Conformal Killing forms on Riemannian manifolds

Habilitationsschrift
zur Feststellung der Lehrbefähigung
für das Fachgebiet Mathematik
in der Fakultät für Mathematik und Informatik
der Ludwig-Maximilians-Universität München.

Vorgelegt im Dezember 2001 von

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Chapter 0

Introduction

0.1 Introduction

A classical object of differential geometry are *Killing vector fields*. These are by definition infinitesimal isometries, i.e. the flow of such a vector field preserves a given metric. The space of all Killing vector fields forms the Lie algebra of the isometry group of a Riemannian manifold and the number of linearly independent Killing vector fields measures the degree of symmetry of the manifold. It is known that this number is bounded from above by the dimension of the isometry group of the standard sphere and, on compact manifolds, equality is attained if and only if the manifold is isometric to the standard sphere. Slightly more generally one can consider *conformal vector fields*, i.e. vector fields with a flow preserving a given conformal class of metrics. There are several geometric conditions which force a conformal vector field to be Killing. These two classes of vector fields are well studied and one has many classical results.

Much less is known about a rather natural generalization of conformal vector fields, the so-called *conformal Killing forms*. These are differential forms ψ satisfying for any vector field X the differential equation

$$\nabla_X \psi \; - \; \tfrac{1}{p+1} \, X \lrcorner \, d\psi \; + \; \tfrac{1}{n-p+1} \, X^* \wedge d^* \psi \; = \; 0 \, , \qquad (0.1.1)$$

where p is the degree of the form ψ and n the dimension of the manifold. Moreover, ∇ denotes the covariant derivative of the Levi-Civita connection, X^* is 1-form dual to X and $\lrcorner$ is the operation dual to the wedge product. It is easy to see that a conformal Killing 1-form is dual to a conformal vector field. Coclosed conformal Killing p-forms are called *Killing forms*. For $p = 1$ they are dual to Killing vector fields.

The left hand side of equation (0.1.1) defines a first order elliptic differential operator T, which was already studied in the context of Stein-Weiss operators. Equivalently one can describe a conformal Killing form as a form in the kernel of T. From this point of view conformal Killing forms are similar to twistor spinors in spin geometry. One shared property is the conformal invariance of the defining equation. In particular, any form which is parallel for some metric g, and thus a Killing form for trivial reasons, induces non-parallel conformal Killing forms for metrics conformally equivalent to g (by a non-trivial change of the metric).

Killing forms, as a generalization of the Killing vector fields, were introduced by K. Yano in [Ya51]. Later S. Tachibana (c.f. [Ta69]), for the case of 2–forms, and more generally T. Kashiwada (c.f. [Ka68], [KaTa69]) introduced conformal Killing forms generalizing conformal vector fields. These articles contain several Weitzenböck formulas and integrability results for Killing resp. conformal Killing forms. Nevertheless, the only given examples of Riemannian manifolds admitting conformal Killing forms are spaces of constant curvature, e.g. the standard sphere.

Already K. Yano noted that a p–form ψ is a Killing form if and only if for any geodesic γ the $(p-1)$–form $\dot{\gamma} \lrcorner \psi$ is parallel along γ. In particular, Killing forms give rise to quadratic first integrals of the geodesic equation, i.e. functions which are constant along geodesics. Hence, they can be used to integrate the equation of motion. This was first done in the article [PW70] of R.Penrose and M. Walker, which initiated an intense study of Killing forms in the physics literature. In particular, there is a local classification of Lorentz manifolds with Killing 2-forms. More recently Killing forms and conformal Killing forms have been successfully applied to define symmetries of field equations (c.f. [BCK97], [BC97]).

Despite this longstanding interest in Killing forms there are only very few global results on the existence or non-existence of (conformal) Killing forms on Riemannian manifolds. The aim of our paper is to fill this gap and to start a thorough study of global properties of conformal Killing forms.

As a first contribution we will show that there are several classes of Riemannian manifolds admitting Killing forms, which did not appear in the literature so far. In particular, we will show that there are Killing forms on nearly Kähler manifolds and on manifolds with a weak G_2–structure. All these examples are related to Killing spinors and nearly parallel vector cross products. Moreover, they are all so-called *special Killing forms*. The restriction from Killing forms to special Killing forms is analogous to the definition of a Sasakian structure as a unit length Killing vector field satisfying an additional equation. One of our main results in this paper is the complete classification of manifolds admitting special Killing forms.

Since conformal Killing forms are sections in the kernel of an elliptic operator it is clear that they span a finite dimensional space in the case of compact manifolds. Our second main result is an explicit upper bound for the dimension of the space of conformal Killing forms on arbitrary connected Riemannian manifolds. The upper bound is provided by the dimension of the corresponding space on the standard sphere. It is also shown that if the upper bound is attained the manifold has to be conformally flat.

There are several non-existence results for conformal Killing forms, e.g. on compact manifolds of negative constant sectional curvature. All of them are trivial consequences of a well-known integrability condition. The only further work in this direction is due to

S. Yamaguchi (c.f. [Y75]). He states that on a compact Kähler manifold, any conformal Killing form has to be parallel (with some restrictions in low dimensions and for low degrees). We completely clarify the situation in the Kähler case. First of all we show that there are two wrong statements in the paper of S. Yamaguchi. It turns out that there are examples of non-parallel conformal Killing 2-forms and of conformal Killing n-forms on $2n$-dimensional Kähler manifolds. The complex projective spaces provide the simplest examples. These forms are closely related to *Hamiltonian 2-forms*, which were recently studied in [ACG01a] in connection with weakly self-dual Kähler surfaces and Bochner flat Kähler manifolds. Moreover, we show that the remaining exceptional cases cannot occur.

As our last main result, we show that on a compact manifold with holonomy G_2 any conformal Killing p–form ($p \neq 3, 4$) has to be parallel. As a first step we prove that on a compact manifold with holonomy G_2 or Spin_7 any closed or coclosed conformal Killing form has to be parallel. On compact G_2-manifolds we then continue to show that for $p \neq 3, 4$ any conformal Killing p-form is either closed or coclosed.

So far we described our most important results. In addition we proved several properties of conformal Killing forms which may be useful in a further study of the subject. In our paper we also tried to collect all that is presently known for conformal Killing forms on Riemannian manifolds. This includes new proofs and new versions of known results.

Acknowledgments

In the first place, I would like to thank Prof. D. Kotschick for many valuable discussions, his support and his interest in my work. I am grateful to A. Moroianu and G. Weingart for many helpful comments, important hints and a continued interest in the topic of conformal Killing forms on Riemannian manifolds. Moreover, I would like to thank V. Apostolov and P. Gauduchon for several interesting discussions.

0.2 Overview

We will now give a more detailed description of our paper.

Chapter 1. The first chapter contains several equivalent definitions and basic properties of conformal Killing forms. In particular, we give the fundamental Weitzenböck formulas and integrability conditions. This generalizes the characterization of Killing vector fields on compact manifolds as divergence-free vector fields in the kernel of $\Delta - 2\,\mathrm{Ric}$. Other interesting properties are the conformal invariance of the defining equation and the fact that the Hodge star operator preserves the space of conformal Killing forms. We also include a short section on Killing k-tensors. These are $(0, k)$-tensors $\mathcal{T}$ such that the complete symmetrization of $\nabla\mathcal{T}$ vanishes. Any Killing 2-form has an associated Killing 2-tensor and any Killing k-tensor defines a k-th order first integral of the geodesic equation.

Chapter 2. The second chapter collects most of the known global examples. We start with a discussion of parallel forms. Here we give a characterisation of conformal Killing forms which are induced from parallel forms via a conformal change of the metric (Proposition 2.1.1). This can be used to show that on a 4-dimensional manifold the complement of the zero set of a self-dual conformal Killing 2-form is conformally equivalent to a Kähler manifold. We continue with a discussion of conformal vector fields, including the fact that they are dual to conformal Killing 1-forms. As the next class of manifolds with Killing forms we describe Sasakian manifolds. We show that starting from the Killing vector field defining the Sasakian structure one can define several Killing forms in higher degree. Killing forms on Sasakian manifolds were also studied in [Y72a]. New examples of conformal Killing forms can be constructed using vector cross products. We recall the definition of vector cross products and show that the fundamental 2-form of a nearly Kähler manifold is Killing and that the same is true for the defining 3-form of a weak G_2-structure. A further interesting result is that the Kähler form of an almost Hermitian manifold is a conformal Killing 2-form if and only if the manifold is nearly Kähler (Proposition 2.4.2). Finally, we recall the construction of conformal Killing forms on the sphere. Here one can explicitly compute the spectrum of the Laplace operator on forms. It turns out that the eigenforms corresponding to the minimal eigenvalues are conformal Killing forms.

Chapter 3. The third chapter contains the classification of special Killing forms (Theorem 3.2.6). These are Killing forms ψ satisfying the additional equation

$$\nabla_X \psi \;=\; c\, X^* \wedge \psi \;, \tag{0.2.2}$$

for any vector field X and some constant c. For Killing 1-forms of length one equation (0.2.2) just defines a Sasakian structure. Hence, special Killing forms appear as a natural generalization of Sasakian structures. The main idea for the classification is to show that special Killing p-forms induce a parallel $(p+1)$-form on the metric cone. If the manifold is not isometric to the sphere the cone is irreducible. Hence, the problem of describing special Killing forms translates into a holonomy problem. The possible holonomies for the cone are $\mathrm{U}(m), \mathrm{SU}(m), \mathrm{G}_2$ and Spin_7, which translate into Sasakian, Einstein-Sasakian, nearly Kähler and weak G_2 structures on M. In particular, we obtain new examples of (special) Killing forms on Einstein Sasakian and on 3-Sasakian manifolds (Proposition 3.2.4).

Chapter 4. In this chapter we consider the space of all conformal Killing forms and prove in Theorem 4.3.2 a sharp upper bound on the dimension of the space of conformal Killing forms. The idea is to construct a vector bundle together with a connection, called *Killing connection*, such that conformal Killing forms are in a 1-1-correspondence to parallel sections for this connection. It then follows that the dimension of the space of conformal Killing forms is bounded by the rank of the constructed vector bundle. Moreover, it turns out that this rank is exactly the number of linearly independent conformal Killing forms on the standard sphere and that a manifold on which the maximal dimension is attained has to be conformally flat. Together with a conformal Killing form ψ we consider $d\psi, d^*\psi$ and $\Delta\psi$. Several elementary but lengthy calculations show that covariant derivatives of each of these four forms can be expressed in terms of the other three forms and zero order curvature terms. Based on the resulting four equations we define the components of the Killing connection.

The fourth chapter also contains an interesting curvature condition satisfied by conformal Killing forms (Proposition 4.2.1) and a characterization of conformal Killing forms ψ for which $d\psi$ or $d^*\psi$ are again conformal Killing forms (Proposition 4.4.9). Moreover, we give here a surprising commutator rule between the twistor operator T and the Laplace operator (Proposition 4.4.6). As a result of this formula we have that on locally symmetric spaces any conformal Killing form can be decomposed into a sum of eigenforms of the Laplace operator which are again conformal Killing forms. The proof of the commutator rule is contained in Appendix C.

Chapter 5. In this chapter we consider the holonomy decomposition of Killing forms. If a Riemannian manifold has reduced holonomy the form bundle splits into parallel subbundles, i.e. bundles preserved by the Levi-Civita connection. Accordingly one has a decomposition of any form. In Proposition 5.1.3 we prove in the case of manifolds with holonomy G_2 or Spin_7 that any component in the holonomy decomposition of a Killing form is again a Killing form. In Proposition 5.1.2 we show the same statement for conformal Killing m-forms on a $2m$-dimensional manifold. The chapter also contains results on the decomposition of Killing forms on Riemannian products.

Chapter 6. Here we prove non-existence results, i.e. we show that under certain conditions conformal Killing forms have to be parallel. We start with the case of compact Kähler manifolds. In Theorem 6.1 we recall the results of S. Yamaguchi, which still leave open several cases and moreover contain wrong statements. Nevertheless, we are able to correct the theorem and to clarify the situation completely. We first prove that on a compact 10-dimensional Kähler manifold any conformal Killing 3-form has to be parallel. This rules out the exceptional case in the theorem of S. Yamaguchi. Our most important result for Kähler manifolds is Theorem 6.1.5. Here we show that on a compact $2n$-dimensional Kähler manifold an n-form u is conformal Killing if and only if it is of the form $u = L^k\, u_0$, where u_0 is the primitive part of an invariant conformal Killing 2-form and L denotes the wedging with the Kähler form. Next, we study conformal Killing 2-forms on compact Kähler manifolds and show how they are related to Hamilton 2-forms which were studied in [ACG01a]. In particular, we see that there are many examples of compact Kähler manifolds with non-parallel conformal Killing 2-forms, which also yield examples of conformal Killing n-forms on $2n$-dimensional Kähler manifolds. As the simplest example we describe the construction on complex projective spaces.

In the second part of Chapter 6 we consider conformal Killing forms on compact manifolds with holonomy G_2. In Theorem 6.2.1 we prove that on these manifolds any closed or coclosed conformal Killing form has to be parallel and that any conformal Killing p-form with $p \neq 3, 4$ has to be either closed or coclosed. For the first part we consider the decomposition of 2- resp. 3-forms as a G_2-representation and derive explicit formulas for the projections onto the irreducible summands. Here we have to prove several elementary but useful formulas.

Chapter 7. In the last chapter we collect several results for conformal Killing forms which are still without an interesting application. In the first part we consider conformal Killing forms on Einstein manifolds and give simplified versions of formulas of Chapter 4. In the second part we study conformal Killing 2-forms. The main result here is Proposition 7.2.3, where we show that a conformal Killing 2-form considered as skew-symmetric endomorphism commutes with the Weyl tensor. Finally, we cite results on conformal Killing spinors on Sasakian manifolds and further integrability conditions.

Appendices. Appendix A contains elementary remarks on the extension of linear maps as derivations on the space of forms. Appendix C gives the proof of the commutator rule used in Chapter 4. The interesting part is Appendix B, where we study the curvature endomorphism $q(R)$ in more detail. In particular, we prove that $2q(R)$ is the zero order term in the Weitzenböck formula for the Laplace operator on forms. It is easy to see that $q(R)$ acts as a scalar multiple of the identity on spaces of constant curvature. In Proposition B.0.8 we show that in some sense also the converse is true. Finally, we present a more general definition of $q(R)$ as an endomorphism of an arbitrary bundle associated to a representation of the holonomy group. In particular, we show that it depends only on the representation defining the bundle.

Chapter 1

Conformal Killing Forms

1.1 Definition

In this section, we will define conformal Killing forms, give integrability conditions and prove some elementary properties, including equivalent characterizations of conformal Killing forms.

Let $(V, \langle\cdot,\cdot\rangle)$ be an n–dimensional Euclidean vector space. Then the $SO(n)$–representation $V^* \otimes \Lambda^p V^*$ has the following decomposition:

$$V^* \otimes \Lambda^p V^* \;\cong\; \Lambda^{p-1}V^* \oplus \Lambda^{p+1}V^* \oplus \Lambda^{p,1}V^* \;, \qquad (1.1.1)$$

where $\Lambda^{p,1}V^*$ is the intersection of the kernels of wedge product and contraction map. The highest weight of the representation $\Lambda^{p,1}V^*$ is the sum of the highest weights of V^* and of $\Lambda^p V^*$. In general, this is a decomposition into irreducible summands.

Elements of $\Lambda^{p,1}V^* \subset V^* \otimes \Lambda^p V^*$ can be considered as 1-forms on V with values in $\Lambda^p V^*$. For any $v \in V$, $\alpha \in V^*$ and $\psi \in \Lambda^p V^*$, the projection $\mathrm{pr}_{\Lambda^{p,1}} : V^* \otimes \Lambda^p V^* \to \Lambda^{p,1}V^*$ is then explicitly given by

$$[\mathrm{pr}_{\Lambda^{p,1}}(\alpha \otimes \psi)]\, v \;:=\; \alpha(v)\,\psi \;-\; \tfrac{1}{p+1}\, v \lrcorner (\alpha \wedge \psi) \;-\; \tfrac{1}{n-p+1}\, v^* \wedge (\alpha^\sharp \lrcorner \psi) \;, \qquad (1.1.2)$$

where v^* denotes the 1-form dual to v, i.e. $v^*(w) = \langle v, w\rangle$, $\alpha^\sharp$ is the vector defined by $\alpha(v) = \langle \alpha^\sharp, v\rangle$ and $v \lrcorner$ denotes the interior multiplication which is dual to the wedge product $v \wedge$.

The construction described above immediately translates to Riemannian manifolds (M^n, g), where we have the decomposition

$$T^*M \otimes \Lambda^p T^*M \;\cong\; \Lambda^{p-1}T^*M \oplus \Lambda^{p+1}T^*M \oplus \Lambda^{p,1}T^*M \qquad (1.1.3)$$

with $\Lambda^{p,1}T^*M$ denoting the vector bundle corresponding to the representation $\Lambda^{p,1}$. The covariant derivative $\nabla\psi$ of a p–form ψ is a section of $T^*M \otimes \Lambda^p T^*M$, projecting it onto the summands $\Lambda^{p+1}T^*M$ and $\Lambda^{p-1}T^*M$ yields $d\psi$ and $d^*\psi$. The projection onto the third summand $\Lambda^{p,1}T^*M$ defines a natural first order differential operator T, which we will call the *twistor operator*. The twistor operator $T : \Gamma(\Lambda^pT^*M) \to \Gamma(\Lambda^{p,1}T^*M) \subset \Gamma(T^*M \otimes \Lambda^pT^*M)$ is given for any vector field X by the following formula

$$[T\psi](X) := [\mathrm{pr}_{\Lambda^{p,1}}(\nabla\psi)](X) = \nabla_X \psi - \tfrac{1}{p+1} X \lrcorner\, d\psi + \tfrac{1}{n-p+1} X^* \wedge d^*\psi \, .$$

The twistor operator T is a typical example of a so-called Stein–Weiss operator and was in this context already considered by T. Branson in [Br97]. The definition is also similar to the definition of the twistor operator in spin geometry. There, one has the decomposition of the tensor product of spinor bundle and cotangent bundle into the sum of spinor bundle and kernel of the Clifford multiplication. The twistor operator is defined as the projection of the covariant derivative of a spinor onto the kernel of the Clifford multiplication, which, as a vector bundle, is associated to the representation given by the sum of highest weights of spin and standard representation.

Definition. A p-form ψ is called a *conformal Killing p-form* if and only if ψ is in the kernel of T, i.e. if and only if ψ satisfies for all vector fields X the equation

$$\nabla_X \psi = \tfrac{1}{p+1} X \lrcorner\, d\psi - \tfrac{1}{n-p+1} X^* \wedge d^*\psi \, . \qquad (1.1.4)$$

If the p-form ψ is in addition coclosed it is called a *Killing p-form*. This is equivalent to $\nabla\psi \in \Gamma(\Lambda^{p+1}T^*M)$ or to $X \lrcorner\, \nabla_X\psi = 0$ for any vector field X. Closed conformal Killing forms will be called $*$*–Killing forms*. Sometimes they are also called *planar*. In the physics literature, equation (1.1.4) defining a conformal Killing form is often called the *Killing–Yano equation.*

In Section 2 we will see that the Killing-Yano equation is indeed a generalization of the Killing vector field equation, i.e. we will show that a conformal Killing 1–form is dual to a conformal vector field, whereas a Killing 1–form is dual to a Killing vector field. Note that parallel forms are conformal Killing forms for trivial reasons. Moreover, conformal Killing forms which are closed and coclosed, e.g. harmonic forms on compact manifolds, have to be parallel.

It follows from the decomposition (1.1.3) that the covariant derivative $\nabla\psi$ splits into three components. Using the twistor operator T we can write the covariant derivative of a p-form ψ as

$$\nabla_X\psi = \tfrac{1}{p+1} X \lrcorner\, d\psi - \tfrac{1}{n-p+1} X^* \wedge d^*\psi + [T\psi](X) \, . \qquad (1.1.5)$$

This formula leads to the following pointwise norm estimate together with a further characterization of conformal Killing forms.

Lemma 1.1.1 *Let* $(M^n,\, g)$ *be a Riemannian manifold and let* ψ *be any* p*–form. Then*

$$|\,\nabla\psi\,|^2 \;\geq\; \tfrac{1}{p+1}\,|\,d\psi\,|^2 \;+\; \tfrac{1}{n-p+1}\,|\,d^*\psi\,|^2 \;, \qquad (1.1.6)$$

with equality if and only if ψ *is a conformal Killing* p*–form.*

Proof. We consider the embeddings $i_{\Lambda^{p\pm1}} : \Lambda^{p\pm1}T^*M \to T^*M \otimes \Lambda^pT^*M$ which are right inverse to the wedge product resp. the contraction map (cf. Section 4.6). Since (1.1.3) is an orthogonal decomposition we have

$$|\,\nabla\psi\,|^2 \;=\; |\,i_{\Lambda^{p+1}}(d\psi)\,|^2 \;+\; |\,i_{\Lambda^{p-1}}(d^*\psi)\,|^2 \;+\; |\,T\psi\,|^2 \;.$$

Hence the lemma immediately follows from (4.6.10). □

The pointwise estimate for $|\nabla\psi|^2$ was also proven in [GM75], where it was used to derive a lower bound for the spectrum of the Laplace operator on p–forms on manifolds with positive curvature operator. It follows immediately from this proof that eigenforms which realize the lower bound have to be conformal Killing forms. Nevertheless, this does not lead to interesting examples, since the condition to have positive curvature operator is very restrictive. In fact, all these manifolds are locally isometric to spheres.

As another application of Lemma 1.1.1 one can prove that the Hodge star-operator $*$ maps conformal Killing p–forms into conformal Killing $(n-p)$–forms.

Corollary 1.1.2 *Any* p*–form* ψ *is a conformal Killing* p*–form if and only if* $*\psi$ *is a conformal Killing (n–p)–form. In particular, the Hodge star* $*$ *interchanges closed and coclosed conformal Killing forms, i.e. Killing and* $*$*–Killing forms.*

Proof. It follows from Lemma 1.1.1 that conformal Killing forms are characterized by the fact that inequality (1.1.6) becomes an equality. Since the Hodge star-operator is an isometry and since $d^* = \pm * d *$, with the sign depending on the degree of the form, we have

$$\tfrac{1}{n-p+1}\,|\,d*\psi\,|^2 \;+\; \tfrac{1}{p+1}\,|\,d^**\psi\,|^2 \;=\; \tfrac{1}{n-p+1}\,|\,d^*\psi\,|^2 \;+\; \tfrac{1}{p+1}\,|\,d\psi\,|^2 \;=\; |\,\nabla\psi\,|^2$$

where we assumed ψ to be a conformal Killing p–form. The Hodge operator commutes with the covariant derivative and in particular it follows $|\,\nabla*\psi\,|^2 = |\,\nabla\psi\,|^2$. Hence, we have an equality in the estimate (1.1.6) for the $(n-p)$–form $*\psi$, i.e. $*\psi$ has to be a conformal Killing $(n-p)$–form. □

We will now derive integrability conditions which characterize conformal Killing forms on compact manifolds. Similar characterizations were obtained in [Ka68]).

At first, we obtain two Weitzenböck formulas by differentiating equation (1.1.5). Their proof also follows from some later calculations. We have

$$\nabla^*\nabla\psi \;=\; \tfrac{1}{p+1}\,d^*d\,\psi \;+\; \tfrac{1}{n-p+1}\,dd^*\,\psi \;+\; T^*T\,\psi \;, \qquad (1.1.7)$$

$$2q(R)\,\psi \;=\; \tfrac{p}{p+1}\,d^*d\,\psi \;+\; \tfrac{n-p}{n-p+1}\,dd^*\,\psi \;-\; T^*T\,\psi \;, \qquad (1.1.8)$$

where $2q(R)$ is the curvature expression appearing in the classical Weitzenböck formula for the Laplacian on p-forms: $\Delta = d^*d + dd^* = \nabla^*\nabla + 2q(R)$. It is the symmetric endomorphism of the bundle of differential forms defined by

$$2q(R) = \sum e_j^* \wedge e_i \lrcorner R_{e_i,e_j},$$

where $\{e_1\}$ is any local ortho-normal frame and R_{e_i,e_j} denotes the curvature of the form bundle. On forms of degree one and two one has an explicit expression for the action of $2q(R)$. Indeed, if ξ is any 1–form, then $2q(R)\,\xi = \mathrm{Ric}\,(\xi)$ and if ω is any 2–form then

$$2q(R)\,\omega = \frac{2\,s}{n}\,\omega - 2\,\mathcal{R}(\omega) + \mathrm{Ric}_0(\omega) , \qquad (1.1.9)$$

where s is the scalar curvature, $\mathcal{R}$ denotes the Riemannian curvature operator defined by $g(\mathcal{R}(X \wedge Y), Z \wedge U) = -g(R(X, Y)Z, U)$ and Ric_0 is the endomorphism induced by the trace-free Ricci tensor. In particular, this last summand vanishes if (M^n, g) is an Einstein manifold. Note that Ric is a symmetric endomorphism of the tangent bundle which can be extended as a derivation to an endomorphism on forms of any degree. Denoting this extension by Ric again we can write equation (1.1.9) also as: $2q(R)\,\omega = \mathrm{Ric}\,(\omega) - 2\mathcal{R}\,(\omega)$. Section B of the appendix contains further properties of the curvature endomorphism $2q(R)$.

Integrating the second Weitzenböck formula (1.1.8) gives rise to an important integrability condition. Indeed we have

Proposition 1.1.3 *Let (M^n, g) a compact Riemannian manifold. Then a p–form is a conformal Killing p–form, if and only if*

$$2q(R)\,\psi = \tfrac{p}{p+1}\,d^*d\,\psi + \tfrac{n-p}{n-p+1}\,dd^*\,\psi . \qquad (1.1.10)$$

As an application of this proposition, we conclude that there are no conformal Killing forms on compact manifolds where $q(R)$ has only negative eigenvalues. This is the case on manifolds with constant negative sectional curvature or on conformally flat manifolds with negative-definite Ricci tensor. Of course, this gives only very few examples. We will see later that there are much bigger classes of Riemannian manifolds which do not admit any non-parallel conformal Killing forms.

For coclosed forms, Proposition 1.1.3 is a generalization of the well-known characterization of Killing vector fields, as divergence free vector fields in the kernel of $\Delta - 2\,\mathrm{Ric}$. In the general case, it can be reformulated as

Corollary 1.1.4 *Let (M^n, g) a compact Riemannian manifold with a coclosed p–form ψ. Then ψ is a Killing form if and only if*

$$\Delta\,\psi = \frac{p+1}{p}\,2q(R)\,\psi .$$

Of course, there is a corresponding result for $*$–Killing forms on compact manifolds. If the manifold is not compact, we still have the equation for $\Delta\psi$. We have a similar characterization for conformal Killing m-forms on a $2m$-dimensional manifold.

Corollary 1.1.5 *Let $(M^{2m},\, g)$ be a compact Riemannian manifold. Then an m–form ψ is a conformal Killing form, if and only if*

$$\Delta\,\psi \;\;=\;\; \tfrac{m+1}{m}\, 2q(R)\,\psi \;.$$

One of the most important properties of the equation defining conformal Killing forms is its conformal invariance (c.f. [BC97]). We note that the same is true for the twistor equation in spin geometry. The precise formulation for conformal Killing forms is

Proposition 1.1.6 *Let $(M^n,\, g)$ be a Riemannian manifold with a conformal Killing p–form ψ. Then $\widehat{\psi} := e^{(p+1)\lambda}\psi$ is a conformal Killing p–form with respect to the conformally equivalent metric $\hat{g} := e^{2\lambda} g$.*

It is known that on a compact manifold of dimension greater than two, every Riemannian metric is conformally equivalent to some metric of constant scalar curvature. Thus, in the study of conformal Killing forms, we may assume the scalar curvature to be constant. Moreover, Proposition 1.1.6 has the following

Corollary 1.1.7 *Let $(M^n,\, g)$ be a Riemannian manifold with a conformal Killing p–form ψ and a conformal vector field ξ with Lie derivative $\mathcal{L}_\xi\, g = 2\lambda\, g$. Then*

$$\mathcal{L}_\xi\, \psi \;-\; (p+1)\,\lambda\,\psi$$

is again a conformal Killing p–form.

Note that the above corollary states in particular the existence of a representation of the isometry group on the space of conformal Killing forms. There is still another characterization of conformal Killing forms which is usually given as the definition.

Proposition 1.1.8 *Let $(M^n,\, g)$ be a Riemannian manifold. A p–form ψ is a conformal Killing form if and only if there exists a $(p-1)$–form θ such that*

$$\begin{aligned}(\nabla_Y\, \psi)(X, X_2, \ldots, X_p) \;\;&+\;\; (\nabla_X\, \psi)(Y, X_2, \ldots, X_p) \\ &= 2g(X,\, Y)\,\theta(X_2, \ldots, X_p) \;-\; \sum_{a=2}^{p} (-1)^a \Big(g(Y,\, X_a)\,\theta(X,\, X_2, \ldots, \hat{X}_a, \ldots,\, X_p) \\ &\qquad +\; g(X,\, X_a)\,\theta(Y,\, X_2, \ldots, \hat{X}_a, \ldots,\, X_p)\Big)\end{aligned}$$

for any vector fields $Y, X, X_1, \ldots X_p$, where $\hat{X}_a$ means that X_a is omitted.

Proof. Evidently the above equation is equivalent to the following one

$$\begin{aligned} X \lrcorner \nabla_Y \psi \; + \; Y \lrcorner \nabla_X \psi \;\; &= \;\; 2\, g(X,\, Y)\, \theta \; - \; Y \wedge (X \lrcorner \theta) \; - \; X \wedge (Y \lrcorner \theta) \\ &= \;\; X \lrcorner (Y \wedge \theta) \; + \; Y \lrcorner (X \wedge \theta) \end{aligned}$$

Assume that such a $(p-1)$–form θ exists, then summation over $X = Y = e_i$, for an ortho-normal basis $\{e_i\}$, leads to $-2d^*\psi = 2(n-(p-1))\theta$, i.e.

$$\theta \;\; = \;\; -\tfrac{1}{n-p+1}\, d^*\psi$$

and the equation may be written as

$$\begin{aligned} 0 \;\; &= \;\; [X \lrcorner \nabla_Y \psi \; + \; \tfrac{1}{n-p+1} X \lrcorner (Y \wedge \; d^*\psi)] \; + \; [Y \lrcorner \nabla_X \psi \; + \; \tfrac{1}{n-p+1} Y \lrcorner (X \wedge \; d^*\psi)] \\ &= \;\; X \lrcorner (\, T\psi\, (Y)) + Y \lrcorner (\, T\psi\, (X)) \; . \end{aligned}$$

Hence, if ψ is a conformal Killing p-form, the characterizing equation is satisfied with $\theta = -\frac{1}{n-p+1}\, d^*\psi$. Conversely, if a p-form ψ satisfies the equation, it follows that $T\psi$ is completely skew-symmetric, i.e. it lies in the $\Lambda^{p+1}(T^*M)$–summand of $T^*M \otimes \Lambda^p(T^*M)$. But by definition, the twistor operator T maps into the complement of this summand. Hence, $T\psi = 0$ and ψ is a conformal Killing p–form. □

1.2 Killing tensors and first integrals

It was already mentioned in the introduction that the interest in Killing forms in relativity theory stems from the fact that they define first integrals of the geodesic equation. At the end of this chapter, we will now describe this construction in more detail.

Let ψ be a Killing p-form and let γ be a geodesic, i.e. $\nabla_{\dot\gamma}\, \dot\gamma = 0$. Then

$$\nabla_{\dot\gamma}\, (\dot\gamma \lrcorner \psi) \; = \; (\nabla_{\dot\gamma}\, \dot\gamma) \lrcorner \psi \; + \; \dot\gamma \lrcorner \nabla_{\dot\gamma}\, \psi \; = \; 0 \; ,$$

i.e. $\dot\gamma \lrcorner \psi$ is a $(p-1)$–form parallel along the geodesic γ and in particular its length is constant along γ. The definition of this constant can be given in a more general context. Indeed for any p-form ψ we can consider a symmetric bilinear form K_ψ defined for any vector fields X, Y as

$$K_\psi(X,\, Y) \; := \; g(X \lrcorner \psi,\, Y \lrcorner \psi\,) \; .$$

For Killing forms the associated bilinear form has a very nice property.

Lemma 1.2.1 *If ψ is a Killing form, then the associated symmetric bilinear form K_ψ is a Killing tensor, i.e. for any vector fields X, Y, Z it satisfies the equation*

$$(\nabla_X K_\psi)(Y,\, Z) \; + \; (\nabla_Y K_\psi)(Z,\, X) \; + \; (\nabla_Z K_\psi)(X,\, Y) \; = \; 0 \; . \tag{1.2.11}$$

In particular, $K_\psi(\dot\gamma,\, \dot\gamma)$ is constant along any geodesic γ.

In general, a $(0,k)$–tensor $\mathcal{T}$ is called *Killing tensor* if the complete symmetrization of $\nabla\mathcal{T}$ vanishes. This is equivalent to $(\nabla_X \mathcal{T})(X,\ldots,X) = 0$. It follows again that for such a Killing tensor, the expression $\mathcal{T}(\dot\gamma,\ldots,\dot\gamma)$ is constant along any geodesic γ and hence defines a k-th order first integral of the geodesic equation. Note that the length of the $(p-1)$–form $X \lrcorner \psi$ is $K_\psi(X,X)$ and that $\mathrm{tr}(K_\psi) = p\,|\psi|^2$.

Chapter 2

Examples

First of all, we have conformal and Killing vector fields, which is of course a very well studied class with a huge number of examples. Then, we can use the conformal invariance to produce examples of conformal Killing forms as scaled parallel forms (with respect to conformally equivalent metrics). Moreover, in the physics literature, one can find many locally defined metrics admitting (conformal) Killing forms. But apart from these, there seems to be only very few known global examples; they are mainly spaces of constant curvature and Sasakian manifolds. In this section we will make some comments on parallel forms and conformal vector fields. We will describe the examples on spheres and Sasakian manifolds, and finally present two new classes of manifolds admitting conformal Killing forms.

2.1 Parallel forms

Parallel forms are obviously in the kernel of the twistor operator, hence they are conformal Killing forms. Using Proposition 1.1.6, we see that with any parallel form ψ, also the form $\widehat{\psi} := e^{(p+1)\lambda}\,\psi$ is a conformal Killing p–form with respect to the conformally equivalent metric $\widehat{g} := e^{2\lambda}\,g$. The next proposition gives a characterization of conformal Killing forms which in this way are related to parallel forms.

Proposition 2.1.1 *Let $(M^n,\,g)$ be a Riemannian manifold with a conformal Killing p–form ψ. There exists a function λ such that $\widehat{\psi} := e^{(p+1)\lambda}\,\psi$ is parallel with respect to the Levi-Civita connection of $\widehat{g} := e^{2\lambda}\,g$, if and only if the following two equations are satisfied*

$$d\psi \;=\; -\,(p+1)\,d\lambda\,\wedge\,\psi \qquad \textit{and} \qquad d^*\psi \;=\; (n-p+1)\,\mathrm{grad}(\lambda)\,\lrcorner\,\psi\ .$$

Proof. Let $\widehat{g} := e^{2\lambda}g$ be a metric conformally equivalent to g. Then the Levi-Civita connection $\widehat{\nabla}$ with respect to $\widehat{g}$ is given on p–forms by the formula

$$\widehat{\nabla}_X\,u \;=\; \nabla_X\,u \;-\; pX(\lambda)\,u \;-\; d\lambda\,\wedge\,X\,\lrcorner\,u \;+\; X\,\wedge\,\mathrm{grad}(\lambda)\,\lrcorner\,u\ ,$$

where u is any p–form. Using this equation for the p–form $\widehat{\psi} := e^{(p+1)\lambda}\,\psi$, we see that $\widehat{\psi}$ is parallel, if and only if for any vector field X the following equation holds

$$\begin{aligned} 0 &= (p+1)\,X(\lambda)\,\psi \;+\; \nabla_X\,\psi \;-\; pX(\lambda)\,\psi \;-\; d\lambda\,\wedge\,X\,\lrcorner\,\psi \;+\; X\,\wedge\,\mathrm{grad}(\lambda)\,\lrcorner\,\psi \\ &= \nabla_X\,\psi \;+\; X\,\lrcorner\,(d\lambda\,\wedge\,\psi) \;+\; X\,\wedge\,\mathrm{grad}(\lambda)\,\lrcorner\,\psi \\ &= X\,\lrcorner\,\left(\tfrac{1}{p+1}\,d\psi \;+\; d\lambda\,\wedge\,\psi\right) \;+\; X\,\wedge\,\left(-\tfrac{1}{n-p+1}\,d^*\psi \;+\; \mathrm{grad}(\lambda)\,\lrcorner\,\psi\right)\;. \end{aligned}$$

In the last equation we used the assumption that ψ is a conformal Killing p–form. The two equations of the proposition follow, if we contract or wedge with $X := e_i$ and sum over a local ortho-normal frame $\{e_i\}, i = 1, \dots, n$. □

Note that the above proof also shows that the modified form $\widehat{\psi}$ is closed if and only if the first equation holds and is coclosed if and only if the second equation holds.

Let ψ be a conformal Killing form which is parallel for some conformally equivalent metric. Then the length function $f := |\psi|^2$ has no zeros and the conformal Killing form $\widehat{\psi}$ corresponding to the metric $\widehat{g} = f^{-2}g$ has constant length 1. Conversely, we have the following application of Proposition 2.1.1 for conformal Killing 2–forms on 4–dimensional manifolds (c.f. [P92] or [ACG01a]). In this situation we can assume that a conformal Killing 2-form ψ is either self-dual or anti-self-dual. If ψ is a self-dual or anti-self-dual 2-form then $\psi^2 = -\frac{|\psi|^2}{2}\,\mathrm{id}$, where we consider ψ as a skew-symmetric endomorphism. Hence, any such form defines (outside its zero set) an almost complex structure: $I_\psi := \frac{\sqrt{2}}{|\psi|}\,\psi$.

