GENERALIZED KILLING SPINORS AND LAGRANGIAN GRAPHS

ANDREI MOROIANU AND UWE SEMMELMANN

Abstract. We study generalized Killing spinors on the standard sphere $\mathbb{S}^3$, which turn out to be related to Lagrangian embeddings in the nearly Kähler manifold $S^3 \times S^3$ and to great circle flows on $\mathbb{S}^3$. Using our methods we generalize a well known result of Gluck and Gu [6] concerning divergence-free geodesic vector fields on the sphere and we show that the space of Lagrangian submanifolds of $S^3 \times S^3$ has at least three connected components.

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1. Introduction

In this article we investigate *generalized Killing spinors* [4] (cf. also [5]) on the standard sphere $\mathbb{S}^3$. Recall that generalized Killing spinors on some spin manifold (M, g) are spinors on M verifying the equation

$$\nabla_X \Psi = A(X) \cdot \Psi \qquad \forall X \in \mathrm{T}M \tag{1}$$

for some endomorphism A symmetric with respect to g. Real Killing spinors (for $A = \lambda \mathrm{id}$, $\lambda \in \mathbb{R}$) or parallel spinors (for $A = 0$) are particular examples of such objects. Generalized Killing spinors arise as restrictions of parallel spinors to hypersurfaces, and the converse is true under some analyticity assumption [2]. This initial problem can thus be understood as an isometric embedding problem for $\mathbb{S}^3$ into some 4-dimensional hyperkähler ambient space. Note that in [10] we gave examples of genuine (i.e. non-Killing) generalized Killing spinors on $\mathbb{S}^3$, showing that the problem is non-trivial.

In our first result we show that generalized Killing spinors on any 3-dimensional spin manifold (M, g) are in one-to-one correspondence with divergence-free orthonormal frames on M. Our examples in [10] are equivalent in this setting with frames made by Hopf (left or right-invariant) vector fields for the Killing spinors, and to reflexions of such Hopf left or right-invariant frames with respect to some fixed right or left-invariant Hopf vector field, for the genuine generalized Killing spinors.

We next interpret generalized Killing spinors on $\mathbb{S}^3$ as maps $f : \mathbb{S}^3 \to \mathbb{S}^3$ whose differential has the following symmetry property: for every $g \in \mathbb{S}^3$, the linear map $M_g : \mathrm{T}_e\mathbb{S}^3 \to \mathrm{T}_e\mathbb{S}^3$

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defined by $X \mapsto df_g(gX)f(g)^{-1}$ is symmetric with respect to the standard scalar product on $\mathrm{T}_e\mathbb{S}^3 = \mathbb{R}^3$. Here $\mathbb{S}^3$ is viewed as a Lie group with Lie algebra $\mathrm{T}_e\mathbb{S}^3$, $f(g)^{-1}$ denotes infinitesimal left translation with the inverse of $f(g)$, and gX is the value at g of the left invariant vector field generated by X.

This point of view is particularly interesting when considering the graph of f as a submanifold of $S^3 \times S^3$ endowed with its 3-symmetric nearly Kähler metric. It turns out that the above symmetry property of f is equivalent to the fact that the graph $\Gamma_{f^{-1}}$ of the map $g \mapsto f(g)^{-1}$ is a Lagrangian submanifold of the nearly Kähler $S^3 \times S^3$, endowed with its fundamental two-form Ω. Note that the terminology is somewhat improper since Ω is not closed. Nonetheless, Lagrangian submanifolds of 6-dimensional nearly Kähler manifolds were intensively studied in the last decades, perhaps motivated by the fact that they are automatically minimal (cf. Ejiri [3] in the case of $\mathbb{S}^6$ and Gutowski-Ivanov-Papadopoulos [8] in general).

Until now, the only known examples (up to isometry) of Lagrangian submanifolds of the nearly Kähler $S^3 \times S^3$ were the factors and the diagonal. Our examples of genuine generalized Killing spinors on $\mathbb{S}^3$ yield in this way new examples of Lagrangian graphs of $S^3 \times S^3$, but we have also found an interesting family of Lagrangian submanifolds of $S^3 \times S^3$ which project onto a strict submanifold on each factor. We computed the metric structure for each of the examples, which, for the usual normalization of the metric on $S^3 \times S^3$, turn out to be round spheres of radius $\frac{2}{3}$ and $\frac{4}{3}$, as well as Berger spheres of some different volume. This last observation shows that the space of Lagrangian submanifolds of $S^3 \times S^3$ has at least three connected components.

We have also investigated generalized Killing spinors on $\mathbb{S}^3$ by comparing them to some fixed Killing spinor. In this way, every generalized Killing spinor on $\mathbb{S}^3$ is characterized by a function α and a vector field ξ satisfying some coupled non-linear differential system. In the particular case where the function α vanishes, the system reduces to the condition that ξ is a geodesic divergence-free vector field. Such objects were studied by Gluck and Gu [6], who showed (using a nice interpretation as holomorphic graphs in the oriented Grassmannian $\tilde{\mathrm{Gr}}_2(\mathbb{R}^4)$) that they are necessarily Hopf vector fields. Translating back into our setting, we obtain as a corollary that every generalized Killing spinor on $\mathbb{S}^3$ whose scalar product with some Killing spinor vanishes is necessarily in the list of our known examples. We finally generalize this result to the case when this scalar product is constant but not necessarily zero, by solving an ODE along the orbits of ξ.

