{1}{x}dt^2 + 2\frac{y}{x}dzdt.$$

with orthonormal basis

$$\begin{aligned} &e_1 = \partial_{x_1}, \quad && e_2 = \partial_{x_2}, \quad && e_3 = \partial_{x_3}, \quad && e_8 = \partial_{x_4}, \\ &e_4 = \tfrac{1}{\sqrt{x}}\partial_y, && e_5 = -\sqrt{x}\partial_t, && e_6 = -\tfrac{1}{\sqrt{x}}\partial_x, && e_7 = \tfrac{1}{\sqrt{x}}(\partial_z - y\partial_t). \end{aligned}$$

The only non-trivial brackets are

$$\begin{aligned} &[e_4, e_6] = -\tfrac{1}{2\sqrt{x}^3} e_4 \quad && [e_5, e_6] = \tfrac{1}{2\sqrt{x}^3} e_5 \\ &[e_4, e_7] = \tfrac{1}{\sqrt{x}^3} e_5 && [e_6, e_7] = \tfrac{1}{2\sqrt{x}^3} e_7. \end{aligned}$$

Since the anti-self-dual 2-forms ω_1, ω_2 and ω_3 are the associated Kähler forms of the hyperkähler structure on M, we have $\nabla\omega_i = 0$. In particular,

$$\nabla(e^6 \wedge e^7) = \nabla(e^4 \wedge e^5) = -\frac{1}{12} \cdot \frac{1}{\sqrt{x^3}} (e_4 \otimes \widetilde{\omega_1} + e_5 \otimes \widetilde{\omega_2})$$

holds. Thus

$$\begin{aligned}
\nabla\rho_1 &= -\frac{1}{4\sqrt{3}}\cdot\frac{1}{\sqrt{x^3}}(e_4\otimes\widetilde{\omega}_1\wedge e_8+e_5\otimes\widetilde{\omega}_2\wedge e_8)\\
\nabla\rho_2 &= -\frac{1}{12\sqrt{2x^3}}(e_4\otimes\widetilde{\omega}_1\wedge(e_1+e_3)+e_5\otimes\widetilde{\omega}_2\wedge(e_1+e_3))\\
\nabla\rho_3 &= -\frac{1}{12\sqrt{x^3}}(e_4\otimes\widetilde{\omega}_1\wedge e_1+e_5\otimes\widetilde{\omega}_2\wedge e_1),
\end{aligned}$$

which shows that the holonomy does not reduce to the corresponding stabiliser subgroup. This example also shows that Ricci-flatness does not imply the vanishing of the torsion for an integrable structure of any type.

(iv) (*non-trivial compact $PSU(3)$-manifold which is not Einstein*) Recall that by defining a basis of (anti)-self-dual forms $\omega_{1\pm}=e^{47}\pm e^{56}$, $\omega_{2\pm}=e^{46}\mp e^{57}$ and $\omega_{3\pm}=e^{45}\pm e^{67}$ we can write

$$\rho=e_{123}+\frac{1}{2}e_1\wedge\omega_{1-}+\frac{1}{2}e_2\wedge\omega_{2-}+\frac{1}{2}e_3\wedge\omega_{3-}+\frac{\sqrt{3}}{2}e_8\wedge\omega_{3+}. \tag{3.14}$$

and

$$\star\rho=e_{45678}-\frac{1}{2}e_{238}\wedge\omega_1+\frac{1}{2}e_{138}\wedge\omega_2-\frac{1}{2}e_{128}\wedge\omega_3+\frac{\sqrt{3}}{2}e_{123}\wedge\omega_{3+}. \tag{3.15}$$

Note that

$$\omega_{i\pm}\wedge\omega_{j\mp}=0,\quad \omega_{i\pm}\wedge\omega_{j\pm}=\pm2\delta_{ij}e_{4567}. \tag{3.16}$$

Take N to be a 6-dimensional nilmanifold associated with the Lie algebra $\mathfrak{g}=\langle e_2,\dots,e_8\rangle$ whose structure constants are determined by

$$de_i=\begin{cases} 0, & i=2,\dots,7\\ \omega_{1+}=e_{47}+e_{56}, & i=8\end{cases}$$

(the labeling of the vectors follows our conventional use of a $PSU(3)$-frame), that is the only non-trivial structure constants are

$$c_{478}=-c_{748}=c_{568}=-c_{658}=1.$$

Let G be the associated simply-connected Lie group. The rationality of the structure constants guarantees the existence of a lattice Γ for which $N = \Gamma\backslash G$ is compact [35]. We let $M = T^2 \times N$ with $e_i = dt_i$, $i = 1, 2$ on the torus, hence $de_i = 0$. We take the basis $e_1, \ldots, e_8$ to be orthonormal on M and let g denote the corresponding metric. Then (3.14), (3.15), (3.16) imply

$$d\rho = \frac{\sqrt{3}}{2} de_8 \wedge \omega_{3+} = \frac{\sqrt{3}}{2} \omega_{1+} \wedge \omega_{3+} = 0$$

and similarly, we have

$$\begin{aligned} d \star \rho &= e_{4567} \wedge de_8 - \frac{1}{2}\omega_{1-} \wedge e_{23} \wedge de_8 + \frac{1}{2}\omega_{2-} \wedge e_{13} \wedge de_8 - \frac{1}{2}\omega_{3-} \wedge e_{12} \wedge de_8 \\ &= e_{4567} \wedge \omega_{1+} - \frac{1}{2}\omega_{1-} \wedge e_{23} \wedge \omega_{1+} + \frac{1}{2}\omega_{2-} \wedge e_{13} \wedge \omega_{1+} - \frac{1}{2}\omega_{3-} \wedge e_{12} \wedge \omega_{1+} \\ &= 0. \end{aligned}$$

Next we compute the covariant derivatives $\nabla_{e_i} e_j$. By Koszul's formula, we have

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g(e_i, [e_k, e_j]) = c_{ijk} + c_{kij} + c_{kji}.$$

It follows that

$$\nabla e_i = \begin{cases} 0, & i = 1,\ 2,\ 3 \\ -\frac{1}{2}(e_7 \otimes e_8 + e_8 \otimes e_7), & i = 4 \\ -\frac{1}{2}(e_6 \otimes e_8 + e_8 \otimes e_6), & i = 5 \\ \frac{1}{2}(e_5 \otimes e_8 + e_8 \otimes e_5), & i = 6 \\ \frac{1}{2}(e_4 \otimes e_8 + e_8 \otimes e_4), & i = 7 \\ \frac{1}{2}(-e_4 \otimes e_7 + e_7 \otimes e_4 - e_5 \otimes e_6 + e_6 \otimes e_5), & i = 8 \\ . & \end{cases}$$

Now

$$\nabla_{e_4}(e_8 \wedge \omega_{3+}) = e_{457}$$

and since the coefficient of $e_8 \wedge \omega_{3+}$ is irrational while all the remaining ones are rational, we deduce

$$\nabla_{e_4}\rho \neq 0$$

and therefore the metric is non-symmetric. Moreover, a straightforward computation shows the diagonal of the Ricci-tensor Ric_{ij} to be given by

$$\mathrm{Ric}_{ii} = \begin{cases} 0, & i = 1,\ 2,\ 3,\ 8 \\ \frac{1}{2}, & i = 4,\ 5,\ 6,\ 7 \end{cases}$$

In particular, it follows that (M, g) is not Einstein, that is, Ric will usually have a non-trivial $[2, 2]$-component (cf. Proposition 3.12).

(v) (*non-trivial compact example of a type* 3_+*-structure*) In [40], Salamon constructed a compact almost quaternionic 8-manifold M whose structure form Ω is closed, but not parallel. The example is of the form $M = N^6 \times T^2$, where N^6 is a compact nilmanifold associated with the Lie algebra given by

$$de_i = \begin{cases} 0, & i = 1,\ 2,\ 3,\ 5 \\ e_{15}, & i = 4 \\ e_{13}, & i = 6 \end{cases}$$

According to Corollary 3.11, this implies the existence of a compact non-trivial structure of type $3_\pm$.

Proposition 3.13.

(i) *Compact integrable* $PSU(3)$*-manifolds with non-vanishing torsion and which are not Einstein exist. On the other hand, vanishing torsion does not imply Ricci-flatness. Moreover, there exist non-trivial integrable local Ricci-flat examples.*

(ii) *Compact integrable structures of type* 3_+ *with non-vanishing torsion exist. Moreover, there exist non-trivial local and Ricci-flat examples of type* 2_0*- and* 3_0*-structures.*

3.3 Integrable variational structures and related geometries in the generalised case

3.3.1 The spinorial solution of the variational problem and related geometries

Having analysed the variational problem for classical structures in terms of a spinorial field equation we want to approach the generalised case in a similar vein. First, we give a somewhat technical definition which has the merit of encapsulating at once the equations coming from the unconstrained, constrained and the twisted variational problem.

Definition 3.4. *Let* H *be a (closed) 3-form and* λ *be a real, non-zero constant.*

(i) *A topological generalised* G_2*-structure* (M, ρ) *is said to be* (closed) strongly integrable *with respect to* H *if and only if*

$$d_H\rho = 0, \quad d_H\hat{\rho} = 0.$$

(ii) *A topological generalised* G_2*-structure* (M, ρ) *is said to be* (closed) weakly integrable with respect to H and with Killing number λ *if and only if*

$$d_H\rho = \lambda\hat{\rho}.$$

We call such a structure even *or* odd *according to the parity of the form* ρ*. If we do not wish to distinguish the type, we will refer to both structures as* (closed) weakly integrable.

Similarly, the structures in (i) and (ii) will also be referred to as integrable *if a condition applies to both (closed) weakly and strongly integrable structures.*

(iii) *A topological generalised* $Spin(7)$*-structure* (M, ρ) *will be called* (closed) integrable with respect to H *if and only if*

$$d_H \rho = 0.$$

Remark: As we shall see in Corollary 3.18, the constant λ describes the Λ^4_0-component of $d\varphi_\pm$, where $\varphi_\pm$ are the underlying stable 3-forms of a weakly integrable structure. A classical G_2-structure having only torsion components in this module is said to be *nearly-parallel* and can be equivalently characterised by the existence of a Killing spinor in the sense of [5] (see also the discussion in Section 3.2.1). The constant λ is usually referred to as the *Killing number* which explains our terminology.