Proposition 2.1.2 *Let $(M^4,\, g)$ be a 4-dimensional manifold admitting a self-dual or anti-self-dual conformal Killing 2–form ψ without zeros. Then $(\widehat{g} := |\psi|^{-2}\,g,\, I_\psi)$ defines a Kähler structure on M with Kähler form $\widehat{\psi} = |\psi|^{-3}\,\psi$.*

Proof. First of all we note that for a self-dual or anti-self-dual 2–form ψ the two equations of Proposition 2.1.1 are equivalent. If we define the function λ by $e^{2\lambda} = |\psi|^{-2}$, we have to verify the equation

$$d^*\psi \;=\; 3\,\mathrm{grad}(\lambda)\,\lrcorner\,\psi \;=\; -\,3\,|\psi|^{-1}\,\mathrm{grad}(|\psi|)\,\lrcorner\,\psi \;=\; -\,\tfrac{3}{2}\,|\psi|^{-2}\,\mathrm{grad}(|\psi|^2)\,\lrcorner\,\psi\;.$$

It remains to compute $\mathrm{grad}(|\psi|^2)$ for the conformal Killing 2-form ψ. Here we find

$$\begin{aligned} d|\psi|^2(X) &= 2\,g(\nabla_X\psi,\,\psi) \;=\; \tfrac{2}{3}\,g(X\,\lrcorner\,d\psi,\,\psi) \;-\; \tfrac{2}{3}\,g(X\,\wedge\,d^*\psi,\,\psi) \\ &= -\,\tfrac{4}{3}\,g(X\,\wedge\,d^*\psi,\,\psi) \;=\; \tfrac{4}{3}\,\psi(d^*\psi,\,X)\;. \end{aligned}$$

Hence, it follows: $\mathrm{grad}(|\psi|^2) = \frac{4}{3}\psi(d^*\psi)$, where we consider the 2–form ψ as a skew-symmetric endomorphism. Using this formula for the gradient of the length function of ψ we obtain

$$-\,\tfrac{3}{2}\,|\psi|^{-2}\,\mathrm{grad}(|\psi|^2)\,\lrcorner\,\psi \;=\; -\,2\,|\psi|^{-2}\,\psi^2(d^*\psi) \;=\; d^*\psi\;,$$

where again ψ is identified with the corresponding endomorphism and we have used the formula $\psi^2 = -\frac{|\psi|^2}{2}\,\mathrm{id}$, valid for self-dual or anti-self-dual 2–forms. We see that the two equations of Proposition 2.1.1 are satisfied, i.e. $\widehat{\psi}$ is parallel and $(\widehat{g} := |\psi|^{-2}\, g,\, I_\psi)$ defines a Kähler structure. $\square$

Later we will see that there are classes of manifolds, e.g. Kähler manifolds or G_2–manifolds, where any conformal Killing form is parallel (possibly with some restrictions on the degree of the forms). For these manifolds we can apply the following

Proposition 2.1.3 *Let $(M,\, g)$ be a connected Riemannian manifold such that any conformal p–form, for $a \le p \le b$, is parallel. Then any metric conformally equivalent to g which admits a parallel p–form, for $a \le p \le b$, has to be a constant multiple of g.*

Proof. Assume that there is a parallel p-form $\widehat{\psi}$ for a conformally equivalent metric $\widehat{g} = e^{2\lambda} g$. Then $\psi := e^{-(p+1)\lambda}\, \widehat{\psi}$ is a conformal Killing p-form with respect to the metric g. But since there are no non-parallel conformal Killing p-forms it has to be parallel. Hence, we are in the situation of Proposition 2.1.1, i.e. we have a parallel form ψ such that the form $\widehat{\psi} = e^{(p+1)\lambda}\, \psi$ is parallel with respect to the conformally equivalent metric $\widehat{g} = e^{2\lambda} g$. Since ψ is parallel we have $d\psi = 0 = d^*\psi$ and we conclude from Proposition 2.1.1 that $d\lambda \wedge \psi = 0 = \mathrm{grad}(\lambda) \lrcorner\, \psi$. It follows

$$0 \;=\; \mathrm{grad}(\lambda) \lrcorner\, (d\lambda \wedge \psi) \;=\; |\mathrm{grad}(\lambda)|^2\, \psi \;-\; d\lambda \,\wedge\, (\mathrm{grad}(\lambda) \lrcorner\, \psi) \;=\; |\mathrm{grad}(\lambda)|^2\, \psi$$

On the other hand, since ψ is parallel, it has no zeros. Thus, $\mathrm{grad}(\lambda)$ has to vanish on M, i.e. λ is constant. $\square$

2.2 Conformal vector fields

Conformal Killing forms were introduced as a generalization of conformal vector fields, i.e. we have the following well-known result.

Proposition 2.2.1 *Let $(M,\, g)$ be a Riemannian manifold. Then a vector field ξ is dual to a conformal Killing 1-form if and only if it is a conformal vector field, i.e. there exists a function f such that $\mathcal{L}_\xi\, g = f\, g$. Moreover, ξ is dual to a Killing 1-form if and only if it is a Killing vector field, i.e. if $\mathcal{L}_\xi\, g = 0$.*

Proof. Let $\eta := \xi^*$ be the 1-form dual to the vector field ξ. Then η is a conformal Killing 1-form if and only if for any vector fields X, Y

$$\begin{aligned} 0 \;&=\; (\nabla_X \eta)(Y) \;-\; \tfrac{1}{2}\, d\eta(X,\, Y) \;+\; \tfrac{1}{n}\, g(X,\, Y)\, d^*\eta \\ &=\; (\nabla_X \eta)(Y) \;-\; \tfrac{1}{2}\, ((\nabla_X \eta)(Y) \;-\; (\nabla_Y \eta)(X)) \;+\; \tfrac{1}{n}\, g(X,\, Y)\, d^*\eta \\ &=\; \tfrac{1}{2}\, ((\nabla_X \eta)(Y) \;+\; (\nabla_Y \eta)(X)) \;+\; \tfrac{1}{n}\, g(X,\, Y)\, d^*\eta \\ &=\; \tfrac{1}{2}\, (\mathcal{L}_\xi g)(X,\, Y) \;+\; \tfrac{1}{n}\, g(X,\, Y)\, d^*\eta \;. \end{aligned}$$

In particular, if η is a Killing 1-form, i.e. if η is in addition coclosed, we obtain $\mathcal{L}_\xi g = 0$. Hence, ξ is a Killing vector field. □

We will now recall several classical results concerning the existence, resp. non-existence of conformal Killing 1-forms. A natural question would be whether it is possible to extend these properties to conformal Killing forms of arbitrary degree. Part of the following proposition is due to M. Obata (c.f. [Ob72]).

Proposition 2.2.2 *Let (M, g) be a compact Riemannian manifold. Then any conformal vector field on M is already a Killing vector field if one of the following conditions is satisfied:*

1. *(M, g) has constant non-positive scalar curvature.*
2. *(M, g) is an Einstein manifold not isometric to the sphere.*
3. *(M, g, J) is a Kähler manifold.*
4. *(M^{2n+1}, g, ξ) is a Sasakian manifold, with $n \geq 2$, which is not isometric to the sphere.*

Corollary 2.2.3 *Let (M^n, g) be a compact irreducible and simply connected Riemannian manifold. If the holonomy group of M is strictly contained in $SO(n)$, then any conformal vector field is already a Killing vector field.*

Proof. It follows from the assumptions of the corollary and the Berger list of possible holonomy groups that M is either a Riemannian symmetric space or the holonomy of M has to be one of the following groups: $\mathrm{U}(m), n = 2m$; $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1), n = 4m$; $\mathrm{SU}(m), n = 2m$; $\mathrm{Sp}(m), n = 4m$; $G_2, n = 7$; $\mathrm{Spin}_7, n = 8$. An irreducible Riemannian symmetric space is automatically Einstein and we can apply the second condition of the proposition. Manifolds with holonomy $\mathrm{U}(m)$ are Kähler, hence the third condition of Proposition 2.2.2 is satisfied. Manifolds with holonomy $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ are called quaternion Kähler and known to be Einstein. Thus, we can again apply the second condition. If the holonomy of the manifolds is one of the remaining groups, then it is automatically Ricci-flat and the scalar curvature is identically zero, i.e. we can apply the first or the second condition. □

Note that four of the possible holonomies imply that the underlying manifolds has to be Ricci-flat. If the manifold is also compact then any Killing vector field is parallel, as follows from the characterization given in Corollary 1.1.4. In fact, the same is true for any compact manifold with non-positive Ricci curvature.

The equation defining a conformal Killing form is conformally invariant (c.f. Proposition 1.1.6). Hence, any vector field which is Killing for some metric gives rise to a conformal vector field with respect to any conformally equivalent metric. It turns out that, with the exception of certain conformal vector fields on the sphere and the Euclidean space, also the converse is true. This is a theorem of D.V. Alekseevskii (c.f. [Al72]) which generalizes the corresponding result for compact manifolds due to M. Obata and J.Lelong-Ferrand (c.f. [LF]).

Theorem 2.2.4 *Let (M, g) be a complete Riemannian manifold which is not conformally equivalent to the sphere or the Euclidean space. Then any conformal vector field is a Killing vector field for some metric conformally equivalent to g.*

2.3 Sasakian manifolds

The first interesting class of manifolds admitting conformal Killing forms are the Sasakian manifolds. These are contact manifolds satisfying a normality (or integrability) condition. In the context of conformal Killing forms, it is convenient to use the following

Definition. A Riemannian manifold (M, g) is called a *Sasakian* manifold, if there exists a unit length Killing vector field ξ satisfying for any vector field X the equation

$$\nabla_X (d\xi^*) \;=\; -2\, X^* \wedge \xi^* \, . \tag{2.3.1}$$

Note that in the usual definition of a Sasakian structure, as a special contact structure one has the additional condition $\phi^2 = -\,\mathrm{id} + \eta \otimes \xi$ for the associated endomorphism $\phi = -\nabla\xi$ and the dual 1-form $\eta := \xi^*$. But this equation follows from (2.3.1), if we write (2.3.1) first as

$$(\nabla_X \phi)(Y) \;=\; g(X, Y)\, \xi \;-\; \eta(Y)\, X \;, \tag{2.3.2}$$

and take then the scalar product with ξ. It follows that the dimension of a Sasakian manifolds has to be odd and if $\dim(M) = 2n + 1$, then $\xi^* \wedge (d\xi^*)^n$ is the Riemannian volume form on M.

There are many examples of Sasakian manifolds, e.g. given as S^1–bundles over Kähler manifolds. Even in the special case of *3-Sasakian* manifolds, where one has three unit length Killing vector fields, each defining a Sasakian structure with the $SO(3)$-commutator relations, one knows that there are infinitely many diffeomorphism types (c.f. [BGM96]).

On a manifold with a Killing vector field ξ we have the Killing 1-form ξ^*. It is then natural to ask whether $d\xi^*$ is also a conformal Killing form. The next proposition shows that for Einstein manifolds this is the case, if and only if ξ defines a Sasakian structure. More generally, we have

Proposition 2.3.1 *Let* $(M,\, g)$ *be a Riemannian manifold with a Sasakian structure defined by a unit length vector field* ξ*. Then the 2–form* $d\,\xi^*$ *is a conformal Killing form. Moreover, if* $(M^n,\, g)$ *is an Einstein manifold with scalar curvature* s *normalized to* $s = n(n-1)$ *and if* ξ *is a unit length Killing vector field such that* $d\,\xi^*$ *is a conformal Killing form, then* ξ *defines a Sasakian structure.*

Proof. We first prove that for a Killing vector field ξ defining a Sasakian structure, the 2-form $d\xi^*$ is a conformal Killing form. From the definition (2.3.1) of the Sasakian structure we obtain: $d^*d\,\xi^* = 2(n-1)\,\xi$. Substituting ξ in (2.3.1) using this formula yields:

$$\nabla_X\,(\,d\,\xi^*) \;=\; -\,2X^* \,\wedge\, \tfrac{1}{2(n-1)}\, d^*d\,\xi^* \;=\; -\tfrac{1}{n-1}\,X^* \,\wedge\, d^*d\,\xi^* \;.$$

But since $d\,\xi$, is closed this equation implies that $d\,\xi^*$ is indeed a conformal Killing form.

To prove the second statement, we first note that $d^*d\,\xi^* = \Delta\,\xi^* = 2\mathrm{Ric}\;(\xi^*) = 2(n-1)\xi^*$ because of equation (1.1.10) for Killing 1-forms and the assumption that $(M,\, g)$ is an Einstein manifold with normalized scalar curvature. Then we can reformulate the condition that $d\xi^*$ is a closed conformal Killing form to obtain

$$\nabla_X\,(\,d\,\xi^*) \;=\; -\frac{1}{n-1}\,X^* \wedge d^*d\,\xi^* \;=\; -\frac{1}{n-1}\,X^* \wedge 2(n-1)\xi^* \;=\; -2\,X^* \wedge \xi^* \;,$$

i.e. the unit length Killing vector field ξ also satisfies the equation (2.3.1) and thus defines a Sasakian structure. $\square$

If we assume the manifold (M,g) to be complete and not of constant positive curvature, we can weaken the assumptions in the second statement of the above theorem, i.e. we do not have to assume the length function $f := |\xi^*|^2$ to be constant. Indeed, if f is not constant, equation (2.3.1) implies that f satisfies the equation

$$\nabla^2_{X,\,Y}(df)Z \;+\; 2X(f)\,g(Y,\,Z) \;+\; Y(f)\,g(X,\,Z) \;+\; Z(f)\,g(X,\,Y) \;=\; 0$$

for any vector fields $X,\, Y,\, Z$. Due to a theorem of M. Obata (c.f. [Ob72]) it then follows that the universal covering of M is isometric to the sphere, which was excluded.

At this point we know that on a Sasakian manifold defined by a Killing vector field ξ, the dual 1-form ξ^* and the 2-form $d\,\xi^*$ are both conformal Killing forms. By the following proposition, all possible wedge products of ξ^* and $d\,\xi^*$ are as well conformal Killing forms. In fact this is part of a more general property which we will further discuss in Chapter 3.

Proposition 2.3.2 *Let* $(M^{2n+1},\, g,\, \xi)$ *be a Sasakian manifold with Killing vector field* ξ*. Then*

$$\omega_k \;\;:=\;\; \xi^* \,\wedge\, (d\xi^*)^k$$

is a Killing $(2k+1)$*–form for* $k = 0, \dots n$*. Moreover,* ω_k *satisfies for any vector field* X *and any* k *the additional equation*

$$\nabla_X(d\omega_k) \;\; = \;\; -\,2\,(k+1)\,X^* \,\wedge\, \omega_k.$$

In particular, ω_k *is an eigenform of the Laplace operator corresponding to the eigenvalue* $4(k+1)(n-k)$*.*

Proof. Since $(M^{2n+1},\, g,\, \xi)$ is a Sasakian manifold, we know that $\omega_n = \xi^* \wedge (d\xi^*)^n$ is the Riemannian volume form. Hence, ω_n is parallel and a conformal Killing form for trivial reasons. For the other cases, $0 < k < n$, we use that ξ^* is a Killing 1-form and the defining equation (2.3.1) to obtain

$$\begin{aligned}
\nabla_X(\xi^* \wedge (d\xi^*)^k) &= (\nabla_X \xi^*) \wedge (d\xi^*)^k \;+\; \xi^* \wedge \nabla_X (d\xi^*)^k \\
&= \tfrac{1}{2}\,(X \lrcorner\, d\xi^*) \wedge (d\xi^*)^k \;+\; k\,\xi^* \wedge \nabla_X(d\xi^*) \wedge (d\xi^*)^{k-1} \\
&= \tfrac{1}{2(k+1)}\, X \lrcorner\, (d\xi^*)^{k+1} \;-\; 2k\,\xi^* \wedge (X \wedge \xi^*) \wedge (d\xi^*)^{k-1} \\
&= \tfrac{1}{2(k+1)}\, X \lrcorner\, d(\xi^* \wedge (d\xi^*)^k)
\end{aligned}$$

From this we conclude that $\xi^* \wedge (d\xi^*)^k$ is coclosed and it immediately follows that ω_k is a Killing form. To prove that ω is an eigenform of the Laplace operator, we first compute the covariant derivative of $d\omega_k = (d\xi^*)^{k+1}$. We find

$$\nabla_X(d\omega_k) \;=\; (k+1)\nabla_X(d\xi^*) \wedge (d\xi^*)^k \;=\; -2(k+1) X \wedge \xi^* \wedge (d\xi^*)^k\;.$$

This proves the additional equation for ω_k and leads to

$$\begin{aligned}
\Delta\,\omega_k &= -\sum e_i \lrcorner\, \nabla_{e_i}(d\omega_k) \;=\; 2(k+1) \sum e_i \lrcorner\, (e_i \wedge \xi^* \wedge (d\xi^*)^k) \\
&= 2(k+1)(\dim M - (2k+1))\,\omega_k \\
&= 4(k+1)(n-k)\,\omega_k\;. \quad \square
\end{aligned}$$

2.4 Vector cross products

We will now describe a general construction which provides examples of Killing forms in degrees 2 and 3. For this aim we have to recall the notion of a vector cross product (c.f. [G69]). Let V be a finite dimensional real vector space and let $\langle \cdot, \cdot \rangle$ be a non-degenerate bilinear form on V. Then a *vector cross product* on V is defined as a linear map $P : V^{\otimes r} \to V$ satisfying the axioms

$$
\begin{array}{rrcl}
(i) & \langle P(v_1, \dots v_r), v_i \rangle & = & 0 \qquad (1 \le i \le r), \\
(ii) & |P(v_1, \dots v_r)|^2 & = & \det(\langle v_i, v_j \rangle) .
\end{array}
$$

Vector cross products are completely classified. There are only four possible types: 1-fold and (n-1)-fold vector cross products on n-dimensional vector spaces, 2-fold vector cross products on 7-dimensional vector spaces and 3-fold vector cross products on 8-dimensional vector spaces. We will consider r-fold vector cross products on Riemannian manifolds (M, g). These are tensor fields of type $(r, 1)$ which are fibrewise r-fold vector cross products. As a special class, one has the so-called *nearly parallel* vector cross products. They satisfy the differential equation

$$(\nabla_{X_1} P)(X_1, \dots, X_r) \;=\; 0$$

for any vector fields $X_1, \dots, X_r$. Together with an r-fold vector cross product P, one has an associated $(r+1)$-form ω defined by

$$\omega(X_1, \dots, X_{r+1}) \;=\; g(P(X_1, \dots, X_r), X_{r+1}) .$$

Lemma 2.4.1 *Let P be a nearly parallel r-fold vector cross product with associated form ω. Then ω is a Killing $(r+1)$-form.*

Proof. The vector cross product P is nearly parallel, if and only if the associated $(r+1)$-form ω satisfies the differential equation

$$(\nabla_{X_1} \omega)(X_1, \dots, X_r) \;=\; 0 ,$$

i.e. $X \lrcorner \nabla_X \omega = 0$ for any vector field X, and we already know that this condition is equivalent to the Killing equation for ω. $\Box$

We will examine the four possible types of vector cross products to see which examples of manifolds with Killing forms one can obtain. We start with 1-fold vector cross products, which are equivalent to almost complex structures compatible with the metric. Hence, a Riemannian manifold (M, g) with a nearly parallel 1-fold vector cross product J is the same as an almost Hermitian manifold, where the almost complex structure J satisfies $(\nabla_X J) X = 0$ for all vector fields X. Such manifolds are also called *nearly Kähler*. It follows from Lemma 2.4.1 that the associated 2-form ω defined by $\omega(X, Y) = g(JX, Y)$ is a Killing 2-form. On a Kähler manifold, ω is the Kähler form and thus parallel by definition. But there are also many non-Kähler, nearly Kähler manifolds, e.g. the 3-symmetric spaces which were classified by A. Gray and J. Wolf (c.f. [GW69]). Due to a result of S. Salamon (c.f. [FFS94]) nearly Kähler, non-Kähler manifolds are never Riemannian symmetric spaces.

Next, we consider 2-fold vector cross products. They are defined on 7 dimensional Riemannian manifolds and exist, if and only if the structure group of the underlying manifold M can be reduced to the group $G_2 \subset O(7)$, i.e. M admits a topological G_2-structure. One can show that this is equivalent to the existence of a spin structure on M. Riemannian manifolds with a nearly parallel 2-fold vector cross product are also called *weak G_2–manifolds*. There are many examples of homogeneous and non-homogeneous G_2–manifolds, e.g. on any 3-Sasakian manifold, there exists a second Einstein metric which is weak-G_2. Again, with the exception of the sphere, weak G_2–manifolds are never Riemannian symmetric spaces. These and further results are contained in [FKMS97].

Finally, we have to consider the $(n-1)$-fold and 3-fold vector cross products. But in these cases, results of A. Gray show that the associated forms have to be parallel (c.f. [G69]). Hence, they yield only trivial examples of conformal Killing forms.

It is interesting to note that compact simply connected manifolds with an Einstein-Sasakian structure, a nearly Kähler structure (in dimension 6) or a weak G_2-structure are always spin and admit Killing spinors, canonically associated to the Killing form. This proves again that in these cases the underlying manifold cannot be a Riemannian symmetric space. The relation between Killing forms and Killing spinors in these examples becomes clear from the results in Chapter 3.

We have seen that nearly Kähler manifolds are special almost Hermitian manifolds where the Kähler form ω, defined by $\omega(X,Y) = g(JX,Y)$, is a Killing 2-form. This leads to the natural question whether there are other almost Hermitian manifolds with a Kähler form, which is a conformal Killing form. The following proposition gives an answer to this question.

Proposition 2.4.2 *Let (M^{2n}, g, J) be an almost Hermitian manifold. Then the Kähler form ω is a conformal Killing 2-form if and only if the manifold is nearly Kähler or Kähler.*

Proof. Let Λ denote the contraction with the 2-form ω, i.e. $\Lambda = \frac{1}{2} \sum Je_i \lrcorner\, e_i \lrcorner$. On an almost Hermitian manifold (with Kähler form ω), one has the following well known formulas:

$$\Lambda(d\omega) \;=\; J(d^*\omega) \qquad \text{and} \qquad d\omega \;=\; (d\omega)_0 \;+\; \tfrac{1}{n-1}\,(Jd^*\omega) \wedge \omega \;,$$

where $(d\omega)_0$ denotes the effective or primitive part, i.e. the part of $d\omega$ in the kernel of Λ.

We will show that if ω is a conformal Killing 2-form, then it has to be coclosed. The defining equation of a Killing 2-form reads

$$(\nabla_X\, \omega)(A,\, B) \;=\; \tfrac{1}{3}\, d\omega(X,\, A,\, B) \;-\; \tfrac{1}{2n-1}\, \big(g(X,\, A)\, d^*\omega(B) \;-\; g(X,\, B)\, d^*\omega(A)\big) \;.$$

Because $\nabla_X J \circ J + J \circ \nabla_X J = 0$ we see that $\nabla_X \omega$ is an anti-invariant 2-form. Setting $X = e_i$ and $A = Je_i$ and summing over an ortho-normal basis $\{e_i\}$ substituted into the above equation yields

$$\begin{aligned} -\, d^*\omega(JB) &= \tfrac{1}{3} \sum d\omega(e_i,\, Je_i,\, B) \;+\; \tfrac{1}{2n-1} \sum g(e_i,\, B)\, d^*\omega(Je_i) \\ &= \tfrac{2}{3}\, \Lambda(d\omega) \;+\; \tfrac{1}{2n-1}\, d^*\omega(JB) \\ &= (\tfrac{1}{2n-1} \;-\; \tfrac{2}{3})\, d^*\omega(JB)\;. \end{aligned}$$

From this equation follows immediately $d^*\omega = 0$, i.e. ω is already a Killing 2-form. But this is equivalent for (M, g, J) to be nearly Kähler, where we consider Kähler manifolds as a special case of nearly Kähler manifolds. $\Box$

2.5 Conformal Killing forms on the sphere

The spectrum and the eigenforms of the p-form Laplacian on the sphere are explicitly known. This leads to an explicit knowledge of the conformal Killing p-forms as well.

Let $(S^n,\, g)$ be the standard sphere with scalar curvature $s = n(n-1)$. The curvature operator on 2-forms is then the identity map and an easy calculation shows that $2q(R)$ acts on p-forms as $p(n-p)\,\mathrm{id}$. The spectrum of the Laplace operator on p-forms consists of two series:

$$\lambda'_k \;=\; (p+k)(n-p+k+1) \qquad \text{and} \qquad \lambda''_k \;=\; (p+k+1)(n-p+k)\;,$$

where $k = 0, 1, 2, \ldots$. The eigenvalues λ'_k correspond to closed eigenforms, whereas the eigenvalues λ''_k correspond to coclosed eigenforms. The multiplicities of the eigenvalues are well–known. For the minimal eigenvalues λ'_0 and λ''_0 we have

$$\lambda'_0 \quad \text{has multiplicity} \quad \binom{n+1}{p} \qquad \text{and} \qquad \lambda''_0 \quad \text{has multiplicity} \quad \binom{n+1}{p+1}\;.$$

The conformal Killing forms turn out to be sums of eigenforms of the Laplacian corresponding to the minimal eigenvalues on $\ker(d)$ resp. $\ker(d^*)$.

Proposition 2.5.1 *A p–form ω on the standard sphere $(S^n,\, g)$ is a conformal Killing form, if and only if it is a sum of eigenforms for the eigenvalue λ'_0 resp. of eigenforms for the eigenvalue λ''_0.*

Proof. We will see later that on locally symmetric spaces, the twistor operator T commutes with the Laplace operator (c.f. Proposition 4.4.6). Hence, we can assume the conformal Killing p–form ω to be an eigenform of the Laplacian for an eigenvalue λ.

Moreover, since $*\omega$ is again a conformal Killing form we can assume $p \leq n/2$. Integrating the equation $\Delta = \nabla^*\nabla + 2q(R)$ we get

$$\lambda \, \| \, \omega \, \|^2 \;=\; \| \, \nabla \, \omega \, \|^2 \,+\, (2q(R) \, \omega, \, \omega) \;=\; \tfrac{1}{p+1} \, \| \, d \, \omega \, \|^2 \,+\, \tfrac{1}{n-p+1} \, \| \, d^* \omega \, \|^2 \,+\, p(n-p) \, \| \, \omega \, \|^2 \, .$$

Using $\frac{1}{n-p+1} \leq \frac{1}{p+1}$ (since $p \leq \frac{n}{2}$) and $\| \, d \, \omega \, \|^2 + \| \, d^* \, \omega \, \|^2 = (\Delta\omega, \, \omega) = \lambda \| \, \omega \, \|^2$, we obtain

$$\tfrac{\lambda}{n-p+1} \, \| \, \omega \, \|^2 \,+\, p(n-p) \, \| \, \omega \, \|^2 \;\leq\; \lambda \, \| \, \omega \, \|^2 \;\leq\; \tfrac{\lambda}{p+1} \, \| \, \omega \, \|^2 \,+\, p(n-p) \, \| \, \omega \, \|^2 \, .$$

This implies

$$\lambda_0' \;=\; p \, (n - p + 1) \;\leq\; \lambda \;\leq\; \lambda_0'' \;=\; (p+1) \, (n-p)$$

with $\lambda = \lambda_0'$ if and only if ω is closed and $\lambda = \lambda_0''$ if and only if ω is coclosed. Since $\lambda_1' = (p+1)(n-p+2) \geq \lambda_0'' = (p+1)\,(n-p)$, we see that there is no eigenvalue between λ_0' and λ_0''. Thus, the conformal Killing form ω is either an eigenform for λ_0' and closed or an eigenform for λ_0'' and coclosed.

Now assume that ω is an eigenform for the eigenvalue $\lambda_0' = p\,(n-p+1)$. Then ω is closed and $dd^*\omega = \Delta\omega = p(n-p+1)\omega$. Hence, it follows from equation (1.1.8)

$$T^*T\omega \;=\; \tfrac{n-p}{n-p+1} \, p(n-p+1) \, \omega \,-\, p(n-p) \, \omega \;=\; 0$$

and integrating this equation yields $T\omega = 0$, i.e. ω is a conformal Killing form. In the case where ω is an eigenform for the eigenvalue λ_0'', it follows in the same way from equation (1.1.8) that ω is a conformal Killing form. $\square$

Chapter 3

Special Killing forms

3.1 Definition and Examples

In [Ta70] S. Tachibana and W. Yu introduced the notion of special Killing forms as Killing forms satisfying an additional equation. This definition seemed to be rather restrictive and indeed the only discussed examples were spaces of constant curvature. Nevertheless, it turns out that almost all examples of Killing forms described in the preceding section are special.

In this section we will give the definition and equivalent versions of it, which in some sense are more natural than the original one. In particular, it becomes clear that the restriction from Killing forms to special Killing forms is analogous to the restriction from Killing vector fields to Sasakian structures. Finally, we give a classification of compact manifolds admitting special Killing forms. It turns out that essentially there are no other examples as the ones discussed so far.

Definition. A *special Killing form* is a Killing form ψ which for some constant c and any vector field X satisfies the additional equation

$$\nabla_X (d\psi) \;=\; c\, X^* \wedge \psi \;. \tag{3.1.1}$$

There is an equivalent version of equation (3.1.1), which gives a definition closer to the original one. Indeed, a special Killing form can be defined equivalently as a Killing form satisfying for some (different) constant c and for any vector fields X, Y the equation

$$\nabla^2_{X,\,Y}\, \psi \;\;=\;\; c\; (g(X,\,Y)\, \psi \;-\; X \,\wedge\, Y \lrcorner\, \psi) \;\; . \tag{3.1.2}$$

From equation (3.1.1) it follows immediately that special Killing p–forms are eigenforms of the Laplacian corresponding to the eigenvalue $-c(n-p)$. Hence, on compact manifolds the constant c has to be negative.

Our first examples of special Killing forms came from Sasakian manifolds. Here the defining equation (2.3.1) coincides with equation (3.1.1) for the constant $c = -2$, i.e. a Killing vector field ξ defining a Sasakian structure is dual to a special Killing 1-form with constant $c = -2$. Moreover, we had seen in Proposition 2.3.2 that on a Sasakian manifold also the forms $\omega_k := \xi^* \wedge (d\xi^*)^k$ are special Killing forms. All other known examples are given in

Proposition 3.1.1 *The following manifolds admit special Killing forms:*

1. *Sasakian manifolds with defining Killing vector field ξ. Here all the Killing forms $\omega_k = \xi^* \wedge (d\xi^*)^k$ are special with constant $c = -2(k+1)$.*
2. *Nearly Kähler non-Kähler manifolds in dimension 6. Here the associated 2–form ω is special with constant $c = -\frac{s}{10}$ and the 3-form $*d\omega$ is special with constant $c = -\frac{2s}{15}$, where s denotes the scalar curvature.*
3. *Weak G_2–manifolds of scalar curvature s. Here the associated 3–form is a special Killing form of constant $c = -\frac{2s}{21}$.*
4. *The standard sphere S^n of scalar curvature $s = n(n-1)$. Here all Killing p–forms, i.e. all coclosed minimal eigenforms of the Laplacian are special with constant $c = -(p+1)$.*

Proof. The proof of this proposition will be an an immediate consequence of our classification of special Killing forms given below. Nevertheless, it is not difficult to give direct proofs without using the classification. □

Note that weak G_2–manifolds and the nearly Kähler non-Kähler manifolds in dimension 6 are Einstein manifolds, hence they have constant scalar curvature. One can easily see that the associated 2–form on nearly Kähler manifolds of dimension different from 6 is not special. More precisely, a nearly-Kähler, non–Kähler manifold has an associated 2–form which is a special Killing form if and only if it is 6-dimensional.

In this section we only consider Killing forms. But since the Hodge star operator maps $*$–Killing forms to Killing forms, we have a condition similar to (3.1.1) for $*$–Killing forms.

Lemma 3.1.2 *Let ψ be a $*$–Killing form. Then $*\psi$ is a special Killing form with the constant c if and only if*

$$\nabla_X(d^*\psi) \;=\; c\, X \lrcorner \psi \tag{3.1.3}$$

In particular, a $*$–Killing p–form satisfying (3.1.3) is an eigenform of the Laplacian for the eigenvalue $c\,p$.

Finally, we cite one of the few known results on special Killing forms. In [Ta70] S. Tachibana and W. Yu used it to prove that a Riemannian manifold, not isometric to the standard sphere, can have at most 3 pairwise orthogonal Sasakian structures.

Proposition 3.1.3 *Let $(M,\, g)$ be a complete simply connected Riemannian manifold admitting special Killing forms α and β. If the scalar product $\langle\alpha,\,\beta\rangle$ defines a non–constant function, then $(M,\, g)$ is isometric to the standard sphere. In particular, on a manifold not isometric to the sphere the length function of any special Killing form has to be constant.*

3.2 Classification

In this section we will give a classification of compact Riemannian manifolds admitting special Killing forms. It turns out that a p-form ψ on M is a special Killing form, i.e. a Killing form satisfying the additional equation (3.1.1), if and only if it induces a $(p+1)$–form on the metric cone $\widehat{M}$ which is parallel. Since the metric cone is either flat or irreducible, the description of special Killing forms is reduced to a holonomy problem, i.e. to the question which holonomies admit parallel forms. This question can be completely answered and retranslated into the existence of special geometric structures on the base manifold. The result will be that special Killing forms can exist only on Sasakian manifolds, nearly Kähler manifolds or weak G_2–manifolds. Our approach here is similar to the one of Ch. Bär in [Bä93] which lead to the classification of Killing spinors.

The metric cone $\widehat{M}$ over a Riemannian manifold $(M,\, g)$ is defined as the warped product $M \times_{r^2} \mathbb{R}_+$ with metric $\widehat{g} := r^2 g + dr^2$. An easy calculation shows that the Levi-Civita connection on 1–forms is given by

$$\widehat{\nabla}_X\, Y^* \;=\; \nabla_X Y^* \,-\, \tfrac{1}{r}\, g(X,\, Y)\, dr, \qquad \widehat{\nabla}_X\, dr \;=\; r\, X^*\;,$$
$$\widehat{\nabla}_{\partial_r}\, X^* \;=\; -\,\tfrac{1}{r}\, X^*, \qquad \widehat{\nabla}_{\partial_r}\, dr \;=\; 0\;,$$

where X, Y are vector fields tangent to M with g–dual 1–forms X^*, Y^*, and where ∂_r is the radial vector field on $\widehat{M}$ with $dr(\partial_r) = 1$. From this we immediately obtain the following useful formulas

$$\widehat{\nabla}_X\, \psi \;=\; \nabla_X\, \psi \,-\, \tfrac{1}{r}\, dr \wedge (X \lrcorner\, \psi), \qquad \widehat{\nabla}_{\partial_r}\, \psi \;=\; -\,\tfrac{p}{r}\, \psi\;,$$

where ψ is a p-form on M considered as p–form on $\widehat{M}$. For any p–form ψ on M, we define an associated $(p+1)$–form $\widehat{\psi}$ on $\widehat{M}$ by

$$\widehat{\psi} \;:=\; r^p\, dr \wedge \psi \,+\, \tfrac{r^{p+1}}{p+1}\, d\psi\;. \tag{3.2.4}$$

The next lemma is our main technical tool for the classification of special Killing forms. It states that special Killing forms are exactly those forms which translate into parallel forms on the metric cone.

Lemma 3.2.1 *Let (M, g) be a Riemannian manifold and ψ a p–form on M. Then the associated $(p+1)$–form $\widehat{\psi}$ on the metric cone $\widehat{M}$ is parallel with respect to $\widehat{\nabla}$ if and only if*

$$\nabla_X \psi \;=\; \tfrac{1}{p+1}\, X \lrcorner\, d\psi \qquad \textit{and} \qquad \nabla_X\,(d\psi) \;=\; -\,(p+1)\, X^* \wedge \psi$$

i.e. $\widehat{\psi}$ is parallel if and only if ψ is a special Killing form with constant $c = -(p+1)$.

Proof. We will first show that a $(p+1)$–form $\hat{\psi}$ defined on the metric cone as in (3.2.4) is always parallel in radial direction. Indeed we have

$$\begin{aligned}
\widehat{\nabla}_{\partial_r} \psi \;&=\; p\, r^{p-1}\, dr \wedge \psi \;+\; r^p\, dr \wedge \widehat{\nabla}_{\partial_r} \psi \;+\; r^p\, d\omega \;+\; \tfrac{r^{p+1}}{p+1}\, \widehat{\nabla}_{\partial_r}(d\psi)\\
&=\; (p\, r^{p-1} - r^p\, \tfrac{p}{r})\, dr \wedge \psi \;+\; (r^p - \tfrac{r^{p+1}}{p+1}\, \tfrac{1}{r}\,(p+1))\, d\psi\\
&=\; 0\;.
\end{aligned}$$

Next, we compute the covariant derivative of $\widehat{\psi}$ in direction of a horizontal vector field X. This yields

$$\begin{aligned}
\widehat{\nabla}_X \psi \;&=\; r^p\, \widehat{\nabla}_X (dr) \wedge \psi \;+\; r^p\, dr \wedge \widehat{\nabla}_X \psi \;+\; \tfrac{r^{p+1}}{p+1}\, \widehat{\nabla}_X (d\psi)\\
&=\; r^{p+1}\, X^* \wedge \psi \;+\; r^p\, dr \wedge \nabla_X \psi \;+\; \tfrac{r^{p+1}}{p+1}\, \nabla_X (d\psi) \;-\; \tfrac{r^p}{p+1}\, dr \wedge (X \lrcorner\, d\psi)\\
&=\; r^{p+1} \Big(X^* \wedge \psi \;+\; \tfrac{1}{p+1}\, \nabla_X (d\psi) \Big) \;+\; r^p\, dr \wedge \Big(\nabla_X \psi \;-\; \tfrac{1}{p+1}\, X \lrcorner\, d\psi \Big)\;.
\end{aligned}$$

From this equation it becomes clear that $\widehat{\psi}$ is parallel, if and only if the two brackets vanish, i.e. if and only if the form ψ on M is a special Killing form. □

We already know that on Sasakian manifolds, the Killing 1-form ξ^* together with all forms $\xi^* \wedge (d\xi^*)^k$ are special Killing forms. As an immediate corollary of Lemma 3.2.1 we see that a similar statement is true for all manifolds admitting special Killing forms of odd degree. Note that we have to assume the Killing form ψ to be of odd degree, since otherwise $d\psi \wedge d\psi = 0$ and we could not obtain a new Killing form.

Lemma 3.2.2 *Let ψ be a special Killing form of odd degree p, then all the forms*

$$\psi_k \;:=\; \psi \wedge (d\psi)^k \qquad\qquad k = 0, \ldots$$

are special Killing forms of degree $p + k(p+1)$.

Proof. Let $\widehat{\psi}$ be the parallel form associated with the special Killing form ψ. Then the form $\widehat{\psi_k}$ associated to ψ_k turns out to be $\frac{(p+1)^k}{k+1}\, \widehat{\psi}^{\,k+1}$ which is of course again parallel. Hence, ψ_k is a special Killing form. □

In the proof of the lemma we have used that the power of the associated form $\widehat{\psi}$ is again parallel and can be written as associated form for some other special Killing form ψ_k. The following lemma will show that this is a general fact, i.e. we have an simple characterization of all parallel forms on the metric cone. It turns out that there are no other parallel forms on the cone as the ones corresponding to special Killing forms on the base manifold.