2. Spinors on 3-manifolds and divergence-free frames

Let (M^3, g) be a 3-dimensional spin manifold. We denote by Σ_3 the irreducible Cl_3 module on which the volume element acts by $-\mathrm{id}$. This sign choice (opposite to the one in [11]) is motivated by the identification with quaternions in the next section (see the discussion after Equation (8)).

Since the spin representation $\mathrm{Spin}(3) \to \mathrm{Aut}(\Sigma_3)$ is isomorphic to the left multiplication of unit quaternions on $\Sigma_3 \simeq \mathbb{H}$, the spinor bundle ΣM has a quaternionic structure, acting from the right. This structure is compatible with the natural scalar product $\langle\cdot,\cdot\rangle$ on the spin bundle, in the sense that $\langle\Psi a,\Phi a\rangle = |a|^2\langle\Psi,\Phi\rangle$ for every $a\in\mathbb{H}$ and $\Psi,\Phi\in\Sigma M$.

In particular, if $a\in\mathrm{Im}\mathbb{H}$ is an imaginary quaternion, then $a^2=-|a|^2$, so for every spinor Ψ we have

$$|a|^2\langle\Psi a,\Psi\rangle = \langle\Psi a^2,\Psi a\rangle = -|a|^2\langle\Psi,\Psi a\rangle,$$

showing that Ψa is orthogonal to Ψ. On the other hand, for every $x\in M$ and non-zero $\Psi\in\Sigma_x M$, the map $\mathrm{T}_x M\to\Psi^\perp\subset\Sigma_x M$, mapping X to $X\cdot\Psi$ is an isomorphism (for dimensional reasons). For every imaginary quaternion a and nowhere vanishing spinor field $\Psi\in C^\infty(\Sigma M)$ we can thus define a vector field ξ_a on M by

$$\xi_a\cdot\Psi = \Psi a. \tag{2}$$

By the above choice of the spin module, the volume form of M acts by $-\mathrm{id}$ on the spin bundle, hence

$$\omega\cdot\Psi = *\omega\cdot\Psi \qquad \forall\ \omega\in C^\infty(\Lambda^2 M),\ \Psi\in C^\infty(\Sigma M). \tag{3}$$

We now give a characterization of generalized Killing spinors in terms of the associated unit vector fields ξ_a. Recall that every generalized Killing spinor Ψ has constant length. Indeed, from (1) we get $X(|\Psi|^2) = 2\langle\nabla_X\Psi,\Psi\rangle = 2\langle A(X)\cdot\Psi,\Psi\rangle = 0$ for every vector field X.

Lemma 2.1. *A spinor $\Psi\in C^\infty(\Sigma M)$ of constant length is a generalized Killing spinor if and only if the vector fields ξ_a are divergence-free for all $a\in\mathrm{Im}\mathbb{H}$.*

Proof. Since Ψ has constant length, $\nabla_X\Psi$ is orthogonal to Ψ at every point, so by the linearity of the covariant derivative, there exists some endomorphism A of $\mathrm{T}M$ such that

$$\nabla_X\Psi = A(X)\cdot\Psi \tag{4}$$

for any vector field X. Taking the covariant derivative in the defining equation (2) yields

$$\nabla_X\xi_a\cdot\Psi + \xi_a\cdot A(X)\cdot\Psi = \nabla_X(\Psi a) = (\nabla_X\Psi)a = A(X)\cdot\Psi a = A(X)\cdot\xi_a\cdot\Psi\ .$$

Hence by (3)

$$\nabla_X\xi_a\cdot\Psi = 2A(X)\wedge\xi_a\cdot\Psi = 2*(A(X)\wedge\xi_a)\cdot\Psi\ ,$$

and it follows

$$\nabla_X\xi_a = 2*(A(X)\wedge\xi_a) = -2A(X)\lrcorner*\xi_a\ . \tag{5}$$

Assume now that Ψ is a generalized Killing spinor, i.e. that the endomorphism A defined in Equation (4) is symmetric. We obtain

$$\delta\xi_a = -e_i\lrcorner\nabla_{e_i}\xi_a = 2e_i\lrcorner A(e_i)\lrcorner*\xi_a = 0\ .$$

(Here and in the following we use Einstein's summation convention over repeated subscripts).

Conversely, if ξ_a are divergence-free, (5) shows that $e_i\lrcorner A(e_i)\lrcorner*\xi_a=0$ for every $a\in\mathrm{Im}\mathbb{H}$, i.e. the 2-form $e_i\wedge A(e_i)$ vanishes. Since this two-form represents the skew-symmetric part of A, the lemma follows. □

Since every oriented orthonormal frame labeled (ξ_i,ξ_j,ξ_k) defines (up to sign) a unique spinor of unit length satisfying (2) for $a=i,j,k$, the previous lemma gives at once:

Corollary 2.2. *Generalized Killing spinors on M are (up to sign) in 1-1 correspondence with oriented orthonormal frames of divergence-free vector fields on M.*

3. Spinors on $\mathbb{S}^3$

In this section we describe spinors on the round sphere $\mathbb{S}^3$ and translate the generalized Killing equation into conditions on the defining function in a left-invariant frame.

We consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{H}$, with the induced Lie group structure. In this way $\mathbb{S}^3$ is identified to SU(2) and the Lie algebra of $\mathbb{S}^3$ is identified with $\mathrm{Im}\mathbb{H} \simeq \mathfrak{su}(2)$. More generally, the tangent space $\mathrm{T}_g\mathbb{S}^3$ is identified to $g\mathrm{Im}\mathbb{H}$ and the infinitesimal left and right translations on $\mathbb{S}^3$ are given by left or right quaternionic products. Let $(e_1, e_2, e_3) = (i, j, k)$ be a fixed basis in $\mathrm{T}_e\mathbb{S}^3 = \mathrm{Im}\mathbb{H}$ (positively oriented by convention). Then left translation defines an orthonormal frame on $\mathbb{S}^3$: $u(g) := (ge_1, ge_2, ge_3)$. We will take u as a reference frame and we endow $\mathbb{S}^3$ with the orientation induced by u. The Levi-Civita connection on left-invariant vector fields $X_g = gx$ and $Y_g = gy$ is given by the well known formula

$$(\nabla_X Y)_g = \tfrac{1}{2} g[x, y], \tag{6}$$

for any $g \in \mathrm{SU}(2)$ and $x, y \in \mathfrak{su}(2)$. More generally, if Y is any vector field on $\mathbb{S}^3$, we can write $Y_g = gy(g)$ for every $g \in \mathbb{S}^3$ where $y : \mathbb{S}^3 \to \mathrm{T}_e\mathbb{S}^3$ is some smooth function, and we have

$$(\nabla_X Y)_g = g\left(\tfrac{1}{2}[x, y(g)] + y_*(X)\right). \tag{7}$$

With respect to the fixed frame u the connection 1-form ω of the Levi-Civita connection is

$$\omega(X) = \tfrac{1}{2}\mathrm{ad}_x = *x \in \mathfrak{so}(3) \cong \mathbb{R}^3 ,$$

for every tangent vector $X \in \mathrm{T}_g\mathbb{S}^3$ written as $X = gx$, $x \in \mathrm{T}_e\mathbb{S}^3 \cong \mathbb{R}^3$. Here and henceforth we identify vectors and 1-forms using the Riemannian metric. We denote by $\tilde{u}$ a lift of the frame u to a section of the spin principal bundle. Any spinor Ψ can then be written as

$$\Psi = [\tilde{u}, f], \tag{8}$$

for some function f defined on $\mathbb{S}^3$ with values in the spin module, which in our case can be identified with $\mathbb{H}$. Since $ijk = -1$ it follows that $X \cdot Y \cdot Z \cdot \Psi = -\Psi$ for every positive orthonormal base X, Y, Z. This shows that the Clifford action of X and $*X$ are related by

$$*X \cdot \Psi = X \cdot \Psi \tag{9}$$

for every tangent vector X and for every spinor Ψ (see also (3) and recall that in [11] the opposite sign convention was used).

The covariant derivative of Ψ with respect to some tangent vector $X = gx = [u(g), x]$ is given by

$$\nabla_X \Psi = [\tilde{u}, X(f) + \widetilde{\omega(X)} \cdot f] . \tag{10}$$

Here $A \mapsto \tilde{A}$ denotes the inverse of the differential of the spin covering, which via the isomorphism $\mathfrak{so}(3) \cong \mathfrak{spin}(3)$ corresponds to the multiplication by $\frac{1}{2}$. We infer

$$[\tilde{u}, \widetilde{\omega(X)} \cdot f] = [\tilde{u}, \widetilde{*x} \cdot f] = \tfrac{1}{2} * X \cdot \Psi = \tfrac{1}{2} X \cdot \Psi . \tag{11}$$

In the particular case of a constant function f we thus obtain from (10) and (11):

$$\nabla_X \Psi = [\tilde{u}, \widetilde{\omega(X)} \cdot f] = \tfrac{1}{2} X \cdot \Psi \ . \tag{12}$$

Hence any spinor given by a constant function with respect to a left-invariant frame is a Killing spinor for the Killing constant $\frac{1}{2}$. Similarly, constant spinors with respect to a right-invariant frame are Killing spinors with Killing constant $-\frac{1}{2}$.

Consider the unit vectors ξ_a defined by Ψ via (2). Our next goal is to interpret the condition $\delta\xi_a = 0$ from Lemma 2.1 in terms of the function f defining the spinor Ψ in the left-invariant frame $\tilde{u}$ (cf. Equation (8)).

Writing $\xi_a(g) = [u, v_a] = g v_a(g)$ for some function $v_a : \mathbb{S}^3 \to \mathrm{Im}\mathbb{H} \cong \mathbb{R}^3$, Equation (2) translates into

$$v_a(g) = f(g) a f(g)^{-1} \qquad \text{or, equivalently,} \qquad \xi_a(g) = g f(g) a f(g)^{-1} \ . \tag{13}$$

Lemma 3.1. *Let $\Psi = [\tilde{u}, f]$ be a spinor on $\mathbb{S}^3$ with associated vector fields ξ_a defined by* (2)*. Then the vector fields ξ_a are divergence-free for all $a \in \mathrm{Im}\mathbb{H}$ if and only if for every $g \in \mathbb{S}^3$, the endomorphism M_g of $\mathrm{Im}\mathbb{H}$ defined by $M_g(x) := df_g(gx) f^{-1}(g)$ is symmetric.*