We want to interpret the equations in Definition 3.4 in terms of the data of Theorem 2.14. The key will be again the interplay between the twisted Dirac operators $\mathcal{D}$ and $\widehat{\mathcal{D}}$ on $\Delta \otimes \Delta$, given in a local basis by

$$\mathcal{D}(\Psi \otimes \Phi) = \sum s_i \cdot \nabla_{s_i} \Psi \otimes \Phi + s_i \cdot \Psi \otimes \nabla_{s_i} \Phi$$

and

$$\widehat{\mathcal{D}}(\Psi \otimes \Phi) = \sum \nabla_{s_i} \Psi \otimes s_i \cdot \Phi + \Psi \otimes s_i \cdot \nabla_{s_i} \Phi,$$

and the corresponding Dirac operators on Λ^p (under the usual identification of $\Delta \otimes \Delta$ with forms) $d + d^\star$ and $(-1)^p(d - d^\star)$. For the generalised version however, we have to use the twisted isomorphism L_b (as defined in Section 1.5) between $\Delta \otimes \Delta$ and $\Lambda^{ev,od}$. In order to describe its effect on the level of differential operators we define for a p-form α the operator

$$d^\Box \alpha^p = (-1)^{n(p+1)+1} \Box \, d \, \Box \alpha^p.$$

This definition ensures that in absence of a B-field, the identity

$$d^{\square} = d^{\star}$$

holds for $n = 7$ or 8 (see Lemma 1.26).

Proposition 3.14. *(Cf. the notation of page 84)*

(i) *For $n = 7$ we have*

$$L_b^{ev,od} \circ \mathcal{D} = dL_b^{od,ev} + d^{\square} L_b^{od,ev} + \frac{1}{2} e^{b/2} \wedge (db \llcorner L^{od,ev} - db \wedge L^{od,ev})$$

and

$$L_b^{ev,od} \circ \widehat{\mathcal{D}} = \pm(dL_b^{od,ev} - d^{\square} L_b^{od,ev}) \mp \frac{1}{2} e^{b/2} \wedge (db \llcorner L^{od,ev} + db \wedge L^{od,ev}).$$

(ii) *For $n = 8$ we have*

$$\begin{aligned} L_{b,\pm}^{ev} \circ \mathcal{D} &= dL_{b,\pm}^{od,ev} + d^{\square} L_{b,\pm}^{od,ev} + \tfrac{1}{2} e^{b/2} \wedge (db \llcorner L_{\pm}^{od,ev} - db \wedge L_{\pm}^{od,ev}) \\ L_{b,\pm}^{od} \circ \mathcal{D} &= dL_{b,\pm}^{od,ev} + d^{\square} L_{b,\pm}^{od,ev} + \tfrac{1}{2} e^{b/2} \wedge (db \llcorner L_{\pm}^{od,ev} - db \wedge L_{\pm}^{od,ev}) \end{aligned}$$

and

$$\begin{aligned} L_{b,\pm}^{ev} \circ \widehat{\mathcal{D}} &= dL_{b,\pm}^{od} - d^{\square} L_{b,\pm}^{od} - \frac{1}{2} e^{b/2} \wedge (db \llcorner L_{\pm}^{od} + db \wedge L_{\pm}^{od}) \\ L_{b,\pm}^{od} \circ \widehat{\mathcal{D}} &= -dL_{b,\pm}^{ev} + d^{\square} L_{b,\pm}^{ev} + \frac{1}{2} e^{b/2} \wedge (db \llcorner L_{\pm}^{ev} + db \wedge L_{\pm}^{ev}) \end{aligned}$$

Proof: For $b = 0$ the assertion

$$L^{ev,od} \circ \mathcal{D} = dL^{od,ev} + d^* L^{od,ev}$$

and

$$L^{ev,od} \circ \widehat{\mathcal{D}} = \pm(dL^{od,ev} - d^* L^{od,ev}).$$

is classical (see, for instance, [7] Subsection 1.I; compare also Section 3.2.1).

Now consider the case of an arbitrary B-field b which gives rise to the twisted maps $L_b^{ev,od}:\Delta\otimes\Delta\to\Lambda^{ev,od}T^*$. Then

$$L_b^{ev,od}\circ\mathcal{D}=e^{b/2}\wedge L^{ev,od}\circ\mathcal{D}=e^{b/2}\wedge\Big(dL^{od,ev}+d^\star L^{od,ev}\Big).$$

The first term on the right hand side equals

$$e^{b/2}\wedge dL^{od,ev}=dL_b^{od,ev}-\frac{1}{2}e^{b/2}\wedge db\wedge L^{od,ev}.$$

For the second term, we have to treat the cases $n=7$ and $n=8$ separately. If $n=7$, then

$$d^\star L^{od,ev}=\mp\star\, d\star L^{od,ev}.$$

Proposition 1.27 and Lemma 1.26 give

$$\begin{aligned}
e^{b/2}\wedge d^*L^{od,ev} &= \mp e^{b/2}\wedge\star d\star L^{od,ev}\\
&= \mp e^{b/2}\wedge\star d\sigma(L^{ev,od})\\
&= \mp e^{b/2}\wedge\star d(e^{b/2}\wedge\sigma(L_b^{ev,od}))\\
&= \mp e^{b/2}\wedge\star(e^{b/2}\wedge(\frac{1}{2}db\wedge\sigma(L_b^{ev,od})+d\sigma(L_b^{ev,od})))\\
&= \mp\frac{1}{2}e^{b/2}\wedge\star(db\wedge\sigma(L^{ev,od}))\mp\square_{g,b}dL_b^{ev,od}\\
&= \frac{1}{2}e^{b/2}\wedge(db\llcorner L^{od,ev})+d^\square L_b^{od,ev},
\end{aligned}$$

hence

$$L_b^{ev,od}\circ\mathcal{D}=dL_b^{od,ev}+d^\square L_b^{od,ev}+\frac{1}{2}e^{b/2}\wedge(db\llcorner L^{od,ev}-db\wedge L^{od,ev}).$$

Similarly, we obtain

$$\begin{aligned}
L_b^{ev,od}\circ\widehat{\mathcal{D}} &= \pm e^{b/2}\wedge(dL^{od,ev}\pm\star d\star L^{od,ev})\\
&= \pm(dL_b^{od,ev}-d^\square L_b^{od,ev})\mp\frac{1}{2}e^{b/2}\wedge(db\llcorner L^{od,ev}+db\wedge L^{od,ev}).
\end{aligned}$$

Next we treat the case $n = 8$. Here we have to take into account the chirality of the spinors. Recall our convention that $L^{ev,od}_{b,\pm}$ denotes the restriction to $\Delta_* \otimes \Delta_{+,-}$ where $* = +$ for L^{ev}_b and $* = -$ for L^{od}_b. So far we know that

$$L^{ev,od}_{b,\pm} \circ \mathcal{D} = e^{b/2} \wedge (dL^{od,ev}_{\pm} - \star d \star L^{od,ev}_{\pm}).$$

Lemma 1.26 and Proposition 1.27 imply

$$\star L^{ev}_{\pm} = \pm\sigma(L^{ev}_{\pm}) \text{ and } \star L^{od}_{\pm} = \mp\sigma(L^{od}_{\pm}).$$

Consequently, we have to deal with both cases separately. Now

$$\begin{aligned}
e^{b/2} \wedge d^* L^{ev}_{\pm} &= -e^{b/2} \wedge \star d \star L^{ev}_{\pm} \\
&= \mp e^{b/2} \wedge \star d\sigma(L^{ev}_{\pm}) \\
&= \mp e^{b/2} \wedge \star d(e^{b/2} \wedge \sigma(L^{ev}_{b,\pm})) \\
&= \mp \frac{1}{2} e^{b/2} \wedge \star(db \wedge \sigma(L^{ev}_{\pm})) \pm e^{b/2} \wedge \star(e^{b/2} \wedge [dL^{ev}_{b,\pm}]^{\wedge}) \\
&= \pm \frac{1}{2} e^{b/2} \wedge (db \llcorner * \sigma(L^{ev}_{\pm})) \pm \Box dL^{ev}_{b,\pm} \\
&= \frac{1}{2} e^{b/2} \wedge (db \llcorner L^{ev}_{\pm}) + d^{\Box} L^{ev}_{b,\pm}.
\end{aligned}$$

Similarly, we find

$$-e^{b/2} \wedge \star d \star L^{od}_{\pm} = \frac{1}{2} e^{b/2} \wedge (db \llcorner L^{od}_{\pm}) + d^{\Box} L^{od}_{b,\pm}.$$

Putting these pieces together yields

$$L^{ev,od}_{b,\pm} \circ \mathcal{D} = dL^{od,ev}_{b,\pm} + d^{\Box} L^{ev,od}_{b,\pm} + \frac{1}{2} e^{b/2} (db \llcorner L^{ev,od}_{\pm} - db \wedge L^{od,ev}_{\pm}).$$

A similar expression holds for $L^{ev,od}_{b,\pm} \circ \widehat{\mathcal{D}}$. ■

Theorem 2.14 states that a generalised structure ρ on M can be represented by $e^{-F} L_b(\Psi_+ \otimes \Psi_-)$. The integrability conditions in Definition 3.4 translate as follows.

Theorem 3.15.

(i) *A generalised G_2-structure (M^7, ρ) is weakly integrable with Killing number λ if and only if $e^{-F}L_b(\Psi_+ \otimes \Psi_-) = \rho + \Box_{g,b}\rho$ satisfies (with $T = db/2 + H$)*

1. *for all vector fields X,*

$$\nabla_X \Psi_\pm \pm \frac{1}{4}(X \llcorner T) \cdot \Psi_+ = 0,$$

2. $$(dF \pm \frac{1}{2}T \pm \lambda) \cdot \Psi_\pm = 0$$

   in case of an even structure and

$$(dF \pm \frac{1}{2}T + \lambda) \cdot \Psi_\pm = 0$$

   in case of an odd structure.

The structure $e^{-F}L_b(\Psi_+ \otimes \Psi_-) = \rho + \Box_{g,b}\rho$ is strongly integrable if and only if these equations hold for $\lambda = 0$.

(ii) *A generalised $Spin(7)$-structure (M^8, ρ) is integrable if and only if $e^{-F}L_b(\Psi_+ \otimes \Psi_-) = \rho$ satisfies (with $T = db/2 + H$)*

1. *for all vector fields X,*

$$\nabla_X \Psi_\pm \pm \frac{1}{4}(X \llcorner T) \cdot \Psi_\pm = 0$$

2. $$(dF \pm \frac{1}{2}T) \cdot \Psi_\pm = 0$$

We refer to the equation involving the covariant derivative of the spinor as the (generalised) Killing equation *and to the equation involving the differential of F as the* dilaton equation*.*

Remark: The generalised Killing equation basically states that we have two metric connections $\nabla^{\pm}$ preserving the underlying $G_{2\pm}$- or $Spin(7)_{\pm}$-structures whose torsion tensor (as it is to be defined in 3.3.2) is *skew*. The dilaton equation then serves to identify the components of the torsion with respect to the decomposition into irreducible $G_{2\pm}$- or $Spin(7)_{\pm}$-modules with the additional data dF and λ.