Lemma 3.2.3 *Let ω be a form on the metric cone $\widehat{M}$. Then ω is parallel with respect to $\widehat{\nabla}$ if and only if there exists a special Killing form ψ on M such that $\omega = \hat{\psi}$.*

Proof. We know already that $\widehat{\psi}$ is parallel on the metric cone, provided that ψ is a special Killing form on M. It remains to verify the opposite direction. Assuming ω to be a parallel form on the cone we write it as

$$\omega \;=\; \omega_0 \;+\; dr \wedge \omega_1 \;,$$

where we consider ω_0 and ω_1 as a r-dependent family of forms on M. It is clear that ω is parallel in the radial direction ∂_r if and only if the same is true for the two forms ω_0 and ω_1. Let $\eta = \eta(r)$ be any horizontal p-form on $\widehat{M}$ considered as family of forms on M. Locally we can write $\eta = \sum r^p f_I(r,x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, with multi index $I = (i_1, \ldots, i_p)$. Then η is parallel in radial direction if and only if

$$\begin{aligned} 0 \;&=\; \partial_r(r^p f_I(r,x)) \;+\; r^p f_I(r,x)\,(-\tfrac{p}{r}\,) \\ &=\; p\, r^{p-1} f_I(r,x) \;+\; r^p\, \partial_r(f_I(r,x)) \;-\; r^{p-1} p\, f_I(r,x) \\ &=\; r^p\, \partial_r(f_I(r,x)) \;. \end{aligned}$$

It follows that $f_I(r,x)$ does not depend on r. Hence, we can write $\eta = r^p\, \eta_0$, where η_0 is a p-form on M. In particular, we have $\omega_0 = r^{p+1}\omega_0^M$ and $\omega_1 = r^p\, \omega_1^M$, where ω_0^M and ω_1^M are forms on M. Next, we consider the covariant derivative of the parallel form ω in direction of a horizontal vector field X. Here we obtain

$$\begin{aligned} \widehat{\nabla}_X\, \omega \;&=\; r^{p+1} \widehat{\nabla}_X\, \omega_0^M \;+\; r^{p+1}\, X^* \wedge \omega_1^M \;+\; r^p dr \wedge \widehat{\nabla}_X\, \omega_1^M \\ &=\; r^{p+1} \big(\nabla_X\, \omega_0^M \;-\; \tfrac{1}{r}\, dr \wedge (X \lrcorner\, \omega_0^M)\big) \\ &\qquad\qquad + r^{p+1} X^* \wedge \omega_1^M \;+\; r^p dr \wedge \nabla_X\, \omega_1^M \;. \end{aligned}$$

From this we conclude that the form $\omega = r^p\, dr \wedge \omega_1^M + r^{p+1}\omega_0^M$ is parallel if and only if the following two equations are satisfied for all vector fields X on M

$$\nabla_X\, \omega_1^M \;=\; X \lrcorner\, \omega_0^M \qquad \text{and} \qquad \nabla_X\, \omega_0^M \;=\; -\,X^* \wedge \omega_1^M \;. \tag{3.2.5}$$

Using these equations we immediately find:

$$d\,\omega_0^M \;=\; 0 \;=\; d^*\omega_1^M, \quad d\,\omega_1^M = (p+1)\,\omega_0^M, \quad d^*\omega_0^M = (n-p)\omega_0^M \;.$$

In particular, we have $\Delta\,\omega_1^M = (p+1)(n-p)\,\omega_1^M$ and it is clear that $\omega = \widehat{\psi}$ for the special Killing p-form $\psi = \omega_1^M$. $\square$

Up to now we know that the map $\psi \mapsto \widehat{\psi}$ defines a 1-1-correspondence between special Killing p-forms on M and parallel $(p+1)$-forms on the metric cone $\widehat{M}$. We will use this fact to describe manifolds admitting special Killing forms. Let M be a compact oriented simply connected manifold, then the metric cone $\widehat{M}$ is either flat, and the manifold M has to be isometric to the standard sphere, or the cone is irreducible (c.f. [Bä93] or [G79]). In the latter case we know from the holonomy theorem of M. Berger that $\widehat{M}$ is either symmetric or its holonomy is one of the the following groups: $\mathrm{SO}(m)$, $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$, G_2 or Spin_7. An irreducible symmetric space as well as a manifold with holonomy $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ is automatically Einstein (c.f. [Be]). But it follows from the O'Neill formulas applied to the cone, that $\widehat{\mathrm{Ric}}\,(\partial_r,\,\partial_r) = 0$, i.e. the metric cone can only be Einstein if it is Ricci-flat. In this case the symmetric space has to be flat and the holonomy $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ restricts further to $\mathrm{Sp}(m)$ (this again can be found in c.f. [Be]).

Let $(M,\,g)$ be a compact oriented simply connected manifold not isometric to the sphere. If ψ is a special Killing form on M then the metric cone $\widehat{M}$ is an irreducible manifold with a parallel form $\widehat{\psi}$. Since any parallel form induces a holonomy reduction, we see that the above list of possible holonomies is further reduced to $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2, or Spin_7. We will now go through this list and determine what are the possible parallel forms and how they translate into special Killing forms on M. The description of possible parallel forms can be found in [Be]. The only exception is the holonomy $\mathrm{Sp}(m)$. Nevertheless, the parallel forms can be described using the realization of $\mathrm{Sp}(m)$–representation due to H. Weyl (the result is also contained in [Fu58]). Concerning the translation from special holonomy on $\widehat{M}$ to special geometric structures on M we refer to [Bä93], where the explicit constructions are described.

The first case, i.e. holonomy $\mathrm{U}(m)$, is equivalent to $\widehat{M}$ being a Kähler manifold. In this case all parallel forms are linear combinations of powers of the Kähler form. On the other hand, it is well-known that $\widehat{M}$ is Kähler, if and only if M is a Sasakian manifold. If the Killing vector field ξ defines the Sasakian structure on M, then $\widehat{\xi} = r dr \wedge \xi^* + \frac{r^2}{2} d\xi^*$ defines the Kähler form on $\widehat{M}$. Hence, all special Killing forms on a Sasakian manifold are spanned by the forms ω_k given in Proposition 2.3.2, and they all correspond to the powers of the Kähler form on $\widehat{M}$.

In the next case, $\widehat{M}$ has holonomy $\mathrm{SU}(m)$ and equivalently is Ricci-flat and Kähler. In this case, there are two additional parallel forms given by the complex volume form and its conjugate. As real forms we obtain the real part resp. the imaginary part of the complex volume form. Because of the O'Neill formulas, the cone is Ricci-flat, if and only if the base manifold is Einstein, i.e. in this case our manifold is Einstein-Sasakian. As special Killing forms we have the forms ω_k and two additional forms of degree m. The two extra forms can also be described using the Killing spinors of an Einstein-Sasakian manifold.

In the third case, $\widehat{M}$ has holonomy $\mathrm{Sp}(m)$ and is by definition a hyper-Kähler manifold, i.e. there are three Kähler forms compatible with the metric and such that the corresponding complex structures satisfy the quaternionic relations. In this case all parallel forms are linear combinations of wedge products of powers of the three Kähler forms. The metric cone is hyper-Kähler if and only if the base manifold has a 3-Sasakian structure and the possible special Killing forms are described by

Proposition 3.2.4 *Let $(M,\, g)$ be a manifold with a 3-Sasakian structure defined by the Killing 1–forms η_1, η_2 and η_3. Then all special Killing forms on M are linear combinations of the forms $\psi_{a,\,b,\,c}$ defined for any integers (a,b,c) by*

$$\begin{aligned}\psi_{a,\,b,\,c} := \tfrac{a}{a+b+c}\,[\eta_1 \wedge (d\eta_1)^{a-1}] \wedge (d\eta_2)^b \wedge (d\eta_3)^c \\ + \tfrac{b}{a+b+c}\,(d\eta_1)^a \wedge [\eta_2 \wedge (d\eta_2)^{b-1}] \wedge (d\eta_3)^c \\ + \tfrac{c}{a+b+c}\,(d\eta_1)^a \wedge (d\eta_2)^b \wedge [\eta_3 \wedge (d\eta_3)^{c-1}]\ .\end{aligned}$$

Proof. Let ϕ_i be the parallel 2-form associated with the Sasakian structure ξ_i, for $i = 1,2,3$, i.e.

$$\phi_i \;=\; r\,dr \wedge \eta_i \;+\; \tfrac{r^2}{2}\,d\eta_i\ .$$

Then it follows from a simple computation that $\phi_1^a \wedge \phi_2^b \wedge \phi_3^c$ is a parallel form which is, up to a factor, associated to the form $\psi_{a,b,c}$ defined above. □

Next, we have to consider the two exceptional holonomies G_2 resp. Spin_7. These holonomies are defined by the existence of a parallel 3– resp. 4–form ψ and the only non-trivial parallel forms on such a manifold are the linear combinations of ψ and $*\psi$. The metric cone has holonomy G_2 if and only if the base manifold is a 6-dimensional nearly Kähler manifold. Here, the parallel 3-form ψ translates into the Kähler form ω and the parallel 4-form $*\psi$ translates, up to a constant, into the 3-form $*d\omega$. To make this more precise, we note the following simple fact

Lemma 3.2.5 *Let ω be a p-form on M considered as p-form on the metric cone $\widehat{M}$. Then the Hodge star operators of M and $\widehat{M}$ are related by*

$$*_{\widehat{M}}\,\omega \;=\; r^{n-2p}(*_M \omega) \wedge dr\ .$$

Now, back to the nearly Kähler case, let $\psi = r^2 dr \wedge \omega + \frac{r^3}{3} d\omega$ be the parallel 3-form associated with the Kähler form ω. As in the proof of Lemma 3.2.3 we conclude

$\Delta\omega = 12\omega$. Hence, the scalar curvature s_M of the 6-dimensional nearly Kähler manifold is normalized to $s_M = 30$. Applying the above lemma yields

$$
\begin{aligned}
*_{\widehat{M}}\psi \;&=\; r^2 *_{\widehat{M}}(dr \wedge \omega) \;+\; \tfrac{r^3}{3} *_{\widehat{M}}(d\omega) \\
&=\; r^2\, \partial_r \lrcorner\, (*_{\widehat{M}}\omega) \;+\; \tfrac{r^3}{3} *_{\widehat{M}}(d\omega) \;=\; r^4 *_M \omega \;+\; \tfrac{r^3}{3}(*_M\, d\omega) \wedge dr\ .
\end{aligned}
$$

Since $\Delta\,\omega = 12\,\omega$ and $d^*\omega = 0$ it follows $d^*d\omega = -*_M\, d *_M d\omega = 12\omega$ and we obtain $d(*_M d\omega) = -12 *_M \omega$. Substituting this into the above equation, we find

$$
*_{\widehat{M}}\psi \;=\; -\tfrac{r^4}{12}\, d(*_M d\omega) \;-\; \tfrac{r^3}{3}\, dr \wedge (*_M d\omega)\ .
$$

From where we conclude that $*_M d\omega$ is the special Killing form on the nearly Kähler manifold M corresponding to the parallel 4-form $-3 *_{\widehat{M}} \psi$ on $\widehat{M}$.

Finally we have to consider the case of holonomy Spin_7. The metric cone has holonomy Spin_7 if and only if M is a 7-dimensional manifold with a weak G_2-structure. Here the parallel 4-form ψ on the cone is self-dual, i.e. $*\psi = \psi$, and the corresponding special Killing form is just the 3-form defining the weak G_2-structure.

Summarizing our description of compact manifolds with special Killing forms we have the following

Theorem 3.2.6 *Let (M^n, g) be a compact, simply connected manifold admitting a special Killing form. Then M is either isometric to S^n or M is a Sasakian, 3-Sasakian, nearly Kähler or weak G_2–manifold. Moreover, on these manifolds any special Killing form is a linear combination of the Killing forms described above.*

Chapter 4

The space of conformal Killing forms

In this chapter we will prove a sharp upper bound on the dimension of the space of conformal Killing spinors. The idea is to construct a vector bundle together with a connection, called *Killing connection*, such that conformal Killing forms are in a 1-1-correspondence to parallel sections for this connection. It then follows immediately that the dimension of the space of conformal Killing forms is bounded by the rank of the constructed vector bundle. Moreover, it turns out that this rank is exactly the number of linearly independent conformal Killing forms on the standard sphere.

4.1 The Killing connection

By definition, the covariant derivative of a conformal Killing p–form ψ involves $d\psi$ and $d^*\psi$. Hence, the first step will be a computation of the covariant derivative of these sections. For $2p \neq n$ we obtain an expression involving only zero order terms and $\Delta\psi$ (c.f. Corollary 4.1.4). The case $2p = n$ has to be treated separately but leads eventually to the same result. The next step will be a computation of the covariant derivative of $\Delta\psi$. Here we obtain an expression involving zero order terms and the sections $d\psi$ and $d^*\psi$ (c.f. Proposition 4.4).

This can be formulated in the following way. Let $\hat{\psi} := (\psi, d\psi, d^*\psi, \Delta\psi)$, then $\hat{\psi}$ is a section of $\mathcal{E}^p(M) := \Lambda^p T^*M \oplus \Lambda^{p+1}T^*M \oplus \Lambda^{p-1}T^*M \oplus \Lambda^p T^*M$ and we have $\nabla_X \hat{\psi} = A(X)\,\hat{\psi}$, where $A(X)$ is a certain 4×4-matrix with coefficients which are endomorphisms of the form bundle, depending on the vector field X. The Killing connection $\widetilde{\nabla}$ is then a connection on $\mathcal{E}^p(M)$ defined as $\widetilde{\nabla}_X := \nabla_X - A(X)$ and the conformal Killing forms are by definition the first component of parallel sections of $\mathcal{E}^p(M)$.

We start with defining the four components of the Killing connection on $\mathcal{E}^p(M)$ under the assumption that $2\,p \neq n$. In the rest of this section and in Section 4.4 we will prove that conformal Killing forms are parallel for this connection. Let $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ be a section of $\mathcal{E}^p(M)$. Then the first component is defined by

$$(\widetilde{\nabla}_X \Psi)_1 \;=\; \nabla_X \psi_1 \;-\; \tfrac{1}{p+1}\, X \lrcorner\, \psi_2 \;+\; \tfrac{1}{n-p+1}\, X \wedge \psi_3 \;.$$

Let ψ be a conformal Killing form. Then it is clear from the definition that we have $(\widetilde{\nabla}_X \Psi)_1 = 0$ for $\Psi = \widehat{\psi} = (\psi,\, d\psi,\, d^*\psi,\, \Delta\psi)$. In order to define the second and third components of the covariant derivative, we introduce the following notation: $R^+(X) := \sum e_j \wedge R_{X, e_j}$ and $R^-(X) := \sum e_j \lrcorner\, R_{X,\, e_j}$, where $\{e_i\}$ is a local ortho-normal basis. We then define:

$$(\widetilde{\nabla}_X \Psi)_2 \;:=\; \nabla_X(\psi_2) \;-\; \tfrac{p+1}{p}\, R^+(X)\, \psi_1 \;+\; \tfrac{p+1}{n-2p}\, X \wedge \psi_4 \;-\; \tfrac{(p+1)^2}{p(n-2p)}\, X \wedge 2q(R)\, \psi_1 \;,$$

$$(\widetilde{\nabla}_X \Psi)_3 \;:=\; \nabla_X(\psi_3) \;+\; \tfrac{n-p+1}{n-p}\, R^-(X)\, \psi_1 \;+\; \tfrac{n-p+1}{n-2p}\, X \lrcorner\, \psi_4 \;-\; \tfrac{(n-p+1)^2}{(n-p)(n-2p)}\, X \lrcorner\, 2q(R)\, \psi_1 \;.$$

The fourth component is a rather lengthy expression, which only simplifies for Einstein manifolds. Here we still need the following notations: $(\delta R)_X := -\sum (\nabla_{e_i} R)_{e_i,\, X}$, where we consider R as an endomorphism-valued 2-form and $\{e_i\}$ is again a local ortho-normal basis. Moreover, we define $\delta R^+\, \psi := \sum e_j \wedge (\delta R)_{e_j}\, \psi$ and $\delta R^-\, \psi := \sum e_j \lrcorner\, (\delta R)_{e_j}\, \psi$. Using this notation, the fourth component of the Killing connection is defined as

$$\begin{aligned}
(\widetilde{\nabla}_X \Psi)_4 \;:=\; & \nabla_X(\psi_4) \;-\; \tfrac{1}{p}\, X \lrcorner \left(\tfrac{1}{p+1}\, \mathrm{Ric}\,(\psi_2) \;+\; \tfrac{p-1}{p+1}\, 2q(R)\, \psi_2 \;+\; \delta R^+\, \psi_1 \right) \\
& + \tfrac{1}{n-p}\, X \wedge \left(\tfrac{1}{n-p+1}\, [s\, \psi_3 \;-\; \mathrm{Ric}\,(\psi_3)] \;+\; \tfrac{n-p-1}{n-p+1}\, 2q(R)\, \psi_3 \;+\; \delta R^-\, \psi_1 \right) \\
& - 2q(\nabla_X R)\, \psi_1 \;+\; \delta R(X)\, \psi_1 \;.
\end{aligned}$$

For Einstein manifolds, Ric acts as a scalar multiple of the identity and the endomorphisms $(\delta R)_X$ resp. $\delta R^{\pm}$ vanish (c.f. Lemma 4.4.3). The simplified expression for $(\widetilde{\nabla}_X \psi)_4$ is contained in Corollary 7.1.1.

We will now prove that the second and third component of the Killing connection vanish on conformal Killing forms. This is a straightforward calculation starting from the definition. The proof of the corresponding statement for the fourth component is contained in Section 4.4. It is also an elementary but somewhat tedious calculation.

We start with proving an expression for $\nabla_X(d\psi)$ in terms of $d\,d^*\psi$ and the zero order term $R^+(X)\,\psi$. Replacing $d\,d^*\psi$ by $d^* d\,\psi$ leads to the additional zero order term $2q(R)\,\psi$.

Proposition 4.1.1 *Let ψ be a conformal Killing p-form, then for all vector fields X*

$$\begin{aligned}
\nabla_X(d\,\psi) \;&=\; \tfrac{p+1}{p}\, R^+(X)\, \psi \;+\; \tfrac{p+1}{p(n-p+1)}\, X \wedge d\,d^*\, \psi \\
&=\; \tfrac{p+1}{p}\, R^+(X)\, \psi \;-\; \tfrac{1}{n-p}\, X \wedge d^* d\, \psi \;+\; \tfrac{p+1}{p(n-p)}\, X \wedge 2q(R)\, \psi \;.
\end{aligned}$$

Proof. Since ψ is a conformal Killing p-form, it satisfies equation (1.1.4). Taking the covariant derivative with respect to X leads to

$$\begin{aligned}\nabla_X(\nabla_Y \psi) &= \tfrac{1}{p+1}\left((\nabla_X Y) \lrcorner\, d\psi \;+\; Y \lrcorner\, \nabla_X(d\psi)\right)\\ &\qquad - \tfrac{1}{n-p+1}\left((\nabla_X Y) \wedge d^*\psi \;+\; Y \wedge \nabla_X(d^*\psi)\right) .\end{aligned}$$

Hence

$$\begin{aligned}\nabla^2_{X,Y}(\psi) &:= \nabla_X(\nabla_Y \psi) - \nabla_{\nabla_X Y}\psi\\ &= \tfrac{1}{p+1}\, Y \lrcorner\, \nabla_X(d\psi) \;-\; \tfrac{1}{n-p+1}\, Y \wedge \nabla_X(d^*\psi) .\end{aligned}$$

We note that taking the trace in the last expression proves the Weitzenböck formula (1.1.7) on conformal Killing forms. However, we will use it here to compute the curvature $R_{X,Y} = \nabla^2_{X,Y} - \nabla^2_{Y,X}$. We get

$$\begin{aligned}R_{X,Y}\psi &= \tfrac{1}{p+1}\left(Y \lrcorner\, \nabla_X(d\psi) \;-\; X \lrcorner\, \nabla_Y(d\psi)\right)\\ &\qquad - \tfrac{1}{n-p+1}\left(Y \wedge \nabla_X(d^*\psi) \;-\; X \wedge \nabla_Y(d^*\psi)\right) .\end{aligned}$$

Applying this equation, we obtain:

$$\begin{aligned}R^+(X)\psi &= \sum e_j \wedge R_{X,e_j}\psi\\ &= \tfrac{1}{p+1}\sum e_j \wedge \left[\, e_j \lrcorner\, \nabla_X(d\psi) \;-\; X \lrcorner\, \nabla_{e_j}(d\psi)\right]\\ &\qquad - \tfrac{1}{n-p+1}\sum e_j \wedge \left[e_j \wedge \nabla_X(d^*\psi) \;-\; X \wedge \nabla_{e_j}(d^*\psi)\right]\\ &= \nabla_X(d\psi) \;-\; \tfrac{1}{p+1}\sum e_j \wedge \left[X \lrcorner\, \nabla_{e_j}(d\psi)\right]\\ &\qquad + \tfrac{1}{n-p+1}\sum e_j \wedge \left[X \wedge \nabla_{e_j}(d^*\psi)\right]\\ &= \nabla_X(d\psi) \;+\; \tfrac{1}{p+1}\sum X \lrcorner\, \left[e_j \wedge \nabla_{e_j}(d\psi)\right] \;-\; \tfrac{1}{p+1}\nabla_X(d\psi)\\ &\qquad - \tfrac{1}{n-p+1}\, X \wedge dd^*\psi\\ &= \tfrac{p}{p+1}\nabla_X(d\psi) \;-\; \tfrac{1}{n-p+1}\, X \wedge dd^*\psi .\end{aligned}$$

This proves the first equation for $\nabla_X(d\psi)$. To prove the second equation we use the Weitzenböck formula (1.1.8) and the assumption that ψ is a conformal Killing form. Hence

$$\begin{aligned}\nabla_X(d\psi) &= \tfrac{p+1}{p} R^+(X)\psi \;+\; \tfrac{p+1}{p(n-p+1)}\, X \wedge dd^*\psi\\ &= \tfrac{p+1}{p} R^+(X)\psi \;+\; \tfrac{p+1}{p}\, X \wedge \left(\tfrac{1}{n-p}\, 2q(R)\psi \;-\; \tfrac{p}{(n-p)(p+1)} d^*d\psi\right)\\ &= \tfrac{p+1}{p} R^+(X)\psi \;-\; \tfrac{1}{n-p}\, X \wedge d^*d\psi \;+\; \tfrac{p+1}{p(n-p)}\, X \wedge 2q(R)\psi . \quad \Box\end{aligned}$$

Next, we do the same for $\nabla_X(d^*\psi)$ and obtain an expression involving $d^*d\psi$ and the zero order term $R^-(X)\psi$. Again, the change from $dd^*\psi$ to $d^*d\psi$ leads to the additional zero order term $2q(R)\psi$.

Proposition 4.1.2 *Let ψ be a conformal Killing p-form, then for all vector fields X:*

$$\begin{aligned}\nabla_X(d^*\psi) &= -\tfrac{n-p+1}{n-p}R^-(X)\psi - \tfrac{n-p+1}{(p+1)(n-p)} X \lrcorner d^*d\psi \\ &= -\tfrac{n-p+1}{n-p}R^-(X)\psi + \tfrac{1}{p} X \lrcorner dd^*\psi - \tfrac{n-p+1}{p(n-p)} X \lrcorner 2q(R)\psi \ .\end{aligned}$$

Proof. We use the expression for $R_{X,Y}\psi$ which we derived in the proof of Proposition 4.1.1. Here we have to compute $R^-(X)\psi$.

$$\begin{aligned}R^-(X)\psi &= \sum e_j \lrcorner R_{X,e_j}\psi \\ &= \tfrac{1}{p+1}\sum e_j \lrcorner [e_j \lrcorner \nabla_X(d\psi) - X \lrcorner \nabla_{e_j}(d\psi)] \\ &\qquad - \tfrac{1}{n-p+1}\sum e_j \lrcorner [e_j \wedge \nabla_X(d^*\psi) - X \wedge \nabla_{e_j}(d^*\psi)] \\ &= -\tfrac{1}{p+1} X \lrcorner d^*d\psi - \tfrac{n}{n-p+1}\nabla_X(d^*\psi) + \tfrac{p-1}{n-p+1}\nabla_X(d^*\psi) + \tfrac{1}{n-p+1}\nabla_X(d^*\psi) \\ &= -\tfrac{1}{p+1} X \lrcorner d^*d\psi - \tfrac{n-p}{n-p+1}\nabla_X(d^*\psi) \ .\end{aligned}$$

This proves the first equation for $\nabla_X(d\psi)$. To prove the second one we use the Weitzenböck formula (1.1.8) and the assumption that ψ is a conformal Killing form. Hence,

$$\begin{aligned}\nabla_X(d^*\psi) &= -\tfrac{n-p+1}{n-p}R^-(X)\psi - \tfrac{n-p+1}{(n-p)(p+1)} X \lrcorner d^*d\psi \\ &= -\tfrac{n-p+1}{n-p}R^-(X)\psi - \tfrac{n-p+1}{n-p} X \lrcorner \left(\tfrac{1}{p}2q(R)\psi - \tfrac{n-p}{p(n-p+1)}dd^*\psi\right) \\ &= -\tfrac{n-p+1}{n-p}R^-(X)\psi + \tfrac{1}{p} X \lrcorner dd^*\psi - \tfrac{n-p+1}{p(n-p)} X \lrcorner 2q(R)\psi \ . \quad \square\end{aligned}$$

From the result and the proof of the preceding two propositions, we obtain in the case where ψ is a Killing or $*$-Killing form a nice formula for $\nabla^2_{X,Y}\psi$. This will be helpful especially in Section 6.1. Moreover, starting at the same point, we will derive in the next section a curvature condition for conformal Killing forms. Here we have

Corollary 4.1.3 *Let (M^n, g) be a Riemannian manifold with a conformal Killing p–form ψ. Then ψ satisfies for any vector fields X, Y the equations:*

$$\begin{aligned}\nabla^2_{X,Y}\psi &= \tfrac{1}{p} Y \lrcorner R^+(X)\psi \qquad &&\textit{if} \quad d^*\psi = 0 \ , \\ \nabla^2_{X,Y}\psi &= \tfrac{1}{n-p} Y \wedge R^-(X)\psi \qquad &&\textit{if} \quad d\psi = 0 \ .\end{aligned}$$

Finally, we want to prove that the second and third component of the Killing connection vanishes for conformal Killing forms. For this, we still have to replace the terms $d^* d\psi$ resp. $d\, d^*\psi$ appearing in the formulas of Propositions 4.1.1 resp. 4.1.2 by $\Delta\psi$. At this point the assumption $2p \neq n$ is necessary since we want to invert the Weitzenböck formulas (1.1.7) and (1.1.8).

Corollary 4.1.4 *Let ψ be a conformal Killing p–form, with $2\,p \neq n$ and let $\Psi = \widehat{\psi}$ be the associated section of $\mathcal{E}^p(M)$. Then $(\widetilde{\nabla}\Psi)_2 = 0$ and $(\widetilde{\nabla}\Psi)_3 = 0$, i.e. the second and the third component of the Killing connection vanish for the conformal Killing form ψ.*

Proof. First of all, we recall that the vanishing of the second and third component of the Killing connection is equivalent to the following two equations.

$$
\begin{aligned}
\nabla_X(d\psi) &= \tfrac{p+1}{p}\, R^+(X)\,\psi \;-\; \tfrac{p+1}{n-2p}\, X \wedge \Delta\,\psi \;+\; \tfrac{(p+1)^2}{p(n-2p)}\, X \wedge 2q(R)\,\psi\;, \\
\nabla_X(d^*\psi) &= -\tfrac{n-p+1}{n-p}\, R^-(X)\,\psi \;-\; \tfrac{n-p+1}{n-2p}\, X \lrcorner\, \Delta\,\psi \;+\; \tfrac{(n-p+1)^2}{(n-p)(n-2p)}\, X \lrcorner\, 2q(R)\,\psi\;.
\end{aligned}
$$

If ψ is a conformal Killing p-form with $2p \neq n$, we can invert the two Weitzenböck formulas (1.1.7) and (1.1.8) in order to obtain expressions for $d^*d\,\psi$ and $dd^*\psi$ in terms of $\nabla^*\nabla\psi$ and $2q(R)\psi$. We get

$$
d^*d\,\psi \;=\; \tfrac{(n-p)(p+1)}{n-2p}\,\nabla^*\nabla\,\psi \;-\; \tfrac{p+1}{n-2p}\,2q(R)\,\psi\;, \tag{4.1.1}
$$

$$
d\,d^*\,\psi \;=\; -\tfrac{p(n-p+1)}{n-2p}\,\nabla^*\nabla\,\psi \;+\; \tfrac{n-p+1}{n-2p}\,2q(R)\,\psi\;. \tag{4.1.2}
$$

Applying these equations and the classical Weitzenböck formula $\Delta = \nabla^*\nabla + 2q(R)$ yields

$$
\begin{aligned}
\nabla_X(d\psi) &= \tfrac{p+1}{p}\, R^+(X)\,\psi \;+\; \tfrac{p+1}{p(n-p+1)}\, X \wedge \Big(-\tfrac{p(n-p+1)}{n-2p}\,\nabla^*\nabla\,\psi \;+\; \tfrac{n-p+1}{n-2p}\,2q(R)\,\psi\Big) \\
&= \tfrac{p+1}{p}\, R^+(X)\,\psi \;-\; \tfrac{p+1}{n-2p}\, X \wedge \nabla^*\nabla\,\psi \;+\; \tfrac{p+1}{p(n-2p)}\, X \wedge 2q(R)\,\psi \\
&= \tfrac{p+1}{p}\, R^+(X)\,\psi \;-\; \tfrac{p+1}{n-2p}\, X \wedge \Delta\,\psi \;+\; \tfrac{(p+1)^2}{p(n-2p)}\, X \wedge 2q(R)\,\psi\;.
\end{aligned}
$$

To prove the equation for $\nabla_X(d^*\psi)$ we use (4.1.1) and proceed similarly.

$$
\begin{aligned}
\nabla_X(d^*\psi) &= -\tfrac{n-p+1}{n-p}\, R^-(X)\,\psi \;-\; \tfrac{n-p+1}{(p+1)(n-p)}\, X \lrcorner\, \Big(\tfrac{(n-p)(p+1)}{n-2p}\,\nabla^*\nabla\,\psi \;-\; \tfrac{p+1}{n-2p}\,2q(R)\,\psi\Big) \\
&= -\tfrac{n-p+1}{n-p}\, R^-(X)\,\psi \;-\; \tfrac{n-p+1}{n-2p}\, X \lrcorner\, \nabla^*\nabla\,\psi \;+\; \tfrac{n-p+1}{(n-p)(n-2p)}\, X \lrcorner\, 2q(R)\,\psi \\
&= -\tfrac{n-p+1}{n-p}\, R^-(X)\,\psi \;-\; \tfrac{n-p+1}{n-2p}\, X \lrcorner\, \Delta\,\psi \;+\; \tfrac{(n-p+1)^2}{(n-p)(n-2p)}\, X \lrcorner\, 2q(R)\,\psi\;. \quad \square
\end{aligned}
$$

So far, we know that for a conformal Killing form ψ, the first three components of the Killing connection have to vanish. It still remains to show that the same is true for the fourth component. The corresponding calculation of $\nabla_X(\Delta\psi)$ is straightforward but rather lengthy. Hence, we postpone it to Section 4.4, where it is given in the proof of Proposition 4.4.1.

4.2 The curvature condition

In this section we digress from the proof of the dimension bound to discuss a remarkable formula for the Riemannian curvature applied to conformal Killing forms. This formula has several applications, e.g. it poses strong restrictions for conformal Killing forms which are at the heart of all the known non-existence theorems. In fact our curvature condition is already contained in [Ka68]. Nevertheless, we present it here in a rather different and much shorter form.

In the proof of Propositions 4.1.1 we obtained an expression for the curvature $R_{X,Y}$ applied to a conformal Killing tensor ψ. This formula still involves $\nabla_X(d\psi)$ and $\nabla_X(d^*\psi)$. But substituting these terms by applying Propositions 4.1.1 and 4.1.2 we obtain

Proposition 4.2.1 *Let (M^n, g) be a Riemannian manifold with a conformal Killing p–form ψ, then for any vector fields X, Y the following equation is satisfied:*

$$\begin{aligned} R(X,Y)\,\psi \;&=\; \tfrac{1}{p(n-p)}\,(Y \wedge X \lrcorner \;-\; X \wedge Y \lrcorner\,)\,2q(R)\,\psi \\ &\quad -\tfrac{1}{p}\big(X \lrcorner\, R^+(Y) \;-\; Y \lrcorner\, R^+(X)\big)\,\psi \;-\; \tfrac{1}{n-p}\big(X \wedge R^-(Y) \;-\; Y \wedge R^-(X)\big)\,\psi\;. \end{aligned}$$

If we assume in addition that the conformal Killing form is either closed or coclosed then the curvature formula becomes even simpler. Using the curvature condition of Proposition 4.2.1 we obtain two equivalent pairs of equations for closed resp. coclosed conformal Killing forms.

Proposition 4.2.2 *Let (M^n, g) be a Riemannian manifold with a conformal Killing p–form ψ. Then for any vector fields X, Y the following equations are satisfied:*

$$\begin{aligned} R(X,Y)\,\psi \;&=\; -\tfrac{1}{p}\,(X \lrcorner\, R^+(Y) \;-\; Y \lrcorner\, R^+(X))\,\psi\;, \qquad &&\textit{if} \quad d^*\psi = 0\;, \\ R(X,Y)\,\psi \;&=\; -\tfrac{1}{n-p}\,(X \wedge R^-(Y) \;-\; Y \wedge R^-(X))\,\psi\;, \qquad &&\textit{if} \quad d\psi = 0\;. \end{aligned}$$

Corollary 4.2.3 *Let (M^n, g) be a Riemannian manifold with a conformal Killing p–form ψ. Then for any vector fields X, Y the following equations are satisfied:*

$$\begin{aligned} \tfrac{1}{p}\,(Y \wedge X \lrcorner \;-\; X \wedge Y \lrcorner\,)\,2q(R)\,\psi \;&=\; (X \wedge R^-(Y) \;-\; Y \wedge R^-(X))\,\psi \quad &&\textit{if} \quad d^*\psi = 0\;, \\ \tfrac{1}{n-p}\,(Y \wedge X \lrcorner \;-\; X \wedge Y \lrcorner\,)\,2q(R)\,\psi \;&=\; (X \lrcorner\, R^+(Y) \;-\; Y \lrcorner\, R^+(X))\,\psi \quad &&\textit{if} \quad d\psi = 0\;. \end{aligned}$$

Considering $R(\cdot,\cdot)\,\psi$ as a section of $\Lambda^p(T^*M)\otimes\Lambda^2(T^*M)$, we can write the above curvature condition in a much shorter form. Indeed, we have a decomposition of the tensor product $\Lambda^p(T^*M)\otimes\Lambda^2(T^*M)$ corresponding to the following isomorphism of $SO(n)$–representations:

$$\Lambda^pV^*\otimes\Lambda^2V^* \;\cong\; \Lambda^pV^*\oplus\Lambda^{p+1,1}V^*\oplus\Lambda^{p-1,1}V^*\oplus\Lambda^{p+2}V^*\oplus\Lambda^{p-2}V^*\oplus\Lambda^{p,2}V^*\ . \quad (4.2.3)$$

The notation is the same as in (1.1.1) and $\Lambda^{p,2}V^*$ is defined as the irreducible representation which has as highest weight the sum of the highest weights of Λ^pV^* and Λ^2V^*. In Section 4.6 we give explicit formulas for the projections onto the six summands on the right hand side of (4.2.3), denoted as pr_{Λ^p}, $\mathrm{pr}_{\Lambda^{p\pm1,1}}$, $\mathrm{pr}_{\Lambda^{p\pm2}}$ and $\mathrm{pr}_{\Lambda^{p,2}}$. In particular, we will see that the projections of $R(\cdot,\cdot)\,\psi$ onto the summands $\Lambda^{p\pm2}T^*M$ vanish because of the Bianchi identity and the projection of $R(\cdot,\cdot)\,\psi$ onto Λ^pT^*M is precisely $2q(R)\,\psi$. Using this notation it follows from the results of Section 4.6 (c.f. Lemma 4.6.5, 4.2.2 and 4.2.3) that we can reformulate the curvature condition of Proposition 4.2.1 as well as the the specialized equations from Proposition 4.2.2 in the following form

Corollary 4.2.4 *Let $(M^n,\,g)$ be a Riemannian manifold with a conformal Killing form ψ. Then*

$$\mathrm{pr}_{\Lambda^{p,2}}(\,R(\cdot,\cdot)\,\psi\,) \;=\; 0\ . \quad (4.2.4)$$

Moreover, if ψ is coclosed, then the additional equation $\mathrm{pr}_{\Lambda^{p-1,1}}(\,R(\cdot,\cdot)\,\psi\,)=0$ is satisfied. Similarly, if ψ is closed then the additional equation $\mathrm{pr}_{\Lambda^{p+1,1}}(\,R(\cdot,\cdot)\,\psi\,)=0$ holds.

There are several situations in which the curvature equation of Proposition 4.2.1 is satisfied for every p–form, e.g. it is true for every 1–form ξ. Here we have $2q(R)\xi=\mathrm{Ric}\,(\xi)$ and an easy calculation shows

$$(X\lrcorner\, R^+(Y)\;-\;Y\lrcorner\, R^+(X))\,\xi \;=\; -\,R_{X,Y}\,\xi$$

$$(X\wedge R^-(Y)\;-\;Y\wedge R^-(X))\,\xi \;=\; (Y\wedge X\lrcorner\;-\;X\wedge Y\lrcorner\,)\,\mathrm{Ric}\,(\xi)$$

Also, it is true for any p–form on a space of constant sectional curvature c. In this case we find $2q(R)=c\,p\,(n-p)\,\mathrm{id}$, $R^+(X)\,\psi=-p\,X\wedge\psi$ and $R^-(X)\,\psi=-(n-p)\,X\lrcorner\,\psi$. Slightly more generally, it is not difficult to prove that the same is true on conformally flat manifolds, i.e. we have

Proposition 4.2.5 *Let $(M^n,\,g)$ be a conformally flat manifold. Then for any p–form ψ the curvature condition 4.2.4 is satisfied, i.e.*

$$\mathrm{pr}_{\Lambda^{p,2}}\,(\,R(\cdot,\cdot)\,\psi\,) \;=\; 0\ .$$

The first non-trivial condition appears for conformal Killing 2-forms, which is further discussed in Section 7.2. Since the decomposition (4.2.3) is again orthogonal we have a pointwise norm estimate similar to the one in Lemma 1.1.1. Using the results of Section 4.6 we find

Proposition 4.2.6 *Let (M^n, g) be a Riemannian manifold and ψ any p–form then*

$$| R(\cdot,\cdot)\psi |^2 \; + \; \tfrac{1}{p(n-p)} | 2q(R) |^2 \;\; \geq \;\; \tfrac{1}{n-p} | R^+(\cdot)\psi |^2 \; + \; \tfrac{1}{p} | R^-(\cdot)\psi |^2 \; .$$

with equality, if ψ is a conformal Killing form. If in addition, the p–form ψ is coclosed, then $p|R(\cdot,\cdot)\psi|^2 = |R^+(\cdot)\psi|^2$. Similarly, if ψ is closed, then $(n-p)|R(\cdot,\cdot)\psi|^2 = |R^-(\cdot)\psi|^2$.