Proof. We compute the covariant derivative of ξ_a in the direction of some vector $X_g = gx$ (with $x \in \mathrm{Im}\mathbb{H}$) and obtain from (7) and (13)

$$(\nabla_X \xi_a)_g = \frac{1}{2} g \left([x, f(g) a f^{-1}(g)] + df_g(gX) a f^{-1}(g) - f(g) a f^{-1}(g) df_g(gX) f^{-1}(g) \right) \ .$$

Using this formula we may calculate the divergence of the vector field ξ_a, where we set $b := f(g) a f^{-1}(g)$:

$$\begin{aligned} -(\delta\xi_a)_g &= \tfrac{1}{2}\langle [e_i, f(g) a f^{-1}(g)], e_i\rangle + \langle df_g(ge_i) f^{-1}(g) b, e_i\rangle - \langle b df_g(ge_i) f^{-1}(g), e_i\rangle \\ &= \langle M_g(e_i), -e_i b + b e_i\rangle = \langle M_g(e_i), [b, e_i]\rangle = \langle [e_i, M_g(e_i)], b\rangle \\ &= 2\langle \mathrm{vol}, e_i \wedge M_g(e_i) \wedge b\rangle \ . \end{aligned}$$

For any fixed g, when a runs through $\mathrm{Im}\mathbb{H}$, b takes any value in $\mathrm{Im}\mathbb{H}$. It follows that $\delta\xi_a = 0$ for all a if and only if $e_i \wedge M_g(e_i) \wedge b = 0$ for all b, which is equivalent to $e_i \wedge M_g(e_i) = 0$ and finally to M_g being symmetric. □

Example 3.2. In [11] we have constructed the following examples of generalized Killing spinors on $\mathbb{S}^3$:

(1) Killing spinors with constant $\frac{1}{2}$.
(2) Killing spinors with constant $-\frac{1}{2}$.
(3) Products $\xi \cdot \Phi$ where ξ is right-invariant and Φ is a Killing spinor with constant $\frac{1}{2}$.
(4) Products $\xi \cdot \Phi$ where ξ is left-invariant and Φ is a Killing spinor with constant $-\frac{1}{2}$.

Lemma 3.3. *Up to right multiplication of f with some constant quaternion, the above examples of generalized Killing spinors correspond to the following functions and vector fields.*

$$\begin{array}{lll} (1) & f(g)=1 & \xi_a(g)=ga \\ (2) & f(g)=g^{-1} & \xi_a(g)=ag \\ (3) & f(g)=g^{-1}bg & \xi_a(g)=bgag^{-1}b^{-1}g \\ (4) & f(g)=bg^{-1} & \xi_a(g)=gbg^{-1}agb^{-1} \end{array}$$

Proof. Consider a left-invariant frame u and its spin lift $\tilde{u}$ as before. We have already seen in Equation (12) that Killing spinors with constant $\frac{1}{2}$ correspond to constant functions f in the frame $\tilde{u}$. Consider the spinor $\Psi := [\tilde{u}, f]$ with $f(g) = g^{-1}$. For every vector $X \in \mathrm{T}_g\mathbb{S}^3$, written as $X = gx = [u, x]$, the derivative of the $\mathbb{H}$-valued function f with respect to X reads $X(f) = -xg^{-1} = -x \cdot f$. Using Equation (10) we can thus compute the covariant derivative of Ψ:

$$\nabla_X\Psi = [\tilde{u}, X(f) + \widetilde{\omega(X)} \cdot f] = [\tilde{u}, X(f)] + \tfrac{1}{2}X \cdot \Psi = -[u, x] \cdot [\tilde{u}, f] + \tfrac{1}{2}X \cdot \Psi = -\tfrac{1}{2}X \cdot \Psi.$$

This proves case (2).

If ξ is right-invariant and Φ is a Killing spinor with constant $\frac{1}{2}$ we can write $\xi_g = bg = [u, g^{-1}bg]$ for some $b \in \mathrm{Im}\mathbb{H}$ and (up to right multiplication with a constant) $\Phi = [\tilde{u}, 1]$, whence $\xi \cdot \Phi = [\tilde{u}, g^{-1}bg]$. Similarly, if ξ is left-invariant and Φ is a Killing spinor with constant $-\frac{1}{2}$ we can write $\xi_g = gb = [u, b]$ for some $b \in \mathrm{Im}\mathbb{H}$ and (up to right multiplication with a constant) $\Psi = [\tilde{u}, g^{-1}]$, whence $\xi \cdot \Phi = [\tilde{u}, bg^{-1}]$. The corresponding formulas for ξ_a follow from Equation (13). ☐

4. Lagrangian graphs

The graph Γ_f of a smooth map $f : \mathbb{S}^3 \to \mathbb{S}^3$ defines a submanifold in $S^3 \times S^3$. In this section we want to show that the symmetry condition in Lemma 3.1 above translates into the the fact that the graph $\Gamma_{f^{-1}}$ of the map $g \mapsto f(g)^{-1}$ is Lagrangian with respect to a certain non-degenerate 2-form on $S^3 \times S^3$.

We identify $S^3 \times S^3$ with the homogeneous space $S^3 \times S^3 \times S^3/\Delta(S^3)$ via the action

$$(g_1, g_2, g_3) \cdot (a_1, a_2) := (g_1 a_1 g_3^{-1}, g_2 a_2 g_3^{-1}).$$

The stabilizer of (e, e) is then the diagonal of $S^3 \times S^3 \times S^3$ and the projection $\pi : S^3 \times S^3 \times S^3 \to S^3 \times S^3$ is given by $\pi(g_1, g_2, g_3) = (g_1 g_3^{-1}, g_2 g_3^{-1})$.