Proof: We first consider the strong integrability condition $d_H\rho = 0$, $d_H\hat{\rho} = 0$ which is equivalent to

$$d\rho = -H \wedge \rho, \quad d\Box\rho = -H \wedge \Box\rho. \tag{3.17}$$

We write $\rho = e^{-F}L^{ev}_b(\Psi_+ \otimes \Psi_-)$, $e^{-F}L^{ev}_{b,+}(\Psi_+ \otimes \Psi_-)$ or $e^{-F}L^{od}_{b,-}(\Psi_+ \otimes \Psi_-)$ according to the cases G_2, $Spin(7)$ even or $Spin(7)$ odd. Taking $\Box$ of the second equation in (3.17) gives

$$\begin{aligned}
d^{\Box}e^{-F}L^{ev}_b &= e^{b/2} \wedge (H \llcorner e^{-F}L^{ev}), && (G_2)\\
d^{\Box}e^{-F}L^{ev}_{b,+} &= e^{b/2} \wedge (H \llcorner e^{-F}L^{ev}_+), && (Spin(7),\ even)\\
d^{\Box}e^{-F}L^{od}_{b,-} &= e^{b/2} \wedge (H \llcorner e^{-F}L^{od}_-), && (Spin(7),\ odd)
\end{aligned}$$

For example,

$$\begin{aligned}
d^{\Box}e^{-F}L^{ev}_{b,+} &= -\Box(H \wedge \Box e^{-F}L^{ev}_{b,+})\\
&= -e^{b/2} \wedge \star(e^{b/2} \wedge \sigma(H \wedge e^{-F}L^{ev}_{b,+}))\\
&= -e^{b/2} \wedge \star(H \wedge \sigma(e^{-F}L^{ev}_+))\\
&= \quad e^{b/2} \wedge (H \llcorner e^{-F}L^{ev}_+).
\end{aligned}$$

From Proposition 3.14 we obtain the following expressions for the Dirac operator $\mathcal{D}$ (where we put $T = db/2 + H$).

$$\begin{aligned} L^{od} \circ \mathcal{D}(e^{-F}\Psi_+ \otimes \Psi_-) &= T \llcorner L^{ev}(e^{-F}\Psi_+ \otimes \Psi_-) - T \wedge L^{ev}(e^{-F}\Psi_+ \otimes \Psi_-) \\ L^{ev}_+ \circ \mathcal{D}(e^{-F}\Psi_+ \otimes \Psi_-) &= T \llcorner L^{od}_+(e^{-F}\Psi_+ \otimes \Psi_-) - T \wedge L^{od}_+(e^{-F}\Psi_+ \otimes \Psi_-) \\ L^{od}_- \circ \mathcal{D}(e^{-F}\Psi_+ \otimes \Psi_-) &= T \llcorner L^{ev}_-(e^{-F}\Psi_+ \otimes \Psi_-) - T \wedge L^{ev}_-(e^{-F}\Psi_+ \otimes \Psi_-). \end{aligned}$$

Applying Corollary 1.25 to these equations entails

$$\begin{aligned} \mathcal{D}(e^{-F}\Psi_+ \otimes \Psi_-) &= D(e^{-F}\Psi_+) \otimes \Psi_- + e^{-F}\sum_i s_i \cdot \Psi_+ \otimes \nabla_{s_i}\Psi_- \\ &= \frac{1}{4}e^{-F}(-T \cdot \Psi_+ \otimes \Psi_- + \sum_i s_i \cdot \Psi_+ \otimes (s_i \llcorner T) \cdot \Psi_-), \end{aligned} \tag{3.18}$$

where D denotes the usual Dirac operator associated with the Clifford bundle $(\Delta,\, q_\Delta)$. Since $q_\Delta(s_i \cdot \Psi_+, s_j \cdot \Psi_+) = \delta_{ij}$, the contraction of this equation with $q_\Delta(\cdot, s_m \cdot \Psi_+)$ yields

$$q_\Delta(D(e^{-F}\Psi_+), s_m \cdot \Psi_+)\Psi_- + e^{-F}\nabla_{s_m}\Psi_- = \frac{1}{4}e^{-F}(-q_\Delta(T \cdot \Psi_+, s_m \cdot \Psi_+)\Psi_- + (s_m \llcorner T) \cdot \Psi_-),$$

implying

$$e^{-F}\nabla_{s_m}\Psi_- = -\frac{1}{4}q_\Delta(4D(e^{-F}\Psi_+) + e^{-F}T \cdot \Psi_+, s_m \cdot \Psi_+)\Psi_- + \frac{1}{4}e^{-F}(s_m \llcorner T) \cdot \Psi_-.$$

However, from this expression of the covariant derivative we deduce

$$s_m.q_\Delta(\Psi_-, \Psi_-) = 2q_\Delta(\nabla_{s_m}\Psi_-, \Psi_-) = -\frac{1}{2}q_\Delta(4e^F D(e^{-F}\Psi_+) + T \cdot \Psi_+, s_m \cdot \Psi_+) = 0$$

as $s_m \llcorner T \in \mathfrak{so}(T, g)$ and consequently $q_\Delta((s_m \llcorner T) \cdot \Psi_-, \Psi_-) = 0$. It follows

$$\nabla_{s_m}\Psi_- = \frac{1}{4}(s_m \llcorner T) \cdot \Psi_-. \tag{3.19}$$

In order to derive the corresponding expression for the spinor Ψ_+, we consider the Dirac operator $\widehat{\mathcal{D}}$. Firstly, we find

$$\begin{aligned} L^{od} \circ \widehat{\mathcal{D}}(\Psi_+ \otimes e^{-F}\Psi_-) &= T\llcorner e^{-F} L^{ev}(\Psi_+ \otimes \Psi_-) + T \wedge e^{-F} L^{ev}(\Psi_+ \otimes \Psi_-) \\ L^{ev}_+ \circ \widehat{\mathcal{D}}(\Psi_+ \otimes e^{-F}\Psi_-) &= -T\llcorner e^{-F} L^{od}_+(\Psi_+ \otimes \Psi_-) - T \wedge e^{-F} L^{od}_+(\Psi_+ \otimes \Psi_-) \\ L^{od}_- \circ \widehat{\mathcal{D}}(\Psi_+ \otimes e^{-F}\Psi_-) &= T\llcorner e^{-F} L^{ev}_-(\Psi_+ \otimes \Psi_-) + T \wedge e^{-F} L^{ev}_-(\Psi_+ \otimes \Psi_-). \end{aligned}$$

We appeal again to Lemma 1.25 to deduce

$$\begin{aligned} \widehat{\mathcal{D}}(\Psi_+ \otimes e^{-F}\Psi_-) &= \Psi_+ \otimes D(e^{-F}\Psi_-) + e^{-F} \sum_i \nabla_{s_i} \cdot \Psi_+ \otimes s_i \cdot \Psi_- \\ &= \frac{1}{4} e^{-F}(\Psi_+ \otimes T \cdot \Psi_- - \sum_i (s_i \llcorner T) \cdot \Psi_+ \otimes s_i \cdot \Psi_-). \end{aligned} \tag{3.20}$$

Now contraction with $q_\Delta(\cdot, s_m \cdot \Psi_-)$ gets

$$e^{-F}\nabla_{s_m}\Psi_+ = \frac{1}{4} q_\Delta(-4D(e^{-F}\Psi_-) + e^{-F} T \cdot \Psi_-, s_m \cdot \Psi_-)\Psi_+ - \frac{1}{4} e^{-F}(s_m \llcorner T) \cdot \Psi_+.$$

We derive the norm of Ψ_+ and obtain

$$s_m . q_\Delta(\Psi_+, \Psi_+) = 2 q_\Delta(\nabla_{s_m}\Psi_+, \Psi_+) = \frac{1}{2} q_\Delta(-4e^F D(e^{-F}\Psi_-) + T \cdot \Psi_-, s_m \cdot \Psi_-) = 0.$$

Hence in all three cases (G_2, $Spin(7)$ even, $Spin(7)$ odd) we find

$$\nabla_{s_m}\Psi_+ = -\frac{1}{4}(s_m \llcorner T) \cdot \Psi_+. \tag{3.21}$$

If we now consider the inhomogeneous equation $d_H \rho = \lambda \hat{\rho}$, then we pick up an extra term in (3.18) and (3.20). For example, (3.20) becomes

$$\widehat{\mathcal{D}}(\Psi_+ \otimes e^{-F}\Psi_-) = -\lambda e^F \Psi_+ \otimes \Psi_- + \frac{1}{4} e^{-F}(\Psi_+ \otimes T \cdot \Psi_- - \sum_i (s_i \llcorner T) \cdot \Psi_+ \otimes s_i \cdot \Psi_-), \tag{3.22}$$

for weak integrability of even type and

$$\widehat{\mathcal{D}}(\Psi_+ \otimes e^{-F}\Psi_-) = \lambda e^F \Psi_+ \otimes \Psi_- + \frac{1}{4} e^{-F}(\Psi_+ \otimes T \cdot \Psi_- - \sum_i (s_i \llcorner T) \cdot \Psi_+ \otimes s_i \cdot \Psi_-) \tag{3.23}$$

for weak integrability of odd type. But these extra terms cancel if we contract by $q_\Delta(\cdot, s_m \cdot \Psi_-)$ and similarly for the expression involving $\mathcal{D}$. Consequently we derive the same condition on the covariant derivative of $\Psi_\pm$. The difference between these integrability conditions is only visible in the dilaton equation to which we turn next.

Here, we contract (3.22) and (3.23) with $q_\Delta(\cdot, \Psi_+)$ and obtain

$$\begin{aligned} De^{-F}\Psi_- &= -\lambda e^{-F}\Psi_- + \frac{1}{4}e^{-F}T \cdot \Psi_-, \quad \text{even case} \\ De^{-F}\Psi_- &= \lambda e^{-F}\Psi_- + \frac{1}{4}e^{-F}T \cdot \Psi_-, \quad \text{odd case.} \end{aligned}$$

On the other hand, equation (3.19) entails

$$De^{-F}\Psi_- = e^{-F}(-dF \cdot \Psi_- + \frac{3}{4}T \cdot \Psi_-),$$

which gives the dilaton equation for Ψ_-. Using $\mathcal{D}$ in conjunction with (3.21) implies the dilaton equation for Ψ_+.