Finally, we note that it is possible to give an alternative proof of the curvature condition of Proposition 4.2.1, using the Killing connection. Indeed, since a conformal Killing form is parallel with respect to the Killing connection, it follows that the curvature of the Killing connection applied to a conformal Killing form has to vanish. This yields four equations corresponding to the four components of the bundle $\mathcal{E}^p(M)$. The first of these equations turns out to be equivalent to the curvature condition of Proposition 4.2.1.

4.3 The dimension bound

In this section we will use the results obtained so far to prove that the space of conformal Killing forms on a connected manifold is finite dimensional. More precisely, we will give a sharp upper bound for its dimension. It is well-known that the twistor operator is elliptic. Hence, the space of conformal Killing forms is finite dimensional on compact manifolds. Indeed the twistor operator T appeared as one of the typical examples of Stein-Weiss operators in the article of T. Branson [Br97], where he in particular proved its ellipticity. Of course it is also no problem to verify that the symbol map, given by the projection $\mathrm{pr}_{\Lambda^{p,1}}$ is injective. It turns out that the twistor operator is elliptic in a much stronger sense, i.e. it has a symbol of finite type.

Lemma 4.3.1 *If $n > 2$ and $\alpha \otimes \psi$ is a decomposable element in the complexified tensor product $(V^* \otimes \Lambda^p V^*) \otimes_{\mathbb{R}} \mathbb{C} \cong (V \otimes_{\mathbb{R}} \mathbb{C})^* \otimes \Lambda^p (V \otimes_{\mathbb{R}} \mathbb{C})^*$ of degree $0 < p < n$ satisfying $\mathrm{pr}_{\Lambda^{p,1}}(\alpha \otimes \psi) = 0$, then $\alpha \otimes \psi = 0$.*

Proof. According to the explicit formula given in Chapter 1, the condition $\mathrm{pr}_{\Lambda^{p,1}}(\alpha \otimes \psi) = 0$ is equivalent to

$$\alpha(v)\,\psi \;\; = \;\; \frac{1}{p+1}\, v \lrcorner\, (\alpha \wedge \psi) \; + \; \frac{1}{n-p+1}\, v^* \wedge (\alpha^\sharp \lrcorner\, \psi) \qquad (4.3.5)$$

for all $v \in V$. Applying $v \lrcorner \alpha^\sharp \lrcorner$ to this equation we get

$$\alpha(v)\, v \lrcorner \alpha^\sharp \lrcorner \psi \;=\; \frac{1}{n-p+1}\, \alpha(v)\, v \lrcorner \alpha^\sharp \lrcorner \psi\,.$$

By assumption, we have $p < n$ and we conclude $v \lrcorner \alpha^\sharp \lrcorner \psi = 0$ for all $v \in V$. So $\alpha^\sharp \lrcorner \psi = 0$, unless $p = 1$. In case $p = 1$, but $n > 2$, we may still conclude $\alpha^\sharp \lrcorner \psi = 0$ by rewriting equation (4.3.5) to read

$$\alpha(v)\, \psi \;=\; \frac{1}{2}\,\alpha(v)\, \psi \;-\; \frac{1}{2}\,\alpha \wedge (v \lrcorner \psi) \;+\; \frac{1}{n}\, v^* \wedge (\alpha^\sharp \lrcorner \psi).$$

Then $\alpha^\sharp \lrcorner \psi = 0$ follows, since we may choose v^* linearly independent from α and ψ.

A completely analogous argument interchanging $\wedge$ and $\lrcorner$ shows $\alpha \wedge \psi = 0$. Consequently, $\mathrm{pr}_{\Lambda^{p,1}}(\alpha \otimes \psi) = 0$ implies $\alpha \wedge \psi = 0 = \alpha^\sharp \lrcorner \psi$ and thus $(\alpha \otimes \psi)(v) = \alpha(v)\, \psi = 0$ for all $v \in V$ by equation (4.3.5). $\square$

Evidently, not only the proof given above breaks down for $n = 2$, but the statement of Lemma 4.3.1 itself becomes wrong. Any two complex isotropic covectors $\alpha,\, \psi \in (V \otimes_{\mathbb{R}} \mathbb{C})^*$ with $\langle \alpha, \psi \rangle \neq 0$ satisfy $\mathrm{pr}_{\Lambda^{p,1}}(\alpha \otimes \psi) = 0$.

It follows already from the theory of differential operators with symbols of finite type that the kernel of the twistor operator, i.e. the space of conformal Killing forms, is finite dimensional on connected Riemannian manifolds. Nevertheless, we will use the Killing connection of Section 4.1 to prove this result. This approach has the advantage of providing an explicit upper bound for the dimension.

Theorem 4.3.2 *Let $(M,\, g)$ be a n–dimensional connected Riemannian manifold and denote with $\mathcal{C}K^p(M)$ the space of conformal Killing p-forms, then*

$$\dim \mathcal{C}K^p(M) \;\leq\; \binom{n+2}{p+1}$$

with equality attained on the standard sphere. Moreover, if a manifold admits the maximal possible number of linear independent conformal Killing forms, then it is conformally flat.

Proof. In Section 1.1.4 we defined the Killing connection on sections of the bundle $\mathcal{E}^p(M)$ in such a way that a p–form ψ is a conformal Killing form, if and only if the associated section $\widehat{\psi} := (\psi,\, d\psi,\, d^*\psi,\, \Delta\psi)$ is parallel with respect to the Killing connection. Hence, the rank of the bundle $\mathcal{E}^p(M)$ is an upper bound on the dimension of the space of conformal Killing forms, i.e.

$$\dim \mathcal{C}K^p(M) \;\leq\; 2\binom{n}{p} \;+\; \binom{n}{p-1} \;+\; \binom{n}{p+1} \;=\; \binom{n+1}{p} \;+\; \binom{n+1}{p+1} \;=\; \binom{n+2}{p+1}\,.$$

It follows from Section 2.5 that this upper bound is attained on the standard sphere. Finally, we use Theorem 7.4.1 to show that manifolds with the maximal possible number of linearly independent conformal Killing forms have to be conformally flat. Indeed one can easily verify that for any $x \in M$ the evaluation map $\mathcal{E}^p(M) \to \Lambda^p(T^*_x M)$ is surjective. $\square$

From the proof of the theorem we also obtain sharp upper bounds for the dimension of the space of closed resp. coclosed conformal Killing forms. Again, the upper bound is provided by the dimension of the corresponding spaces on the sphere and manifolds with the maximal possible number of closed resp. coclosed conformal Killing spinors have to be conformally flat.

4.4 The fourth component of the Killing connection

In this section we will prove that also the fourth component of the Killing connection applied to a conformal Killing form has to vanish, i.e. the main result of this section is

Proposition 4.4.1 *Let ψ be a conformal Killing p-form, with $2\,p \neq n$ and let $\Psi = \widehat{\psi}$ be the associated section of $\mathcal{E}^p(M)$. Then $(\widetilde{\nabla}\Psi)_4 = 0$, i.e. for any vector field X a conformal Killing p-form ψ satisfies the equation*

$$\begin{aligned}\nabla_X(\Delta\psi) \;&=\; \tfrac{1}{p}\, X \lrcorner \left(\tfrac{1}{p+1}\, \mathrm{Ric}\,(d\psi) \;+\; \tfrac{p-1}{p+1}\, 2q(R)\, d\psi \;+\; \delta R^+ \,\psi \right) \\ &\quad -\; \tfrac{1}{n-p}\, X \wedge \left(\tfrac{1}{n-p+1}\, [s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;+\; \tfrac{n-p-1}{n-p+1}\, 2q(R)\, d^*\psi \;+\; \delta R^- \,\psi \right) \\ &\quad +\; 2q(\nabla_X R)\, \psi \;-\; \delta R(X)\, \psi \;.\end{aligned}$$

The proof of the formula for $\nabla_X(\Delta\psi)$ consist of a sequence of elementary calculations starting from the definitions. They are straight forward but rather lengthy. We derive the expression for $\nabla_X(\Delta\psi)$ by using the decomposition of the covariant derivative given in (1.1.5). In the present case it reads

$$\nabla_X(\Delta\psi) \;=\; \tfrac{1}{p+1}\, X \lrcorner\, d(\Delta\,\psi) \;-\; \tfrac{1}{n-p+1}\, X \wedge d^*(\Delta\,\psi) \;+\; T(\Delta\,\psi)(X)\;. \tag{4.4.6}$$

Hence we have to compute $d(\Delta\psi) = \Delta(d\psi)$, $d^*(\Delta\psi) = \Delta(d^*\psi)$ and $T(\Delta\psi)$. For the computation of the first two terms we will need the following elementary lemma. Recall that the Ricci tensor can be extend as a derivation to $\Lambda^k(T^*M)$, which is locally given as $\mathrm{Ric} \;=\; \sum e_j \wedge \mathrm{Ric}\,(e_j) \lrcorner$. If $(M,\, g)$ is an Einstein manifold, i.e. $\mathrm{Ric} \;=\; \lambda\,\mathrm{id}$ and Lemma A.0.3 implies that $\mathrm{Ric}\,(\omega) = \deg(\omega)\,\lambda\,\omega$. Using this notation we have

Lemma 4.4.2 *Let $(M,\, g)$ be a Riemannian manifold with scalar curvature s and let ψ be a conformal Killing p-form, then*

$$\sum e_j \wedge R_{e_i,e_j}(\nabla_{e_i}\psi) \;=\; -\tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi) \;+\; \tfrac{1}{p+1}\,2q(R)\,d\psi \;,$$

$$\sum e_j \lrcorner\, R_{e_i,e_j}(\nabla_{e_i}\psi) \;=\; \tfrac{1}{n-p+1}\,[s\,d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;-\; \tfrac{1}{n-p+1}\,2q(R)\,d^*\psi \;.$$

Proof. Let us denote the two sums of the proposition by $R^+(\psi) := \sum e_j \wedge R_{e_i,e_j}(\nabla_{e_i}\psi)$ and $R^-(\psi) := \sum e_j \lrcorner\, R_{e_i,e_j}(\nabla_{e_i}\psi)$. Then

$$\begin{aligned}
R^+(\psi) \;&=\; \sum e_j \wedge R_{e_i,e_j}\Big(\tfrac{1}{p+1}\,e_i \lrcorner\, d\psi \;-\; \tfrac{1}{n-p+1}\,e_i \wedge d^*\psi\Big)\\
&=\; \tfrac{1}{p+1}\sum e_j \wedge \big[(R_{e_i,e_j}\,e_i)\lrcorner\, d\psi \;+\; e_i \lrcorner\, R_{e_i,e_j}(d\psi)\big]\\
&\qquad -\; \tfrac{1}{n-p+1}\sum e_j \wedge \big[(R_{e_i,e_j}\,e_i) \wedge d^*\psi \;+\; e_i \wedge R_{e_i,e_j}(d^*\psi)\big]\\
&=\; \tfrac{1}{p+1}\sum e_j \wedge \big[-\mathrm{Ric}\,(e_j)\lrcorner\, d\psi \;+\; e_i \lrcorner\, R_{e_i,e_j}(d\psi)\big]\\
&\qquad -\; \tfrac{1}{n-p+1}\sum e_j \wedge \big[-\mathrm{Ric}\,(e_j) \wedge d^*\psi \;+\; e_i \wedge R_{e_i,e_j}(d^*\psi)\big]\\
&=\; -\tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi) \;+\; \tfrac{1}{p+1}\,2q(R)\,d\psi \;.
\end{aligned}$$

Note that $\sum e_j \wedge \mathrm{Ric}\,(e_j) = 0$ because of the symmetry of the Ricci tensor. Moreover, we used the 1. Bianchi identity, which can be written as

$$\sum e_j \wedge e_i \wedge R_{e_i,e_j} \;=\; 0 \;.$$

A similar calculation proves the second statement of the lemma.

$$\begin{aligned}
R^-(\psi) \;&=\; \sum e_j \lrcorner\, R_{e_i,e_j}\Big(\tfrac{1}{p+1}\,e_i \lrcorner\, d\psi \;-\; \tfrac{1}{n-p+1}\,e_i \wedge d^*\psi\Big)\\
&=\; \tfrac{1}{p+1}\sum e_j \lrcorner\, \big[(R_{e_i,e_j}\,e_i)\lrcorner\, d\psi \;+\; e_i \lrcorner\, R_{e_i,e_j}(d\psi)\big]\\
&\qquad -\; \tfrac{1}{n-p+1}\sum e_j \lrcorner\, \big[(R_{e_i,e_j}\,e_i) \wedge d^*\psi \;+\; e_i \wedge R_{e_i,e_j}(d^*\psi)\big]\\
&=\; \tfrac{1}{p+1}\sum e_j \lrcorner\, \big[-\mathrm{Ric}\,(e_j)\lrcorner\, d\psi \;+\; e_i \lrcorner\, R_{e_i,e_j}(d\psi)\big]\\
&\qquad -\; \tfrac{1}{n-p+1}\sum e_j \lrcorner\, \big[-\mathrm{Ric}\,(e_j) \wedge d^*\psi \;+\; e_i \wedge R_{e_i,e_j}(d^*\psi)\big]\\
&=\; \tfrac{1}{n-p+1}\sum\,[g(\mathrm{Ric}\,(e_j),\,e_j)\,d^*\psi \;-\; \mathrm{Ric}\,(e_j) \wedge e_j \lrcorner\, d^*\psi\,] \;-\; \tfrac{1}{n-p+1}\,2q(R)\,d^*\psi\\
&=\; \tfrac{1}{n-p+1}\,(s\,d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)) \;-\; \tfrac{1}{n-p+1}\,2q(R)\,d^*\psi \;. \quad \Box
\end{aligned}$$

For the definition of the fourth component of the Killing connection we introduced the notation $(\delta R)_X \; := \; -\sum (\nabla_{e_i} R)_{e_i, X}$, where we consider R as an endomorphism-valued 2-form. It is well known that the second Bianchi identity links δR to the covariant derivative of the Ricci curvature:

Lemma 4.4.3 *Let X, Y, Z be any vector fields, then*

$$g((\delta R)_X Y, Z) \;=\; (\nabla_Y \mathrm{Ric})(\,Z,\, X\,) \;-\; (\nabla_Z \mathrm{Ric})(\,Y,\, X\,)$$

Proof. The proof uses the second Bianchi identity. We write the right-hand side of the equation as

$$\begin{aligned}
&(\nabla_Y \mathrm{Ric})(\,Z,\, X\,) \;-\; (\nabla_Z \mathrm{Ric})(\,Y,\, X\,)\\
&\quad= \quad \sum (\nabla_Y\, R)(Z, e_i, e_i, X) \;-\; (\nabla_Z\, R)(Y, e_i, e_i, X)\\
&\quad= \; -\sum (\nabla_Z\, R)(e_i, Y, e_i, X) \;+\; (\nabla_{e_i}\, R)(Y, Z, e_i, X) \;+\; (\nabla_Z\, R)(Y, e_i, e_i, X)\\
&\quad= \; -\sum (\nabla_{e_i}\, R)(Y, Z, e_i, X) \;=\; -\sum (\nabla_{e_i}\, R)(e_i, X, Y, Z) \;=\; g((\delta R)_X Y, Z)\;. \quad \Box
\end{aligned}$$

In particular, $\delta R \;=\; 0$, for manifolds with parallel Ricci tensor, e.g. Einstein manifolds of dimension greater than two. We also defined: $\delta R^+ \psi \;:=\; \sum e_j \wedge (\delta R)_{e_j} \psi$ and $\delta R^- \psi \;:=\; \sum e_j \lrcorner\, (\delta R)_{e_j} \psi$. Using this notation and applying Lemma 4.4.2 we derive the following

Proposition 4.4.4 *Let $(M,\, g)$ be a Riemannian manifolds with scalar curvature s and let ψ be a conformal Killing p-form, then*

$$\begin{aligned}
\Delta(d\psi) \;&=\; \frac{1}{p}\,\mathrm{Ric}\,(d\psi) \;+\; \frac{p-1}{p}\, 2q(R)\, d\psi \;+\; \frac{p+1}{p}\,\delta R^+\, \psi\;,\\
\Delta(d^*\psi) \;&=\; \frac{1}{n-p}\,[s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;+\; \frac{n-p-1}{n-p}\, 2q(R)\, d^*\psi \;-\; \frac{n-p+1}{n-p}\,\delta R^-\, \psi\;.
\end{aligned}$$

Proof. Let ψ be a conformal Killing p-form. Then we have already an equation for $\nabla_Y(d\psi)$. We take the covariant derivative with respect to X and obtain

$$\begin{aligned}
\nabla_X \nabla_Y\,(d\,\psi) \;&=\; \tfrac{p+1}{p(n-p+1)}\,(\nabla_X Y \,\wedge\, dd^*\,\psi \;+\; Y \,\wedge\, \nabla_X(dd^*\,\psi))\\
&\qquad + \; \tfrac{p+1}{p} \sum \left(\nabla_X\, e_j \,\wedge\, R_{Y, e_j}\psi \;+\; e_j \,\wedge\, \nabla_X(R_{Y,\, e_j}\,\psi)\right)\;.
\end{aligned}$$

From this we conclude:

$$\begin{aligned}
\nabla^2_{X,Y}(d\psi) \;&=\; \tfrac{p+1}{p(n-p+1)}\, Y \,\wedge\, \nabla_X(dd^*\,\psi)\\
&\qquad + \; \tfrac{p+1}{p} \sum e_j \,\wedge\, \left[\nabla_X(R_{Y,\, e_j}\,\psi) \;-\; R_{\nabla_X Y,\, e_j}\,\psi \;-\; R_{Y, \nabla_X e_j}\,\psi\right]\;.
\end{aligned}$$

Taking the trace we get an expression for $\nabla^*\nabla(d\psi)$:

$$\begin{aligned}\nabla^*\nabla\,(d\psi) &= -\tfrac{p+1}{p}\sum e_j \wedge \Big[\nabla_{e_i}(R_{e_i,\,e_j}\,\psi) \;-\; R_{\nabla_{e_i}e_i,\,e_j}\,\psi \;-\; R_{e_i,\nabla_{e_i}e_j}\,\psi\Big] \\ &= -\tfrac{p+1}{p}\sum e_j \wedge \big[(\nabla_{e_i}R)_{e_i,\,e_j}\,\psi \;+\; R_{e_i,\,e_j}(\nabla_{e_i}\psi)\big] \\ &= \tfrac{p+1}{p}\,\delta R^+\,\psi \;-\; \tfrac{p+1}{p}\Big[-\tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi) \;+\; \tfrac{1}{p+1}\,2q(R)\,(d\psi)\Big] \\ &= \tfrac{p+1}{p}\,\delta R^+\,\psi \;+\; \tfrac{1}{p}\,\mathrm{Ric}\,(d\psi) \;-\; \tfrac{1}{p}\,2q(R)\,d\psi\ .\end{aligned}$$

This proves the statement for $\Delta(d\psi) = \nabla^*\nabla d\psi + 2q(R)d\psi$. The proof of the equation for $\Delta(d^*\psi)$ is similar. We start with

$$\begin{aligned}\nabla_X\nabla_Y\,(d^*\psi) &= -\tfrac{n-p+1}{(n-p)(p+1)}\,(\nabla_X Y \lrcorner\, d^*d\,\psi \;+\; Y \lrcorner\, \nabla_X(d^*d\,\psi)) \\ &\qquad -\; \tfrac{n-p+1}{n-p}\sum \big(\nabla_X e_j \lrcorner\, R_{Y,\,e_j}\psi \;+\; e_j \lrcorner\, \nabla_X(R_{Y,\,e_j}\,\psi)\big)\ .\end{aligned}$$

From this we get

$$\begin{aligned}\nabla^2_{X,Y}(d^*\psi) &= -\tfrac{n-p+1}{(n-p)(p+1)}\,Y \lrcorner\, \nabla_X(d^*d\,\psi) \\ &\qquad -\; \tfrac{n-p+1}{n-p}\sum e_j \lrcorner\, \big[\nabla_X(R_{Y,\,e_j}\psi) \;-\; R_{\nabla_X Y,\,e_j}\,\psi \;-\; R_{Y,\nabla_X e_j}\,\psi\big]\ .\end{aligned}$$

Finally we take the trace to obtain an expression for $\nabla^*\nabla(d^*\psi)$:

$$\begin{aligned}\nabla^*\nabla\,(d^*\,\psi) &= \tfrac{n-p+1}{n-p}\sum e_j \lrcorner\, \big[(\nabla_{e_i}R)_{e_i,\,e_j}\,\psi \;+\; R_{e_i,\,e_j}(\nabla_{e_i}\psi)\big] \\ &= -\tfrac{n-p+1}{n-p}\,\delta R^-\psi \;+\; \tfrac{n-p+1}{n-p}\Big[\tfrac{1}{n-p+1}\,(s\,d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)) \;-\; \tfrac{1}{n-p+1}\,2q(R)\,d^*\psi\Big] \\ &= -\tfrac{n-p+1}{n-p}\,\delta R^-\psi \;+\; \tfrac{1}{n-p}\,[s\,d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;-\; \tfrac{1}{n-p}\,2q(R)\,d^*\psi\ . \quad \square\end{aligned}$$

At the end of this section we will prove a second formula for $\Delta(d\psi)$ resp. $\Delta(d^*\psi)$. Comparing it with the corresponding one from the proposition above provides us with an expression for $\delta R^{\pm}$, which we then can use in the final computation of $T(\Delta\psi)$, i.e. we obtain

Proposition 4.4.5 *Let $(M,\,g)$ be a Riemannian manifold and ψ be a conformal Killing p-form, then*

$$\delta R^+\,\psi \;=\; \sum e_k \wedge 2q(\nabla_{e_k}R)\,\psi \qquad \textit{and} \qquad \delta R^-\,\psi \;=\; \sum e_k \lrcorner\, 2q(\nabla_{e_k}R)\,\psi\ .$$

So far we have expressions for $\Delta(d\psi)$ and for $\Delta(d^*\psi)$. It still remains to compute $T(\Delta\psi)$, for which we need a commutator rule between the covariant derivative ∇ and the operator $\nabla^*\nabla + 2q(R)$. The proof of it is again a lengthy calculation, which is contained in Appendix C. Nevertheless, it is surprising to see that after a series of cancellations it is possible to obtain a rather short formula. In our case we will need the following special case.

Proposition 4.4.6 *Let ψ be a conformal Killing form, then for any vector field X*

$$T(\Delta\psi)(X) \;=\; 2\,q(\nabla_X R)\,\psi \;-\; \delta R(X)\,\psi \;.$$

In particular, $\Delta\psi$ is again a conformal Killing p-form, if the manifold is locally symmetric. If the manifold is Einstein then $T(\Delta\,\psi)(X) = 2q(\nabla_X R)\,\psi$ for any vector field X.

Proof. In Appendix C we prove a formula for the commutator of the twistor operator T and the Laplace operator Δ. This formula still involves the projection onto $\Lambda^{p,1}$, which is explicitly given in (1.1.2). Together with the equations for $\delta R^{\pm}\psi$ of Proposition 4.4.5 we find

$$\begin{aligned}
T(\Delta\psi)(X) \;&=\; \sum \mathrm{pr}_{\Lambda^{p,1}}\left(e_j \otimes [2q(\nabla_{e_j}R) \;-\; \delta R(e_j)]\,\psi\right)\\
&=\; 2\,q(\nabla_X R)\,\psi \;-\; \delta R(X)\,\psi\\
&\qquad -\tfrac{1}{p+1}\,X \lrcorner \sum e_j \wedge [2q(\nabla_{e_j}R) \;-\; \delta R(e_j)]\,\psi\\
&\qquad -\tfrac{1}{n-p+1}\,X \wedge \sum e_j \lrcorner\,[2q(\nabla_{e_j}R) \;-\; \delta R(e_j)]\,\psi\\
&=\; 2\,q(\nabla_X R)\,\psi \;-\; \delta R(X)\,\psi \;.
\end{aligned}$$

For locally symmetric spaces we have $\nabla R = 0$. Thus, $2\,q(\nabla_X R)\,\psi$ and $\delta R(X)\,\psi$ vanish. If the manifold is Einstein we know that $\delta R(X)\,\psi$ has to vanish. □

From the above proposition we see that on a locally symmetric space the Laplace operator preserves the space of conformal Killing forms. In such a situation we have the following

Corollary 4.4.7 *Let $(M,\,g)$ be a compact Riemannian manifold such that, the Laplace operator preserves the space of conformal Killing forms, e.g. for locally symmetric spaces. Then any conformal Killing form decomposes into a sum of eigenforms of the Laplace operator, which are again conformal Killing forms.*

Proof. On compact manifolds the space of conformal Killing forms can be equipped with the induced L^2–scalar product and the Laplace operator acts as a symmetric operator on this finite dimensional space. Hence, it is completely diagonalizable, i.e. any conformal

Killing form can be decomposed into a sum of eigenforms which, by definition, are again conformal Killing forms. □

Finally we compute $\nabla_X(\Delta\psi)$ by substituting the expressions for $d(\Delta\psi)$, $d^*(\Delta\psi)$ and $T(\Delta\psi)(X)$ into equation (4.4.6). This will then finish the proof of Proposition 4.4.1. Note that we have some simplification in the case of Einstein manifolds (c.f. Corollary 7.1.1).

$$
\begin{aligned}
\nabla_X(\Delta\psi) \;=\; & \tfrac{1}{p+1}\, X \lrcorner\, d(\Delta\,\psi) \;-\; \tfrac{1}{n-p+1}\, X \wedge d^*(\Delta\,\psi) \;+\; T(\Delta\,\psi)(X) \\
= \;& \tfrac{1}{p+1}\, X \lrcorner \left(\tfrac{1}{p}\,\mathrm{Ric}\,(d\psi) \;+\; \tfrac{p-1}{p}\, 2q(R)\, d\psi \;+\; \tfrac{p+1}{p}\, \delta R^+\, \psi \right) \\
& - \tfrac{1}{n-p+1}\, X \wedge \left(\tfrac{1}{n-p}\, [s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;+\; \tfrac{n-p-1}{n-p}\, 2q(R)\, d^*\psi \;-\; \tfrac{n-p+1}{n-p}\, \delta R^-\, \psi \right) \\
& + 2q(\nabla_X R)\, \psi \;-\; \delta R(X)\, \psi \\
= \;& \tfrac{1}{p}\, X \lrcorner \left(\tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi) \;+\; \tfrac{p-1}{p+1}\, 2q(R)\, d\psi \;+\; \delta R^+\, \psi \right) \\
& - \tfrac{1}{n-p}\, X \wedge \left(\tfrac{1}{n-p+1}\, [s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;+\; \tfrac{n-p-1}{n-p+1}\, 2q(R)\, d^*\psi \;+\; \delta R^-\, \psi \right) \\
& \qquad + 2q(\nabla_X R)\, \psi \;-\; \delta R(X)\, \psi\ .
\end{aligned}
$$

In the last part of this section we will describe an alternative way to compute $\Delta(d\psi)$ and $\Delta(d^*\psi)$. The idea is to write the Laplace operator as $\Delta = \nabla^*\nabla + 2q(R)$ and then to replace $\nabla^*\nabla$ using the first Weitzenböck formula (1.1.7). It remains to prove commuting rules between $2q(R)$ and d resp. d^*, as well a formula for $T^*T(d\psi)$ resp. $T^*T(d^*\psi)$. In the end we can compare the two equations for $\Delta(d\psi)$ resp. $\Delta(d^*\psi)$ and thus obtain a proof of Proposition 4.4.5. Moreover, we obtain further information on conformal Killing forms which we can apply later. In particular we will see that several formulas become much easier on Einstein manifolds. We start with a formula for the twistor operator applied to $d\psi$ resp. $d^*\psi$.

Lemma 4.4.8 *Let ψ be a conformal Killing p-form and X any vector field, then*

$$
\begin{aligned}
T(d\psi)(X) \;&=\; \tfrac{p+1}{p}\, R^+(X)\, \psi \;+\; \tfrac{p+1}{p(n-p)}\, X \wedge 2q(R)\, \psi\ , \\
T(d^*\psi)(X) \;&=\; -\tfrac{n-p+1}{n-p}\, R^-(X)\, \psi \;-\; \tfrac{n-p+1}{p(n-p)}\, X \lrcorner\, 2q(R)\, \psi\ .
\end{aligned}
$$

Proof. Using the formula for the twistor operator T on $(p+1)$–forms we find

$$
\begin{aligned}
T(d\psi) \;&=\; \nabla(d\psi) \;+\; \tfrac{1}{n-p} \sum e_i \otimes e_i \wedge d^*d\,\psi \;=\; \sum e_i \otimes \left[\nabla_{e_i}(d\psi) \;+\; \tfrac{1}{n-p}\, e_i \wedge d^*d\,\psi \right] \\
&=\; \sum e_i \otimes \left[\tfrac{p+1}{p}\, R^+(e_i)\, \psi \;+\; \tfrac{p+1}{p(n-p)} e_i \wedge 2q(R)\psi \right]\ .
\end{aligned}
$$

The corresponding formula for T on $(p-1)$–forms yields

$$T(d^*\psi) \;=\; \nabla(d^*\psi) \;-\; \tfrac{1}{p}\sum e_i\otimes e_i \lrcorner\, dd^*\psi \;=\; \sum e_i\otimes\Big[\nabla_{e_i}(d^*\psi) \;-\; \tfrac{1}{p}\, e_i \lrcorner\, dd^*\psi\Big]$$

$$=\; \sum e_i\otimes\Big[-\tfrac{n-p+1}{n-p}\, R^-(e_i)\,\psi \;-\; \tfrac{n-p+1}{p(n-p)} e_i \lrcorner\, 2q(R)\psi\Big] \;. \qquad \Box$$

This Lemma gives a criterion to decide whether for a given conformal Killing form ψ the forms $d\psi$ or $d^*\psi$ are again conformal Killing forms.

Corollary 4.4.9 *Let $(M,\,g)$ be a manifold with a conformal Killing p–form ψ. Then $d\psi$ is again a conformal Killing form if and only if*

$$R^+(X)\,\psi \;=\; -\,\frac{1}{n-p}\,X\,\wedge\,2q(R)\,\psi$$

for any vector field X. Similarly, $d^*\psi$ is again a conformal Killing form if and only if for any vector field X

$$R^-(X)\,\psi \;=\; -\,\frac{1}{p}\,X \lrcorner\; 2q(R)\,\psi \;.$$

Next we have to compute T^* of $T(d\psi)$ resp. $T(d^*\psi)$. The adjoint operator T^* is described by the following

Lemma 4.4.10 *Let $\phi\in\Gamma(\Lambda^{p,1}T^*M)\subset\Gamma(T^*M\otimes\Lambda^pT^*M)$, then*

$$T^*(\phi) \;=\; -\sum\nabla_{e_i}\,(\phi(e_i)) \;+\; \sum\,((\nabla_{e_i}e_i)\lrcorner\,\otimes\,1)\,\phi \;.$$

By combining the last two lemmas we obtain

Proposition 4.4.11 *Let ψ be a conformal Killing tensor, then*

$$T^*T(d\psi) \;=\; -\,\frac{p+1}{p(n-p)}\,d(2q(R)\,\psi) \;-\; \frac{1}{p}\,2q(R)\,d\psi \;+\; \frac{1}{p}\,\mathrm{Ric}\,(d\psi) \;+\; \frac{p+1}{p}\,\delta R^+\,\psi \;,$$

$$T^*T(d^*\psi) \;=\; -\,\frac{n-p+1}{p(n-p)}\,d^*(2q(R)\,\psi) \;-\; \frac{1}{n-p}\,2q(R)\,d^*\psi \;+\; \frac{1}{n-p}\,[s\,d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;-\; \frac{n-p+1}{n-p}\,\delta R^-\,\psi \;.$$

Proof. Using the preceding lemmas we compute

$$
\begin{aligned}
T^*T(d\psi) &= T^*\Big(\tfrac{p+1}{p}\sum e_i\otimes[R^+(e_i)\,\psi\;+\;\frac{1}{n-p}\,e_i\wedge 2q(R)\,\psi\,]\Big)\\
&= -\tfrac{p+1}{p}\sum\Big(\nabla_{e_i}(R^+(e_i)\,\psi)\;+\;\tfrac{1}{n-p}\,\nabla_{e_i}(e_i\wedge 2q(R)\,\psi)\Big)\\
&\qquad +\;\tfrac{p+1}{p}\sum\Big(R^+(\nabla_{e_i}e_i)\,\psi\;+\;\tfrac{1}{n-p}\,(\nabla_{e_i}e_i)\wedge 2q(R)\,\psi\Big)\\
&= -\tfrac{p+1}{p}\sum e_j\wedge[\nabla_{e_i}(R_{e_i,e_j}\,\psi\,)\;-\;R_{\nabla_{e_i}e_i,e_j}\,\psi\;-\;R_{e_i,\nabla_{e_i}e_j}\,\psi]\\
&\qquad -\;\tfrac{p+1}{p(n-p)}\sum e_i\wedge\nabla_{e_i}(2q(R)\,\psi\,)\\
&= -\tfrac{p+1}{p(n-p)}\,d(2q(R)\,\psi\,)\;-\;\tfrac{p+1}{p}\sum\;e_j\wedge[(\nabla_{e_i}R)_{e_i,e_j}\,\psi\;+\;R_{e_i,e_j}(\nabla_{e_i}\psi\,)]\;.
\end{aligned}
$$

A similar calculation gives the formula for $T^*T(d^*\psi)$. Indeed we have

$$
\begin{aligned}
T^*T(d^*\psi) &= T^*\Big(-\tfrac{n-p+1}{n-p}\sum e_i\otimes[R^-(e_i)\,\psi\;+\;\frac{1}{p}\,e_i\lrcorner\,2q(R)\,\psi\,]\Big)\\
&= \tfrac{n-p+1}{n-p}\sum\Big(\nabla_{e_i}(R^-(e_i)\,\psi\,)\;+\;\tfrac{1}{p}\,\nabla_{e_i}(e_i\lrcorner\,2q(R)\,\psi\,)\Big)\\
&\qquad -\;\tfrac{n-p+1}{n-p}\sum\Big(R^-(\nabla_{e_i}e_i)\,\psi\;+\;\tfrac{1}{p}\,(\nabla_{e_i}e_i)\lrcorner\,2q(R)\,\psi\Big)\\
&= \tfrac{n-p+1}{n-p}\sum e_j\lrcorner\,[\nabla_{e_i}(R_{e_i,e_j}\,\psi\,)\;-\;R_{\nabla_{e_i}e_i,e_j}\,\psi\;-\;R_{e_i,\nabla_{e_i}e_j}\,\psi]\\
&\qquad +\;\tfrac{n-p+1}{p(n-p)}\sum e_i\lrcorner\,\nabla_{e_i}(2q(R)\,\psi\,)\\
&= -\tfrac{n-p+1}{p(n-p)}\,d^*(2q(R)\,\psi\,)\;+\;\tfrac{n-p+1}{n-p}\sum\;e_j\lrcorner\,[(\nabla_{e_i}R)_{e_i,e_j}\,\psi\;+\;R_{e_i,e_j}(\nabla_{e_i}\psi)]\;.\quad\Box
\end{aligned}
$$

Finally we need the commutator rule for $2q(R)$ with the operators d resp. d^*, which we state in the following proposition. The proof is completely elementary.