The tangent space at (e, e) is identified with

$$\mathfrak{m} := \{(X_1, X_2, X_3) | X_i \in \mathfrak{su}(2), X_1 + X_2 + X_3 = 0\}.$$

In this identification, a tangent vector (Y_1, Y_2) corresponds to

$$\pi^{\mathfrak{m}}(Y_1, Y_2, 0) - \tfrac{1}{3}(Y_1 + Y_2, Y_1 + Y_2, Y_1 + Y_2) = \tfrac{1}{3}(2Y_1 - Y_2, 2Y_2 - Y_1, -Y_1 - Y_2)\ . \tag{14}$$

Let B be the Killing form of $\mathfrak{su}_2$, and denote by $B_0 := \frac{1}{12}B$ its rescaling. Then

$$g((X_1, X_2, X_3), (Y_1, Y_2, Y_3)) := -(B_0(X_1, Y_1) + B_0(X_2, Y_2) + B_0(X_3, Y_3))$$

defines a homogeneous nearly Kähler metric of scalar curvature scal = 30 on $S^3 \times S^3$ (cf. [9], Lemma 5.4). Denoting $-B_0$ simply by $\langle \cdot, \cdot \rangle$, and using the identification (14), the induced metric on $S^3 \times S^3$ reads

$$g((X_1, X_2), (Y_1, Y_2)) = \tfrac{1}{3}(2\langle X_1, Y_1\rangle + 2\langle X_2, Y_2\rangle - \langle X_1, Y_2\rangle - \langle X_2, Y_1\rangle) \ . \tag{15}$$

The manifold $S^3 \times S^3$ has the structure of a 3-symmetric space. The corresponding almost complex structure is defined as

$$J(X_1, X_2, X_3) = \tfrac{2}{\sqrt{3}}(X_3, X_1, X_2) + \tfrac{1}{\sqrt{3}}(X_1, X_2, X_3) \ ,$$

which by (14) can be rewritten as

$$J(X_1, X_2) = \tfrac{1}{\sqrt{3}}(X_1 - 2X_2, 2X_1 - X_2) \ .$$

Let Ω be the fundamental 2-form $\Omega(A, B) = g(JA, B)$, then

$$\Omega((X_1, X_2), (Y_1, Y_2)) = \tfrac{1}{\sqrt{3}}(\langle X_1, Y_2\rangle - \langle X_2, Y_1\rangle) \ .$$

At an arbitrary point $(g_1, g_2) \in S^3 \times S^3$, the 2-form Ω is defined by

$$\Omega((X_1, X_2), (Y_1, Y_2)) := \tfrac{1}{\sqrt{3}}(\langle g_1^{-1}X_1, g_2^{-1}Y_2\rangle - \langle g_2^{-1}X_2, g_1^{-1}Y_1\rangle) \ . \tag{16}$$

A 3-dimensional submanifold L of the nearly Kähler manifold $S^3 \times S^3$ is called *Lagrangian* if $\Omega(A, B) = 0$ for all tangent vectors $A, B \in \mathrm{T}L$. Notice that this is a generalization of the usual concept of the a Lagrangian submanifold since the fundamental 2-form Ω is not closed.

Lemma 4.1. *Let $f : \mathbb{S}^3 \to \mathbb{S}^3$ be a smooth map. Then the endomorphism M_g defined in Lemma 3.1 is symmetric for all g if and only if the graph $\Gamma_{f^{-1}}$ of f^{-1} is a Lagrangian submanifold of $S^3 \times S^3$ with respect to the 2-form Ω.*

Proof. The tangent space to the graph $\Gamma_{f^{-1}}$ at $(g, f(g)^{-1})$ is the set of vectors of the form $(gx, -f(g)^{-1}df_g(gx)f(g)^{-1})$ for $x \in \mathfrak{su}(2)$.

By (16), the value of Ω at the point $(g, f(g))$ on two such tangent vectors is

$$-\tfrac{1}{\sqrt{3}}(\langle x, df_g(gy)f(g)^{-1}\rangle - \langle df_g(gx)f(g)^{-1}, y\rangle) = -\tfrac{1}{\sqrt{3}}(\langle x, M_g(y)\rangle - \langle M_g(x), y\rangle) \ ,$$

which shows that $\Gamma_{f^{-1}}$ is Lagrangian with respect to Ω if and only if M_g is symmetric for all $g \in \mathbb{S}^3$. □