Conversely assume that the two spinors $\Psi_\pm$ satisfy the generalised Killing equations. We consider the G_2-case first. Since

$$d_H(e^{-F}L_b^{ev,od}) = e^{-F}e^{b/2} \wedge (T \wedge L^{ev,od} - dF \wedge L_b^{ev,od} + dL^{ev,od})$$

it suffices to show that

$$dL^{ev,od} - dF \wedge L^{ev,od} = \lambda L^{od,ev} - T \wedge L^{ev,od}. \tag{3.24}$$

Let us consider a weak structure of even type (the odd case, again, is analogous). Then

$$\begin{aligned}(dL^{ev} - dF \wedge L^{ev})(\Psi_+ \otimes \Psi_-) &= \frac{1}{2}L^{od}(-dF \cdot \Psi_+ \otimes \Psi_- + \Psi_+ \otimes dF \cdot \Psi_-) + \\ &\sum_i s_i \wedge \nabla_{s_i} L^{ev}(\Psi_+ \otimes \Psi_-) \\ &= \lambda L^{od}(\Psi_+ \otimes \Psi_-) + \frac{1}{4}L^{od}(T \cdot \Psi_+ \otimes \Psi_- + \Psi_+ \otimes T \cdot \Psi_-) + \\ &\frac{1}{2}\sum_i L^{od}(s_i \cdot \nabla_{s_i}\Psi_+ \otimes \Psi_- - \nabla_{s_i}\Psi_+ \otimes s_i \cdot \Psi_- + \\ &s_i \cdot \Psi_+ \otimes \nabla_{s_i}\Psi_- - \Psi_+ \otimes s_i \cdot \nabla_{s_i}\Psi_-) \\ &= \lambda L^{od}(\Psi_+ \otimes \Psi_-) - \frac{1}{8}L^{od}(T \cdot \Psi_+ \otimes \Psi_- + \Psi_+ \otimes T \cdot \Psi_- - \\ &\sum_i (s_i \llcorner T) \cdot \Psi_+ \otimes s_i \cdot \Psi_- - \sum_i s_i \cdot \Psi_+ \otimes (s_i \llcorner T) \cdot \Psi_-) \\ &= (\lambda L^{od} - T \wedge L^{ev})(\Psi_+ \otimes \Psi_-)\end{aligned}$$

which proves (3.24).

It remains to establish the $Spin(7)$-case. Here, we have to show that the forms $e^{-F}L^{ev}_{b+}$ and $e^{-F}L^{od}_{b-}$ are closed under d_H. Equivalently, we can show that

$$dL^{ev}_+ - dF \wedge L^{ev}_+ = -T \wedge L^{ev}_+$$

and

$$dL^{od}_- - dF \wedge L^{od}_- = -T \wedge L^{od}_-,$$

which is proven in the same way as in the G_2-case. ■

Remark:

(i) Similarly, we can introduce a 1-form α and consider the twisted differential operator d_α. This would lead to a substitution of dF by $dF + \alpha$ in the dilaton equation. If α is in addition closed, such a geometry would naturally appear as the twisted variational problem with a 1-form. However for later applications the twisted problem with a 3-form is more natural as it fits into the natural setup of T-duality.

(ii) The generalised Killing and the dilaton equation with closed T occur in physics as supersymmetric solutions of type IIA/B supergravity in the NS-NS sector (see, for instance, [20]).

3.3.2 Geometrical properties

As in the classical case, the spinorial formulation is useful to deduce global information about the geometrical properties of integrable structures. In particular we want to compute the Ricci tensor and to that end, we need to understand the previous theorem again from a rather representation theoretic point of view.

Recall that the *torsion tensor* measures the difference between an arbitrary metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ and is defined by [19]

$$g(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}\mathrm{Tor}(X, Y, Z).$$

In the situation of Theorem 3.15, we have two principal subbundles induced by Ψ_+ and Ψ_- inside the spin bundle which carry metric connections ∇^+ and ∇^- such that

$$\nabla^{\pm}_X \Psi_{\pm} = \nabla_X \Psi_{\pm} \pm \frac{1}{4}(X \llcorner T) \cdot \Psi_{\pm}.$$

Hence these connections are induced by the metric linear connections

$$\nabla^{\pm}_X Y = \nabla_X Y \pm \frac{1}{2} T(X, Y, \cdot)$$

and the theorem entails the torsion $\mathrm{Tor}_{\pm}$ of $\nabla^{\pm}$ to be

$$\mathrm{Tor}_{\pm} = \pm T.$$

Starting with spinors, it makes therefore sense to consider a broader class of geometries, namely geometries with two linear metric connections whose torsion tensors are skew and sum up to 0. In the form picture, we then ask for closed-ness of the B-field free form $\rho_0 = L(\Psi_+ \otimes \Psi_-)$ with respect to the twisted differential operator d_T. The untwisted variational problem singles out those geometries with exact torsion as $d_{db}\rho_0 = 0$ holds if and only if $d(e^b\rho_0) = 0$. Hence these geometries encapsulate all the structures we introduced and conveniently avoid distinguishing between "internal" torsion db coming from the ubiquitous B-field and "external" torsion H which might or might not be present. Consequently, we refer to T as the *torsion* of the generalised exceptional structure and regard it as a part of the intrinsic data of an integrable structure. Connections with skew-symmetric torsion have been systematically studied in the recent mathematical literature, e.g. [1], [19], [17] and [31]. The case of a G_2-connection with closed skew-symmetric torsion has been considered in [14], where such G_2-structures are called *strong* (conflicting with our notion of strong integrability which is why we refer to these structures as *closed*).

Again, we call such a geometry *non-trivial* if it has non-trivial torsion. A generalised geometry which is trivial can be completely understood in terms of classical geometries. If the torsion vanishes, then both spinors Ψ_+ and Ψ_- are parallel with respect to the Levi-Civita connection. Consequently, the holonomy of the underlying Riemannian manifold reduces to G_2 or $Spin(7)$. In the case of a generalised G_2-structure, it follows in particular that the underlying topological generalised G_2-structure cannot be exotic (cf. Section 2.2). If the spinors were to be linearly dependent at one point, covariant constancy would imply global linear dependency and we would have an ordinary manifold of holonomy G_2. If the two spinors are linearly independent at some (and hence at any) point, then the holonomy reduces to an $SU(3)$-principal fibre bundle which is the intersection of the two G_2-structures. In this case, M is locally isometric to $CY^3 \times N^1$ where CY^3 is a Calabi-Yau 3-fold. Similarly, if the two spinors defining a generalised $Spin(7)$-structure of even type are linearly dependent at one point, then the holonomy of the underlying Riemannian manifold is contained in $Spin(7)$. Otherwise, the holonomy reduces to a subgroup of $Spin(6) = SU(4)$. For structures of odd type, vanishing torsion means that the holonomy reduces to G_2 and thus M^8 is locally isometric to $N^7 \times N^1$ where N^7 has holonomy G_2.

For clarity of exposition, we will deal with the G_2- and the $Spin(7)$-cases separately.

Closed integrable generalised $\mathbf{G_2}$-structures

In order to fix the notation, we state the

Proposition 3.16. [17] *For any G_2-structure with stable form φ there exist unique differential forms $\lambda \in \Omega^0(M)$, $\theta \in \Omega^1(M)$, $\xi \in \Omega^2_{14}(M,\varphi)$, and $\tau \in \Omega^3_{27}(M,\varphi)$ so that the differentials of φ and $\star\varphi$ are given by*

$$\begin{aligned} d\varphi &= -\lambda \star \varphi + \tfrac{3}{4}\theta \wedge \varphi + \star\tau \\ d\star\varphi &= \theta \wedge \star\varphi + \xi \wedge \varphi. \end{aligned}$$

To specify the torsion tensor of a connection is in general not sufficient to guarantee its uniqueness. However, this is true for G_2-connections with skew-torsion. Using the notation of the previous proposition, we can assert the following result.

Proposition 3.17. [19], [17] *For a G_2-structure with stable form φ the following statements are equivalent.*

(i) *The G_2-structure is* integrable, *i.e.* $\xi = 0$.

(ii) *There exists a unique linear connection* $\widetilde{\nabla}$ *whose torsion tensor* Tor *is skew and which preserves the* G_2*-structure, i.e.*

$$\widetilde{\nabla}\varphi = 0.$$

The torsion can be expressed by

$$\begin{aligned} \mathrm{Tor} &= -\star d\varphi - \frac{7}{6}\lambda \cdot \varphi + \star(\theta \wedge \varphi) \\ &= -\frac{1}{6}\lambda \cdot \varphi + \frac{1}{4} \star (\theta \wedge \varphi) - \star\tau. \end{aligned} \tag{3.25}$$

Moreover, the Clifford action of the torsion 3-form on the induced spinor Ψ *is*

$$\mathrm{Tor} \cdot \Psi = \frac{7}{6}\lambda\Psi - \theta \cdot \Psi. \tag{3.26}$$

Using the labeling of Proposition 3.16 with additional subscripts $\pm$ to indicate the torsion forms of $\nabla^{\pm}$, equations (3.25) and (3.26) read

$$\mathrm{Tor}_{\pm} = \pm T = -\frac{1}{6}\lambda_{\pm} \cdot \varphi_{\pm} + \frac{1}{4} \star (\theta_{\pm} \wedge \varphi_{\pm}) - \star\tau_{\pm}$$

and

$$\mathrm{Tor}_{\pm} \cdot \Psi_{\pm} = \pm T \cdot \Psi_{\pm} = \frac{7}{6}\lambda_{\pm}\Psi_{\pm} - \theta_{\pm} \cdot \Psi_{\pm}.$$

Note also the general G_2-identity $(X \lrcorner \star\varphi) \cdot \Psi = 4X \cdot \Psi$. In view of Theorem 3.15 we can use the dilaton equation to relate the torsion components to the additional parameters dF and λ. We have

$$\pm T \cdot \Psi_{\pm} = \mp 2\lambda\Psi_{\pm} - 2dF \cdot \Psi_{\pm}$$

if the structure is even and

$$\pm T \cdot \Psi_{\pm} = -2\lambda\Psi_{\pm} - 2dF \cdot \Psi_{\pm}$$

if the structure is odd.