Proposition 4.4.12 *Let $(M,\,g)$ be a Riemannian manifold of scalar curvature s and let ψ be a conformal Killing p-form, then*

$$
\begin{aligned}
d(2q(R)\,\psi) &= \tfrac{p-1}{p+1}\,2q(R)\,d\psi\;+\;\tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi)\;+\;\sum e_k\wedge 2q(\nabla_{e_k}R)\,\psi\;,\\
d^*(2q(R)\,\psi) &= \tfrac{n-p-1}{n-p+1}\,2q(R)\,d^*\psi\;+\;\tfrac{1}{n-p+1}\,[s\,d^*\psi\;-\;\mathrm{Ric}\,(d^*\psi)]\;-\;\sum e_k\lrcorner\,2q(\nabla_{e_k}R)\,\psi\;.
\end{aligned}
$$

Proof. We compute $d(2q(R)\,\psi)$ using a local ortho-normal frame $\{e_i\}$.

$$\begin{aligned}
d(2q(R)\,\psi) &= \sum e_k \wedge \nabla_{e_k}(e_j \wedge e_i \lrcorner R_{e_i,e_j}\,\psi\,) \\
&= \sum e_k \wedge [\nabla_{e_k}(e_j) \wedge e_i \lrcorner R_{e_i,e_j}\,\psi \;+\; e_j \wedge \nabla_{e_k}(e_i) \lrcorner R_{e_i,e_j}\,\psi \\
&\qquad\qquad + \; e_j \wedge e_i \lrcorner \nabla_{e_k}(R_{e_i,e_j}\,\psi\,)] \\
&= \sum e_k \wedge e_j \wedge e_i \lrcorner [(\nabla_{e_k}R)_{e_i,e_j}\,\psi \;+\; R_{e_i,e_j}(\nabla_{e_k}\,\psi)] \\
&= \sum e_k \wedge 2q(\nabla_{e_k}R)\,\psi \;+\; \sum e_k \wedge e_j \wedge e_i \lrcorner R_{e_i,e_j}(\nabla_{e_k}\psi)\;.
\end{aligned}$$

Since ψ is a conformal Killing p-form we can further simplify the second sum. We obtain

$$\begin{aligned}
&\sum e_k \wedge e_j \wedge e_i \lrcorner R_{e_i,e_j}(\nabla_{e_k}\psi) \\
&= -\sum e_j \wedge e_k \wedge e_i \lrcorner R_{e_i,e_j}(\nabla_{e_k}\psi) \\
&= \sum e_j \wedge e_i \lrcorner [\,e_k \wedge R_{e_i,e_j}(\nabla_{e_k}\psi\,)] \;-\; \sum e_j \wedge R_{e_i,e_j}(\nabla_{e_i}\psi) \\
&= \sum e_j \wedge e_i \lrcorner [\,e_k \wedge R_{e_i,e_j}(\tfrac{1}{p+1}\,e_k \lrcorner d\psi \;-\; \tfrac{1}{n-p+1}\,e_k \wedge d^*\psi)] \\
&\qquad + \tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi) \;-\; \tfrac{1}{p+1}\,2q(R)\,d\psi \\
&= \tfrac{1}{p+1}\sum e_j \wedge e_i \lrcorner \big[\,e_k \wedge (R_{e_i,e_j}e_k) \lrcorner d\psi \;+\; e_k \wedge e_k \lrcorner (R_{e_i,e_j}d\psi)\big] \\
&\quad - \tfrac{1}{n-p+1}\sum e_j \wedge e_i \lrcorner \big[\,e_k \wedge (R_{e_i,e_j}e_k) \wedge d^*\psi \;+\; e_k \wedge e_k \wedge (R_{e_i,e_j}d^*\psi)\big] \\
&\qquad + \tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi) \;-\; \tfrac{1}{p+1}\,2q(R)\,d\psi \\
&= \tfrac{1}{p+1}\sum e_j \wedge e_i \lrcorner \big[-R_{e_i,e_j}(d\psi) \;+\; (p+1)\,R_{e_i,e_j}(d\psi)\,\big] \\
&\qquad + \tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi) \;-\; \tfrac{1}{p+1}\,2q(R)\,d\psi \\
&= \tfrac{p-1}{p+1}\,2q(R)\,d\psi \;+\; \tfrac{1}{p+1}\,\mathrm{Ric}\,(d\psi)\;.
\end{aligned}$$

At the end we still did the following calculation, using the symmetry of the Ricci tensor and the 1. Bianchi identity:

$$\begin{aligned}
&\sum e_j \wedge e_i \lrcorner \, [\, e_k \wedge (R_{e_i,\, e_j} e_k) \wedge d^*\psi \,] \\
&\quad = \quad \sum e_j \wedge (R_{e_i,\, e_j} e_i) \wedge d^*\psi \; - \; e_j \wedge e_k \wedge g(R_{e_i,\, e_j} e_k,\, e_i)\, d^*\psi \\
&\qquad\qquad + \; e_j \wedge e_k \wedge (R_{e_i,\, e_j} e_k) \wedge (e_i \lrcorner \, d^*\psi\,) \\
&\quad = \; -2 \sum e_j \wedge \mathrm{Ric}\,(e_j) \wedge d^*\psi \; - \; e_j \wedge e_k \wedge e_r \wedge [(R_{e_k,\, e_r} e_j) \lrcorner \, d^*\psi\,] \\
&\quad = \; -\sum e_k \wedge e_r \wedge R_{e_k,\, e_r}(d^*\psi\,) \;\; = \;\; 0 \, .
\end{aligned}$$

The proof of the second statement, i.e. the formula for $2q(R)(d^*\psi)$, is quite similar. Here we have

$$\begin{aligned}
d^*(2q(R)\,\psi) \;\; &= \;\; -\sum e_k \lrcorner \, \nabla_{e_k}(e_j \wedge e_i \lrcorner \, R_{e_i,\, e_j}\, \psi\,) \\
&= \;\; -\sum e_k \lrcorner \, [\nabla_{e_k}(e_j) \wedge e_i \lrcorner \, R_{e_i,\, e_j}\, \psi \; + \; e_j \wedge \nabla_{e_k}(e_i) \lrcorner \, R_{e_i,\, e_j}\, \psi \\
&\qquad\qquad + \; e_j \wedge e_i \lrcorner \, \nabla_{e_k}(R_{e_i,\, e_j}\, \psi\,)] \\
&= \;\; -\sum e_k \lrcorner \, e_j \wedge e_i \lrcorner \, [(\nabla_{e_k} R)_{e_i,\, e_j}\, \psi \; + \; R_{e_i,\, e_j}(\nabla_{e_k}\, \psi)] \\
&= \;\; -\sum e_k \lrcorner \, 2q(\nabla_{e_k} R)\, \psi \;\; - \;\; \sum e_k \lrcorner \, e_j \wedge e_i \lrcorner \, R_{e_i,\, e_j}(\nabla_{e_k}\psi) \, .
\end{aligned}$$

Since ψ is a conformal Killing p-form we can further simplify the second sum. We obtain

$$
\begin{aligned}
&- \sum e_k \lrcorner e_j \wedge e_i \lrcorner R_{e_i,e_j}(\nabla_{e_k}\psi) \\
&\quad = \; - \sum e_i \lrcorner R_{e_i,e_j}(\nabla_{e_j}\psi) \; - \; \sum e_j \wedge e_i \lrcorner [e_k \lrcorner R_{e_i,e_j}(\nabla_{e_k}\psi)] \\
&\quad = \; - \sum e_j \wedge e_i \lrcorner [e_k \lrcorner R_{e_i,e_j}(\tfrac{1}{p+1}\, e_k \lrcorner d\psi \; - \; \tfrac{1}{n-p+1}\, e_k \wedge d^*\psi)] \\
&\qquad\qquad + \; \tfrac{1}{n-p+1}\, [s\, d^*\psi \; - \; \mathrm{Ric}\,(d^*\psi)] \; - \; \tfrac{1}{n-p+1}\, 2q(R)\, d^*\psi \\
&\quad = \; - \tfrac{1}{p+1} \sum e_j \wedge e_i \lrcorner \big[e_k \lrcorner (R_{e_i,e_j} e_k) \lrcorner d\psi \; + \; e_k \lrcorner e_k \lrcorner (R_{e_i,e_j} d\psi) \big] \\
&\qquad + \tfrac{1}{n-p+1} \sum e_j \wedge e_i \lrcorner \big[e_k \lrcorner (R_{e_i,e_j} e_k) \wedge d^*\psi \; + \; e_k \lrcorner e_k \wedge (R_{e_i,e_j} d^*\psi) \big] \\
&\qquad\qquad + \; \tfrac{1}{n-p+1}\, [s\, d^*\psi \; - \; \mathrm{Ric}\,(d^*\psi)] \; - \; \tfrac{1}{n-p+1}\, 2q(R)\, d^*\psi \\
&\quad = \; \tfrac{1}{n-p+1} \sum e_j \wedge e_i \lrcorner \big[- R_{e_i,e_j}(d^*\psi) \; + \; (n-p+1)\, R_{e_i,e_j}(d^*\psi) \big] \\
&\qquad\qquad + \; \tfrac{1}{n-p+1}\, [s\, d^*\psi \; - \; \mathrm{Ric}\,(d^*\psi)] \; - \; \tfrac{1}{n-p+1}\, 2q(R)\, d^*\psi \\
&\quad = \; \tfrac{n-p-1}{n-p+1}\, 2q(R)\, d\psi \; + \; \tfrac{1}{n-p+1}\, [s\, d^*\psi \; - \; \mathrm{Ric}\,(d^*\psi)] \; . \quad \square
\end{aligned}
$$

Finally we can apply Propositions 4.4.11 and 4.4.12 to derive once again an expression for $\Delta(d\psi)$ resp. $\Delta(d\psi)$. We will then compare the resulting equations with the corresponding ones of Proposition 4.4.4 to obtain the proof of Proposition 4.4.5. Computing $\Delta(d\psi)$ first we find:

$$
\begin{aligned}
\Delta(d\psi) \; &= \; 2q(R)\, d\psi \; + \; \nabla^*\nabla\,(d\psi) \\
&= \; 2q(R)\, d\psi \; + \; \tfrac{1}{n-p}\, dd^*(d\psi) \; + \; T^*T(d\psi) \\
&= \; 2q(R)\, d\psi \; + \; \tfrac{1}{n-p}\, \Delta(d\psi) \; - \; \tfrac{p+1}{p(n-p)}\, d(2q(R)\,\psi) \; - \; \tfrac{1}{p}\, 2q(R)\, d\psi \\
&\qquad + \; \tfrac{1}{p}\, \mathrm{Ric}\,(d\psi) \; + \; \tfrac{p+1}{p}\, \delta R^+\, \psi \\
&= \; \tfrac{(n-p)(p-1)}{p(n-p-1)}\, 2q(R)\, d\psi \; - \; \tfrac{p+1}{p(n-p-1)}\, d(2q(R)\,\psi) \; + \; \tfrac{n-p}{p(n-p-1)}\, \mathrm{Ric}\,(d\psi) \\
&\qquad + \; \tfrac{(n-p)(p+1)}{p(n-p-1)}\, \delta R^+\, \psi \\
&= \; \tfrac{(n-p)(p-1)}{p(n-p-1)}\, 2q(R)\, d\psi \; - \; \tfrac{p+1}{p(n-p-1)} \Big(\tfrac{p-1}{p+1}\, 2q(R)\, d\psi \; + \; \tfrac{1}{p+1}\, \mathrm{Ric}\,(d\psi) \\
&\qquad + \; \sum e_k \wedge 2q(\nabla_{e_k} R)\, \psi \Big) \; + \; \tfrac{n-p}{p(n-p-1)}\, \mathrm{Ric}\,(d\psi) \; + \; \tfrac{(n-p)(p+1)}{p(n-p-1)}\, \delta R^+\, \psi \\
&= \; \tfrac{p-1}{p}\, 2q(R)\, d\psi \; + \; \tfrac{1}{p}\, \mathrm{Ric}\,(d\psi) \; - \; \tfrac{p+1}{p(n-p-1)} \sum e_k \wedge 2q(\nabla_{e_k} R)\, \psi \\
&\qquad + \; \tfrac{(n-p)(p+1)}{p(n-p-1)}\, \delta R^+\, \psi \; .
\end{aligned}
$$

The calculation for $\Delta(d^*\psi)$ is of course almost the same. In this case we obtain

$$\begin{aligned}
\Delta(d^*\psi) &= 2q(R)\, d^*\psi \;+\; \nabla^*\nabla\,(d^*\psi) \\
&= 2q(R)\, d^*\psi \;+\; \tfrac{1}{p}\, d^*d(d^*\psi) \;+\; T^*T(d^*\psi) \\
&= 2q(R)\, d^*\psi \;+\; \tfrac{1}{p}\, \Delta(d^*\psi) \;-\; \tfrac{n-p+1}{p(n-p)}\, d^*(2q(R)\,\psi) \;-\; \tfrac{1}{n-p}\, 2q(R)\, d^*\psi \\
&\qquad\qquad + \tfrac{1}{n-p}\,[s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;-\; \tfrac{n-p+1}{n-p}\, \delta R^-\,\psi \\
&= \tfrac{p(n-p-1)}{(p-1)(n-p)}\, 2q(R)\, d^*\psi \;-\; \tfrac{n-p+1}{(p-1)(n-p)}\, d^*(2q(R)\,\psi) \\
&\qquad\qquad + \tfrac{p}{(p-1)(n-p)}\,[s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \;-\; \tfrac{p(n-p+1)}{(p-1)(n-p)}\, \delta R^-\,\psi \\
&= \tfrac{p(n-p-1)}{(p-1)(n-p)}\, 2q(R)\, d^*\psi \\
&\quad - \tfrac{n-p+1}{(p-1)(n-p)}\, \Big(\tfrac{n-p-1}{n-p+1}\, 2q(R)\, d\psi + \tfrac{1}{n-p+1}\,[s\, d^*\psi - \mathrm{Ric}\,(d^*\psi)] \\
&\qquad\qquad - \sum e_k \lrcorner\, 2q(\nabla_{e_k} R)\,\psi \Big) \\
&\quad + \tfrac{p}{(p-1)(n-p)}\,[s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] - \tfrac{p(n-p+1)}{(p-1)(n-p)}\, \delta R^-\,\psi \\
&= \tfrac{n-p-1}{n-p}\, 2q(R)\, d^*\psi + \tfrac{1}{n-p}\,[s\, d^*\psi \;-\; \mathrm{Ric}\,(d^*\psi)] \\
&\qquad\qquad + \tfrac{n-p+1}{(p-1)(n-p)} \sum e_k \lrcorner\, 2q(\nabla_{e_k} R)\,\psi \;-\; \tfrac{p(n-p+1)}{(p-1)(n-p)}\, \delta R^-\psi \;.
\end{aligned}$$

We see that for locally symmetric manifolds the new formulas for $\Delta(d\psi)$ resp. $\Delta(d^*\psi)$ already coincide with the formulas from Proposition 4.4.4. Since this has to be true also for an arbitrary Riemannian manifold we obtain an expression for $\delta R^{\pm}\psi$, which proves Proposition 4.4.5. On Einstein manifolds we have $(\delta R)(X) = 0$ for any vector field X, which implies $\delta R^{\pm}\psi = 0$. This leads to a nice simplification of all formulas involving these terms, which we will discuss in the next section.

4.5 Forms in the middle dimension

In this part we return to the situation of conformal Killing n-forms on a $2n$-dimensional manifold. Let (M^{2n}, g) be a Riemannian manifold, then the n-form bundle splits corresponding to the eigenspaces of the Hodge star operator. If $n \equiv 2 \mod 4$ one has to consider the complexifications of the form bundles. We introduce the notation $\Lambda^n T^*M = \Lambda^n_+ T^*M \oplus \Lambda^n_- T^*M$ with the corresponding projections $\mathrm{pr}_{\Lambda^\pm}$ and an n–form ψ is decomposed accordingly as $\psi = \psi_+ + \psi_-$. Recall from Corollary 1.1.5 that conformal Killing

n-forms on a compact manifold M^{2n} are characterized by the equation

$$\Delta \psi \;=\; \tfrac{n+1}{n}\, 2q(R)\, \psi \;, \tag{4.5.7}$$

which also holds on non-compact manifolds. Moreover, for a conformal Killing n-form ψ its components $\psi_\pm$ are again conformal Killing n-forms. This follows immediately from Corollary 1.1.2 or the fact that the Hodge star operator commutes with the Laplacian. Assume that $n \equiv 0 \mod 4$, then the projections are given by $\mathrm{pr}_{\Lambda^\pm}(\psi) = \frac{1}{2}(\psi \pm *\psi)$ and we define $d_\pm := \mathrm{pr}_{\Lambda^\pm} \circ d$. Because of

$$*d\, d^* \;=\; d^* d\, * \qquad \text{and} \qquad *\, d^* d \;=\; d\, d^* *$$

we derive for a form $\psi \in \Lambda^n_+$, i.e. for a self-dual form, the equations

$$d_+\, d^* \psi \;=\; \tfrac{1}{2}\, \Delta\psi \qquad \text{and} \qquad d_-\, d^*\psi \;=\; \tfrac{1}{2}\, (dd^*\psi \;-\; d^* d\psi) \;.$$

For a form in $\Lambda^n_- T^*M$ we have to interchange $+$ and $-$ and if $n \not\equiv 0 \mod 4$ we need additional coefficients $\pm i$. Hence, in the case of forms in the middle dimension, we have to consider the following modified Killing bundle

$$\mathcal{C}K^\pm(M) \;:=\; \Lambda^\pm T^*M \oplus \Lambda^{n-1}T^*M \oplus \Lambda^n_\pm T^*M \;.$$

Note that since $\Lambda^{n-1}T^*M \cong \Lambda^{n+1}T^*M$, the two bundles $\mathcal{C}K^\pm(M)$ sum up to the Killing bundle $\mathcal{C}K^n(M)$ defined in the beginning of this section. For any conformal Killing n-form $\psi = \psi_+ + \psi_-$ we associate the sections $\Psi_\pm := (\psi_\pm,\, d^*\psi_\pm,\, d_\mp\, d^*\psi_\pm)$. The next task is to introduce a connection on $\mathcal{C}K^n(M)$ such that its parallel sections are in 1-1 correspondence to conformal Killing n-forms. Equivalently, we have to show that, starting with a conformal Killing form ψ, the covariant derivative of the components $\psi_\pm$, $d^*\psi_\pm$ and $d_\mp\,(d^*\psi_\pm)$ can be expressed in these components and zero order terms involving the curvature. For $\psi_\pm$ resp. $d^*\psi_\pm$ we know this already from the definition resp. Proposition 4.1.1, where we still have to replace the summand $d\, d^*\psi_\pm$ by a linear combination of $2q(R)\psi$ and $d_\mp\,(d^*\psi_\pm)$. This is done using the formula (4.5.7). A little bit more complicated is the third component. We note that because of $d = \pm * d^* *$, the covariant derivative of $d_\mp\,(d^*\psi_\pm)$ has only 2 components. These are

$$d(d_\mp d^*\psi) \;=\; \mp\, d(d_\pm d^*\psi) \;=\; \mp\, d(\tfrac{n+2}{2n}\, q(R)\, \psi) \qquad \text{and} \qquad T(d_\mp d^*\psi) \;.$$

Hence, it remains to compute $T(d_\mp d^*\psi)$, which can be achieved using the twistor Weitzenböck formula given in Corollaries 4.6.6 and 4.6.7. After a lengthy calculation the result is again that the covariant derivative of $T(d_\mp d^*\psi)$ only depends of $\psi_\pm$, $d^*\psi_\pm$ and zero order curvature terms. Summarizing this discussion we see that the statement of Theorem 4.3.2 remains true also for forms in the middle dimension.

4.6 Projections and embeddings

In this section we will consider the tensor product decomposition of $V^* \otimes \Lambda^p V^*$ and of $\Lambda^2 V^* \otimes \Lambda^p V^*$. We give explicit formulas for the projections onto the summands and for the embeddings of these summands into the tensor product. This will explain the constants appearing in the definition of the twistor operator and in the curvature condition. Moreover, it provides new proofs for several formulas. All computations are done for $\mathrm{SO}(n)$–representations. But of course they immediately translate into the corresponding statements on manifolds. We start with the decomposition of $V^* \otimes \Lambda^p V^*$. Here we have

$$V^* \otimes \Lambda^p V^* \;\cong\; \Lambda^{p-1} V^* \oplus \Lambda^{p+1} V^* \oplus \Lambda^{p,1} V^* \;. \tag{4.6.8}$$

In general, this is a decomposition into irreducible summands. We define projections $\pi_{\Lambda^{p\pm1}} : V^* \otimes \Lambda^p V^* \to \Lambda^{p\pm1} V^*$ as

$$\pi_{\Lambda^{p+1}}(X \otimes \psi) \;:=\; X \wedge \psi \qquad \text{and} \qquad \pi_{\Lambda^{p-1}}(X \otimes \psi) \;:=\; -\, X \lrcorner\, \psi \;.$$

Next, we define embeddings $i_{\Lambda^{p\pm1}} : \Lambda^{p\pm1} V^* \to V^* \otimes \Lambda^p V^*$ which are right inverses for the projection maps, i.e $\pi_{\Lambda^{p+1}} \circ i_{\Lambda^{p\pm1}} = \mathrm{id}$. The suitable definition is

$$i_{\Lambda^{p+1}}(\psi) \;:=\; \tfrac{1}{p+1} \sum e_i^* \otimes (e_i \lrcorner\, \psi) \qquad \text{and} \qquad i_{\Lambda^{p-1}}(\psi) \;:=\; \tfrac{1}{n-p+1} \sum e_i^* \otimes (e_i^* \wedge \psi) \;,$$

where $\{e_i\}$ is an ortho-normal basis of V. Finally we define the maps $\widetilde{\pi}_{\Lambda^{p\pm1}} := i_{\Lambda^{p\pm1}} \circ \pi_{\Lambda^{p+1}}$, which turn out to be projections from $V^* \otimes \Lambda^p V^*$ onto the summands $\Lambda^{p\pm1} V^*$ embedded into the tensor product. For the projection $\pi_{\Lambda^{p,1}}$ onto the summand $\Lambda^{p,1} V^* \subset V^* \otimes \Lambda^p V^*$ it then immediately follows that $\pi_{\Lambda^{p,1}} = \mathrm{id} - \widetilde{\pi}_{\Lambda^{p+1}} - \widetilde{\pi}_{\Lambda^{p-1}}$. The explicit expression of this definition was already given in Chapter 1. The following formulas are direct consequences of Lemma A.0.3.

$$\pi_{\Lambda^{p\pm1}} \circ i_{\Lambda^{p\pm1}} \;=\; \mathrm{id}_{\,\Lambda^{p\pm1}} \qquad \text{and} \qquad (\widetilde{\pi}_{\Lambda^{p\pm1}})^2 \;=\; \widetilde{\pi}_{\Lambda^{p\pm1}} \;, \tag{4.6.9}$$

$$|i_{\Lambda^{p+1}}(\psi)|^2 \;=\; \tfrac{1}{p+1}\, |\psi|^2 \qquad \text{and} \qquad |i_{\Lambda^{p-1}}(\psi)|^2 \;=\; \tfrac{1}{n-p+1}\, |\psi|^2 \;. \tag{4.6.10}$$

All these definitions and formulas translate to Riemannian manifolds, where we use the same notation. Let ψ be any p–form, then $\nabla\psi$ is a section of $\Lambda^1 T^*M \otimes \Lambda^p T^*M$ and we have of course $\pi_{\Lambda^{p+1}}(\nabla\psi) = d\psi$, $\pi_{\Lambda^{p-1}}(\nabla\psi) = d^*\psi$ and $\pi_{\Lambda^{p,1}}(\nabla\psi) = T\psi$.

In the rest of this section we will study the decomposition of $\Lambda^2 V^* \otimes \Lambda^p V^*$. This involves six summands and thus the calculations become slightly more complicated. Nevertheless, they remain completely elementary and are again based on Lemma A.0.3. It is interesting to study this decomposition since $R(\cdot,\cdot)\,\psi$ is a section of the vector bundle associated to the representation $\Lambda^2 V^* \otimes \Lambda^p V^*$. Hence, we can reformulate the curvature condition (4.2.1) (c.f. Corollary 4.2.4). We consider the the following decomposition:

$$\Lambda^2 V^* \otimes \Lambda^p V^* \;\cong\; \Lambda^p V^* \oplus \Lambda^{p+1,1} V^* \oplus \Lambda^{p-1,1} V^* \oplus \Lambda^{p+2} V^* \oplus \Lambda^{p-2} V^* \oplus \Lambda^{p,2} V^* \;. \tag{4.6.11}$$

The notation is the same as above, i.e. $\Lambda^{r,s}V^*$ is defined as the irreducible representation which has as highest weight the sum of the highest weights of $\Lambda^r V^*$ and $\Lambda^s V^*$. We will now define projections and embeddings as we did for the above decomposition.

We start with the summand $\Lambda^p V^*$, there the natural projection is given by

$$\mathrm{pr}_{\Lambda^p} : \Lambda^2 V^* \otimes \Lambda^p V^* \to \Lambda^p V^* \quad \text{with} \qquad \mathrm{pr}_{\Lambda^p}(\omega \otimes \psi) := \omega \bullet \psi \, .$$

Recall that $\bullet$ denotes the natural action of $\Lambda^2 V^* \cong \mathfrak{so}(n)$ on the space of forms $\Lambda^* V^*$, i.e. for any vectors $X, Y \in V$ we have $(X \wedge Y) \bullet \psi := (Y^* \wedge X \lrcorner - X^* \wedge Y \lrcorner)\, \psi$ (c.f. Section A). The embedding j_{Λ^p} of $\Lambda^p V^*$ into the tensor product, which is right inverse to pr_{Λ^p}, is given as:

$$j_{\Lambda^p} : \Lambda^p V^* \to \Lambda^2 V^* \otimes \Lambda^p V^* \qquad \text{with} \quad j_{\Lambda^p}(\psi) := -\tfrac{1}{2p(n-p)} \sum (e_i^* \wedge e_j^*) \otimes (e_i^* \wedge e_j^*) \bullet \psi \, .$$

A similar calculation as for the decomposition (4.6.8) shows that $\widetilde{\mathrm{pr}}_{\Lambda^p} := j_{\Lambda^p} \circ \mathrm{pr}_{\Lambda^p}$ is a projection map onto the summand $\Lambda^p V^*$ embedded into the tensor product. As above we can translate all these definitions to Riemannian manifolds and if we consider $R(\cdot,\cdot)\,\psi$ as section of $\Lambda^2 T^*M \otimes \Lambda^p T^*M$ we obtain the following

Lemma 4.6.1 *Let (M^n, g) be a Riemannian manifold and ψ any p-form, then*

$$\mathrm{pr}_{\Lambda^p}(R(\cdot,\cdot)\,\psi) = 2q(R)\,\psi \quad \textit{and} \quad \widetilde{\mathrm{pr}}_{\Lambda^p}(R(\cdot,\cdot)\,\psi)_{X,Y} = -\tfrac{1}{p(n-p)}(X \wedge Y) \bullet 2q(R)\,\psi \, .$$

As the next summand in the decomposition (4.6.11) we consider $\Lambda^{p+1,1}V^*$. Here the natural projection is a Plücker differential, i.e. we define it as

$$\begin{aligned} \mathrm{pr}_{\Lambda^{p+1,1}} : \Lambda^2 V^* \otimes \Lambda^p V^* &\to \Lambda^{p+1,1} V^* \subset \Lambda^1 V^* \otimes \Lambda^{p+1} V^* \\ \omega \otimes \psi &\mapsto \pi_{\Lambda^{p+1,1}} \Big(\sum (e_i \lrcorner\, \omega) \otimes (e_i^* \wedge \psi) \Big) \end{aligned}$$

and obtain as right inverse the embedding $j_{\Lambda^{p+1,1}}$ of $\Lambda^{p+1,1}V^*$ into the tensor product which is defined by

$$\begin{aligned} j_{\Lambda^{p+1,1}} : \Lambda^{p+1,1} V^* \hookrightarrow \Lambda^1 V^* \otimes \Lambda^{p+1} &\to \Lambda^2 V^* \otimes \Lambda^p V^* \\ \lambda \otimes \psi &\mapsto -\tfrac{1}{p} \sum (e_i^* \wedge \lambda) \otimes (e_i \lrcorner\, \psi) \, . \end{aligned}$$

Again, we get a projection map $\widetilde{\mathrm{pr}}_{\Lambda^{p+1,1}} := j_{\Lambda^{p+1,1}} \circ \mathrm{pr}_{\Lambda^{p+1,1}}$, this time onto the summand $\Lambda^{p+1,1}V^*$ embedded into the tensor product. Applied to the curvature section $R(\cdot,\cdot)\,\psi$ we find

Lemma 4.6.2 *Let (M^n, g) be a Riemannian manifold and ψ any p-form. Then for any vector fields X, Y:*

$$\begin{aligned} \mathrm{pr}_{\Lambda^{p+1,1}}(R(\cdot,\cdot)\,\psi)_X &= -R^+(X)\,\psi - \tfrac{1}{n-p} X \wedge 2q(R)\,\psi \, , \\ \widetilde{\mathrm{pr}}_{\Lambda^{p+1,1}}(R(\cdot,\cdot)\,\psi)_{X,Y} &= \tfrac{1}{p} \Big(\tfrac{1}{n-p}(X \wedge Y) * 2q(R)\psi - \big[X \lrcorner\, R^+(Y) - Y \lrcorner\, R^+(X) \big]\, \psi \Big) \, . \end{aligned}$$

The projection onto the summand $\Lambda^{p-1,1}V^*$ is similar. Here we define it as

$$\begin{aligned} \mathrm{pr}_{\Lambda^{p-1,1}} : \Lambda^2 V^* \otimes \Lambda^p V^* &\to \Lambda^{p-1,1}V^* \subset \Lambda^1 V^* \otimes \Lambda^{p-1}V^* \\ \omega \otimes \psi &\mapsto \pi_{\Lambda^{p-1,1}} \left(\sum (e_i \lrcorner\, \omega) \otimes (e_i \lrcorner\, \psi) \right) . \end{aligned}$$

The right inverse embedding $j_{\Lambda^{p-1,1}}$ of $\Lambda^{p-1,1}V^*$ into the tensor product is defined as

$$\begin{aligned} j_{\Lambda^{p-1,1}} : \Lambda^{p-1,1}V^* \hookrightarrow \Lambda^1 V^* \otimes \Lambda^{p-1} &\to \Lambda^2 V^* \otimes \Lambda^p V^* \\ \lambda \otimes \psi &\mapsto -\tfrac{1}{n-p} \sum (e_i^* \wedge \lambda) \otimes (e_i^* \wedge \psi) . \end{aligned}$$

As above we get a projection map $\widetilde{\mathrm{pr}}_{\Lambda^{p-1,1}} := j_{\Lambda^{p-1,1}} \circ \mathrm{pr}_{\Lambda^{p-1,1}}$, in this case onto the summand $\Lambda^{p-1,1}V^*$ embedded into the tensor product. Applied to the curvature section $R(\cdot,\cdot)\,\psi$ we obtain

Lemma 4.6.3 *Let (M^n, g) be a Riemannian manifold and ψ any p-form. Then for any vector fields X, Y:*

$$\begin{aligned} \mathrm{pr}_{\Lambda^{p-1,1}} (R(\cdot,\cdot)\,\psi)_X &= -R^-(X)\,\psi - \tfrac{1}{p} X \lrcorner\, 2q(R)\,\psi , \\ \widetilde{\mathrm{pr}}_{\Lambda^{p-1,1}} (R(\cdot,\cdot)\,\psi)_{X,Y} &= \tfrac{1}{n-p} \left(\tfrac{1}{p}(X \wedge Y) * 2q(R)\psi - \left[X \wedge R^-(Y) - Y \wedge R^-(X)\right] \psi \right) . \end{aligned}$$

For the next two summands of the decomposition (4.6.11) we only need to define the projections. In fact it turns out that under these projections the curvature section $R(\cdot,\cdot)\,\psi$ is mapped to zero. As natural definitions for $\mathrm{pr}_\pm : \Lambda^2 V^* \otimes \Lambda^p V^* \to \Lambda^{p\pm 2}V^*$ we take

$$\mathrm{pr}_+(\omega \otimes \psi) := \omega \wedge \psi \qquad \text{and} \qquad \mathrm{pr}_-(\omega \otimes \psi) := \omega \lrcorner\, \psi .$$

The following lemma is then a well-known reformulation of the first Bianchi identity

Lemma 4.6.4 *Let ψ be any p-form, then*

$$\mathrm{pr}_\pm (R(\cdot,\cdot)\,\psi) = 0 .$$

The last summand, $\Lambda^{p,2}V^*$ is already a subset of the tensor product $\Lambda^2 V^* \otimes \Lambda^p V^*$, i.e. the projection $\mathrm{pr}_{\Lambda^{p,2}}$ is given as:

$$\mathrm{pr}_{\Lambda^{p,2}} = \mathrm{id} - \widetilde{\mathrm{pr}}_{\Lambda^p} - \widetilde{\mathrm{pr}}_{\Lambda^{p+1,1}} - \widetilde{\mathrm{pr}}_{\Lambda^{p-1,1}} .$$

Using the explicit formulas of Lemmas 4.6.1, 4.6.2 and 4.6.3 we obtain an expression for $\mathrm{pr}_{\Lambda^{p,2}}$ applied to the curvature section.

Lemma 4.6.5 *Let (M^n, g) be a Riemannian manifold and ψ any p-form. Then for any vector fields X, Y:*

$$\mathrm{pr}_{\Lambda^{p,2}}(R(\cdot,\cdot)\,\psi) = R(X,Y)\,\psi - \tfrac{1}{p(n-p)}\,(X \wedge Y) * 2q(R)\,\psi$$
$$+ \tfrac{1}{p}\,\big(X \lrcorner R^+(Y) - Y \lrcorner R^+(X)\big)\,\psi + \tfrac{1}{n-p}\,\big(X \wedge R^-(Y) - Y \wedge R^-(X)\big)\,\psi\ .$$

Along the same lines we can give alternative proofs of several results of Chapter 4. For this we have to define further projections, which applied to $\nabla^2 u$ yield compositions of first order differential operators, e.g. $Td\psi$ or Td^*u. It is then easy to deduce relations between the projections which translates into formulas for the corresponding differential operators. As first projection we define

$$\begin{array}{rcccc} \mathrm{pr}_1^+ \;:\; T^*M \otimes T^*M \otimes \Lambda^p T^*M & \to & T^*M \otimes \Lambda^{p+1}T^*M & \to & \Lambda^{p+1,1}T^*M \\ e_1 \otimes e_2 \otimes \psi & \mapsto & e_1 \otimes (e_2 \wedge \psi) & \mapsto & \mathrm{pr}_{\Lambda^{p+1,1}}(e_1 \otimes (e_2 \wedge \psi))\ . \end{array}$$

Let ψ be any p-form then $\nabla^2\psi$ is a section of $T^*M \otimes T^*M \otimes \Lambda^p T^*M$ and it is easy to show that $\mathrm{pr}_1^+(\nabla^2\psi) = T(d\psi)$. Next we need the map

$$\begin{array}{rcccc} \mathrm{pr}_2^+ \;:\; T^*M \otimes T^*M \otimes \Lambda^p T^*M & \to & T^*M \otimes \Lambda^{p,1}T^*M & \to & \Lambda^{p+1,1}T^*M \\ e_1 \otimes e_2 \otimes \psi & \mapsto & e_1 \otimes \mathrm{pr}_{\Lambda^{p,1}}(e_2 \otimes \psi) & \mapsto & \mathrm{pr}_{\Lambda^{p+1,1}}(e_1 \wedge \mathrm{pr}_{\Lambda^{p,1}}(e_2 \otimes \psi)). \end{array}$$

In this case there appears a new first order differential operator, which we denote by θ^+. We have

$$\mathrm{pr}_1^+(\nabla^2\psi) = \theta^+ T(\psi)\ .$$

The operator θ^+ is defined as $\mathrm{pr} \circ \nabla$, where pr is the projection $T^*M \otimes \Lambda^{p,1}T^*M \to \Lambda^{p+1,1}T^*M$ defined above. Then we need a third projection which will produce the curvature term. We define it as

$$\begin{array}{rcccc} \pi^+ \;:\; T^*M \otimes T^*M \otimes \Lambda^p T^*M & \to & \Lambda^2 T^*M \otimes \Lambda^p T^*M & \to & \Lambda^{p+1,1}T^*M \\ e_1 \otimes e_2 \otimes \psi & \mapsto & (e_1 \wedge e_2) \otimes \psi & \mapsto & \mathrm{pr}_{\Lambda^{p+1,1}}\big(\sum e_i \lrcorner (e_1 \wedge e_2) \otimes (e_1 \wedge \psi)\big)\ . \end{array}$$

By the first projection $\nabla^2\psi$ is mapped to the curvature $R(\cdot,\cdot)\psi$ and the second is, up to a constant, the projection discussed above. Hence, we obtain

$$\pi^+(\nabla^2\psi) = -\tfrac{1}{p}\,R^+(\cdot)\,\psi - \tfrac{1}{p(n-p)}\,\cdot \wedge 2q(R)\,\psi\ .$$

Having defined these three projections it is very easy to prove that they satisfy the following linear relation

$$(p+1)\,\pi^+ + \mathrm{pr}_1^+ = \tfrac{p+1}{p}\,\mathrm{pr}_2^+\ .$$

To obtain a twistor Weitzenböck formula we only have to apply this relation to $\nabla^2\psi$ and substitute the formulas for the three different projections of $\nabla^2\psi$. The result is

Corollary 4.6.6 *Let ψ be any p-form then for any vector field X:*

$$T(d\psi)\,X \;=\; \tfrac{p+1}{p}\,\theta^+(T\psi)\,X \;+\; \tfrac{p+1}{p}\,R^+(X)\,\psi \;+\; \tfrac{p+1}{p(n-p)}\,X\wedge 2q(R)\,\psi\ ,$$

$$\nabla_X(d\psi) \;=\; -\tfrac{1}{n-p}\,X\wedge d^*d\psi \;+\; \tfrac{p+1}{p}\,\theta^+(T\psi)(X) \;+\; \tfrac{p+1}{p}\,R^+(X)\,\psi \;+\; \tfrac{p+1}{p(n-p)}\,X\wedge 2q(R)\,\psi\ .$$

For a conformal Killing p-form ψ this yields exactly the first equation of Lemma 4.4.8. To obtain the second equation, i.e. the formula for $Td^*\psi$, we have again to define three projections and to find a linear relation between them. In this case we have

$$\mathrm{pr}_1^- \;:\; T^*M\otimes T^*M\otimes\Lambda^pT^*M \;\to\; T^*M\otimes\Lambda^{p-1}T^*M \;\to\; \Lambda^{p-1,1}T^*M$$

$$e_1\otimes e_2\otimes\psi \;\mapsto\; e_1\otimes(e_2\lrcorner\,\psi) \;\mapsto\; \mathrm{pr}_{\Lambda^{p-1,1}}\,(e_1\otimes(e_2\lrcorner\,\psi))\ .$$

This is the projection with $\mathrm{pr}_1^-(\nabla^2)\psi = T(d^*\psi)$. As second projection we define

$$\mathrm{pr}_2^- \;:\; T^*M\otimes T^*M\otimes\Lambda^pT^*M \;\to\; T^*M\otimes\Lambda^{p,1}T^*M \;\to\; \Lambda^{p-1,1}T^*M$$

$$e_1\otimes e_2\otimes\psi \;\mapsto\; e_1\otimes\mathrm{pr}_{\Lambda^{p,1}}(e_2\otimes\psi) \;\mapsto\; \mathrm{pr}_{\Lambda^{p-1,1}}\,(e_1\lrcorner\,\mathrm{pr}_{\Lambda^{p,1}}(e_2\otimes\psi))$$

This projection yields $\mathrm{pr}_2^-(\nabla^2)\psi = \theta^-T(\psi)$, where the differential operator θ^- is defined similar to θ^+. Finally, we again have a curvature part. In this case it is given as

$$\pi^- \;:\; T^*M\otimes T^*M\otimes\Lambda^pT^*M \;\to\; \Lambda^2T^*M\otimes\Lambda^pT^*M \;\to\; \Lambda^{p-1,1}T^*M$$

$$e_1\otimes e_2\otimes\psi \;\mapsto\; (e_1\wedge e_2)\otimes\psi \;\mapsto\; \mathrm{pr}_{\Lambda^{p-1,1}}\,(\textstyle\sum e_i\lrcorner\,(e_1\wedge e_2)\otimes(e_1\lrcorner\,\psi))\ .$$

From these definitions it is then easy to conclude that the projections satisfy the following linear relations

$$(n-p+1)\,\pi^- \;+\; \mathrm{pr}_1^- \;=\; \tfrac{n-p+1}{n-p}\,\mathrm{pr}_2^-\ .$$

Substituting the expressions obtained by applying the three projections to $\nabla^2\psi$ we conclude

Corollary 4.6.7 *Let ψ be any p-form then for any vector field X*

$$T(d^*\psi)\,X \;=\; \tfrac{n-p+1}{n-p}\,\theta^-(T\psi)\,X \;+\; -\,\tfrac{n-p+1}{n-p}\,R^-(X)\,\psi \;-\; \tfrac{n-p+1}{p(n-p)}\,X\lrcorner\,2q(R)\,\psi\ ,$$

$$\nabla_X(d^*\psi) \;=\; \tfrac{1}{p}\,X\lrcorner\,dd^*\psi \;+\; \tfrac{n-p+1}{n-p}\,\theta^-(T\psi) \;-\; \tfrac{n-p+1}{n-p}\,R^-(X)\,\psi \;-\; \tfrac{n-p+1}{p(n-p)}\,X\lrcorner\,2q(R)\,\psi\ .$$

Considering a conformal Killing form this proves the second equation of Lemma 4.4.8. There are still three further projections to define, leading from $T^*M\otimes T^*M\otimes\Lambda^pT^*M$ to Λ^pT^*M. The linear relation for these projections then proves the Weitzenböck formula (1.1.7).