Proposition 4.2.
(1) *The submanifolds $\Gamma_1 := \{(g, 1) | g \in \mathbb{S}^3\}$ and $\Gamma_2 := \{(g, g) | g \in \mathbb{S}^3\}$ are Lagrangian submanifolds of $S^3 \times S^3$ isometric to the round sphere $\mathbb{S}^3(\frac{2}{3})$, of volume $\frac{8}{27}\mathrm{vol}(\mathbb{S}^3)$.*
(2) *For every $b \in \mathbb{S}^2 \subset \mathbb{S}^3$, the submanifolds and $\Gamma_3(b) := \{(g, g^{-1}bg) | g \in \mathbb{S}^3\}$ and $\Gamma_4(b) := \{(g, gb) | g \in \mathbb{S}^3\}$ are Lagrangian submanifolds of $S^3 \times S^3$ isometric to the Berger sphere obtained from $\mathbb{S}^3(\frac{2}{\sqrt{3}})$ by rescaling the metric on the fibres of the Hopf fibration by a factor $\frac{1}{\sqrt{3}}$. Their volume is equal to $\frac{24}{27}\mathrm{vol}(\mathbb{S}^3)$.*
(3) *For every $a, b \in \mathbb{S}^2 \subset \mathbb{S}^3, a \perp b$, the submanifolds $L(a, b) := \{gag^{-1}, gbg^{-1}) | g \in \mathbb{S}^3\}$ are are Lagrangian submanifolds of $S^3 \times S^3$ isometric to the round sphere $\mathbb{S}^3(\frac{4}{3})$, of volume $\frac{64}{27}\mathrm{vol}(\mathbb{S}^3)$.*

Proof. The metric structure of the above submanifolds is an immediate consequence of (15). The Lagrangian property follows from Lemmas 3.3 and 4.1 in the first two cases, as Γ_1, Γ_2, $\Gamma_3(b)$ and $\Gamma_4(-b)$ are the graphs $\Gamma_{f^{-1}}$ for f in one of the cases (1)–(4) of Lemma 3.3 respectively. The verification of the fact that $L(a,b)$ are Lagrangian is straightforward using (16). □

Since the volume is constant on connected components of the space of Lagrangian submanifolds, we obtain directly the following:

Corollary 4.3. *The space of Lagrangian submanifolds of $S^3 \times S^3$ has at least 3 connected components.*

We end up this section with a remark concerning the possible radii of round Lagrangian spheres in $S^3 \times S^3$.

Proposition 4.4. *The radius of a round Lagrangian sphere in $S^3 \times S^3$ is necessarily of the form $\frac{k}{3}$ for some integer $k \geq 2$. The values $k = 2$ and $k = 4$ are realized by the examples (1) and (3) in Proposition 4.2.*

Proof. Let $L \subset S^3 \times S^3$ be a 3-dimensional Lagrangian submanifold. Then the isometries of $S^3 \times S^3$ define deformations of L. Indeed the isometry group of the 3-symmetric space $S^3 \times S^3$ is 9-dimensional, whereas the isometry group of L is at most 6-dimensional, obtained in the case of the round 3-sphere. Hence there remains an at least 3-dimensional space of normal deformations.

In [12] (see also [8]) it was shown that for every infinitesimal deformation transversal to the diffeomorphisms of L, the so-called variation 1-form is a co-closed eigenform of the Hodge-Laplace operator of L for the eigenvalue 9, when the scalar curvature of the nearly Kähler manifold $S^3 \times S^3$ is normalized to 30.

It is well known that the Laplace spectrum on co-closed 1-forms on the round sphere $\mathbb{S}^3(r)$ is given by $\{\frac{k^2}{r^2} | k = 2, 3, \ldots\}$. Note that the first eigenspace, corresponding to the eigenvalue $\frac{4}{r^2}$, is exactly the space of (1-forms dual to) Killing vector fields.

Assume now that L is isometric to a round sphere $\mathbb{S}^3(r)$. Since 9 is eigenvalue of the Hodge-Laplace operator of L, there exists some integer $k \geq 2$ such that $9 = \frac{k^2}{r^2}$, thus $r = \frac{k}{3}$. □

5. Geodesic vector fields

Let us fix throughout this section the unit length spinor $\Phi := [\tilde{u}, 1]$. By (12), Φ is a Killing spinor with Killing constant $\frac{1}{2}$. Compared to Φ, any spinor Ψ on $\mathbb{S}^3$ is determined by a vector field V and a function α. Indeed the map $\mathrm{T}_g\mathbb{S}^3 \times \mathbb{R} \to \Sigma_g\mathbb{S}^3$ defined by $(V, \alpha) \mapsto V \cdot \Phi + \alpha\Phi$ is bijective at every point g, so the spinor Ψ can be uniquely written as

$$\Psi = V \cdot \Phi + \alpha\Phi \ . \tag{17}$$

The generalized Killing equation for Ψ translates into a system of equations for V and α:

Proposition 5.1. *The spinor* $\Psi := V\cdot\Phi + \alpha\Phi$ *is a generalized Killing spinor of unit length if and only if the following system holds:*

$$\begin{array}{ll} \text{(i)} & \alpha^2 + |V|^2 = 1 \\ \text{(ii)} & -V\lrcorner * \nabla_V V - V(\alpha)V + \alpha\nabla_V V + d\alpha = 0 \\ \text{(iii)} & \alpha * (V\wedge dV) + (2\alpha - \delta V)(1-\alpha^2) + \alpha V(\alpha) = 0 \end{array}$$

Proof. We assume that Ψ has unit length, which is equivalent to (i). It remains to show that the generalized Killing condition is equivalent to (ii)-(iii). We compute the covariant derivative of Ψ using Equation (9):

$$\begin{aligned} \nabla_X\Psi &= \nabla_X V\cdot\Phi + X(\alpha)\Phi + \tfrac{1}{2}V\cdot X\cdot\Phi + \tfrac{1}{2}\alpha X\cdot\Phi \\ &= \big(\nabla_X V + \tfrac{1}{2} * (V\wedge X) + \tfrac{1}{2}\alpha X\big)\cdot\Phi + \big(X(\alpha) - \tfrac{1}{2}\langle V, X\rangle\big)\,\Phi. \end{aligned}$$