Corollary 3.18. *If the generalised G_2-structure is weakly integrable, then there exist two linear connections $\nabla^{\pm}$ preserving the $G_{2\pm}$-structure with skew torsion $\pm T$. These connections are uniquely determined. The differentials of the stable forms $\varphi_{\pm}$ and $\star\varphi_{\pm}$ are given as follows. If the structure is weakly integrable and of even type, then*

$$\begin{aligned} d\varphi_+ &= \tfrac{12}{7}\lambda \star \varphi_+ + \tfrac{3}{2} dF \wedge \varphi_+ - \star T_{27+} \\ d\star\varphi_+ &= 2dF \wedge \star\varphi_+ \end{aligned}$$

and

$$\begin{aligned} d\varphi_- &= -\tfrac{12}{7}\lambda \star \varphi_- + \tfrac{3}{2} dF \wedge \varphi_- + \star T_{27-} \\ d\star\varphi_- &= 2dF \wedge \star\varphi_-, \end{aligned}$$

where $T_{27\pm}$ denotes the projection of T onto $\Omega^3_{27}(M, \varphi_{\pm})$. Moreover, the torsion can be expressed by the formula

$$\mathrm{Tor}_{\pm} = \pm T = -e^{2F} \star de^{-2F} \varphi_{\pm} \pm 2\lambda \cdot \varphi_{\pm}. \tag{3.27}$$

If the structure is weakly integrable and of odd type, then

$$\begin{aligned} d\varphi_+ &= \tfrac{12}{7}\lambda \star \varphi_+ + \tfrac{3}{2} dF \wedge \varphi_+ - \star T_{27+} \\ d\star\varphi_+ &= 2dF \wedge \star\varphi_+ \end{aligned}$$

and

$$\begin{aligned} d\varphi_- &= \tfrac{12}{7}\lambda \star \varphi_- + \tfrac{3}{2} dF \wedge \varphi_- + \star T_{27-} \\ d\star\varphi_- &= 2dF \wedge \star\varphi_-. \end{aligned}$$

The torsion can be expressed by the formula

$$\mathrm{Tor}_{\pm} = \pm T = -e^{2F} \star de^{-2F} \varphi_{\pm} + 2\lambda \cdot \varphi_{\pm}. \tag{3.28}$$

The strongly integrable case follows if we set $\lambda = 0$.

Conversely, if we are given two G_2-structures defined by the stable forms φ_+ and φ_- inducing the same metric, a constant λ and a function F such that (3.27) or (3.28) defines *a (closed) 3-form T, then the corresponding spinors $\Psi_\pm$ satisfy the integrability condition of Theorem 3.15 and hence define a (closed) integrable generalised G_2-structure of even or odd type.*

Depending on the torsion components which are turned on, we will also say that the $G_{2\pm}$-structure has torsion in the components $\mathbf{1}_\pm$, $\mathbf{7}_\pm$ or $\mathbf{27}_\pm$.

Next we investigate closed structures only. The expressions (3.27) and (3.28) for the torsion form T have an interesting consequence (see also [20] and [32]). Assume M to be compact and endowed with a closed weakly integrable structure of even type. Then (3.27) and Stokes' Theorem imply

$$\int_M e^{-2F} T \wedge \star T = \mp \int_M T \wedge d(e^{-2F}\varphi_\pm) + 2\lambda \int_M T \wedge e^{-2F} \star \varphi_\pm = \frac{4}{7}\lambda^2 \int_M e^{-2F} vol_M.$$

Here we have used that $dT = 0$ and the fact that the projection of T on $\varphi_\pm$ is given by $T_{1\pm} = 2\lambda\varphi_\pm$. Since the left hand side is strictly positive (unless $T \equiv 0$), we need $\lambda \neq 0$. If we start with an odd structure instead, we also find

$$\int_M e^{-2F} T \wedge \star T = \mp \int_M T \wedge d(e^{-2F}\varphi_\pm) \pm 2\lambda \int_M T \wedge e^{-2F} \star \varphi_\pm = \frac{4}{7}\lambda^2 \int_M e^{-2F} vol_M,$$

since now $T_{1\pm} = \pm 2\lambda\varphi_\pm$.

Consequently, we obtain the following no-go theorem.

Corollary 3.19. *If M is compact and carries a closed integrable generalised G_2-structure, then $T = 0$ if and only if $\lambda = 0$. In that case the spinors $\Psi_\pm$ are parallel with respect to the Levi-Civita connection.*

Next we compute the Ricci tensor. As a corollary, we will be able to exclude the existence of interesting closed strongly integrable examples on homogeneous spaces.

To begin with, let Ric and $\mathrm{Ric}^{\pm}$ denote the Ricci tensor associated with the Levi-Civita connection and the connections $\nabla^{\pm}$. The relationship between Ric and $\mathrm{Ric}^{\pm}$ was analysed in [19]. Generally speaking, if we have a G-structure with a G-preserving, metric linear connection $\widetilde{\nabla}$ that has closed, skew torsion, and a G-invariant spinor Ψ, then the following identities hold.

Proposition 3.20. [19] *The Ricci tensor associated with $\widetilde{\nabla}$ is determined by the equation*

$$\widetilde{\mathrm{Ric}}(X) \cdot \Psi = (\widetilde{\nabla}_X \mathrm{Tor} + \frac{1}{2} X \llcorner dT) \cdot \Psi$$

and relates to the metric Ricci tensor through

$$\mathrm{Ric}(X,Y) = \widetilde{\mathrm{Ric}}(X,Y) + \frac{1}{2} d^* \mathrm{Tor}(X,Y) + \frac{1}{4} g(X \llcorner \mathrm{Tor}, Y \llcorner \mathrm{Tor}).$$

Theorem 3.21. *The Ricci-tensor of a closed integrable generalised G_2-structure is given by*

$$\mathrm{Ric}(X,Y) = -\frac{7}{2} H^F(X,Y) + \frac{1}{4} g(X \llcorner T, Y \llcorner T),$$

where $H^F(X,Y) = X.Y.F - \nabla_X Y.F$ is the Hessian of the dilaton F.

Proof: According to the previous proposition we obtain

$$\mathrm{Ric}(X,Y) = \frac{1}{2}(\mathrm{Ric}^+(X,Y) + \mathrm{Ric}^-(X,Y)) + \frac{1}{4} g(X \llcorner T, Y \llcorner T).$$

Consequently, it remains to show that

$$\mathrm{Ric}^+(X,Y) + \mathrm{Ric}^-(X,Y) = -7 H^F(X,Y). \tag{3.29}$$

Since $\nabla^{\pm}$ preserves the $G_{2\pm}$-structures and $dT = 0$, only $T_{7\pm}$, the $\Lambda^3_{7\pm}$ component of T, impacts on $\mathrm{Ric}^{\pm}$. Hence we can write

$$\gamma_{X^{\pm}} \llcorner \star \varphi_{\pm} = \nabla^{\pm}_X T_{7\pm} = \pm \frac{1}{2} \nabla^{\pm}_X \star (dF \wedge \varphi_{\pm})$$

and thus

$$\mathrm{Ric}^{\pm}(X)\cdot\Psi_{\pm} = \pm\nabla^{\pm}_{X}T\cdot\Psi_{\pm} = \pm(\gamma_{X^{\pm}}\llcorner\star\varphi_{\pm})\cdot\Psi_{\pm} = \pm 4\gamma_{X^{\pm}}\cdot\Psi_{\pm}.$$

Moreover, for any G_2-structure (φ, g) we have $g(X\wedge\varphi, X\wedge\varphi) = 4g(X,X)$ and therefore,

$$\begin{aligned}
\mathrm{Ric}^{\pm}(X,Y) &= \pm 4g(\gamma_{X^{\pm}}, Y)\\
&= \pm g(\star(\gamma_{X^{\pm}}\wedge\varphi_{\pm}), \star(Y\wedge\varphi_{\pm}))\\
&= \mp g(\gamma_{X^{\pm}}\llcorner\star\varphi_{\pm}, \star(Y\wedge\varphi_{\pm}))\\
&= -\frac{1}{2}g(\nabla^{\pm}_{X}\star(dF\wedge\varphi_{\pm}), \star(Y\wedge\varphi_{\pm}))\\
&= \frac{1}{2}g(Y\wedge\nabla^{\pm}_{X}\star(dF\wedge\varphi_{\pm}), \star\varphi_{\pm}).
\end{aligned} \tag{3.30}$$

We shall now derive (3.29) by a pointwise computation in a frame that satisfies $\nabla_{e_i}e_j = 0$, or equivalently, $\nabla^{\pm}_{e_i}e_j = \pm\frac{1}{2}\sum_k T_{ijk}e_k$ in a fixed point x. Continuing with (3.30),

$$\begin{aligned}
\mathrm{Ric}^{\pm}(e_i,e_j) &= \frac{1}{2}g(e_j\wedge\nabla^{\pm}_{e_i}\star(dF\wedge\varphi_{\pm}), \star\varphi_{\pm})\\
&= \frac{1}{2}g(\nabla^{\pm}_{e_i}(e_j\wedge\star(dF\wedge\varphi_{\pm})) - \nabla^{\pm}_{e_i}e_j\wedge\star(dF\wedge\varphi_{\pm}), \star\varphi_{\pm})\\
&= \frac{1}{2}e_i.g(e_j\wedge\star(dF\wedge\varphi_{\pm}), \star\varphi_{\pm}) - \frac{1}{2}g(\nabla^{\pm}_{e_i}e_j\wedge\star(dF\wedge\varphi_{\pm}), \star\varphi_{\pm})\\
&= -\frac{7}{2}e_i.e_j.F \pm \frac{7}{4}\sum_k T_{ijk}e_k.F.
\end{aligned}$$

On the other hand, the Hessian at the point x evaluated in this basis is

$$H^F(e_i,e_j) = e_i.e_j.F - \nabla_{e_i}e_j.F = e_i.e_j.F,$$

which yields the assertion. ■

Corollary 3.22. *For a closed integrable generalised G_2-structure, the scalar curvature of $\nabla^{\pm}$ is*

$$\mathrm{Scal}^{\pm} = \frac{7}{2}\Delta F,$$

where $\Delta(\cdot) = -\mathrm{Tr}_{\mathrm{g}} H^{(\cdot)}$ *is the Laplacian associated with the metric* g*. Hence, the scalar curvature of the metric* g *is given by*

$$\mathrm{Scal} = \mathrm{Tr}(\mathrm{Ric}) = \frac{7}{2}\Delta F + \frac{1}{4}\, \|T\|^2 \,.$$

Since $T = 0$ implies the covariant constancy of the spinors $\Psi_\pm$ with respect to the Levi-Civita connection, an integrable structure with vanishing torsion is necessarily Ricci-flat. It is interesting to note however, that already the vanishing of the dilaton F is sufficient.