Chapter 5

Holonomy decomposition

If the holonomy group of an oriented n–dimensional Riemannian manifold restricts to some subgroup of $\mathrm{SO}(n)$ the bundle of forms splits into parallel sub-bundles, i.e. bundles which are invariant under the Levi-Civita connection. These sub-bundles correspond to the irreducible summands into which the form representations splits after restriction to the holonomy group. It follows that on a manifold with restricted holonomy every differential form has a decomposition as a sum of forms lying in different parallel sub-bundles of the form bundle. In this chapter we will discuss the question whether the components of a conformal Killing form, appearing in such a decomposition, are again conformal Killing forms. Our results are mainly for Killing forms resp. $*$–Killing forms. In the first part we will consider irreducible manifolds with restricted holonomy, while in the second part we will study reducible manifolds with conformal Killing forms.

5.1 Irreducible manifolds with restricted holonomy

Let (M^n, g) be an oriented Riemannian manifold with holonomy group $G := \mathrm{Hol}\,(M, g)$ which is assumed to be a proper subgroup of $\mathrm{SO}(n)$. The bundle of p–form decomposes into a sum of parallel sub-bundles:

$$\Lambda^p(T^*M) \;\cong\; \sum V_i \; . \tag{5.1.1}$$

We assume that the vector bundles V_i correspond to the different isotypic components in the decomposition of the representation Λ^p restricted to G. As an immediate consequence we have

Lemma 5.1.1 *Let ψ be a Killing p-form and let $\psi = \sum \psi_i$ be the holonomy decomposition according to (5.1.1). Then*

$$\Delta\,\psi_i \;=\; \tfrac{p+1}{p}\,2q(R)\,\psi_i \; .$$

Proof. Since the decomposition (5.1.1) is parallel, the Laplace operator Δ and the curvature endomorphism $2q(R)$ commute with the projections onto the summands V_i, i.e. if ψ_i is a section of V_i then the same is true for $\Delta\psi_i$ resp. $2q(R)\psi_i$. But we know already $\Delta\psi = \frac{p+1}{p} 2q(R)\psi$ for a Killing form ψ. Hence, the proof of the lemma follows by projecting this equation onto the different summands V_i. $\square$

Note that the analogous result is true for $*$–Killing p–forms, which, because of the second Weitzenböck formula (1.1.8), satisfy the equation $\Delta\,\psi = \frac{n-p+1}{n-p}\,2q(R)\psi\,.$

On a compact manifold we can use Corollary 1.1.4 to conclude that a component ψ_i is again a Killing form if and only if it is coclosed. Unfortunately it is in general not clear whether a coclosed form splits under the holonomy decomposition into a sum of coclosed forms. Nevertheless, we have the following special case, which at the same time is more general since it is a statement for conformal Killing forms.

Proposition 5.1.2 *Let $(M^{2m},\, g)$ be a compact Riemannian manifold with restricted holonomy and let ψ be any m–form with holonomy decomposition $\psi = \sum \psi_i\,.$ Then ψ is a conformal Killing form if and only if the same is true for all components ψ_i.*

Proof. In the special case where the dimension of the manifold M is twice the degree of the form ψ, the integrability condition of Corollary 1.1.3 tells us that ψ is a conformal Killing m–form if and only if $2q(R)\psi = \frac{m}{m+1}\,\Delta\psi$. Hence, we can argue as in the proof of the lemma above. $\square$

A similar statement is true for compact manifolds with holonomy G_2 or Spin_7. In these cases we have

Proposition 5.1.3 *Let $(M,\, g)$ be a compact manifold with holonomy G_2 or Spin_7 and let ψ be any form with holonomy decomposition $\psi = \sum \psi_i\,.$ Then ψ is a Killing form or a $*$–Killing form if and only if the same is true for all components ψ_i.*

Proof. Since manifolds with holonomy G_2 or Spin_7 are Ricci flat we do not have to consider Killing 1–forms, which are automatically parallel. We start with the G_2–case, where it is enough to consider Killing forms of degree 2 and 3. The proof for $*$–Killing forms of degree 2 and 3 is then completely the same and by duality under the Hodge star operator this covers all possible cases. We have the following decompositions of the relevant form spaces:

$$\Lambda^2 \;=\; \Lambda^2_7 \oplus \Lambda^2_{14} \qquad \text{and} \qquad \Lambda^3 \;=\; \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}\;.$$

Here Λ^p_r denotes the irreducible summand of dimension r in the decomposition of the G_2–representation Λ^p. In particular, Λ^3_1 is the 1-dimensional space spanned by the invariant 3–form ω, which gives rise to a parallel form on M defining the holonomy reduction to

G_2. Let $\psi = \psi_7 + \psi_{14}$ be the holonomy decomposition of a Killing 2-form ψ. Then we know already

$$\Delta\, \psi_7 \;=\; \tfrac{3}{2}\, 2q(R)\, \psi_7 \;=\; \tfrac{3}{2}\, \mathrm{Ric}\,(\psi_7) \;=\; 0 \;.$$

Note that the action of the curvature endomorphism $q(R)$ only depends of the representation defining the vector bundle on which it acts. Hence, ψ_7 has to be parallel, which implies that the holonomy is a proper subgroup of G_2. In any case, $\psi_{14} = \psi - \psi_7$ is coclosed and we have proved the proposition for 2-forms. The case of 3-forms is similar here we have $\psi = \psi_1 + \psi_7 + \psi_{27}$. As above we see immediately that ψ_1 and ψ_7 have to be parallel. Thus, ψ_{27} is coclosed, which completes the proof for 3–forms.

We now turn to the case of holonomy Spin_7. The case of 4-forms already follows from Proposition 5.1.2. It remains to consider 2- and 3-forms, where we have the decompositions

$$\Lambda^2 \;=\; \Lambda^2_7 \oplus \Lambda^2_{21} \qquad \text{and} \qquad \Lambda^3 \;=\; \Lambda^3_8 \oplus \Lambda^3_{48} \;.$$

The representation Λ^3_8 is isomorphic to the standard representation, defining the bundle of 1-forms, i.e. the curvature endomorphism $2q(R)$ acts as the Ricci tensor and has to vanish. As above we see that the Λ^3_8–components of Killing forms have to be parallel. There is still another argument to show that $2q(R)$ acts trivially on Λ^3_8. Indeed, the spinor bundle of a manifold with Spin_7–holonomy splits into the sum of a trivial line bundle, corresponding to the parallel spinor, and the sum of a bundle of rank 7 and a bundle of rank 8. These two bundles are induced by the 8-dimensional standard representation and by Λ^2_7. It is well-known that $2q(R)$ acts as $\frac{s}{16}\mathrm{id}$ on the summands of the spinor bundle. But for Spin_7–manifolds the scalar curvature s is zero and we conclude that $2q(R)$ acts trivially on Λ^2_7 and Λ^2_8. The rest of the argument is the same as in in the G_2–case. □

5.2 Conformal Killing forms on Riemannian products

The decomposition of a manifold as a Riemannian product is of course a special case of a holonomy reduction. It is known for the isometry group of a Riemannian product that it is the product of the isometry groups of the factors, i.e. any Killing 1-form on such a product can be decomposed as a sum of 1-forms, which are Killing 1-forms on the different factors. The following proposition gives the generalization of this property for Killing forms of higher degree.

Proposition 5.2.1 *Let $(M,\, g)$ be a compact manifold, which is the Riemannian product of $(M_1,\, g_1)$ and $(M_2,\, g_2)$. Then $\Lambda^p(T^*M) = \sum_{r=0}^p \Lambda^r(T^*M_1) \otimes \Lambda^{p-r}(T^*M_2)$ and $\psi \in \Lambda^p(T^*M)$ decomposes as $\psi = \sum_{r=0}^p \psi_r$. If ψ is a Killing p–form, then the same is true for all components ψ_r. Moreover, the components ψ_0 resp. ψ_p projects to Killing p–forms on $(M_1,\, g_1)$ resp. $(M_2,\, g_2)$.*

Proof. Let $\psi = \sum \psi_r$ be the decomposition, where ψ_r is a section of $\Lambda^r(T^*M_1) \otimes \Lambda^{p-r}(T^*M_2)$. Since (M, g) is a Riemannian product of (M_1, g_1) and (M_2, g_2) we conclude that for any vector field X also $\nabla_X(\psi_r)$ is a section of $\Lambda^r(T^*M_1) \otimes \Lambda^{p-r}(T^*M_2)$. We will use the characterization of a Killing form as a form ψ with $X \lrcorner \nabla_X(\psi) = 0$. Assuming that ψ is a Killing p–form we have

$$0 \;=\; X \lrcorner \nabla_X(\psi) \;=\; \sum X \lrcorner \nabla_X(\psi_r).$$

If X is defined by a vector field on M_1 or M_2 then all summands on the right hand side belong to different spaces and we conclude $X \lrcorner \nabla_X(\psi_r) = 0$. Since we can choose an adapted local basis, such that the first vectors form a local basis of TM_1 and the rest a local basis of TM_2, we also have $d^*\psi_r = 0$ for all r. Finally, we conclude as in the proof of Lemma 5.1.1, that all components satisfy the equation $\Delta\psi_r = \frac{p+1}{p} 2q(R)\psi_r$, i.e. according to Corollary 1.1.3 they are all Killing p–forms.

For the proof of the second statement we consider the component $\psi_0 \in \Lambda^0(T^*M_1) \otimes \Lambda^p(T^*M_2)$ and vector fields $X_a \in \Gamma(TM_a)$, $a = 1, 2$. Since ψ_0 is a Killing form it follows

$$X_2 \lrcorner \nabla_{X_1}(\psi_0) \;=\; -X_1 \lrcorner \nabla_{X_2}(\psi_0) \;=\; 0\ .$$

Locally we can write ψ_0 as $f\,\omega$, where $\omega \in \Lambda^p(T^*M_2)$ and $\nabla_{X_1}\omega = 0$. This yields $X_1(f) = 0$ for any vector field $X_1 \in \Gamma(TM_1)$, i.e. ψ_0 projects onto a p–form on M_2, which is again a Killing p–form. The proof for the component ψ_p is completely analogous.
□

Up to now we only have considered Killing forms. We close this section by citing an interesting result of S. Sato, which states that on a reducible manifold any conformal Killing form of small degree is already a Killing form. More precisely we have

Theorem 5.2.2 (S. Sato, [S70]) *Let (M^n, g) be a compact Riemannian manifold, which is locally isometric to the product of two Riemannian manifolds. Let $p \le n/2$ be the dimension of one of the factors. Then any conformal Killing r form, with $3(r-1) < 2p$ is already a Killing form.*

Chapter 6

Non-existence results

We already mentioned in Chapter 1 that there are no conformal Killing forms on a compact manifold on which the curvature endomorphism $2q(R)$ has only negative eigenvalues, e.g. on manifolds with constant negative sectional curvature. Slightly more generally we could say, that on a compact manifold the condition $2q(R) \leq 0$ implies that any conformal Killing form has to be parallel. This was an immediate consequence of the second Weitzenböck formula (c.f. Corollary 1.1.3). In this chapter we will show that such non-existence phenomena occur on a much wider class of manifolds, e.g. on Kähler and on G_2–manifolds.

6.1 Conformal Killing forms on Kähler manifolds

Results for Kähler manifolds have already been known for some time. It was first shown in [Y75] that on a compact Kähler manifold any Killing form has to be parallel. In fact, this is very easy to see, so we include it here with a short proof. Some years later S. Yamaguchi et al. showed in [JAY85] that with a few exceptions any conformal Killing form on a compact Kähler manifold has to be parallel. We will start with the case of Killing forms and give afterwards the result for conformal Killing forms. At first we have the following simple lemma which is true not only for Kähler manifolds.

Lemma 6.1.1 *Let $(M,\, g)$ be a compact Riemannian manifold admitting a parallel form ω. Then contraction with ω maps Killing forms to parallel forms, i.e. if ψ is a Killing form then for any vector field X:*

$$\nabla_X \, (\omega \lrcorner \, \psi) \;\; \equiv \;\; 0 \; .$$

Proof. Let ω be a parallel r–form and let ψ be a Killing p–form, with $p > r$. We first show that $\omega \lrcorner \, \psi$ is again a Killing form. For this we use the characterization of Killing forms as forms satisfying $X \lrcorner \, \nabla_X(\psi) = 0$ for any vector field X. In the present case we have

$$X \lrcorner \nabla_X(\omega \lrcorner \psi) \;=\; X \lrcorner ((\nabla_X \omega) \lrcorner \psi \;+\; \omega \lrcorner \nabla_X \psi) \;=\; X \lrcorner \omega \lrcorner \nabla_X \psi \;=\; 0 \;.$$

Since ψ is a Killing p–form we know that $\Delta\psi = \frac{p+1}{p}\, 2q(R)\psi$. The contraction with a parallel form commutes with the Laplace operator and with the curvature endomorphism $2q(R)$. Thus, we find $\Delta(\omega \lrcorner \psi) = \frac{p+1}{p} 2q(R)(\omega \lrcorner \psi)$. On the other hand we obtained that $\omega \lrcorner \psi$ is a Killing $(p-r)$–form and hence satisfies the equation $\Delta(\omega \lrcorner \psi) = \frac{p-r+1}{p-r} 2q(R)(\omega \lrcorner \psi)$. Comparing these two equations for $\Delta(\omega \lrcorner \psi)$, shows that $\omega \lrcorner \psi$ has to be a harmonic Killing form, which on a compact manifold implies that it has to be parallel.

In the case $p = r$ we have that $\omega \lrcorner \psi = g(\omega, \psi)$ is a function, i.e. $0 = 2q(R)(\omega \lrcorner \psi) = \frac{p}{p+1}\, \Delta(\omega \lrcorner \psi)$. Hence, the function $\omega \lrcorner \psi$ has to be constant, i.e. parallel. □

For compact Kähler manifolds it is now only a small step to show that any Killing form is parallel. It seems to be a reasonable conjecture that the same is true for any irreducible compact Riemannian manifold admitting a non-trivial parallel form.

Proposition 6.1.2 *Let (M^{2n}, g, J) be a compact Kähler manifold. Then any Killing p-form and any $*$–Killing p-form on M, with $2 \le p \le 2n-2$, is parallel.*

Proof. Let ω be the Kähler form, which is a parallel 2-form. We know already that for a Killing p–form ψ the $(p-2)$–form $\omega \lrcorner \psi$ is parallel. Applying Corollary 4.1.3 for the Killing form ψ we get $p\, \nabla^2_{X,Y}\, \psi = Y \lrcorner R^+(X)\psi$. Contracting this equation with ω leads to $0 = Y \lrcorner \omega \lrcorner R^+(X)\psi$, which has to be true for any vector field X and Y. Hence,

$$0 \;=\; \omega \lrcorner R^+(X)\, \psi \;=\; R^+(X)(\omega \lrcorner \psi) \;+\; R^-(JX)\, \psi \;=\; R^-(JX)\, \psi \;.$$

Because of $2q(R) = -\sum e_i \wedge R^-(e_i)$, this equation implies $2q(R)\psi = 0$. Since ψ is a Killing form it also follows $\Delta\psi = 0 = \nabla^*\nabla\psi$, which on a compact manifold is equivalent to $\nabla\psi \equiv 0$. Finally, duality under the Hodge star operator implies the statement for $*$–Killing forms. □

To get a similar statement for conformal Killing forms is much more difficult. The proof only simplifies for Kähler–Einstein manifolds or for *primitive* forms, i.e. forms in the kernel of the contraction with the Kähler form. The following result was proved in [JAY85]

Theorem 6.1.3 *Let (M^{2n}, g, J) be a compact Kähler manifold.*

1. *If $2n \ge 10$ then any conformal Killing p–form $(n > p \ge 4)$ is parallel.*
2. *If $2n \ge 6$ and $2n \ne 10$ then any conformal Killing 3-form is parallel.*
3. *Any primitive conformal Killing p–form is parallel $(n \ge p \ge 3)$.*

Note that the theorem cannot hold for conformal Killing 1-forms. Indeed, on compact Kähler manifolds conformal vector fields are automatically Killing vector fields and there are of course compact Kähler manifolds with non-parallel Killing vector fields. Not included in the statement of the theorem are conformal Killing 2-forms, conformal Killing n-forms on $2n$-dimensional Kähler manifolds and conformal Killing 3-forms on 10-dimensional Kähler manifolds. In fact [JAY85] contains wrong statements on primitive conformal Killing 2-forms on 4-dimensional Kähler manifolds and on non-primitive conformal Killing n-forms on $2n$-dimensional Kähler manifolds. As we will see it is not possible to exclude these cases. Moreover, it possible to prove that the exceptional case $(2n, p) = (10, 3)$ cannot occur.

Proposition 6.1.4 *Let $(M,\, g,\, J)$ be a 10-dimensional compact Kähler manifold, then any conformal Killing 3–form is parallel.*

The following theorem clarifies the situation of conformal Killing n-forms on $2n$-dimensional Kähler manifolds. More details of it and also an index-free proof of Theorem 6.1.3 will appear in the forthcoming paper [MS01].

Theorem 6.1.5 *Let $(M^{2n},\, g,\, J)$ be a compact Kähler manifold and let u be any n-form. Then u is a conformal Killing n-form if and only if $u = L^k(u_0)$, where u_0 is the primitive part of an invariant conformal Killing 2-form. Here, L denotes the wedging with the Kähler form.*

We will now give the proof of Proposition 6.1.4. Unfortunately it involves several elementary but rather lengthy calculations. Nevertheless, it also indicates how the proof in general case can be achieved. At first we have to recall several basic definitions and facts.

On a complex manifold the differential splits as $d = \partial + \bar{\partial}$. Moreover, one has the following real differential operator

$$d^c \;=\; i\,(\bar{\partial} \,-\, \partial) \;=\; \sum Je_i \wedge \nabla_{e_i} \;.$$

Its adjoint is denoted by δ^c. If ω is the Kähler form then Λ denotes the contraction with ω and L the wedge product with ω. The following commutator relations are essential for all calculations of this section

$$d^c \;=\; -\,[\,d^*,\, L\,] \;=\; -\,[\,d,\, J\,]\,, \qquad \delta^c \;=\; [\,d,\, \Lambda\,] \;=\; -\,[\,d^*,\, J\,]\;,$$
$$d \;=\; [\,\delta^c,\, L\,] \;=\; [\,d^c,\, J\,]\;, \qquad d^* \;=\; -\,[\,d^c,\, \Lambda\,] \;=\; [\,\delta^c,\, J\,]\;.$$

Then, the following commutators vanish

$$0 \;=\; [\,d,\, L\,] \;=\; [\,d^c,\, L\,] \;=\; [\,d^*,\, \Lambda\,] \;=\; [\,\delta^c,\, \Lambda\,] \;=\; [\,\Lambda,\, J\,] \;=\; [\,J,\, *\,]\;.$$

Moreover, the following anti-commutators vanish

$$0 \;=\; d^*d^c \;+\; d^cd^* \;=\; dd^c + d^cd \;=\; d^*\delta^c \;+\; \delta^cd^* \;+\; d\delta^c \;+\; \delta^cd \;.$$

Combining these elementary identities we get

$$\begin{aligned} [\,\Lambda,\, dd^*\,] \;&=\; -\,\delta^cd^* \;, \qquad [\,\Lambda,\, d^*d\,] \;=\; -\,d^*\delta^c \;, \qquad [\,\Lambda,\, dd^c\,] \;=\; dd^* \;-\; \delta^cd^c \;, \\ 0 \;&=\; [\,\Lambda,\, d\delta^c\,] \;=\; [\,\Lambda,\, d^cd^*\,] \;, \\ [\,J,\, dd^*\,] \;&=\; d\delta^c \;+\; d^cd^* \;, \qquad [\,J,\, d^*d\,] \;=\; d^*d^c \;+\; \delta^cd \;, \qquad [\,J,\, dd^c\,] \;=\; 0 \;. \end{aligned}$$

Having all these relations we can now consider the conformal Killing equation on Kähler manifolds. We start to compute $\delta^c u$ for a conformal Killing p-form u. We obtain

$$\begin{aligned} \delta^c u \;&=\; -\sum Je_i \,\lrcorner\, \nabla_{e_i} u \;=\; -\sum Je_i \,\lrcorner\, (\tfrac{1}{p+1}\, e_i \,\lrcorner\, du \;-\; \tfrac{1}{n-p+1}\, e_i \,\wedge\, d^*u) \\ &=\; -\,\tfrac{2}{p+1}\,\Lambda(du) \;+\; \tfrac{1}{n-p+1}\,J(d^*u) \;=\; -\,\tfrac{2(n-p+1)}{(n-p)(p+1)}\,\Lambda(du) \;+\; (n-p)\,d^*Ju \\ &=\; -\,\tfrac{2}{p+1}\,d\Lambda(u) \;+\; \tfrac{2}{p+1}\,\delta^c u \;+\; \tfrac{1}{n-p+1}\,J(d^*u) \\ &=\; -\,\tfrac{2}{p-1}\,d\Lambda(u) \;+\; \tfrac{p+1}{(p-1)(n-p+1)}\,J(d^*u) \;. \end{aligned}$$

From this we conclude

$$\begin{aligned} \delta^c\, d\,u \;&=\; -\,d\,\delta^c\,u \;=\; -\,\tfrac{p+1}{(p-1)(n-p+1)}\,J(dd^*u) \;+\; \tfrac{p+1}{(p-1)(n-p+1)}\,d^cd^*\,u \;, \\ d^*\delta^c\,u \;&=\; -\,\delta^cd^*u \;=\; -\,\tfrac{2(n-p+1)}{(n-p)(p+1)}\,\Lambda(d^*du) \;. \end{aligned}$$

A similar calculation we can do for the operator d^c. Here we obtain

$$\begin{aligned} d^c\,u \;&=\; -\sum e_i \,\wedge\, \nabla_{Je_i}\, u \;=\; -\sum e_i \,\wedge\, (\tfrac{1}{p+1}\, Je_i \,\lrcorner\, du \;-\; \tfrac{1}{n-p+1}\, Je_i \,\wedge\, d^*u) \\ &=\; \tfrac{1}{p+1}\,J(du) \;+\; \tfrac{2}{n-p+1}\,L(d^*u) \;=\; \tfrac{1}{p+1}\,J(du) \;+\; \tfrac{2}{n-p+1}\,d^*(Lu) \;+\; \tfrac{2}{n-p+1}\,d^cu \\ &=\; \tfrac{n-p+1}{(p+1)(n-p-1)}\,J(du) \;+\; \tfrac{2}{n-p+1}\,d^*(Lu) \;. \end{aligned}$$

We apply this to conclude

$$d^*d^c\,u \;=\; \tfrac{n-p+1}{(p+1)(n-p-1)}\,(\,Jd^*du \;-\; \delta^cdu\,) \;.$$

Using the various commutator rules we obtain

$$\begin{aligned} J\Lambda(dd^*u) \;&=\; J(dd^*\Lambda u \;-\; \delta^cd^*u) \\ &=\; dd^*(J\Lambda u) \;+\; d\delta^c(\Lambda u) \;+\; d^cd^*(\Lambda u) \;-\; J\delta^cd^*u \end{aligned}$$

and similarly

$$\begin{aligned} J\Lambda(d^*du) &= J(d^*d(\Lambda u) - d^*\delta^c u) \\ &= d^*d(J\Lambda u) + d^*d^c(\Lambda u) + \delta^c d(\Lambda u) - Jd^*\delta^c u \ . \end{aligned}$$

Next, we compute $\mathrm{Ric}\,(Ju)$ for a conformal Killing p-form u by using the curvature identity. The result is the following Lemma which was also proved in [JAY85].

Lemma 6.1.6 *Let (M^n, g, J) be a Kähler manifold with a conformal Killing p-form u. Then*

$$[\,p(n-p) - n\,]\,\mathrm{Ric}\,(Ju) + 2q(R)\,(Ju) = 0 \ .$$

On the other side, we can contract the curvature identity with the Kähler form. This yields for a conformal Killing form u the equation

$$(n-p)\,\mathrm{Ric}\,(J\Lambda u) + 2q(R)(J\Lambda u) = 0 \ .$$

Comparing this with the equation obtained by contracting the equation of Lemma 6.1.6 we arrive at (c.f. [JAY85])

Corollary 6.1.7 *Let (M^n, g, J) be a Kähler manifold. If u is a conformal Killing p-form with $p \neq n-1$, then*

$$\mathrm{Ric}\,(J\Lambda u) = 0 = 2q(R)(J\Lambda u) \ .$$

Let us introduce the following notation

$$\begin{aligned} x &:= J\Lambda(dd^*u), \qquad y := J\Lambda(d^*du), \qquad a := dd^*(J\Lambda u), \\ b &:= d^*d(J\Lambda u), \qquad \alpha := \delta^c d(\Lambda u), \qquad \beta := d^*d^c(\Lambda u) \ . \end{aligned}$$

Summarizing all the equations obtained so far we have

$$\begin{aligned} &(1) \qquad \tfrac{p}{p+1}\, y + \tfrac{n-p}{n-p+1}\, x = 0 \ , \\ &(2) \qquad x = a - [\alpha + \beta] - \tfrac{2(n-p+1)}{(n-p)(p+1)}\, y \ , \\ &(3) \qquad y = b + [\alpha + \beta] + \tfrac{2(n-p+1)}{(n-p)(p+1)}\, y \ , \\ &(4) \qquad \alpha = -\tfrac{p+1}{(p-1)(n-p+1)}\, x - \tfrac{p+1}{(p-1)(n-p+1)}\, \beta \ , \\ &(5) \qquad \beta = \tfrac{n-p+1}{(p+1)(n-p-1)}\, y - \tfrac{n-p+1}{(p+1)(n-p-1)}\, \alpha \ . \end{aligned}$$

Using the last four equations we can express x and y in terms of a and b. We find

$$y = \tfrac{(p+1)(n-p)}{(p+1)(n-p)-(3n-2p+4)}\, b \ , \qquad x = \tfrac{p(n-p+1)}{p(n-p+1)-(3n-2p+4)}\, a \ .$$

Substituting this back into the first equation and taking the scalar product with $J\Lambda u$ we eventually obtain

$$0 \;=\; \tfrac{p}{p+1}\,\tfrac{(p+1)(n-p)}{(p+1)(n-p)-(3n-2p+4)}\,|\,d(J\Lambda u)\,|^2 + \tfrac{n-p}{n-p+1}\,\tfrac{p(n-p+1)}{p(n-p+1)-(3n-2p+4)}\,|\,d^*(J\Lambda u)\,|^2 \,. \quad (6.1.1)$$

With the exception of some cases (for small p and n) the coefficients in this equation have the same sign. Hence, on a compact Kähler manifolds one conclude $\nabla(J\Lambda u) = 0$, which then can be used to prove Theorem 6.1. One of these exceptions is the case of a conformal Killing 3-form on a compact Kähler manifold of real dimension 10, i.e. $n = 10$ and $p = 3$. Here the denominator in the formula for y vanishes. Nevertheless, we still have the following equations

$$\begin{aligned} y \;&=\; -\tfrac{7}{6}\,x, \qquad \tfrac{1}{3}\,x \;=\; a \;-\; [\alpha \,+\, \beta], \qquad -\tfrac{1}{2}\,x \;=\; b \;+\; [\alpha \,+\, \beta], \\ \alpha \;&=\; -\tfrac{1}{4}\,x \;-\; \tfrac{1}{4}\,\beta, \qquad \beta \;=\; -\tfrac{7}{18}\,x \;-\; \tfrac{1}{3}\,\alpha \,. \end{aligned}$$

Solving this system of equations in terms of x leads to

$$\alpha \;=\; -\tfrac{1}{6}\,x, \qquad \beta \;=\; -\tfrac{1}{3}\,x, \qquad a \;=\; -\tfrac{1}{6}\,x, \qquad b \;=\; 0\,.$$

Taking the scalar product with $J\Lambda u$, the definition of b immediately implies that $d(J\Lambda u)$ vanishes. But since Λu is in this special case a 1-form, we obtain also $d(\Lambda u) = 0$. Hence, we have in addition $\alpha = 0$ and it follows $x = 0$ and in the end $a = 0$, which is equivalent to $d^*(\Lambda u) = 0$. Altogether this means that Λu is a parallel 1-form, i.e. u is either primitive and it follows from Theorem 6.1.3 that it is parallel or we have a non-trivial parallel 1-form Λu. The proof that under this assumption u has to be parallel is contained in [JAY85]. But it is not difficult to give a direct proof. First of all we note, that $\delta^c d^* u = 0$ follows from the equations for δ^c. Then, contracting the defining equation of a conformal Killing form with ω and using the assumption $\nabla \Lambda u = 0$ leads to

$$0 \;=\; \tfrac{1}{p+1}\,J(\Lambda du) \;-\; \tfrac{p-1}{n-p+1}\,d^*u \;=\; -\tfrac{1}{p+1}\,J(\delta^c u) \;-\; \tfrac{p-1}{n-p+1}\,d^*u \,.$$

Hence, we have $J\delta^c u = -\,\tfrac{p^2-1}{n-p+1}\,d^*u$. Next, we contract the equation for $\nabla^2_{X,Y}u$ with the Kähler form and obtain

$$\begin{aligned} 0 \;&=\; (n-p+1)\,Y \lrcorner\, \Lambda(R^+(X)u) \;+\; g(X,Y)\,\Lambda(dd^*u) \;-\; \Lambda(X \,\wedge\, Y \lrcorner\, dd^*u) \\ &\qquad\qquad -\, p\,\Lambda(Y \,\wedge\, \nabla_X(d^*u)) \\ &=\; (n-p+1)\,Y \lrcorner\, R^-(JX)\,u \;-\; g(X,Y)\,\delta^c d^*u \;-\; JX \lrcorner\, Y \lrcorner\, dd^*u \\ &\qquad\qquad +\, X \,\wedge\, Y \lrcorner\, \delta^c d^*u \;-\; p\,JY \lrcorner\, \nabla_X(d^*u) \\ &=\; (n-p+1)\,Y \lrcorner\, R^-(JX)\,u \;-\; JX \lrcorner\, Y \lrcorner\, dd^*u \;-\; p\,JY \lrcorner\, \nabla_X(d^*u) \\ &=\; (n-p+1)\,(p-1)\,R^-(JX)\,u \;+\; (p-1)\,JX \lrcorner\, dd^*u \;+\; p\,J(\nabla_X(d^*u)) \,. \end{aligned}$$

For the last equality we wedged with $Y = e_i$ and summed over a local ortho-normal basis $\{e_i\}$. Doing the same with respect to X leads to

$$\begin{aligned} 0 &= (n-p+1)\,(p-1)\,2q(R)\,u \;+\; p\,(p-1)\,dd^*u \;+\; p\,d^c\,J(d^*u) \\ &= (n-p+1)\,(p-1)\,(\,\tfrac{p}{p+1}\,d^*du \;+\; \tfrac{n}{n-p+1}\,dd^*u\,) \;+\; p\,d^c\,J(d^*u)\;. \end{aligned}$$

Finally, we can take the scalar product with u and apply the formula for $d^c\,J(d^*u)$. Note that J extends to a skew-symmetric map of $\Lambda^2 T^*M$. This gives

$$0 \;=\; (n-p+1)(p-1)\,\tfrac{p}{p+1}\,\|\,du\,\|^2 \;+\; n(p-1)\,\|\,d^*u\,\|^2 \;+\; p\,\tfrac{p^2-1}{n-p+1}\,\|\,d^*u\,\|^2$$

and, since all coefficients are positive, we can conclude as above and obtain that u has to be parallel. This completes the proof of Proposition 6.1.6.

We will now consider conformal Killing n-forms on compact Kähler manifolds of real dimension $2n$ and prove eventually Theorem 6.1.5. In this case we know from Proposition 5.1.2 that any component of the holonomy decomposition is again a conformal Killing form. In particular, we can assume that our conformal Killing form is a (p,q)-form. We start with fixing some notation.

Let (M^{2n}, g, J) be a Kähler manifold, with Kähler form ω defined by $\omega(X,Y) = g(JX,Y)$. We fix a local ortho-normal basis e_i with $Je_{2j-1} = e_{2j}$. Then

$$f_j := \tfrac{1}{2}(e_{2j-1} - ie_{2j}) \in T^{1,0}M \qquad\qquad f^j := (e^{2j-1} + ie^{2j}) \in \Lambda^{1,0}M\;.$$

The dual $(f_j)^*$ of f_j with respect to the $\mathbb{C}$-linear extended metric turns out to be $\frac{1}{2}\,\bar f^j$, i.e.

$$(f_j)^* \;=\; \tfrac{1}{2}(e^{2j-1} - ie^{2j}) \;=\; \tfrac{1}{2}\,\bar f^j \in \Lambda^{0,1}M \qquad \text{and} \qquad (f^j)(f_k) = \delta_{jk}\;.$$

With these definitions it is clear that we have

$$f_j\,\lrcorner\; : \Lambda^{p,q}M \to \Lambda^{p-1,q}M \qquad\qquad f^j\wedge\; : \Lambda^{p,q}M \to \Lambda^{p+1,q}M\;.$$

Using the special basis $\{f_j\}$ resp. $\{f^j\}$ we can also give local expressions for the components ∂ and $\bar\partial$ of the differential d, as well as for their adjoint maps. We find

$$\begin{aligned} \partial &= \sum f^j \wedge \nabla_{f_j}\;, & \bar\partial &= \sum \bar f^j \wedge \nabla_{\bar f_j}\;, \\ \partial^* &= -2\sum f_j\,\lrcorner\,\nabla_{\bar f_j}\;, & \bar\partial^* &= -2\sum \bar f_j\,\lrcorner\,\nabla_{f_j}\;. \end{aligned}$$

Next, we need an explicit formula for the Kähler form and the contraction with it. We have

$$\omega \;=\; \tfrac{i}{2}\sum f^j \wedge \bar f^j\;, \qquad \Lambda \;=\; \omega\,\lrcorner \;=\; 2i\sum f_j\,\lrcorner\,\bar f_j\,\lrcorner\;.$$

Recall, that the following identities hold on Kähler manifolds

$$[\Lambda,\bar\partial\,] \;=\; -i\,\partial^*\;, \qquad [\,\Lambda,\,\partial\,] \;=\; i\,\bar\partial^*\;, \qquad [\,\Lambda,\,L\,] \;=\; (n-p-q)\,\mathrm{id}_{\,\Lambda^{p,q}}\;.$$

Extending the third equation we obtain another useful relation.

Lemma 6.1.8 *Let (M^{2n}, g, J) be a Kähler manifold and let α be a (p,q)-form on M, then*

$$[\,\Lambda,\, L^k\,]\,\alpha \;=\; k\,(n-p-q-k+1)\,L^{k-1}\alpha\ .$$

Proof. We prove this formula by induction with respect to k. For $k=1$ it is just the well-known commutator relation cited above. Assume we know the formula for $k-1$, then

$$\begin{aligned}
[\,\Lambda,\, L^k\,]\,\alpha \;&=\; (\,\Lambda\circ L^k - L^k\circ\Lambda\,)\,\alpha \;=\; \Lambda\circ L^{k-1}\,(L\alpha)\;-\;L^{k-1}\,(L\circ\Lambda)\,\alpha\\
&=\; L^{k-1}\circ\Lambda\,(L\alpha)\;+\;(k-1)\,(n-(p+1)-(q+1)-(k-1)+1)\,L^{k-2}\,(L\alpha)\\
&\quad\; -\,L^{k-1}\circ\Lambda\,(L\alpha)\;+\;(n-p-q)\,L^{k-1}\alpha\\
&=\; ((k-1)(n-p-q-k)\;+\;(n-p-q))\,L^{k-1}\alpha\\
&=\; k\,(n-p-q-k+1)\,L^{k-1}\alpha\ . \quad\Box
\end{aligned}$$

Finally we still need the following elementary lemma.