On the other hand, for every vector field Y we have

$$Y\cdot\Psi = Y\cdot V\cdot\Phi + \alpha Y\cdot\Phi = (*(Y\wedge V) + \alpha Y)\cdot\Phi - \langle Y, V\rangle\Phi.$$

Consequently, the tensor A defined in Equation (4) satisfies

$$\begin{aligned} \langle A(X), Y\rangle &= \langle\nabla_X\Psi, Y\cdot\Psi\rangle \\ &= \langle\nabla_X V + \tfrac{1}{2} * (V\wedge X) + \tfrac{1}{2}\alpha X, *(Y\wedge V) + \alpha Y\rangle - \langle Y, V\rangle\big(X(\alpha) - \tfrac{1}{2}\langle V, X\rangle\big) \end{aligned}$$

Using the standard properties of the Hodge adjoint $*$ we readily obtain

$$\begin{aligned} A(X) &= -V\lrcorner * \nabla_X V + \alpha\nabla_X V - \tfrac{1}{2}|V|^2 X + \tfrac{1}{2}\langle X, V\rangle V + \tfrac{1}{2}\alpha * (V\wedge X) \\ &\quad -\tfrac{1}{2}\alpha V\lrcorner * X + \tfrac{1}{2}\alpha^2 X - VX(\alpha) + \tfrac{1}{2}\langle X, V\rangle V \\ &= -V\lrcorner * \nabla_X V + \alpha\nabla_X V - \tfrac{1}{2}|V|^2 X + \langle X, V\rangle V - \alpha V\lrcorner * X + \tfrac{1}{2}\alpha^2 X - VX(\alpha). \end{aligned}$$

It was already noticed that A is symmetric if and only if $e_i\wedge A(e_i) = 0$ for some local orthonormal basis e_i. From the previous formula we get:

$$\begin{aligned} e_i\wedge A(e_i) &= -e_i\wedge(V\lrcorner * \nabla_{e_i}V) + \alpha dV - \alpha e_i\wedge(V\lrcorner * e_i) + V\wedge d\alpha \\ &= V\lrcorner(e_i\wedge *\nabla_{e_i}V) - *\nabla_V V + \alpha dV + 2\alpha * V + V\wedge d\alpha \\ &= -\delta V * V - *\nabla_V V + \alpha dV + 2\alpha * V + V\wedge d\alpha =: \omega. \end{aligned}$$

The 2-form ω vanishes identically if and only if its wedge and interior product with V vanish. This is clear on the support of V, and ω is zero anyway outside the support of V. We now compute:

$$\begin{aligned} V\wedge\omega &= (2\alpha - \delta V) * |V|^2 - V\wedge *\nabla_V V + \alpha V\wedge dV \\ &= (2\alpha - \delta V) * |V|^2 - *\langle V, \nabla_V V\rangle + \alpha V\wedge dV \\ &= *\big((2\alpha - \delta V)(1-\alpha^2) + \alpha V(\alpha)\big) + \alpha V\wedge dV, \end{aligned}$$

and

$$\begin{aligned} V\lrcorner\omega &= -V\lrcorner * \nabla_V V + \alpha V\lrcorner dV + |V|^2 d\alpha - V(\alpha)V \\ &= -V\lrcorner * \nabla_V V + \alpha\nabla_V V - \tfrac{1}{2}\alpha d(|V|^2) + (1-\alpha^2)d\alpha - V(\alpha)V \\ &= -V\lrcorner * \nabla_V V + \alpha\nabla_V V + \alpha^2 d\alpha + (1-\alpha^2)d\alpha - V(\alpha)V \\ &= -V\lrcorner * \nabla_V V + \alpha\nabla_V V + d\alpha - V(\alpha)V. \end{aligned}$$

This proves that the symmetry of A is equivalent to (ii)-(iii). □

The main result in this section is the following

Theorem 5.2. *If the function α associated via* (17) *to a generalized Killing spinor Ψ on $\mathbb{S}^3$ is constant, then Ψ is one of the spinors described in cases* (1) *and* (3) *in Example 3.2 above.*

Proof. Assume first that the function α is identically zero. Then Lemma 5.1 implies that V is a unit length divergence-free vector field satisfying $V\lrcorner * \nabla_V V = 0$. As $\langle V, \nabla_V V\rangle = 0$, it follows that $\nabla_V V = 0$. Using a result of Gluck and Gu ([6], Theorem A) we conclude that V has to be a Hopf vector field, i.e. a left or right-invariant unit vector field on $\mathbb{S}^3$. If V is left-invariant then the function representing $\Psi = V\cdot\Phi$ in the frame $\tilde{u}$ is constant, so Ψ is a Killing spinor with Killing constant $\frac{1}{2}$. If V is right-invariant, then we are in case (3) of Example 3.2.

If α is identically 1, then $V = 0$ so $\Psi = \Phi$ and we are in case (1) of Example 3.2.