Proposition 3.23. *The metric* g *of a closed strongly integrable generalised* G_2*-structure is*

$$\text{Ricci-flat} \Leftrightarrow dF = 0 \Leftrightarrow T = 0.$$

Proof: The only non-trivial statement to prove is that a constant dilaton causes Ricci-flatness. But by Corollary 3.18 the condition $dF = 0$ implies that the underlying $G_{2\pm}$-structures are co-calibrated which means $d{\star}\varphi_\pm = 0$. As we can see earlier in the proof of Theorem 3.21, the Ricci tensors $\mathrm{Ric}^\pm$ also vanish. Consequently, we can appeal to Theorem 5.4 of [19] which asserts that under these circumstances the underlying $G_{2\pm}$-structures actually have holonomy contained in G_2 if $T_{1\pm}$ or $T_{27\pm}$ vanishes. Since we assumed strong integrability, $T_{1\pm} = 0$ and the assertion follows. ■

Remark:

(i) The proposition fails if the torsion is not closed as an example in Section 3.3.3 will show.

(ii) We can always get rid of the dilaton F by the conformal transformation $\tilde{g} = e^{-F} g$, $\tilde{\varphi}_\pm = e^{-\frac{3}{2}F}\varphi_\pm$. The transformed torsion 3-form $\widetilde{\mathrm{Tor}}_\pm$ is given by

$$\widetilde{\mathrm{Tor}}_\pm = \pm\widetilde{T} = e^{-2F} \cdot (T - \frac{1}{2} \star dF \wedge \varphi_\pm)$$

([17], Corollary 4.2). Since the resulting $G_{2\pm}$-structures are co-calibrated, they have non-positive scalar curvature. However, the previous proposition does not apply, because $\widetilde{T}$ is no longer closed.

Since for a homogeneous generalised G_2-structure the dilaton F is necessarily constant, the previous proposition implies the following

Corollary 3.24. *Any homogeneous closed strongly integrable generalised G_2-structure is Ricci-flat and hence both underlying topological G_2-structures have actually holonomy G_2.*

Closed integrable Spin(7)-structures

We now turn to integrable generalised $Spin(7)$-structures which we discuss along the lines of the generalised G_2-case. We summarise the results we need in the next proposition.

Proposition 3.25. [31] *Let (M^8, Ω) be a $Spin(7)$-manifold. There exist unique forms $\theta \in \Omega^1(M)$ and $\tau \in \Omega^5_{48}(M)$ such that*

$$d\Omega = \theta \wedge \Omega + \tau.$$

In particular,

$$\theta = \frac{1}{7} \star (d^*\Omega \wedge \Omega).$$

There exists a unique linear metric connection $\widetilde{\nabla}$ with skew-symmetric torsion which preserves the $Spin(7)$-structure, i.e. $\widetilde{\nabla}\Omega = 0$. It is determined through

$$\mathrm{Tor} = -d^*\Omega - \frac{7}{6} \star (\theta \wedge \Omega) \tag{3.31}$$

and the action of Tor *on the corresponding spinor Ψ is given by*

$$\mathrm{Tor} \cdot \Psi = -\frac{7}{6}\theta \cdot \Psi.$$

Finally, the following expressions hold for the Riemannian scalar curvature Scal *and the scalar curvature* $\widetilde{\mathrm{Scal}}$ *of* $\widetilde{\nabla}$

$$\mathrm{Scal} = \frac{49}{18}\,\|\theta\|^2 - \frac{1}{12}\,\|\mathrm{Tor}\|^2 + \frac{7}{2}d^*\theta, \quad \widetilde{\mathrm{Scal}} = \frac{49}{18}\,\|\theta\|^2 - \frac{1}{3}\,\|\mathrm{Tor}\|^2 + \frac{7}{2}d^*\theta. \tag{3.32}$$

For an integrable generalised $Spin(7)$-structure we have again, according to Theorem 3.15, two metric connections $\nabla^{\pm}$ preserving the $Spin(7)_{\pm}$-structure, and whose torsion is $\mathrm{Tor}_{\pm} = \pm T$. The dilaton equation then implies

$$(dF \pm \frac{1}{2}T)\cdot\Psi_{\pm} = (dF - \frac{7}{12}\theta_{\pm})\cdot\Psi_{\pm} = 0,$$

and hence

$$\theta_{\pm} = \frac{12}{7}dF.$$

As a consequence, formula (3.31) can be written in the more succinct form as given in

Corollary 3.26. *If the generalised* $Spin(7)$*-structure is integrable, then*

$$\pm T = e^{2F} \star de^{-2F}\Omega_{\pm} \tag{3.33}$$

and

$$d\Omega_{\pm} = \frac{12}{7}dF \wedge \Omega_{\pm} \pm \star T_{48\pm}, \tag{3.34}$$

where $T_{48\pm}$ *denotes the projection of* T *onto* $\Omega^3_{48}(M, \Omega_{\pm})$*.*

Conversely, if we are given two $Spin(7)_{\pm}$*-invariant forms inducing the same metric* g*, a function* F *and a closed 3-form* T *such that (3.33) and (3.34) hold, then the corresponding spinors* $\Psi_{\pm}$ *satisfy Theorem 3.15 and hence define an integrable generalised* $Spin(7)$*-structure.*

This implies again a vanishing theorem over compact manifolds if T is closed since

$$\int_M e^{-2F} T \wedge \star T = \pm \int_M T \wedge d(e^{-2F}\Omega_\pm).$$

Corollary 3.27. *If M is compact and carries a closed integrable generalised $Spin(7)$-structure, then $T = 0$ and consequently, the spinors $\Psi_\pm$ are parallel with respect to the Levi-Civita connection.*

Next we compute the Ricci tensor.

Theorem 3.28. *The Ricci-tensor of an integrable generalised $Spin(7)$-structure is given by*

$$\mathrm{Ric}(X,Y) = -2H^F(X,Y) + \frac{1}{4} g(X\llcorner T, Y\llcorner T),$$

where $H^F(X,Y) = X.Y.F - \nabla_X Y.F$ is the Hessian of the dilaton F.

Proof: The proof is essentially the same as for Theorem 3.21, that is we need to compute the sum $\mathrm{Ric}^+ + \mathrm{Ric}^-$. Again we fix a local frame $e_1, \ldots, e_8$ that satisfies $\nabla_{e_i} e_j = 0$ at a point. By proposition 7.2. of [31],

$$\mathrm{Ric}^\pm(X) = \mp \star (\nabla^\pm_X T \wedge \Omega_\pm).$$

Since the connections $\nabla^{\pm}$ preserve the $Spin(7)_{\pm}$ structure,

$$\begin{aligned}
\mathrm{Ric}^{\pm}(e_i, e_j) &= \pm g(\nabla^{\pm}_{e_i} T \wedge \Omega_{\pm}, \star e_j) \\
&= \pm \frac{1}{7} g(\nabla^{\pm}_{e_i} T \wedge \Omega_{\pm}, (e_j \llcorner \Omega_{\pm}) \wedge \Omega_{\pm}) \\
&= \pm g(\nabla^{\pm}_{e_i} T, e_j \llcorner \Omega_{\pm}) \\
&= -\frac{2}{7} g(e_j \wedge \nabla^{\pm}_{e_i} \star (dF \wedge \Omega_{\pm}), \Omega_{\pm}) \\
&= -\frac{2}{7} e_i.g(e_j \wedge \star(dF \wedge \Omega_{\pm}), \Omega_{\pm}) + \frac{2}{7} g(\nabla^{\pm}_{e_i} e_j \wedge \star(dF \wedge \Omega_{\pm}), \Omega_{\pm}) \\
&= -\frac{2}{7} e_i.g(dF \wedge \Omega_{\pm}, e_j \wedge \Omega_{\pm}) + \frac{2}{7} \sum_k T_{ijk} g(dF \wedge \Omega_{\pm}, e_k \wedge \Omega_{\pm}) \\
&= -2 e_i.e_j.F \pm \sum_k T_{ijk} e_k.F,
\end{aligned}$$

and the result follows as in the G_2-case. ■

Corollary 3.29. *For a closed integrable generalised $Spin(7)$-structure, the scalar curvature of $\nabla^{\pm}$ is*

$$\mathrm{Scal}^{\pm} = 2\Delta F = \frac{7}{6} d^* \theta_{\pm},$$

where $\Delta(\cdot) = -\mathrm{Tr}_g H^{(\cdot)}$ is the Laplacian on functions associated with the metric g. Hence, the scalar curvature of g is given by

$$\mathrm{Scal} = \mathrm{Tr}(\mathrm{Ric}) = 2\Delta F + \frac{1}{4} \|T\|^2 .$$

Finally we can again exclude the possibility of any interesting homogeneous structures.

Proposition 3.30. *A closed integrable generalised $Spin(7)$-structure is*

$$\text{Ricci-flat} \Leftrightarrow dF = 0 \Leftrightarrow T = 0.$$

Proof: The previous corollary shows that if $dF = 0$, then the scalar curvature $\mathrm{Scal}^{\pm}$ vanishes and $\mathrm{Scal} = \|T\|^2 /4 \geq 0$. But (3.32) implies $\mathrm{Scal} = -\|T\|^2 /12 \leq 0$, hence $\|T\| = 0$. The remaining implications are obvious. ■

Remark: Note that as in the G_2-case, we can scale dF away by the conformal transformation $\tilde{g} = e^{-\frac{3}{7}F}g$, $\widetilde{\Omega}_{\pm} = e^{-\frac{12}{7}F}$, but we obtain again a torsion form which is not closed [31].

Corollary 3.31. *Any homogeneous closed integrable generalised Spin(7)-structure is Ricci-flat and thus both underlying topological Spin(7)-structures have actually holonomy Spin(7).*

3.3.3 Examples

Straight structures

If we want to construct explicit examples of the various geometries we introduced, then it is most natural to adopt the form point of view and to write the structure forms in terms of the underlying $SU(3)$- or G_2-structures as at the end of Section 1.5. Imposing the integrability condition on the structure form yields differential conditions on the $SU(3)$- or G_2-invariants which can then be discussed in the terms of the respective representation theory.

Example:(*strongly integrable straight structures*)

We first look for strongly integrable generalised structures which are *straight*, that is structures defined by one spinor $\Psi = \Psi_+ = \Psi_-$ (cf. Section 2.2).

(i) (*straight generalised G_2-structures*) Let us consider a generalised G_2-structure which is induced by a classical G_2-structure (M^7, φ). Given a (closed) 3-form T and a function F, we want to solve the equations of strong integrability

$$d_T e^F(1 - \star\varphi) = 0, \quad d_T e^F(-\varphi + vol_g) = 0.$$

Expanding and ordering by degree, these equations hold if and only if

$$\begin{aligned} dF + T - dF \wedge \star\varphi - T \wedge \star\varphi &= 0 \\ dF \wedge \varphi + d\varphi + T \wedge \varphi &= 0 \end{aligned}$$

which is equivalent to

$$dF = 0,\ T = 0,\ d\varphi = 0 \text{ and } d\star\varphi = 0.$$

In particular, the holonomy group of (M, g_φ) is contained in G_2 [16].