Lemma 6.1.9 *Let* deg *denote the degree of a form, then*

$$\begin{aligned}
\sum f^j\,\wedge\,f_j\,\lrcorner \;&=\; \tfrac12\,(\deg + iJ)\;=\;p\,\mathrm{id}_{\Lambda^{p,q}}\ ,\\
\sum \bar f^j\,\wedge\,\bar f_j\,\lrcorner \;&=\; \tfrac12(\deg - iJ)\;=\;q\,\mathrm{id}_{\Lambda^{p,q}}\ .
\end{aligned}$$

After these preparations we can return to the case of conformal Killing forms. Let u be a (p,q)-form which satisfies the equation for conformal Killing forms. It is of course no problem to consider this equation also for complex forms. Comparing the types of the different summands we obtain

$$\begin{aligned}
\nabla_{f_j}u \;&=\; \tfrac{1}{p+q+1}\,f_j\,\lrcorner\,\partial u\;-\;\tfrac{1}{2(2n-p-q+1)}\,\bar f^j\,\wedge\,\bar\partial^* u\ ,\\
0 \;&=\; \tfrac{1}{p+q+1}\,f_j\,\lrcorner\,\bar\partial u\;-\;\tfrac{1}{2(2n-p-q+1)}\,\bar f^j\,\wedge\,\partial^* u\ ,\\
\nabla_{\bar f_j}u \;&=\; \tfrac{1}{p+q+1}\,\bar f_j\,\lrcorner\,\bar\partial u\;-\;\tfrac{1}{2(2n-p-q+1)}\,f^j\,\wedge\,\partial^* u\ ,\\
0 \;&=\; \tfrac{1}{p+q+1}\,\bar f_j\,\lrcorner\,\partial u\;-\;\tfrac{1}{2(2n-p-q+1)}\,f^j\,\wedge\,\bar\partial^* u\ .
\end{aligned}$$

This leads to the following equations

$$\begin{aligned}
\bar\partial^* u \;&=\; -\,\tfrac{i(2n-p-q+1)}{(p+q+1)(n-p)}\,\Lambda(\partial u)\ , &\qquad \partial u \;&=\; \tfrac{i(p+q+1)}{(2n-p-q+1)q}\,L(\bar\partial^* u)\ ,\\
\bar\partial u \;&=\; -\,\tfrac{i(p+q+1)}{(2n-p-q+1)p}\,L(\partial^* u)\ , &\qquad \partial^* u \;&=\; \tfrac{i(2n-p-q+1)}{(p+q+1)(n-q)}\,\Lambda(\bar\partial u)\ .
\end{aligned}$$

Here we are interested in conformal Killing forms of degree n, i.e. we can specialize to the case of a (p,q)-form with $p+q=n$. We only need the first two equations, which combine to yield one further equation

$$\bar\partial^* u \;=\; -\tfrac{i}{q}\,\Lambda(\partial u)\;, \qquad\qquad \partial u \;=\; \tfrac{i}{q}\,L(\bar\partial^* u) \;=\; \tfrac{1}{q^2}\,L(\Lambda\partial u)\;.$$

As already mentioned, we can conclude from Proposition 5.1.2 that for a form u in the middle dimension all its components in the holonomy decomposition are again conformal Killing forms. On Kähler manifolds this means that we can restrict to the case where $u=L^k(u_0)$ for some primitive $(p-k,q-k)$-form u_0. Starting from our equation for ∂u and using Lemma 6.1.8 and 6.1.9 we obtain

$$\begin{aligned}\partial u &= \tfrac{1}{q^2}\,L\,\Lambda\,(\partial u) \;=\; \tfrac{1}{q^2}\,L\,\partial\,\Lambda\,(L^k\,u_0) \;+\; \tfrac{i}{q^2}\,L\,(\bar\partial^* u)\\ &= \tfrac{1}{q^2}\,L\,\partial\,L^k\,(\Lambda u_0) \;+\; \tfrac{1}{q^2}\,k\,(n-(p-k)-(q-k)-k+1)\,L\,\partial\,L^{k-1}\,u_0 \;+\; \tfrac{i}{q^2}\,\tfrac{q}{i}\,\partial u\\ &= \left(\tfrac{1}{q^2}\,k\,(k+1) \;+\; \tfrac{1}{q}\right)\,\partial u\;.\end{aligned}$$

This is the case if and only if $k(k+1)+q=q^2$. If we set $q=k+\epsilon$ we immediately see that the only non-trivial solution is $\epsilon=1$, i.e. $q=k+1$. Using the other two equations we obtain the same condition for p. Hence, we proved

Lemma 6.1.10 *Let $(M^{2n},g,\,J)$ be a Kähler manifold and let u be a conformal Killing (p,q)-form with $p+q=n$. Then $p=q$ and it exists a primitive $(1,1)$-form u_0 with $u=L^{p-1}u_0$.*

Next, we will investigate how the conformal Killing equation for $u=L^{p-1}u_0$ translates into an equation for u_0. Let u be a (real) conformal Killing (p,p)–form on a compact Kähler manifold M of complex dimension $n=2p$ which satisfies the equations

$$\begin{aligned}0 &= \tfrac{1}{n+1}\,f_j\,\lrcorner\,\bar\partial u \;-\; \tfrac{1}{2(n+1)}\,\bar f^j\,\wedge\,\partial^* u\;,\\ \nabla_{\bar f_j}\,u &= \tfrac{1}{n+1}\,\bar f_j\,\lrcorner\,\bar\partial u \;-\; \tfrac{1}{2(n+1)}\,f^j\,\wedge\,\partial^* u\;.\end{aligned}$$

Note that every (p,p)–form satisfying these equations has to be a conformal Killing form. In terms of real operators, the above equations become

$$\begin{aligned}0 &= (X-iJX)\,\lrcorner\,(d-id^c)\,u \;-\; (X-iJX)\,\wedge\,(d^*-i\delta^c)u\;,\\ 2(n+1)\nabla_{X+iJX}\,u &= (X+iJX)\,\lrcorner\,(d-id^c)u-(X+iJX)\wedge(d^*-i\delta^c)u\;.\end{aligned}$$

The real and imaginary part of these equations are equivalent, so it suffices to consider the real parts:

$$0 \;=\; X\,\lrcorner\,du \;-\; JX\,\lrcorner\,d^c u \;-\; X\,\wedge\,d^*u \;+\; JX\,\wedge\,\delta^c u, \tag{6.1.2}$$

$$2(n+1)\nabla_X\,u \;=\; X\,\lrcorner\,du+JX\,\lrcorner\,d^c u-X\,\wedge\,d^*u-JX\,\wedge\,\delta^c u. \tag{6.1.3}$$

After contracting (6.1.2) with X and summing for $X = e_i$ over an orthonormal basis $\{e_i\}$ we obtain

$$2\,\Lambda\, d^c\, u \;=\; (n+1)\, d^*\, u \;+\; J\,\delta^c\, u.$$

Applying the Kähler identities, this equation becomes

$$2\, d^c\Lambda\, u \;+\; 2\, d^*\, u \;=\; (n+1)\, d^*\, u \;+\; \delta^c\, J\, u \;-\; d^*\, u,$$

so $d^c\Lambda u = (p-1)\, d^* u$ (as $Ju = 0$). We already know that $u = L^{p-1}u_0$, where u_0 is a primitive (1,1) form. Thus, the last equation reads

$$d^c\Lambda\, L^{p-1}u_0 \;=\; (p-1)\, d^*\, L^{p-1}\, u_0. \tag{6.1.4}$$

Now, it is easy to check that (c.f. Lemma 6.1.8)

$$\Lambda\, L^{p-1}u_0 \;=\; p(p-1)\, L^{p-2}\, u_0$$

and

$$d^*\, L^{p-1}u_0 \;=\; L^{p-1}d^*u_0 - (p-1)d^cL^{p-2}u_0.$$

So (6.1.4) becomes

$$p(p-1)\, d^cL^{p-2}u_0 \;=\; (p-1)\,(L^{p-1}d^*u_0 - (p-1)d^cL^{p-2}u_0),$$

and finally, since L^{p-2} is injective,

$$(n-1)\, d^cu_0 \;=\; L\, d^*\, u_0. \tag{6.1.5}$$

Applying J to this equation gives

$$(n-1)\, du_0 \;=\; -L\delta^cu_0. \tag{6.1.6}$$

We now check that conversely (6.1.5) implies (6.1.2). Using again (intensively) the Kähler commutator relations we obtain

$$\begin{aligned}
&X \lrcorner\, du - JX \lrcorner\, d^cu - X\wedge d^*u + JX\wedge\delta^cu\\
&\quad= L^{p-1}(X \lrcorner\, du_0 - JX \lrcorner\, d^cu_0) \;+\; (p-1)L^{p-2}(JX\,\wedge\, du_0 \;+\; X\,\wedge\, d^cu_0)\\
&\qquad -L^{p-1}(X\,\wedge\, d^*u_0 \;-\; JX\,\wedge\,\delta^cu_0) \;+\; (p-1)L^{p-2}(JX\,\wedge\, du_0 \;+\; X\,\wedge\, d^cu_0)\\
&\quad= L^{p-1}(X \lrcorner\, du_0 \;-\; JX \lrcorner\, d^cu_0) \;+\; 2(p-1)L^{p-2}(JX\,\wedge\, du_0 \;+\; X\,\wedge\, d^cu_0)\\
&\qquad -L^{p-2}(X\,\wedge\, Ld^*u_0 \;-\; JX\,\wedge\, L\delta^cu_0)\\
&\quad= L^{p-1}(X \lrcorner\, du_0 \;-\; JX \lrcorner\, d^cu_0) \;-\; \frac{1}{n-1}L^{p-1}(X\,\wedge\, d^*u_0 - JX\,\wedge\,\delta^cu_0)\\
&\quad= \frac{1}{n-1}L^{p-1}\,(-X \lrcorner\, L\delta^cu_0 \;-\; JX \lrcorner\, Ld^*u_0 \;-\; X\,\wedge\, d^*u_0 + JX\,\wedge\,\delta^cu_0)\\
&\quad= \frac{1}{n-1}L^{p}(-X \lrcorner\,\delta^cu_0 \;-\; JX \lrcorner\, d^*u_0) \;=\; 0\,.
\end{aligned}$$

Here we again used $\delta^c u_0 = Jd^* u_0$. Next, we translate formula (6.1.3) in terms of u_0. The result is

Lemma 6.1.11 *Let u_0 be a primitive $(1,1)$-form on a $4n$–dimensional compact Kähler manifold M and define $u := L^{p-1}u_0$, with $n = 2p$. Then u is a conformal Killing form, if and only if there exists a 1-form β such that u_0 satisfies*

$$\nabla_X u_0 \;=\; -\frac{2}{n}\,\beta(X)\,\omega \;+\; \beta \wedge JX - J\beta \wedge X, \tag{6.1.7}$$

for all vector fields X. In this case β equals $\frac{n}{2(n^2-1)} Jd^* u_0$.

Proof. We first show that equation (6.1.7) is equivalent to equation (6.1.3). For $u = L^{p-1}u_0$ we compute as before

$$\begin{aligned}
&2(n+1)\nabla_X u - (X \lrcorner\, du \;+\; JX \lrcorner\, d^c u \;-\; X \wedge d^* u \;-\; JX \wedge \delta^c u) \\
&\quad=\; 2(n+1)L^{p-1}\nabla_X u_0 \;-\; L^{p-1}X \lrcorner\, du_0 \;-\; (p-1)JX \wedge L^{p-2}du_0 \\
&\qquad - L^{p-1}JX \lrcorner\, d^c u_0 \;+\; (p-1)X \wedge L^{p-2}d^c u_0 \\
&\qquad - X \wedge L^{p-1}d^* u_0 \;-\; (p-1)X \wedge L^{p-2}d^c u_0 \\
&\qquad + JX \wedge L^{p-1}\delta^c u_0 \;+\; (p-1)JX \wedge L^{p-2}du_0 \\
&\quad=\; L^{p-1}(2(n+1)\nabla_X u_0 \;-\; X \lrcorner\, du_0 \;-\; JX \lrcorner\, d^c u_0 \;+\; X \wedge d^* u_0 \;+\; JX \wedge \delta^c u_0) \\
&\quad=\; L^{p-1}(2(n+1)\nabla_X u_0 \;+\; \tfrac{1}{n-1}(LX \lrcorner\, \delta^c u_0 \;+\; JX \wedge \delta^c u_0) \\
&\qquad - \tfrac{1}{n-1}(LJX \lrcorner\, d^* u_0 \;-\; X \wedge d^* u_0) \;+\; X \wedge d^* u_0 \;+\; JX \wedge \delta^c u_0) \\
&\quad=\; L^{p-1}\Big(2(n+1)\nabla_X u_0 \;+\; \tfrac{2}{n-1}\delta^c u_0(X)\,\omega \;-\; \tfrac{n}{n-1}(\delta^c u_0 \wedge JX \;-\; J\delta^c u_0 \wedge X)\Big)\,.
\end{aligned}$$

In the last equality we used again the relation $\delta^c u_0 = Jd^* u_0$. As L^{p-1} is injective, u satisfies (6.1.3) if and only if u_0 satisfies (6.1.7), with $\beta = -\frac{n}{2(n^2-1)}\,\delta^c u_0$. On the other hand, computing du_0 from equation (6.1.7) we obtain

$$\begin{aligned}
du_0 \;&=\; -\tfrac{2}{n}\sum e_i \wedge \beta(e_i)\,\omega \;+\; \sum e_i \wedge \beta \wedge Je_i \\
&=\; -\tfrac{2}{n}\,\omega \wedge \beta - 2\,\omega \wedge \beta \;=\; -\tfrac{2(n+1)}{n}\,L\,(\,\beta\,) \\
&=\; -\tfrac{1}{n-1}\,L\,J\,d^* u_0\;.
\end{aligned}$$

Since L is injective this shows that given equation (6.1.7) β has to be $\frac{n}{2(n^2-1)} Jd^* u_0$. Moreover, we see that equation (6.1.7) implies equations (6.1.6) and (6.1.5).

Now, let $u = L^{p-1}u_0$ be a conformal Killing form. From the calculations above we know that this leads to equation (6.1.3) and thus to equation (6.1.7) satisfied by u_0.

Conversely, if u_0 is a primitive $(1,1)$-form satisfying equation (6.1.7), then we have seen that u_0 also satisfies equations (6.1.6) and (6.1.5). This is equivalent, as shown above, to equation (6.1.2) for $u := L^{p-1}\, u_0$. Moreover, we know already that equation (6.1.7) is equivalent to equation (6.1.3). Hence, the (p,p)-form u satisfies both the equations (6.1.2) and (6.1.3) which implies that u is a conformal Killing form and finishes the proof of the lemma. $\Box$

Finally, to prove the 1-1-relation between conformal Killing n-forms and invariant conformal Killing 2-forms, we still need the following

Lemma 6.1.12 *Let $(M^{2n},\, g,\, J,\, \omega)$ be a Kähler manifold. Then an effective 2-form u_0 is the primitive part of an invariant conformal Killing 2-form if and only if there exists a 1-form β with*

$$\nabla_X\, u_0 \;=\; (\beta \wedge JX^* \;-\; J\beta \wedge X^*) \;-\; \tfrac{2}{n}\, \beta(X)\, \omega\ . \tag{6.1.8}$$

Proof. Let u be an invariant conformal Killing 2-form. The primitive part of u is given as $u_0 = u - \frac{1}{n}\operatorname{tr}(u)\,\omega$, where $\operatorname{tr}(u) := g(u,\omega)$. Hence, using the characterization of Lemma 6.1.15, we find

$$\begin{aligned}
\nabla_X\, u_0 \;&=\; \nabla_X\, u \;-\; \tfrac{1}{n}\, g(\nabla_X u, \omega)\, \omega \\
&=\; (\beta \wedge JX^* \;-\; J\beta \wedge X^*) \;-\; \beta(X)\, \omega \;-\; \tfrac{1}{n}\, g(\nabla_X u,\, \omega)\, \omega \\
&=\; (\beta \wedge JX^* \;-\; J\beta \wedge X^*) \;-\; \big(1 \;+\; \tfrac{1}{n}\,(2 \;-\; n)\big)\, \beta(X)\, \omega \\
&=\; (\beta \wedge JX^* \;-\; J\beta \wedge X^*) \;-\; \tfrac{2}{n}\, \beta(X)\, \omega\ .
\end{aligned}$$

Conversely, let u_0 be a primitive 2-form satisfying equation (6.1.8). We have to find a function f such that $u := u_0 + f\,\omega$ is a conformal Killing 2-form. First of all we will show that there is some function f_1 with $\delta^c u_0 = df_1$. Indeed, equation (6.1.8) implies (6.1.6), i.e. $L\delta^c u_0 = (n-1) du_0$. It follows $Ld\delta^c u_0 = 0$ and, since L is injective, we conclude that $\delta^c u_0$ is closed. Hence, $\delta^c u_0 = h + df_1$ for some function f_1 and a harmonic 1-form h. We still have to show that h vanishes. First, note that since the manifold is compact h is closed and coclosed. Moreover, h is in the kernel of d^c and δ^c since the manifold is Kähler. Computing we L^2-norm we obtain

$$(h,\, h) \;=\; (h,\, h + df_1) \;=\; (h,\, \delta^c u_0) \;=\; (d^c h,\, u_0) \;=\; 0\ .$$

We have already seen in the proof of the preceding Lemma that β equals $\frac{n}{2(n^2-1)}\delta^c u_0$. Hence, it follows $\beta = df_2$, where f_2 is the corresponding multiple of f_1. We now claim that $u_0 - \frac{n-2}{n}\, f_2\, \omega$ is a conformal Killing 2-form. Indeed,

$$\begin{aligned}
\nabla_X(u_0 + \tfrac{n}{n-2}\, f_2\, \omega\,) \;&=\; \beta \wedge JX^* \;-\; J\beta \wedge X^* \;-\; \tfrac{2}{n}\, \beta(X)\, \omega \;-\; \tfrac{n-2}{n}\, df_2(X)\, \omega \\
&=\; \beta \wedge JX^* \;-\; J\beta \wedge X^* \;-\; (\tfrac{2}{n} \;+\; \tfrac{n-2}{n})\, \beta(X)\, \omega \\
&=\; \beta \wedge JX^* \;-\; J\beta \wedge X^* \;-\; \beta(X)\, \omega\ .
\end{aligned}$$

Hence, $u_0 - \frac{n-2}{n} f_2 \omega$ is a conformal Killing 2-form according to the characterization of Lemma 6.1.15. $\square$

In the remainder of this section we will consider the case of conformal Killing 2-forms on Kähler manifolds. Let (M^{2n}, g, J) be an almost Hermitian manifold with Kähler form ω, i.e. $\omega(X,Y) := g(JX,Y)$. The space of 2-forms has the orthogonal decomposition: $\Lambda^2 = \Lambda^{inv} \oplus \Lambda^{anti}$ into invariant resp. anti-invariant forms, i.e. $u \in \Lambda^{inv}$ if and only if $u(JX,JY) = u(X,Y)$ and $u \in \Lambda^{anti}$ if and only if $u(JX,JY) = -u(X,Y)$. In addition one has for 4-dimensional almost Hermitian manifolds: $\Lambda^2_+ = \mathbb{R}\omega \oplus \Lambda^{anti}$ and $\Lambda^2_- = \Lambda^{inv}_0$, where Λ^{inv}_0 denotes the primitive invariant 2-forms, i.e. the orthogonal complement of the Kähler form ω, and $\Lambda^2_\pm$ are the self-dual resp. anti-self-dual 2-forms.

If u is a conformal Killing 2-form we can apply Proposition 1.1.8 to obtain the equation:

$$(\nabla_X u)(Y, Z) \; + \; (\nabla_Y u)(X, Z) \; = \; 2\, g(X, Y)\, \theta(Z) \; - \; g(Y, Z)\, \theta(X) \; - \; g(X, Z)\, \theta(Y) \; ,$$

where $\theta = -\frac{1}{2n-1} d^* u$. Setting $X := e_i$, $Z := Je_i$ and summing over an orthonormal basis $\{e_i\}$ leads on a Kähler manifold to

$$2Y(g(u, \omega)) \;=\; (4-2n)\, \theta(JY) \quad \text{for} \quad u \in \Lambda^{inv} \; , \tag{6.1.9}$$

$$2Y(g(u, \omega)) \;=\; (2+2n)\, \theta(JY) \quad \text{for} \quad u \in \Lambda^{anti} \; . \tag{6.1.10}$$

Hence, we have the following proposition, which can also be found in [Se].

Proposition 6.1.13 *Let (M^{2n}, g, J) be a Kähler manifold. Then any anti-invariant conformal Killing 2-form is coclosed, and parallel if the manifold is compact. The same is true for any invariant primitive conformal Killing 2-form, if $n \neq 2$. On a 4-dimensional Kähler manifold any conformal Killing 2-form is of the form $u_0 + c\,\omega$ for some constant c and a primitive conformal Killing 2-form u_0.*

Proof. For the proof it suffices to remark that a 2-form is effective if and only if it is orthogonal to the Kähler form. Moreover, the decomposition $\Lambda^2 = \Lambda^{inv} \oplus \Lambda^{anti}$ is orthogonal, i.e. $g(u, \omega) = 0$ for any anti-invariant 2-form u. $\square$

We see that Proposition 6.1.13 still allows the possibility of non-primitive invariant conformal Killing 2-forms, if $n \neq 2$, and of primitive invariant conformal Killing 2-forms for $n = 2$. Moreover, there still could exist non-parallel conformal Killing 2-forms given as the sum of an invariant and an anti-invariant 2-form. In the following we will concentrate on the case of invariant conformal Killing 2-forms. It turns out that there is a close relation to *Hamiltonian 2-forms* which were recently investigated in [ACG01b]. In particular, there is a complete local classification in dimension 4 and a partial classification together with many examples in higher dimensions.

An invariant 2-form u can be written as $u = u_0 + \frac{1}{n}\operatorname{tr}(u)\,\omega$, where u_0 is the primitive part of u and the trace is defined as the scalar product with the Kähler form, i.e. $\operatorname{tr}(u) = g(u,\omega)$. The next lemma gives first properties of such forms (c.f. [Se]).

Lemma 6.1.14 *Let* (M^{2n}, g, J) *be a Kähler manifold with* $n \geq 3$ *and let* u *be a non-primitive invariant conformal Killing 2-form, then*

$$d\,\mathrm{tr}(u) \;=\; -\,\tfrac{n-2}{2n-1}\,Jd^*u\ .$$

Moreover, $\mathrm{grad}(\mathrm{tr}(u))$ *is a real holomorphic vector field and the 1–form* d^*u *is dual to a Killing vector field.*

Proof. The equation for $d\mathrm{tr}(u)$ is of course an immediate consequence of equation (6.1.9). A vector field ξ is real holomorphic if and only if $L_\xi J = 0$. On Kähler manifolds this is equivalent to $\nabla_{JX}\,\xi = \nabla_X\,J\xi$ for all vector fields X. We will show that d^*u is dual to a real holomorphic vector field which is then real holomorphic and divergence free, hence it is a Killing vector field. Since the complex structure preserves real holomorphic vector fields this also implies the statement for $\mathrm{grad}(\mathrm{tr}(u))$. From Proposition 4.1.2 we have $\nabla_X\,(d^*u) = -\frac{2n-1}{2n-2}R^-(X)u - \frac{2n-1}{3(2n-2)}X \lrcorner\, d^*du$. If u is an invariant form then also $\nabla^*\nabla u$ and $2q(R)u$ are invariant 2-forms. With the assumption $n \geq 3$ we are in the case where it is possible to invert the two Weitzenböck formulas (1.1.7) and (1.1.7) (as in the proof of Corollary 4.1.4), i.e. we can express d^*du in terms of $\nabla^*\nabla u$ and $2q(R)u$. Hence, d^*du is again an invariant 2-form and we conclude

$$\begin{aligned}
\nabla_{JX}\,d^*u(Y) - \nabla_X\,Jd^*u(Y) \;&=\; \nabla_{JX}\,d^*u(Y) + \nabla_X\,d^*u(JY)\\
&=\; -\tfrac{2n-1}{2n-2}(R^-(JX)u(Y) + R^-(X)u(JY))\\
&=\; -\tfrac{2n-1}{2n-2}\sum((R_{e_i,JX}u)(e_i,Y) \;+\; (R_{e_i,X}u)(e_i,JY))\\
&=\; -\tfrac{2n-1}{2n-2}\sum((R_{Je_i,JX}u)(Je_i,Y) \;+\; (R_{e_i,X}u)(e_i,JY)\\
&=\; -\tfrac{2n-1}{2n-2}\sum((-R_{e_i,X}u)(e_i,JY) \;+\; (R_{e_i,X}u)(e_i,JY)\\
&=\; 0\ . \quad \Box
\end{aligned}$$

The link between conformal Killing 2–forms and Hamiltonian 2–forms is based on the following characterization of conformal Killing 2–forms on Kähler manifolds (c.f. [ACG01b]).

Lemma 6.1.15 *Let* (M^{2n}, g, J, ω) *be a Kähler manifold. Then an invariant 2-form* u *is a conformal Killing 2-form if and only if there exists a 1-form* β *with*

$$\nabla_X\,u \;=\; (\beta \wedge JX^* \;-\; J\beta \wedge X^*) \;-\; \beta(X)\,\omega\ . \tag{6.1.11}$$

In this case the 1-form β *is given as* $\beta = \frac{1}{2n-1}\,Jd^*u$. *Moreover,* $\beta = -\,\frac{1}{n-2}\,d\,\mathrm{tr}(u)$ *if* $n > 2$.

Proof. First of all we can rewrite equation (6.1.11) as $\nabla_X u = -X \lrcorner (\beta \wedge \omega) + X^* \wedge J\beta$. Setting $X = e_i$ and summing over an ortho-normal frame $\{e_i\}$ we obtain

$$J\beta \;=\; -\tfrac{1}{2n-1}\, d^*u \qquad \text{and} \qquad \beta \wedge \omega \;=\; -\tfrac{1}{3}\, du \ .$$

Substituting this back into (6.1.11) yields the defining equation for a conformal Killing 2-form, i.e.:

$$\nabla_X\, u \;=\; \tfrac{1}{3}\, X \lrcorner\, du \;-\; \tfrac{1}{2n-1}\, X^* \,\wedge\, d^*u \ .$$

The second equation for β immediately follows from equation (6.1.9). Conversely, let first u be an anti-self-dual conformal Killing 2-form on a 4-dimensional almost Hermitian manifold and define $\beta := \frac{1}{3} J d^* u$. Then

$$\beta \wedge \omega \;=\; -\,*\,(\beta^\flat \lrcorner\, \omega) \;=\; -\,*\,J\beta \;=\; \tfrac{1}{3}\,*\,d^*u \;=\; \tfrac{1}{3}\, d * u \;=\; -\,\tfrac{1}{3}\, du \ ,$$

and we obtain that the 2-form u satisfies equation (6.1.11). Now, let u be an invariant conformal Killing 2-form on a Kähler manifold M^{2n} with $n > 2$, then we again define $\beta := \frac{1}{3} J d^* u$ and rewrite the defining equation of a conformal Killing 2-form as

$$\nabla_X\, u \;=\; \tfrac{1}{3}\, X \lrcorner\, du \;-\; \tfrac{1}{2n-1}\, X^* \,\wedge\, d^*u \;=\; \tfrac{1}{3}\, X \lrcorner\, du \;+\; X \,\wedge\, J\beta \ .$$

Hence, to obtain equation (6.1.11) we have to prove $du = -3\, \beta \wedge \omega$, or equivalently

$$du \;=\; -\,\tfrac{3}{2n-1}\, J d^*u \,\wedge\, \omega \ . \tag{6.1.12}$$

Since M is a Kähler manifold we know that with u also $\nabla_X\, u$ is for any vector field X an invariant 2-form, i.e. a form of complex type $(1,1)$. Hence, $(\nabla_X u)(Z_1, Z_2) = 0$ for any $Z_1, Z_2 \in T^{1,0}M$ and the definition of a conformal Killing 2-form immediately implies

$$du(X, Z_1, Z_2) \;=\; \tfrac{3}{2n-1} (X \,\wedge\, d^*u)\,(Z_1, Z_2) \;=\; -\,\tfrac{3}{(2n-1)} (J d^*u \wedge \omega)\,(X, Z_1, Z_2).$$

Since du is a real 3-form of complex type $(2,1) + (1,2)$, the latter equality holds for any vector fields X, Y, Z. This proves equation (6.1.12) and finishes the proof of the lemma. □

As a first application of Lemma 6.1.11 we will describe how to construct conformal Killing 2-forms on the complex projective space (c.f. [ACG01b]). Let $M = \mathbb{C}P^n$ be equipped with the Fubini-Study metric and the corresponding Kähler form ω. Then the Riemannian curvature is given as

$$R_{X,Y} Z \;=\; -\,(X \wedge Y \;+\; JX \wedge JY)\, Z \;-\; 2\,\omega(X,\, Y)\, JZ$$

for any vector fields X, Y, Z. This implies for the Ricci curvature $\mathrm{Ric} = 2(n+1)\mathrm{id}$. Let K be any Killing vector field on $\mathbb{C}P^n$. Then there exists a function f with $\Delta f = 4(n+1) f$ and $K = J\mathrm{grad}(f)$, i.e. f is an eigenfunction of the Laplace operator for the first non-zero eigenvalue. Now, consider the 2–form $\phi := dK^* = dJdf = dd^c(f)$. Since K is a Killing vector field it follows:

$$\begin{aligned}\nabla_X \phi &= \nabla_X(dK) = 2\,\nabla_X(\nabla K) = 2\,\nabla^2_{X,\cdot}K = -\,2\,R(K,\,X)\\ &= -\,2\,(df \wedge JX^* - Jdf \wedge X^*) - 4\,df(X)\,\omega\ .\end{aligned}$$

It is clear that ϕ is an invariant 2-form and an eigenform of the Laplace operator for the minimal eigenvalue $4(n+1)$. A small modification of ϕ defines a conformal Killing 2-form. Indeed, defining $\hat{\phi} := \phi + 6f\omega$ one obtains

$$\nabla_X \hat{\phi} = -\,2\,(df \wedge JX^* - Jdf \wedge X^*) + 2\,df(X)\,\omega\ .$$

Using Lemma 6.1.15 for $\beta := -2\,df$ one concludes that $\hat{\phi}$ is a conformal Killing 2-form. It is not difficult to show that indeed any invariant conformal Killing 2-form on $\mathbb{C}P^n$ has to be of this form. Summarizing the construction one has

Proposition 6.1.16 ([ACG01b]) *Let $K = J\mathrm{grad}(f)$ be any Killing vector field on the complex projective space $\mathbb{C}P^n$ then*

$$\hat{\phi} := dd^c(f) + 6\,f\,\omega = (dd^c(f))_0 + \tfrac{2n-4}{n}\,f\,\omega$$

defines an invariant, non-parallel conformal Killing 2-form. Moreover, in dimension 4, $\hat{\phi} = (dd^c(f))_0$ is a primitive 2-form.

We will now describe the correspondence between Hamiltonian 2-forms and invariant conformal Killing 2-forms. *Hamiltonian 2-forms* are defined as invariant 2-forms $\tilde{u}$ which for any vector field X satisfy the equation

$$\nabla_X \tilde{u} = \tfrac{1}{2}\,(d\mathrm{tr}(\tilde{u}) \wedge JX^* - Jd\mathrm{tr}(\tilde{u}) \wedge X^*)\,.$$

With $\tilde{u}$ one associates two other 2-forms:

$$u := \tilde{u} + \mathrm{tr}(\tilde{u})\,\omega \qquad \text{and} \qquad \hat{u} := \tilde{u} - \tfrac{\mathrm{tr}(\tilde{u})}{2}\,\omega.$$

Note that $2\,\mathrm{tr}(\hat{u}) = (n-2)\,\mathrm{tr}(\tilde{u})$. Hence, on a 4-dimensional manifold $\hat{u} = \tilde{u}_0$ is a primitive invariant 2-form, i.e. it is anti-self-dual. If $n > 2$ one has $\hat{u} = \tilde{u} + \frac{\mathrm{tr}(\hat{u})}{n-2}\,\omega$. The relation between Hamiltonian 2-forms and invariant conformal Killing 2-forms is given by the following

Proposition 6.1.17 *Let $(M^{2n},\,g,\,J,\,\omega)$ be a Kähler manifold with a Hamiltonian 2-form $\tilde{u}$. Then the associated 2-form u is closed and $\hat{u}$ is a conformal Killing 2-form. Conversely, if $\hat{u}$ is a conformal Killing 2-form and $n > 2$, then $\tilde{u} := \hat{u} - \frac{\mathrm{tr}(\hat{u})}{n-2}\,\omega$ is a Hamiltonian 2-form.*

Proof. Let $\tilde{u}$ be a Hamiltonian 2-form. Then the definition implies

$$d\tilde{u} \;=\; \tfrac{1}{2} \sum e_i^* \wedge (d\mathrm{tr}(\tilde{u}) \wedge Je_i^* \;-\; Jd\mathrm{tr}(\tilde{u}) \wedge e_i^*) \;=\; -\,d\,\mathrm{tr}(\tilde{u}) \wedge \omega\;.$$

Hence, $u := \tilde{u} \,+\, \mathrm{tr}(\tilde{u})\,\omega$ is a closed invariant 2-form. For $\hat{u} := \tilde{u} \,-\, \frac{\mathrm{tr}(\tilde{u})}{2}\,\omega$ one obtains

$$\begin{aligned}\nabla_X \hat{u} \;&=\; \nabla_X \tilde{u} \,-\, \tfrac{1}{2}\, d\,\mathrm{tr}(\tilde{u})(X)\,\omega \\ &=\; \tfrac{1}{2}\, (d\,\mathrm{tr}(\tilde{u}) \wedge JX^* \;-\; Jd\,\mathrm{tr}(\tilde{u}) \wedge X^*) \;-\; \tfrac{1}{2}\, d\,\mathrm{tr}(\tilde{u})(X)\,\omega\;.\end{aligned}$$

Thus, the invariant 2-form $\hat{u}$ satisfies the equation (6.1.11) of Lemma 6.1.15 with, $\beta := \frac{1}{2}\, d\,\mathrm{tr}(\tilde{u})$, and it follows that $\hat{u}$ is a conformal Killing 2-form. □

The situation is somewhat different in dimension 4. If $\tilde{u}$ is a Hamiltonian 2-form then $\hat{u} = \tilde{u} - \frac{\mathrm{tr}(\tilde{u})}{2}\omega = \tilde{u}_0$ is an anti-self-dual conformal Killing 2-form. Conversely, if we start with an anti-self-dual conformal Killing 2-form u_0 and ask for which functions f the invariant 2-form $u := u_0 + f\omega$ is Hamiltonian 2-form we have

Lemma 6.1.18 *Let $(M^4,\, g,\, J)$ be a Kähler manifold with an anti-self-dual conformal Killing 2-form u_0. Then $u := u_0 + f\omega$ is Hamiltonian 2-form if and only if*

$$d^* u_0 \;=\; -\,3\,Jdf. \tag{6.1.13}$$

In particular, this is the case on simply connected manifolds where d^*u_0 is dual to a Killing vector field, e.g. on simply connected Einstein manifolds.

Proof. Since $f = \frac{\mathrm{tr}(u)}{2}$ we conclude from the definition that u is a Hamiltonian 2-form if and only if $\nabla_X u = df \wedge JX^* - Jdf \wedge X^*$ for any vector field X. On the other hand u_0 is assumed to be an invariant conformal Killing 2-form. Hence, we have from Lemma 6.1.15 the following equation $\nabla_X\, u_0 = -X \lrcorner\, (\beta \wedge \omega) - J\beta \wedge X^*$, where $\beta = \frac{1}{3}Jd^*u_0$. This implies that $u = u_0 + f\omega$ is a Hamiltonian 2-form if and only if

$$-X \lrcorner\, (\beta \wedge \omega) - J\beta \wedge X^* \;=\; df \wedge JX^* - Jdf \wedge X^* - df(X)\omega \;=\; -X \lrcorner\, (df \wedge \omega) - Jdf \wedge X^*.$$

This is the case if and only if $\beta = df$, or equivalently if $d^*u_0 \;=\; -\,3\,Jdf$. It remains to show that equation (6.1.13) has a solution if d^*u_0 is dual to a Killing vector field and the manifold is simply connected. Let X be any Killing vector field on an arbitrary Kähler manifold, then

$$0 \;=\; L_X\,\omega \;=\; d\,X \lrcorner\,\omega \;+\; X \lrcorner\, d\,\omega \;=\; d\,J\,X^*\;,$$

i.e. the 1-form $J\,X^*$ is closed. Hence, since the manifold is simply connected, there exists some function f with $JX^* = df$ and also $X^* = -Jdf$. Finally, we know from Proposition 7.2.4 that on Einstein manifolds, or more generally under the condition that $\mathrm{Ric} \circ u_0 = u_0 \circ \mathrm{Ric}$ holds, for any conformal Killing 2-form u_0, the 2-form d^*u_0 is dual to a Killing vector field. □

The article [ACG01a] contains a complete local classification of Kähler surfaces with Hamiltonian 2-forms and many compact examples. The examples include all the Hirzebruch surfaces. Moreover, in [ACG01b] the same authors give a partial classification in higher dimensions as well as further examples. It is interesting to note that the main motivation in [ACG01a] and [ACG01b] for the study of Hamiltonian 2-forms comes from the fact that a Kähler surface is weakly self-dual, i.e. its anti-self-dual Weyl tensor is harmonic, if and only if the trace free part of the Ricci form is a Hamiltonian 2-form. Similarly, in higher dimension a Kähler manifold is weakly Bocher flat, i.e. its Bochner tensor is coclosed, if and only if the normalized Ricci form is a Hamiltonian 2-forms.

6.2 Conformal Killing forms on G_2–manifolds

In this section we will show that besides the Kähler manifolds there is still another class of manifolds, where the existence of conformal Killing forms is rather restricted. We will show this for compact manifolds with holonomy G_2. Due to results of D. Joyce (c.f. [J96a], [J96b]) this is a rather rich class and, as recent developments in string theory showed, also a class of manifolds of special interest in physics. As in the Kähler case we will first show that any Killing form or $*$–Killing form is parallel and then prove that any conformal Killing p–form, with $p \neq 3, 4$, is either closed or coclosed.

Let (M^7, g) be a manifold with a topological G_2–structure, i.e. there is a reduction of the frame bundle of M to a G_2-principal bundle. Equivalently there is a differentiable two-fold vector cross product P. The existence of a topological G_2–structure is also equivalent to the existence of a spin structure on M (c.f. [FKMS97]). The manifold (M^7, g) has holonomy G_2, and is called a G_2–manifold, if the Levi-Civita connection reduces to the G_2-principal bundle. Equivalently, the vector cross product P and the associated 3–form ω are parallel. It is easy to see that a manifold with holonomy G_2 is Ricci–flat. Given a compact manifold with holonomy contained in G_2, it was shown in [J96a] that the holonomy group is exactly G_2 if and only if the manifold is simply connected. In this case one can see that the space of parallel forms on M is spanned by $\{1, \omega, *\omega, *1\}$ (c.f. [Br87]). In particular, there are no parallel vector fields.

Theorem 6.2.1 *Let (M^7, g) be a compact manifold with holonomy G_2. Then any Killing form and any $*$–Killing form is parallel. Moreover, any conformal Killing p–form, with $p \neq 3, 4$, is parallel.*

Proof. We will first show that any conformal Killing p–form, with $p \neq 3, 4$, is either closed or coclosed. Since any conformal vector field on a manifold with constant non-positive scalar curvature has to be a Killing vector field (c.f. [Ob72]) we know the statement already for $p = 1$ and for $p = 6$. For the cases $p = 2$ and $p = 5$ we use Proposition 7.2.4, which states that a conformal Killing 2-form is either coclosed or defines a non-trivial Killing vector field ξ. But a G_2-manifold is Ricci-flat so it follows $\Delta\xi = 0$. Hence, since M is compact, the vector field ξ is parallel, which contradicts the assumption that the holonomy is G_2. The statement for a conformal Killing 5-form follows after applying the Hodge star operator.