Finally, in the case where α is a constant different from 0 and 1, Lemma 5.1 (ii) reads $V\lrcorner * \nabla_V V = \alpha\nabla_V V$ and since these two vectors are orthogonal, they both vanish, showing that V is a geodesic vector field. Using Lemma 5.1 (iii) we obtain that the normalized vector field $\xi := \frac{V}{|V|}$ satisfies the equation

$$*(\xi\wedge d\xi) + 2 - \delta\xi\tfrac{\sqrt{1-\alpha^2}}{\alpha} = 0\ . \tag{18}$$

Let us write $\nabla\xi = \phi+\psi$ with ϕ symmetric and ψ skew-symmetric. As $\nabla_\xi\xi = 0$, both ϕ and ψ vanish on ξ. In dimension 2 every trace-free symmetric endomorphism anti-commutes with every skew-symmetric endomorphism, consequently the trace-free part ϕ_0 of ϕ anti-commutes with ψ. Writing $\phi = \phi_0+\frac{1}{2}(\mathrm{tr}\phi)\mathrm{id}$ we infer $(\phi+\psi)^2 = \phi^2+\psi^2+(\mathrm{tr}\phi)\psi$, so the skew-symmetric part of $(\phi+\psi)^2$ equals $(\mathrm{tr}\phi)\psi = -(\delta\xi)\psi$. This can be written as follows:

$$e_i\wedge\nabla_{\nabla_{e_i}\xi}\xi = e_i\wedge(\phi+\psi)^2(e_i) = -2(\delta\xi)\psi = -(\delta\xi)d\xi, \tag{19}$$

where e_i is any local orthonormal frame. Using (19) and the fact that the sectional curvature of $\mathbb{S}^3$ is 1, we compute in a local orthonormal frame e_i parallel at some point:

$$\begin{aligned}\nabla_\xi d\xi &= \nabla_\xi(e_i\wedge\nabla_{e_i}\xi) = e_i\wedge\big(R_{\xi,e_i}\xi+\nabla_{e_i}\nabla_\xi\xi+\nabla_{[\xi,e_i]}\xi\big)\\ &= e_i\wedge\big(\langle e_i,\xi\rangle\xi - e_i - \nabla_{\nabla_{e_i}\xi}\xi\big) = -e_i\wedge\nabla_{\nabla_{e_i}\xi}\xi = (\delta\xi)d\xi,\end{aligned}$$

thus $\nabla_\xi(\xi\wedge d\xi) = (\xi\wedge d\xi)\delta\xi$ and from (18) we get

$$\nabla_\xi\delta\xi = (\delta\xi)^2 - \tfrac{2\alpha}{\sqrt{1-\alpha^2}}\delta\xi.$$

Every orbit of the flow of ξ is a great circle, so is closed. The restriction of $\delta\xi$ to such an orbit is thus a periodic solution of the equation $y' = y^2 - \frac{2\alpha}{\sqrt{1-\alpha^2}}y$. The only periodic solution of this equation being the constants $y = 0$ and $y = \frac{2\alpha}{\sqrt{1-\alpha^2}}$, we obtain that either $\delta\xi$ vanishes on $\mathbb{S}^3$ or $\delta\xi = \frac{2\alpha}{\sqrt{1-\alpha^2}}$. This last case cannot occur since by the Stokes' Theorem the integral over $\mathbb{S}^3$ of $*\delta\xi$ vanishes. By Theorem A in [6] again, ξ is a Hopf vector field on $\mathbb{S}^3$. Moreover, (18) gives $*(\xi\wedge d\xi) = -2$. It is easy to check from (6) that $d\xi = -2*\xi$ when ξ is left-invariant and $d\xi = 2*\xi$ when ξ is right-invariant. Consequently ξ is a left-invariant vector field and finally $\Psi = \alpha\Phi+\sqrt{1-\alpha^2}\,\xi\cdot\Phi$ is a Killing spinor with Killing constant $\frac{1}{2}$. □

Coming back to our description of generalized Killing spinors on $\mathbb{S}^3$ in terms of Lagrangian embeddings, we obtain the following:

Corollary 5.3. *Let $f : \mathbb{S}^3 \to \mathbb{S}^2 \subset \mathbb{S}^3$ be a map whose graph Γ_f is a Lagrangian submanifold of the nearly Kähler manifold $S^3 \times S^3$. Then the map f is either constant, or satisfies $f(g) = g^{-1}bg$ for some fixed $b \in \mathbb{S}^2 \subset \mathbb{S}^3$.*

Proof. The map $g \mapsto f(g)^{-1}$ takes values in $\mathbb{S}^2 \subset \mathbb{R}^3$. Consider the unit vector field on $\mathbb{S}^3$ defined by $V := [u, f^{-1}]$ and the Killing spinor $\Phi := [\tilde{u}, 1]$. Theorem 5.2 applied to the generalized Killing spinor $\Psi := [\tilde{u}, f^{-1}] = V \cdot \Phi$ shows that V is a Hopf vector field. If V is left-invariant then f is constant. If V is right-invariant, $V_g = ag$ for some fixed $a \in \mathbb{S}^2 = \mathbb{S}^3 \cap \mathbb{R}^3$, which yields $gf(g)^{-1} = ag$ and finally $f(g) = g^{-1}a^{-1}g$. □

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Andrei Moroianu, Université de Versailles-St Quentin, UMR 8100 du CNRS, 78035 Versailles Cedex, 45 avenue des États-Unis

E-mail address: andrei.moroianu@math.cnrs.fr

Uwe Semmelmann, Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: uwe.semmelmann@mathematik.uni-stuttgart.de