(ii) (*straight generalised* $Spin(7)$*-structures, even type*) Similarly, we can consider a generalised $Spin(7)$-structure of even type coming from a $Spin(7)$-manifold (M^8, Ω) where strong integrability

$$d_T e^F(1 - \Omega + vol_g) = 0$$

is equivalent to

$$dF = 0,\ T = 0 \text{ and } d\Omega = 0.$$

Again this causes the holonomy of (M, g_Ω) to reduce to $Spin(7)$ [10].

(iii) (*straight generalised* $Spin(7)$*-structures, odd type*) A trivial example of odd type is provided by product manifolds of the form $N^7 \times \mathbb{R}$, where N^7 carries a metric of holonomy G_2. According to Corollary 1.30, the form $dt \wedge (-1 + \star\varphi) - \varphi + vol_N$ defines a generalised $Spin(7)$-structure of odd type which is clearly closed. Multiplying with a dilaton and considering the twisted differential d_T implies once more $dF = 0$ and $T = 0$.

As a result, we obtain the

Proposition 3.32. *Any straight generalised* G_2*- or* $Spin(7)$*-structure is strongly integrable if and only if the holonomy of the underlying Riemannian manifold is contained in* G_2 *or* $Spin(7)$*. Such a structure is necessarily trivial (i.e.* $T = 0$*) and therefore closed.*

Compact examples of strongly integrable generalised structures

Here is a non-trivial compact example with constant dilaton.

Example:(*compact strongly integrable generalised G_2-structures*) Consider the nilmanifold N of dimension 6 associated with the Lie algebra $\mathfrak{g}$ spanned by the orthonormal basis $e_2, \ldots, e_7$ determined by the relations

$$de_i = \begin{cases} e_{37}, & i = 4 \\ -e_{35}, & i = 6 \\ 0, & \text{else.} \end{cases}$$

We then define $M = N \times S^1$ which we endow with the product metric $g = g_N + dt \otimes dt$. On N we choose the $SU(3)$-structure coming from

$$\omega = -e_{23} - e_{45} + e_{67}, \quad \psi_+ = e_{356} - e_{347} - e_{257} - e_{246}$$

(cf. Section 1.2.1) which induces a generalised G_2-structure on M with $\alpha = e_1 = dt$. We let $T = e_{167} + e_{145}$ and consider the equations of strong integrability

$$d_T\rho = 0, \quad d_T\hat{\rho} = 0,$$

where $\rho = \omega + \psi_+ \wedge \alpha - \omega^3/6$ and $\hat{\rho} = \alpha - \psi_- - \omega^2 \wedge \alpha/2$ (using the notation of Section 1.5). These are equivalent to

$$d\omega = 0, \quad d\psi_+ \wedge \alpha = -T \wedge \omega, \quad T \wedge \alpha = d\psi_-, \quad T \wedge \psi_- = 0.$$

By design, $T \wedge \alpha = 0$ and $T \wedge \psi_- = 0$. Moreover,

$$d\omega = -de_4 \wedge e_5 + de_6 \wedge e_7 = 0$$

and

$$d\psi_- = -e_3 \wedge de_4 \wedge e_6 + e_{34} \wedge de_6 + e_{25} \wedge de_6 + e_2 \wedge de_4 \wedge e_7 = 0.$$

Finally, we have

$$d\psi_+ \wedge \alpha = -e_{12367} - e_{12345} = -T \wedge \omega.$$

Note that $dT = 2e_{1357}$ in accordance with Corollary 3.19.

From the next lemma and the previous example we immediately infer the existence of a compact integrable generalised $Spin(7)$-structures of even type.

Lemma 3.33. *If (M, ρ_0, T) is a (closed) strongly integrable generalised G_2-manifold with ρ_0 even, then $(M^7 \times S^1, dt \wedge \hat{\rho}_0 + \rho_0)$ is a (closed) integrable generalised $Spin(7)$-manifold of even type.*

Proof: According to Corollary 1.30, $\rho = dt \wedge \hat{\rho}_0 + \rho_0$ defines a generalised $Spin(7)$-structure and it is immediate to check that $d_T\rho = 0$ and $d_T\hat{\rho} = 0$ imply $d_T\rho = 0$. ■

In order to construct an odd example we start with a compact calibrated G_2-manifold (M^7, φ), i.e. $d\varphi = 0$. An example can be easily obtained by taking the nilmanifold N used for the compact generalised G_2-example. We swap orientations so that $d\psi_+ = 0$ and $d\psi_- \wedge dt \neq 0$, hence

$$d\varphi = d\omega \wedge dt + d\psi_+ = 0, \quad d \star \varphi = d\psi_- \wedge dt + d\omega \wedge \omega = d\psi_- \wedge dt.$$

Write $d \star \varphi = \xi \wedge \varphi$ and put $T = -dt \wedge \xi$. Corollary 1.30 implies that

$$\rho = dt \wedge (-1 + \star_M \varphi) - \varphi + vol_M$$

defines an odd generalised $Spin(7)$-structure on $M^7 \times S^1$ equipped with the product metric $g = g_M + dt \otimes dt$. Now

$$\begin{aligned} d_T\rho &= -dt \wedge d\star_M \varphi - d\varphi + T \wedge \rho \\ &= -dt \wedge \xi \wedge \varphi - T \wedge \varphi \\ &= 0. \end{aligned}$$

In particular, it follows that the torsion of a compact calibrated G_2-manifold can never be closed.

Proposition 3.34. *There exist compact strongly integrable generalised G_2- and $Spin(7)$-manifolds of either type. In particular, we see that in Corollaries 3.19, 3.27 and Propositions 3.23, 3.30 it is necessary to assume T to be closed.*

Weakly integrable structures

Next we look for weakly integrable G_2-structures. The natural idea would be to use nearly parallel G_2-manifolds to produce weak analogues of straight strongly integrable structures. However, any such construction results to be integrable if and only if it is strongly integrable. In this sense weak structures really define a new type of geometry without a classical, i.e. straight counterpart.

To see this, assume we were given a straight weakly integrable structure of even type, i.e. $\varphi_+ = \varphi_-$ and $\pm T = -e^{2F} \star d(e^{-2F}\varphi) \pm 2\lambda\varphi$ (Corollary 3.18). In particular, we have $\star de^{-2F}\varphi = 0$ which is equivalent to $d\varphi = 2dF \wedge \varphi$. It follows $\lambda = 0$ in contradiction to our initial assumption. For a straight structure of odd type, Corollary 3.18 implies at once $T = 0$ and hence $\lambda = 0$.

The absence of a straight example already hints at the fact that examples, if they exist, are quite hard to find. It is instructive to see where the difficulties arise.

Example:(*ansatz for weakly integrable structures with constant dilaton*)

(i) (*even type, homogeneous case*) First we consider the even case, that is ρ is even and satisfies $d_T\rho = \lambda\hat{\rho}$ for some closed 3-form T. Moreover, we require the generalised G_2-structure to be homogeneous. Since the only invariant 0-form is a constant, the 1-component of $\hat{\rho}$ has to vanish which in turn implies that $\sin(a) = 0$ (cf. Proposition 1.29). The equation for weak integrability becomes

$$d_T(1 - \star\varphi) = \lambda(-\varphi + vol_g),$$

implying $T = -\lambda\varphi$ and $-\lambda\varphi \wedge \star\varphi = \lambda vol_g$. Since $\varphi \wedge \star\varphi = 7vol_g$, we have $\lambda = 0$ and therefore $T = 0$ and $d\star\varphi = 0$. Hence we obtain a classical co-calibrated G_2-structure with $\lambda = 0$ which does not define a weakly integrable structure.

(ii) (*odd type*) For any odd structure $\rho \in \Omega^{od}(M)$ we have $d_T\rho = \lambda\hat{\rho}$ and it follows immediately that the 0-form of $\hat{\rho}$ vanishes. Hence $\cos(a) \equiv 0$ and the two spinors Ψ_+ and Ψ_- are orthogonal. Written in terms of the underlying $SU(3)$-structure, the corresponding stable 3-forms $\varphi_\pm$ are given by

$$\varphi_\pm = \omega \wedge \alpha \pm \psi_+,$$

cf. (1.17). The odd and even forms ρ and $\hat{\rho}$ are now $\rho = \alpha - \psi_- - \omega^2 \wedge \alpha/2$ and $\hat{\rho} = \omega + \psi_+ \wedge \alpha - \omega^3/6$. The weak integrability condition becomes

$$d\alpha - d\psi_- - \frac{1}{2}d(\omega^2 \wedge \alpha) + T \wedge (\alpha - \psi_- - \frac{1}{2}\omega^2 \wedge \alpha) = \lambda(\omega + \psi_+ \wedge \alpha - \frac{1}{6}\omega^3)$$

which is equivalent to

$$\begin{aligned} d\alpha &= \lambda\omega \\ T\wedge\alpha - d\psi_- &= \lambda\psi_+\wedge\alpha \\ T\wedge\psi_- + \frac{1}{2}d(\omega^2\wedge\alpha) &= \frac{\lambda}{6}\omega^3. \end{aligned}$$

Using $d\alpha = \lambda\omega$ finally yields

$$\begin{aligned} d\alpha &= \lambda\omega \\ T\wedge\alpha &= \lambda\psi_+\wedge\alpha + d\psi_- \\ T\wedge\psi_- &= -\frac{1}{3}\lambda\omega^3. \end{aligned} \tag{3.35}$$

The shape of these equations suggest to start with a compact Calabi-Yau manifold with integral Kähler class in order to construct a weakly integrable structure over the associated S^1-principal fibre bundle P_ω endowed with a connection 1-form $\theta = \alpha/\lambda$ such that $d\theta = \omega$. However, the second equation of (3.35) implies $T = \lambda\psi_+$ (modulo possible components in the kernel of $\wedge\alpha$), but this fails to satisfy the last equation as $\psi_+\wedge\psi_- = 2\omega^3/3$. Again, the algebraic constraints imply $\lambda = 0$.

The examples we have found so far are, except for the compact one, rather trivial. However, they are still valuable as the device of T-duality permits us to go from a trivial straight solution existing over a non-trivial principal S^1-bundle to a non-trivial solution over a trivial S^1-bundle. This tool will occupy us next.

T-duality

T-duality is the name of a procedure in string theory which consists in changing the topology of a model which is an S^1-fibration (or more generally, a T^n-fibration – the "T" therefore referring to a "torus") by replacing this fibre by a different one of same type, but without destroying integrability. It thus interchanges type IIA with type IIB string theory. A basic instance of T-duality is the $R \to 1/R$ invariance in string theory compactified on a circle of radius R. In this way it relates strongly coupled string theories to weakly coupled ones and opens the way for perturbative methods in the investigation of strongly coupled models. Mathematically speaking, T-duality transforms the data (g, b, F), that is, a generalised metric consisting of an ordinary (usually Lorentzian) metric g and a 2-form b as well as the dilaton F, all living on a principal S^1-bundle $P \to M$ with connection form θ, into a generalised metric (g^t, b^t) and a dilaton F^t living on a *new* principal S^1-fiber bundle $P^t \to M$ with connection form θ^t. The coordinate description of this dualising procedure is well-known in physics under the name of *Buscher rules* (see, for instance Appendix A in [33]). Topological issues of T-duality are discussed in [8] which will be our standard reference.

We will briefly introduce the formalism and set up the notation as far as we will need it here. Consider a principal S^1-bundle $p : P \to M$ which carries a connection form θ. We denote by X the vertical vector field dual to θ, so $X \llcorner \theta = 1$, and by $\mathcal{F}$ the curvature 2-form which we regard as a 2-form on M, that is $d\theta = p^*\mathcal{F}$. Moreover, we assume to be given an S^1-invariant closed integral 3-form T such that the 2-form $\mathcal{F}^t$ defined by

$$p^*\mathcal{F}^t = -X \llcorner T$$

is also closed and integral. In practice we shall often assume $T = 0$. Integrality of $\mathcal{F}^t$ ensures the existence of another principal S^1-bundle P^t, the *T-dual* of P defined by the choice of a connection form θ^t with $d\theta^t = p^*\mathcal{F}^t$. Here and from now on, we ease notation and drop the pull-back p^* bearing in mind that the forms $\mathcal{F}$ and $\mathcal{F}^t$ etc. live on M. Writing

$T = -\theta \wedge \mathcal{F}^t + \mathcal{T}$ for a 3-form $\mathcal{T} \in \Omega^3(M)$, we define the T-dual of T by

$$T^t = -\theta^t \wedge \mathcal{F} + \mathcal{T}.$$

Note that T^t is closed and integral if and only if T is closed and integral.

To make contact with our situation, consider an S^1-invariant exceptional generalised structure defined by ρ which we write

$$\rho = \theta \wedge \rho_0 + \rho_1.$$

The *T-dual of* ρ is defined to be

$$\rho^t = \theta^t \wedge \rho_1 + \rho_0.$$

In particular, T-duality reverses the parity of forms and maps even to odd and odd to even forms. It is enacted by multiplication with the element $X \oplus \theta \in Pin(n,n)$ on ρ (giving the form $(X \oplus \theta) \bullet \rho = \theta \wedge \rho_1 + \rho_0$) followed by the substitution $\theta \to \theta^t$.

The crucial feature of T-duality we shall need in the sequel is that it preserves the $Spin(n,n)$-orbit structure on $\Lambda^{ev,od}$. To see this, we decompose

$$TP \oplus T^*P \cong T \oplus \mathbb{R}X \oplus T^*M \oplus \mathbb{R}\theta, \quad TP \oplus T^*P \cong T \oplus \mathbb{R}X^t \oplus T^*M \oplus \mathbb{R}\theta^t.$$

Then consider the map $\tau : TP \oplus T^*P \to TP^t \oplus T^*P^t$ [12] defined with respect to this splitting by

$$\tau(V + uX \oplus \xi + v\theta) = -V + vX^t \oplus -\xi + u\theta^t.$$

It satisfies

$$(a \bullet \rho)^t = \tau(a) \bullet \rho^t$$

for any $a \in TP \oplus T^*P$ and in particular, $\tau(a)^2 = -(a,a)$:

$$\begin{aligned}
((V+uX\oplus\xi+v\theta)\bullet\rho)^t &= (-\theta\wedge(V\llcorner\rho_0)+V\llcorner\rho_1+u\rho_0-\theta\wedge\xi\wedge\rho_0+\xi\wedge\rho_1+v\theta\wedge\rho_1)^t\\
&= (\theta\wedge(-V\llcorner\rho_0-\xi\wedge\rho_0+v\rho_1)+V\llcorner\rho_1+u\rho_0+\xi\wedge\rho_1)^t\\
&= \theta^t\wedge(V\llcorner\rho_1+u\rho_0+\xi\wedge\rho_1)-V\llcorner\rho_0-\xi\wedge\rho_0+v\rho_1\\
&= (-V+vX^t\oplus-\xi+u\theta^t)\bullet\rho^t.
\end{aligned}$$

Hence this map extends to an isomorphism $\mathit{Cliff}(TP\oplus T^*P)\cong\mathit{Cliff}(TP^t\oplus T^*P^t)$ and any orbit of the form $Spin(TP\oplus T^*P)/G$ gets mapped to an equivalent orbit $Spin(TP^t\oplus T^*P^t)/G^t$ where G and G^t are isomorphic as abstract groups.

As an illustration of this, consider a generalised G_2-structure over P with structure form $\rho=\theta\wedge\rho_0+\rho_1$ and companion $\hat\rho=\theta\wedge\hat\rho_0+\hat\rho_1$ (note the abuse of notation: while $\hat\rho$ denotes the $\wedge$-operation on $\Omega^*(P)$ as introduced in Section 1.6, $\hat\rho_i$ is a mere notation for the forms pulled back from the basis M). These have T-duals ρ^t and $\hat\rho^t$. Since ρ and $\hat\rho$ have the same stabiliser G inside $Spin(TP\oplus T^*P)\cong Spin(7,7)$, it follows that ρ^t and $\hat\rho^t$ are stabilised by the same G^t inside $Spin(TP^t\oplus T^*P^t)\cong Spin(7,7)$ which is hence isomorphic to $G_2\times G_2$. By invariance, $\hat\rho^t$ and $\hat{\rho^t}$ coincide up to a constant (which we henceforth ignore). The integrability condition transforms as follows.

Lemma 3.35.

$$d_T\rho=\lambda\hat\rho \textit{ if and only if } d^t_T\rho^t=-\lambda\widehat{\rho^t}$$

Proof: Since $T=-\theta\wedge\mathcal{F}^t+\mathcal{T}$, expansion of the left hand side yields

$$\begin{aligned}
d_T\rho &= d\theta\wedge\rho_0-\theta\wedge d\rho_0+d\rho_1-\theta\wedge\mathcal{F}^t\wedge\rho_1+\mathcal{T}\wedge\theta\wedge\rho_0+\mathcal{T}\wedge\rho_1\\
&= \mathcal{F}\wedge\rho_0+d\rho_1+\mathcal{T}\wedge\rho_1+\theta\wedge(-\mathcal{F}^t\wedge\rho_1-d\rho_0-\mathcal{T}\wedge\rho_0)\\
&= \theta\wedge\lambda\hat\rho_0+\lambda\hat\rho_1,
\end{aligned}$$

and this is equivalent to

$$\lambda\hat{\rho}_0 = -\mathcal{F}^t \wedge \rho_1 - d\rho_0 - \mathcal{T} \wedge \rho_0 \text{ and } \lambda\hat{\rho}_1 = \mathcal{F} \wedge \rho_0 + d\rho_1 + \mathcal{T} \wedge \rho_1. \tag{3.36}$$

On the other hand, we have

$$\begin{aligned} d_{T^t}\rho^t &= d\theta^t \wedge \rho_1 - \theta^t \wedge d\rho_1 + d\rho_0 - \theta^t \wedge \mathcal{F} \wedge \rho_0 + \mathcal{T} \wedge \theta^t \wedge \rho_1 + \mathcal{T} \wedge \rho_0 \\ &= \mathcal{F}^t \wedge \rho_1 + d\rho_0 + \mathcal{T} \wedge \rho_0 + \theta^t \wedge (-\mathcal{F} \wedge \rho_0 - d\rho_1 - \mathcal{T} \wedge \rho_1) \\ &= -\theta^t \wedge \lambda\hat{\rho}_1 - \lambda\hat{\rho}_0 \end{aligned}$$

which gives precisely (3.36). ■

Corollary 3.36. *Assume that $p : P \to M$ is a principal S^1-bundle with connection form θ, X the vertical vector field with $\theta(X) = 1$, and that T is a closed integral 3-form such that $X \llcorner T$ is also closed and integral.*

(i) *A generalised G_2-structure ρ is closed strongly integrable with torsion T if and only if ρ^t is a closed strongly integrable G_2-structure with torsion T^t,*

closed strongly int. gen. G_2 with torsion T on P

$$\overset{t}{\longleftrightarrow}$$

closed strongly int. gen. G_2 with torsion T^t on P^t.

(ii) *A generalised G_2-structure ρ is closed weakly integrable of even or odd type with torsion T and Killing number λ if and only if ρ^t is a closed weakly integrable G_2-structure of odd or even type with torsion T^t and Killing number $-\lambda$,*

closed even/odd weakly int. gen. G_2 with torsion T, Killing number λ on P

$$\overset{t}{\longleftrightarrow}$$

closed odd/even weakly int. gen. G_2 with torsion T^t, Killing number $-\lambda$ on P^t.

(iii) *A generalised $Spin(7)$-structure ρ is closed integrable of even or odd type with torsion T if and only if ρ^t is a closed integrable $Spin(7)$-structure of odd or even type with torsion T^t,*

$$\textit{closed even/odd int. gen. } Spin(7) \textit{ with torsion } T \textit{ on } P$$

$$\stackrel{t}{\longleftrightarrow}$$

$$\textit{closed odd/even int. gen. } Spin(7) \textit{ with torsion } T^t \textit{ on } P^t.$$

We put the previous corollary into action as follows. Start with a non-trivial principal S^1-fibre bundle P with connection form θ and which admits a metric of holonomy G_2 or $Spin(7)$. We put $T = 0$ so that $\mathcal{F}^t$ – the curvature of the T-dual bundle P^t – vanishes. This implies the triviality of P^t. The resulting straight generalised structure is strongly integrable and so is its dual, but according to the T-duality rules, we acquire non-trivial torsion given by $T^t = \theta^t \wedge \mathcal{F}$. Local examples of such G_2-structures exist in abundance [2]. Applying Lemma 3.33 yields local examples of closed integrable generalised $Spin(7)$-structures of even type over a trivial S^1-fibration $P = M^7 \times S^1$. The torsion T is integral and contracting with the dual of a connection form on P yields 0. According to the T-duality rules we obtain a closed integrable generalised $Spin(7)$-manifold of odd type whose torsion equals $T^t = T$.

Proposition 3.37. *There exist local examples of non-trivial closed strongly integrable generalised G_2- and $Spin(7)$-manifolds of either type.*

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