The next step will be to show that any Killing p-form and any $*$–Killing p-form on a compact manifold with holonomy G_2 is parallel. It is of course sufficient to prove this for $p = 2$ and $p = 3$. In view of Proposition 5.1.3 we can assume that the form is a section of one of the parallel sub-bundles of $\Lambda^2(T^*M)$ resp. $\Lambda^3(T^*M)$. Indeed, the components in the G_2-decomposition of any Killing or $*$-Killing form are again Killing or $*$-Killing forms. Recall that we have the decompositions

$$\Lambda^2 \;=\; \Lambda^2_7 \oplus \Lambda^2_{14} \qquad \text{and} \qquad \Lambda^3 \;=\; \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

and, since $2q(R)$ acts trivially on Λ^3_1 and all the seven dimensional summands, we know already that any conformal Killing form in Λ^2_7, Λ^3_1 or Λ^3_7 has to be parallel. Hence, we only have to consider the case of Killing forms or $*$–Killing forms in the summands Λ^2_{14} or Λ^3_{27}.

We will use the norm characterization of Lemma 1.1.1 to prove that a Killing resp. $*$-Killing form in one of these summands has to be parallel. For this we have to compute the norm of the covariant derivative and to compare it with the norm of the differential resp. codifferential. On an arbitrary Riemannian manifold the covariant derivative of a p–form ψ splits into the embeddings of $d\psi$ and $d^*\psi$, and the twistor operator part $T\psi$. This corresponds to the splitting of the tensor product $T^*M \otimes \Lambda^p(T^*M)$ into three components. In the case of restricted holonomy this decomposition becomes finer, leading to a definition of new first order differential operator adding up to the covariant derivative. We will call these operators *twistor operators*.

At first we consider the 2-form component Λ^2_{14} and define twistor operators corresponding to the decomposition

$$\Lambda^1 \otimes \Lambda^2_{14} \;\;=\;\; \Lambda^1 \oplus \Lambda^3_{27} \oplus V_{64}. \tag{6.2.14}$$

Here V_{64} denotes the irreducible G_2–representation of dimension 64. Let π_a, $a = 1, 2$, denote the projections onto Λ^1 resp. Λ^3_{27}. The precise definition of these projections as well as their properties are given bellow. Next, we have the corresponding twistor operators $T_a\psi := \pi_a(\nabla\psi)$, $a = 1, 2$. It suffices to consider these two operators since for a conformal Killing form ψ the projection of $\nabla\psi$ onto $V_{64} \subset \Lambda^{2,1}$ is obviously zero. It is also clear from the definition that $d^*\psi = T_1\psi$. Next, we will express $d\psi$ in terms of the twistor operators. Let $(\alpha)_r$ denote the component of α in the corresponding irreducible r–dimensional summand. Then for a 2-form ψ:

$$d\psi \;\;=\;\; (d\psi)_1 \;+\; (d\psi)_7 \;+\; (d\psi)_{27}\;.$$

We will now consider a conformal Killing 2-form ψ in Λ^2_{14}. It follows $(d\psi)_1 = 0$ since the 1-dimensional summand does not appear in the decomposition (6.2.14) and it is again clear from the definition that $(d\psi)_{27} = T_2\psi$. For the Λ^3_7–component we use Lemma 6.2.9 and obtain

$$(d\psi)_7 \;=\; \sum (e_i \wedge \nabla_{e_i}\psi)_7 \;=\; -\tfrac{1}{12}\sum p^2(e_i \lrcorner \nabla_{e_i}\psi) \;=\; \tfrac{1}{12}\, p^2(d^*\psi)\ .$$

Since p^2 is injective on vectors, we conclude that $d\psi = 0$ implies $d^*\psi = 0$ and thus a $*$–Killing form in Λ^2_{14} has to be parallel. If we assume ψ to be a Killing form, then $d\psi = T_2\psi = (d\psi)_{27}$ and $d^*\psi = T_1\psi = 0$. Computing the norm, according to Lemma 6.2.11, we obtain

$$|\nabla\psi|^2 \;=\; |j_1(T_1\psi)|^2 \;+\; |j_2(T_2\psi)|^2 \;=\; \tfrac{1}{4}\,|T_1\psi|^2 \;+\; \tfrac{3}{7}\,|T_2\psi|^2 \;=\; \tfrac{3}{7}\,|d\psi|^2 \;\geq\; \tfrac{1}{3}\,|d\psi|^2$$

with equality only if $d\psi = 0$, i.e. only if the Killing 2-form ψ is parallel.

The proof for a closed or coclosed conformal Killing form ψ in Λ^3_{27} is very similar. Here we have the decomposition

$$\Lambda^1 \otimes \Lambda^3_{27} \;=\; \Lambda^2_7 \oplus \Lambda^2_{14} \oplus \Lambda^4_{27} \oplus (V_{64} \oplus V^-_{77})$$

where V^-_{77} is one of the two 77-dimensional irreducible G_2–representation. Corresponding to the first three summands we define three twistor operators $T_a\psi := \pi_a(\nabla\psi)$, $a = 1,2,3$ (again, the details are given bellow). As above, we have that for a conformal Killing form the projections of the covariant derivative onto the last two summands vanish. Moreover, $d^*\psi = T_1\psi + T_2\psi$ holds by definition and

$$d\psi \;=\; (d\psi)_1 \;+\; (d\psi)_7 \;+\; (d\psi)_{27} \;=\; (d\psi)_7 \;+\; (d\psi)_{27} \;=\; \tfrac{1}{12}\, p^2(d^*\psi) \;+\; T_3\psi\ .$$

If we assume $d^*\psi = 0$, then $T_1\psi = 0 = T_2\psi$ and $d\psi = T_3\,\psi$. In this case Lemma 6.2.12 yields

$$|\nabla\psi|^2 \;=\; |j_3(T_3\psi)|^2 \;=\; \tfrac{1}{3}\,|d\psi|^2 \;\geq\; \tfrac{1}{4}\,|d\psi|^2$$

with equality only for $d\psi = 0$, which again implies that the Killing 3–form ψ has to be parallel. If we assume $d\psi = 0$ then $T_3\psi = (d\psi)_{27} = 0$ and $0 = (d\psi)_7 = \frac{1}{12}p^2(d^*\psi) = \frac{1}{12}p^2(T_1\psi)$, i.e. also $T_1\psi = 0$, since p^2 is injective on the 7-dimensional summands. Using Lemma 6.2.12 it then follows

$$|\nabla\psi|^2 \;=\; |j_2(T_2\psi)|^2 \;=\; \tfrac{2}{9}\,|d^*\psi|^2 \;\geq\; \tfrac{1}{5}\,|d^*\psi|^2$$

with equality only for $d^*\psi = 0$, i.e. if ψ is parallel. □

We note that it is possible to give a different proof for the fact that on a G_2–manifold any Killing form has to be parallel. This is done by using a different set of naturally defined twistor operators together with special Weitzenböck formulas adapted to the G_2–holonomy. Similar Weitzenböck formulas exist also in the case of Spin_7–holonomy and again one can show that any Killing form has to be parallel. With the additional conclusion that any conformal Killing p–form, for $p = 1, 2, 5$ and 6, has to be parallel. The idea of considering all possible Weitzenböck formulas adapted to a given holonomy was first developed in [KSW99] in the case of holonomy $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, i.e. for quaternion Kähler manifolds. The general approach is contained in the forthcoming paper [SW01].

At the end of this section we will give the details on G_2–representations which we needed in the above proofs. The main reference is [FG82], from where we take several definitions and formulas. Nevertheless, we also add a few new results which are helpful in our investigation of Killing forms on G_2-manifolds. Moreover, they might be interesting for other applications of the G_2-representation theory.

We consider a 7-dimensional Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ with a two-fold vector cross product $P : V \otimes V \to V$, i.e. P is a linear map which for any vectors X, Y satisfies the equations:

$$\langle P(X, Y), X\rangle = 0 \qquad \text{and} \qquad |\, P(X \wedge Y)\,|^2 = |\, X \wedge Y\,|^2 .$$

Associated with P we have a 3-form ω defined by $\omega(X, Y, Z) = \langle P(X, Y), Z\rangle$. A simple explicit construction for P can be given via the Cayley numbers. It follows that there is an ortho-normal basis $\{e_i\}$, $i = 0, \ldots, 6$, such that $P(e_i, e_{i+1}) = e_{i+3}$ with $i \in \mathbb{Z}_7$. Such a basis is called *Cayley basis*. With respect to a Cayley basis the associated 3-form ω and its Hodge dual $*\omega$ take the form

$$\begin{aligned}
\omega \;&=\; e_0 \wedge e_1 \wedge e_3 \;+\; e_1 \wedge e_2 \wedge e_4 \;+\; e_2 \wedge e_3 \wedge e_5 \;+\; e_3 \wedge e_4 \wedge e_6 \\
&\qquad\qquad + e_0 \wedge e_4 \wedge e_5 \;+\; e_1 \wedge e_5 \wedge e_6 \;+\; e_0 \wedge e_2 \wedge e_6 \; , \\
*\omega \;&=\; -\, e_2 \wedge e_4 \wedge e_5 \wedge e_6 \;+\; e_0 \wedge e_3 \wedge e_5 \wedge e_6 \;-\; e_0 \wedge e_1 \wedge e_4 \wedge e_6 \\
&\qquad + e_0 \wedge e_1 \wedge e_2 \wedge e_5 \;+\; e_1 \wedge e_2 \wedge e_3 \wedge e_6 \;-\; e_0 \wedge e_2 \wedge e_3 \wedge e_4 \\
&\qquad\qquad - e_1 \wedge e_3 \wedge e_4 \wedge e_5 \; .
\end{aligned}$$

In the following we will fix a Cayley basis $\{e_i\}$ and this explicit representation of ω. Note that $|\omega|^2 = |*\omega|^2 = 7$ and $\omega \wedge *\omega = 7 e_0 \wedge \ldots \wedge e_6 = 7\mathrm{vol}$. The group G_2 is defined as the group of all orthogonal transformations under which the 3–form ω is invariant. It is a 14-dimensional subgroup of $\mathrm{SO}(7)$, which naturally acts on the space of forms. Our next aim is to describe the decomposition of the G_2–representation $\Lambda^p V$, for $p = 2, 3$, and to find explicit expressions for the projections onto the irreducible summands. As a first step we , introduce a map p which turns out to be the adjoint of P. We define

$$p : V \to \Lambda^2(V) \qquad \text{with} \qquad p(v) \;:=\; -\tfrac{1}{2} \sum e_i \wedge P(e_i, v) \; .$$

Equivalently, p can be defined as $p(v) = v \lrcorner \omega$. It is then not difficult to show, that p is indeed the adjoint of P, i.e. we have $\langle p(v), \eta\rangle = \langle v, P(\eta)\rangle$ for any vector v and any 2-vector η. Using this notation we can write ω and $*\omega$ in a different way:

$$\omega \;=\; \tfrac{1}{3} \sum e_i \wedge p(e_i) \qquad \text{and} \qquad *\,\omega \;=\; -\tfrac{1}{6} \sum p(e_i) \wedge p(e_i)\;. \tag{6.2.15}$$

An interesting property of the two-fold vector cross product P is that it completely determines the scalar product $\langle\cdot,\cdot\rangle$. This is expressed by the equation $p(X) \wedge p(Y) \wedge \omega = -6\langle X, Y\rangle \,\mathrm{vol}$. The next step is to extend p and P to maps $p : \Lambda^k(V) \to \Lambda^{k+1}(V)$ and $P : \Lambda^k(V) \to \Lambda^{k-1}(V)$. They are defined as

$$P(v_1 \wedge \ldots \wedge v_k) \;=\; \sum_{i<j} (-1)^{i+j+1}\, P(v_i, v_j) \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_k \;,$$

$$p(v_1 \wedge \ldots \wedge v_k) \;=\; \sum_{i} (-1)^{i+1}\, p(v_i) \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k \;.$$

Again, it is easy to show that p is the adjoint of P. Other elementary properties are the following

$$p(\alpha \wedge \beta) \;=\; p(\alpha) \wedge \beta \;+\; (-1)^{|\alpha|}\, \alpha \;\wedge\; p(\beta) \tag{6.2.16}$$

$$P(x \wedge y \wedge z) \;=\; P(x \wedge y) \wedge z \;+\; P(y \wedge z) \wedge x \;+\; P(z \wedge x) \wedge y \tag{6.2.17}$$

$$p(\alpha) \;=\; (-1)^{|\alpha|+1} \;*\; P \;*\; \alpha \tag{6.2.18}$$

where α, β are any forms and x, y, z any vectors. This and the following useful lemma can be found in [FG82].

Lemma 6.2.2 *Let X, Y, Z be any vectors, then*

(1) $\langle\, P(X \wedge Y),\, P(X \wedge Z)\,\rangle \;=\; \langle\, X \wedge Y,\, X \wedge Z\,\rangle \;,$

(2) $P(X \wedge P(X \wedge Y)) \;=\; -\,|\,X\,|^2\, Y \;+\; \langle X, Y\rangle\, X \;,$

(3) $$P^2(X \wedge Y \wedge Z) \;=\; P(P(X \wedge Y) \wedge Z) \;+\; P(P(Y \wedge Z) \wedge X) \;+\; P(P(Z \wedge X) \wedge Y)$$
$$=\; 3\,P(P(X \wedge Y) \wedge Z) \;-\; 3\,\langle X, Z\rangle\, Y \;+\; 3\,\langle Y, Z\rangle\, X \;,$$

(4) $P \circ p\,(X) \;=\; 3\,X \;.$

As an immediate consequence of the lemma we obtain

Corollary 6.2.3 *Let X, Y be any vectors, then*

$$\langle\, p(X),\, p(Y)\,\rangle \;=\; 3\,\langle\, X,\, Y\,\rangle \qquad \textit{and} \qquad \langle\, X \wedge \omega,\, Y \wedge \omega\,\rangle \;=\; 4\,\langle\, X,\, Y\,\rangle \;.$$

Proof. Since P is the adjoint of p we have $\langle\, p(X),\, p(Y)\,\rangle = \langle P\, p(X),\, Y\,\rangle = 3\,\langle\, X,\, Y\,\rangle$. For the second equation we compute

$$\begin{aligned}\langle\, X \wedge \omega,\, Y \wedge \omega\,\rangle \;&=\; \langle\, \omega,\, \langle X, Y\rangle\, \omega \,-\, Y \wedge X \lrcorner\, \omega\,\rangle \;=\; \langle X, Y\rangle\, |\,\omega\,|^2 \,-\, \langle\, Y \lrcorner\, \omega,\, X \lrcorner\, \omega\,\rangle \\ &=\; 7\,\langle X, Y\rangle \,-\, \langle\, p(X),\, p(Y)\,\rangle \;=\; 4\,\langle X, Y\rangle\;. \quad \Box\end{aligned}$$

The maps p and P are by definition G_2–invariant. If we decompose the spaces Λ^p into irreducible G_2–summands, we know by the Lemma of Schur, that p and P are trivial between non-isomorphic summands and a multiple of identity between isomorphic ones. We will consider the decompositions:

$$\Lambda^2 \;=\; \Lambda^2_7 \,\oplus\, \Lambda^2_{14} \qquad \text{and} \qquad \Lambda^3 \;=\; \Lambda^3_1 \,\oplus\, \Lambda^3_7 \,\oplus\, \Lambda^3_{27}\;.$$

Here the subscript denotes the dimension of the corresponding irreducible component, which in low dimensions uniquely describes the representation. In particular, Λ^p_7, $p = 1, 2$ is isomorphic to the standard representation of $G_2 \subset \mathrm{SO}(7)$ and Λ^2_{14} is isomorphic to the Lie algebra of G_2 equipped with the adjoint action. The 1-dimensional representation Λ^3_1 is spanned by the invariant form ω. The following descriptions of the summands are well-known:

$$\Lambda^2_7 \;=\; \{\, X \lrcorner\, \omega \,|\, X \in V\} \;=\; \mathrm{im}(p), \qquad \Lambda^2_{14} \;=\; \{\, \alpha \,|\, \alpha \wedge *\omega = 0\,\} \;=\; \ker(p) \;=\; \ker(P),$$

$$\Lambda^3_7 \;=\; \{\, X \lrcorner\, *\,\omega \,|\, X \in V\,\} \;=\; \mathrm{im}(p), \qquad \Lambda^3_{27} \;=\; \{\, \alpha \,|\, \langle \alpha,\, \omega\rangle = 0 = \alpha \wedge \omega\}\;.$$

Based on these characterizations of the irreducible summands of the form representations we can derive explicit formulas for several maps between isomorphic summands. As a first application we consider the powers of p. The article [FG82] contains explicit formulas for $p^k(e_i)$, where e_i is any vector from the Cayley basis and k is any number. This we can reformulate as

Lemma 6.2.4 *Let X be any vector, then*

$$\begin{aligned}&(1) \qquad p^2(X) \;=\; p\,(X \lrcorner\, \omega) \;=\; 3\, X \lrcorner\, *\,\omega\;,\\ &(2) \qquad p^3(X) \;=\; 3\, p(X \lrcorner\, *\,\omega) \;=\; 9\, X \wedge \omega\;,\\ &(3) \qquad p^4(X) \;=\; 9\, p(X \wedge \omega) \;=\; 36\, X \wedge *\omega\;.\end{aligned}$$

Dualizing these equations we obtain formulas for P on the irreducible 7-dimensional summands in the space of 3-forms resp. in the space of 4-forms.

Lemma 6.2.5 *Let X be any vector, then*

$$P\,(X \lrcorner\, *\,\omega) \;=\; 4\, X \lrcorner\, \omega \qquad \textit{and} \qquad P\,(X \wedge \omega) \;=\; 3\, X \lrcorner\, *\,\omega\;.$$

Proof. For the first equation we note, that the image of P is $\Lambda^2_7 = \mathrm{im}(p)$, i.e. there is some vector ξ with $P\,(X \lrcorner \, *\omega) = p(\xi)$. Taking the scalar product with $p(Z)$, where Z is any vector and applying Corollary 6.2.3 we obtain

$$\begin{aligned}\langle p(\xi), p(Z)\rangle &= 3\,\langle \xi, Z\rangle \\ &= \langle P\,(X \lrcorner \, *\omega), p(Z)\rangle \;=\; \langle X \lrcorner \, *\omega, p^2(Z)\rangle \;=\; 3\,\langle X \lrcorner \, *\omega, Z \lrcorner \, *\omega\rangle \\ &= 3\,\langle X \wedge \omega, Z \wedge \omega\rangle \;=\; 12\,\langle X, Z\rangle \;.\end{aligned}$$

Hence, since Z was an arbitrary vector, we conclude $\xi = 4\,X$, which proves the first equation. The second equation follows from

$$\begin{aligned}P\,(X \wedge \omega) &= P * (*(X \wedge \omega)) \;=\; -\,P * (X \lrcorner \, *\omega) \;=\; -\, * p\,(X \lrcorner \, *\omega) \\ &= -\,3 * (X \wedge \omega) \;=\; 3\,X \lrcorner \, *\omega \;.\end{aligned}$$

Note that $*^2$ is the identity on 4-forms and $P \circ * = * \circ p$ on 3-forms. Moreover, it is well-known that for a p-form α one has $*(X \lrcorner \alpha) = -(-1)^p\, X \wedge *\alpha$ and $*(X \wedge \alpha) = (-1)^p\, X \lrcorner * \alpha$. □

Lemma 6.2.6 *Let X be any vector and let η be any 2-form, then*

$$\begin{aligned}&(1) \qquad P^2(X \wedge \eta) \;=\; 3\,P(P(\eta) \wedge X) \;-\; 3\,X \lrcorner \, \eta \;, \\ &(2) \qquad P\,(X \wedge \eta) \;=\; -\,p\,(X \lrcorner \, \eta) \qquad \textit{for} \quad \eta \in \Lambda^2_{14} \;, \\ &(3) \qquad P^2(X \wedge \eta) \;=\; -\,3\,X \lrcorner \, \eta \qquad \textit{for} \quad \eta \in \Lambda^2_{14} \;.\end{aligned}$$

Proof. If $\eta \in \Lambda^2_{14}$ then obviously $P(\eta) = 0$. Thus, equation (3) follows from equation (1). On the other hand, we know that P maps $X \wedge \eta$ into the image of p, i.e. there is some vector ξ with $P\,(X \wedge \eta) = p(\xi)$. Hence, applying P to both sides and using equation (3) yields $-3\,X \lrcorner \, \eta = P\,p(\xi) = 3\,\xi$, i.e. $\xi = -X \lrcorner \, \eta$, which proves the second equation. The proof of the formula for P^2 is an application of Lemma 6.2.2. We first write

$$\begin{aligned}P^2(X \wedge Y \wedge Z) &= P^2(Y \wedge Z \wedge X) \;=\; 3\,P(P(Y \wedge Z) \wedge X) \\ &= 3\;(\langle Y, X\rangle\, Z \;-\; \langle Z, X\rangle\, Y) \\ &= 3\,P(P(Y \wedge Z) \wedge X) \;-\; 3\,X \lrcorner \,(Y \wedge Z) \;.\end{aligned}$$

Writing the formula this way it becomes clear that we can substitute $Y \wedge Z$ by any 2-form η, which then proves the first equation. □

We also want to remark that $p(*\omega) = 0 = P(\omega)$, which is clear since there is no trivial summand in $\Lambda^2 \cong \Lambda^5$. It is not difficult to determine $p(\omega)$ and $P(*\omega)$ as well. We find

Lemma 6.2.7

$$p(\omega) \;=\; -6 * \omega \qquad\qquad \textit{and} \qquad\qquad P(*\omega) \;=\; -6\,\omega$$

Proof. We start with the proof for $p(\omega)$. Using the realization of ω and $*\omega$ given in (6.2.15) and applying equation (6.2.16) and Lemma 6.2.6 we obtain

$$\begin{aligned} p(\omega) \;&=\; \tfrac{1}{3} \sum (p(e_i) \wedge p(e_i) \;-\; e_i \wedge p^2(e_i)) \\ &=\; -2 * \omega \;-\; \sum e_i \wedge e_i \lrcorner * \omega \;=\; -6 * \omega \;. \end{aligned}$$

Since $P \circ *$ is a G_2-equivariant map which preserves the space of 3-forms we conclude $P(*\omega) = c\,\omega$ for some constant c. Taking the scalar product with ω gives

$$c\,|\,\omega\,|^2 \;=\; \langle P(*\omega),\, \omega\rangle \;=\; \langle *\omega,\, p(\omega)\rangle \;=\; -6\,|*\omega\,|^2 \;=\; -6\,|\,\omega\,|^2 \;.$$

Hence, the constant c is again -6 which finishes the proof of the lemma. □

In the proof of the preceding lemma we used already that $P \circ *$ is a G_2-equivariant map which preserves the space of 3-forms. Similarly, $* \circ P$ preserves the space of 4-forms. Both maps have to be multiples of the identity on the irreducible summands. The corresponding constants are given in

Lemma 6.2.8

$$\begin{array}{lllllllll} P \circ * &=& -6\,\mathrm{id} & \textit{on} & \Lambda^3_1 & \textit{and} & * \circ P \;=\; -6\,\mathrm{id} & \textit{on} & \Lambda^4_1 \\ P \circ * &=& -3\,\mathrm{id} & \textit{on} & \Lambda^3_7 & \textit{and} & * \circ P \;=\; -3\,\mathrm{id} & \textit{on} & \Lambda^4_7 \\ P \circ * &=& \mathrm{id} & \textit{on} & \Lambda^3_{27} & \textit{and} & * \circ P \;=\; \mathrm{id} & \textit{on} & \Lambda^4_{27} \end{array}$$

Proof. The equations on Λ^3_1 resp. Λ^3_1 are of course already contained in Lemma 6.2.7. Moreover, the equations on Λ^3_7 resp. Λ^3_7 are direct consequences of Lemma 6.2.5. The remaining equations can be checked by computing $P \circ *$ on some special element of Λ^3_{27}. □

Using the maps p and P we can now give explicit formulas for the projections onto the 7-dimensional summands. The projection $\pi_{\Lambda^3_1}$ onto Λ^3_1 is of course: $\pi_{\Lambda^3_1}(\alpha) := \frac{1}{7}\,\langle \alpha,\, \omega\rangle\,\omega$ and the remaining projections are obtained by taking the difference with the identity. We have

$$\pi_{\Lambda^2_7}(\alpha) := \tfrac{1}{3}\, p \circ P(\alpha) \qquad\qquad \text{and} \qquad\qquad \pi_{\Lambda^3_7}(\beta) := \tfrac{1}{12}\, p \circ P(\beta) \;,$$

where α is any 2-form and β any 3-form. It is clear that the images of these two maps are the corresponding 7-dimensional summands. To check whether these are indeed projections, we still have to show $(\pi_{\Lambda^2_7})^2 = \pi_{\Lambda^2_7}$, which follows from $P \circ p = 3\,\mathrm{id}_V$, and

$(\pi_{\Lambda^3_7})^2 = \pi_{\Lambda^3_7}$, which follows from $P \circ p = 12 \, \mathrm{id}_{\Lambda^2_7}$. Indeed, any element of Λ^2_7 is given as $X \lrcorner \omega$ for some vector X and because of Lemma 6.2.4 and Lemma 6.2.5

$$P(p(X \lrcorner \omega)) \;=\; 3\, P(X \lrcorner *\omega) \;=\; 12\, X \lrcorner \omega \;.$$

The space of 4-forms Λ^4 is via the Hodge star operator isomorphic to Λ^3. Hence, it has the same decomposition and $\pi_{\Lambda^4_7} := * \, \pi_{\Lambda^3_7} *$. Above we used the following

Lemma 6.2.9 *Let α be a 2-form in Λ^2_{14} and β a 3–form in Λ^3_{27}, then for any vector X*

$$\pi_{\Lambda^3_7}(X \wedge \alpha) \;=\; -\tfrac{1}{12}\, p^2(X \lrcorner \alpha) \qquad \textit{and} \qquad \pi_{\Lambda^4_7}(X \wedge \beta) \;=\; -\tfrac{1}{12}\, p^2(X \lrcorner \beta) \;.$$

Proof. The projection onto the 7-dimensional summand of Λ^3 is defined as $\frac{1}{12}\, pP$. Hence, applying Lemma 6.2.6 for $\eta = \alpha \in \Lambda^2_{14}$ we obtain

$$\pi_{\Lambda^3_7}(X \wedge \alpha) \;=\; \tfrac{1}{12}\, p\, P\, (X \wedge \alpha) \;=\; -\tfrac{1}{12}\, p^2\, (X \lrcorner \alpha)$$

which proves the formula for $\pi_{\Lambda^3_7}$.

The formula for second projection needs a little more work. We first note, that the 4-forms $\{\frac{1}{2}\,(e_i \wedge \omega)\}$ define an ortho-normal basis in Λ^4_7 according to Corollary 6.2.3. Hence, we can write $\pi_{\Lambda^4_7}(X \wedge \beta) = \frac{1}{2} \sum c_i\,(e_i \wedge \omega)$ for some constants c_i, which are determined by

$$\begin{aligned} c_i \;&=\; \langle\, \pi_{\Lambda^4_7}(X \wedge \beta),\, \tfrac{1}{2}\,(e_i \wedge \omega)\,\rangle \;=\; \tfrac{1}{2}\,\langle\, X \wedge \beta,\, e_i \wedge \omega)\,\rangle \;=\; \tfrac{1}{2}\,\langle\, \beta,\, \langle X,\, e_i \rangle\, \omega \;-\; e_i \wedge p(X)\,\rangle \\ &=\; -\tfrac{1}{2}\,\langle\, \beta,\, e_i \wedge p(X)\,\rangle \;. \end{aligned}$$

For the last equation we used the fact that Λ^3_{27} is orthogonal to Λ^3_1. On the other hand, we know that for a 3-form β also $*p(X \lrcorner \beta)$ is a form in Λ^4_7, i.e. there are constants d_i such that $*p(X \lrcorner \beta) = \frac{1}{2} \sum d_i\,(e_i \wedge \omega)$. In this case we find

$$\begin{aligned} d_i \;&=\; \tfrac{1}{2}\,\langle\, *p(X \lrcorner \beta),\, e_i \wedge \omega\,\rangle \;=\; \tfrac{1}{2}\,\langle\, p(X \lrcorner \beta),\, *(e_i \wedge \omega)\,\rangle \\ &=\; -\tfrac{1}{2}\,\langle\, \beta,\, X \wedge P(e_i \lrcorner *\omega)\,\rangle \;=\; -2\,\langle\, \beta,\, X \wedge p(e_i)\,\rangle \\ &=\; 2\;\langle\, \beta,\, p(X \wedge e_i) \;-\; p(X) \wedge e_i\,\rangle \\ &=\; -2\;\langle\, \beta, p(X) \wedge e_i\,\rangle \;. \end{aligned}$$

Here we used $P(\beta) = 0$, since $\beta \in \Lambda^3_{27}$. Moreover, we applied equation (6.2.16) and Lemma 6.2.5. Comparing the coefficients c_i and d_i we obtain

$$\pi_{\Lambda^4_7}(X \wedge \beta) \;=\; \tfrac{1}{4} * p(X \lrcorner \beta) \;.$$

To prove the Lemma we still have to show $-3 * p(X \lrcorner \beta) = p^2(X \lrcorner \beta)$. For this let α be any 3-form, then

$$\langle *p^2(X \lrcorner \beta), \alpha \rangle = \langle \beta, X \wedge P^2(*\alpha) \rangle = -3 \langle \beta, X \wedge P(\alpha) \rangle = -3 \langle p(X \lrcorner \beta), \alpha \rangle.$$

Indeed, Lemma 6.2.8 states $P* = -3\,\mathrm{id}$ on Λ^3_7 and obviously $P* = 0$ on the other summands of Λ^3. The last equation yields $*p^2(X \lrcorner \beta) = -3\,p(X \lrcorner \beta)$ and the formula for $\pi_{\Lambda^4_7}$ follows after applying the Hodge star operator. $\square$

Since $V = \Lambda^1(V)$ is irreducible as G_2–representation and since p is a G_2–equivariant map, it is clear that p and p^2 are injective on V. More precisely we have

Lemma 6.2.10 *The maps p and p^2 are injective on V and for any vector X in V:*

$$|\,p(X)|^2 = 3\,|X|^2 \qquad \textit{and} \qquad |\,p^2(X)|^2 = 36\,|X|^2 .$$

Proof. The formula for the norm of $p(X)$ is already contained in Corollary 6.2.3. To compute the norm of $p^2(X)$ we use in addition Lemma 6.2.4 and conclude

$$|p^2(X)|^2 = 9\,|X \lrcorner *\omega\,|^2 = 9\,|*(X \wedge \omega)\,|^2 = 9\,|X \wedge \omega\,|^2 = 36\,|X|^2 ,$$

which also shows that p^2 is injective on V. $\square$

In Section 4.6 we considered the decomposition of the tensor product $V^* \otimes \Lambda^p(V^*)$. The definition of the corresponding projections and embeddings lead to the definition of the twistor operator and clarified the constants in the norm estimate. This was in the case of SO(n)-representations, we will now specialize to G_2–representations and define projections and embeddings for the following tensor product decompositions:

$$\Lambda^1 \otimes \Lambda^2_{14} = \Lambda^1 \oplus \Lambda^3_{27} \oplus V_{64} \qquad \text{and} \qquad \Lambda^1 \otimes \Lambda^3_{27} = \Lambda^2_7 \oplus \Lambda^2_{14} \oplus \Lambda^4_{27} \oplus (V_{64} \oplus V^-_{77}) ,$$

where V_{64} denotes the 64-dimensional irreducible G_2–representation and V^-_{77} is one of the two 77-dimensional irreducible G_2–representation. Note that Λ^4_{27} is isomorphic to Λ^3_{27}. Similar to Section 4.6 we define projections $\pi_1 : \Lambda^1 \otimes \Lambda^2_{14} \to \Lambda^1$ and $\pi_2 : \Lambda^1 \otimes \Lambda^2_{14} \to \Lambda^3_{27}$ for $\alpha, \beta \in \Lambda^2_{14}$ by

$$\pi_1\,(X \otimes \alpha) = -X \lrcorner \alpha \qquad \text{and} \qquad \pi_2\,(X \otimes \beta) = (X \wedge \beta)_{27} ,$$

where $(\cdot)_{27}$ denotes the projection $\pi_{\Lambda^3_{27}}$. Next, we introduce embeddings $j_1 : \Lambda^1 \to \Lambda^1 \otimes \Lambda^2_{14}$ and $j_2 : \Lambda^3_{27} \to \Lambda^1 \otimes \Lambda^2_{14}$. For $X \in \Lambda^1$ and $\beta \in \Lambda^3_{27}$ we define

$$j_1(X) := -\tfrac{1}{4} \sum e_i \otimes (e_i \wedge X)_{14} \qquad \text{and} \qquad j_2(\beta) := \tfrac{3}{7} \sum e_i \otimes (e_i \lrcorner \beta)_{14}$$

where $\{e_i\}$ is the fixed Cayley basis and $(\cdot)_{14} = \pi_{\Lambda^2_{14}}$.

Lemma 6.2.11 *The embeddings* j_a *are right inverses for* π_a*, i.e.* $\pi_a \circ j_a = \mathrm{id}$, *for* $a = 1, 2$. *Moreover, if* X *is a vector in* V *and* β *is a 3–form in* Λ^3_{27}*, then*

$$|j_1(X)|^2 = \tfrac{1}{4}|X|^2 \qquad \text{and} \qquad |j_2(\beta)|^2 = \tfrac{3}{7}|\beta|^2 .$$

Proof. We start with proving $\pi_1 \circ j_1 = \mathrm{id}$. Let X be any vector, then

$$\begin{aligned} 4\,\pi_1 \circ j_1\,(X) &= \sum e_i \lrcorner\,(e_i \wedge X)_{14} \\ &= \sum e_i \lrcorner\,[e_i \wedge X - \tfrac{1}{3}\,p\,P(e_i \wedge X)] \\ &= 6\,X - \tfrac{1}{3}\sum e_i \lrcorner\,p\,P(e_i \wedge X) . \end{aligned}$$

The sum in the last equation defines a G_2-equivariant map on Λ^1. Hence, by Schur's lemma, it has to be a scalar multiple of the identity. In Lemma 6.2.13 we compute this constant to be 6, which then proves $\pi_1 \circ j_1 = \mathrm{id}$. The formula for the norm of $j_1(X)$ immediately follows from

$$\begin{aligned} 16\,|\,j_1(X)\,|^2 &= \sum |(e_i \wedge X)_{14}|^2 = \sum \langle\, e_i \wedge X, (e_i \wedge X)_{14}\,\rangle \\ &= \sum \langle\, X, e_i \lrcorner\,(e_i \wedge X)_{14}\,\rangle \\ &= \langle\, X, 4\,\pi_1 \circ j_1(X)\,\rangle = 4\,|X|^2 . \end{aligned}$$

The proof of the formulas for j_2 resp. π_2 similar. As we have seen, it suffices to prove that j_2 is right inverse for π_2. Here we find for $\beta \in \Lambda^3_{27}$

$$\begin{aligned} \tfrac{7}{3}\,\pi_2 \circ j_2\,(\beta) &= \pi_{\Lambda^3_{27}}\Big(\sum e_i \wedge [\,e_i \lrcorner\,\beta - \tfrac{1}{3}\,p\,P\,(e_i \lrcorner\,\beta)\,]\Big) \\ &= 3\,\beta - \tfrac{1}{3}\,\pi_{\Lambda^3_{27}}\Big(\sum e_i \wedge p\,P\,(e_i \lrcorner\,\beta)\Big) . \end{aligned}$$

Again, the sum in the last equation defines a G_2–equivariant map, this time on Λ^3_{27}. In Lemma 6.2.13 we prove that it is twice the identity, which then finishes the proof of $\pi_2 \circ j_2 = \mathrm{id}$. $\square$

Finally, we have to define embeddings and projections for the first three summands in the decomposition of $\Lambda^1 \otimes \Lambda^3_{27}$. The three projections π_a, $a = 1, 2, 3$ are for any vector $X \in V$ and any $\beta \in \Lambda^3_{27}$ defined as

$$\pi_1(X \otimes \beta) := -\,(X \lrcorner\,\beta)_7 \qquad \pi_2(X \otimes \beta) := -\,(X \lrcorner\,\beta)_{14} \qquad \pi_3(X \otimes \beta) := (X \wedge \beta)_{27},$$

where $(\cdot)_7 = \pi_{\Lambda^2_7}$, $(\cdot)_{14} = \pi_{\Lambda^2_{14}}$ and $(\cdot)_{27} = \pi_{\Lambda^4_{27}}$. Above we only used the last two embeddings, i.e. the embeddings j_2 resp. j_3 of Λ^2_{14} resp. Λ^3_{27}, which we define as

$$j_2(\alpha) = -\tfrac{2}{9}\sum e_i \otimes (e_i \wedge \alpha)_{27}, \qquad j_3(\gamma) = \tfrac{1}{3}\sum e_i \otimes (e_i \lrcorner\,\gamma)_{27} .$$

Lemma 6.2.12 *The embeddings* j_a *are right inverses for* π_a *i.e.* $\pi_a \circ j_a = \mathrm{id}$, *for* $a = 2, 3$. *Moreover, if* α *is a 2-form in* Λ^2_{14} *and if* γ *is a 4-form in* Λ^4_{27}, *then*

$$|j_2(\alpha)|^2 \;=\; \tfrac{2}{9}\,|\alpha|^2, \qquad |j_3(\gamma)|^2 \;=\; \tfrac{1}{3}\,|\gamma|^2 \;.$$

Proof. As above it suffices to prove that with our definition j_a are right inverses for π_a for $a = 2, 3$. Again, this leads to G_2–equivariant maps of Λ^2_{14} resp. Λ^4_{27} which we compute in Lemma 6.2.13. Starting with $\pi_2 \circ j_2$ and $\alpha \in \Lambda^2_{14}$ we have

$$\begin{aligned}
\tfrac{9}{2}\,\pi_2 \circ j_2(\alpha) \;&=\; \pi_{\Lambda^2_{14}} \Big(\sum e_i \lrcorner\, (e_i \wedge \alpha)_{27} \Big) \\
&=\; \pi_{\Lambda^2_{14}} \Big(\sum e_i \lrcorner\, [e_i \wedge \alpha \;-\; \langle\, e_i \wedge \alpha, \, \omega \,\rangle\, \tfrac{\omega}{7} \;-\; \tfrac{1}{12}\, p\, P\,(e_i \wedge \alpha)\,] \Big) \\
&=\; 5\,\alpha \;-\; \tfrac{1}{12} \sum e_i \lrcorner\, p\, P\,(e_i \wedge \alpha) \;.
\end{aligned}$$

It is easy to see that the sum is equal to $6\,\alpha$, i.e. $\pi_2 \circ j_2 = \mathrm{id}$ (c.f. Lemma 6.2.13). Next, we have to do the calculation for $\pi_3 \circ j_3$ and $\gamma \in \Lambda^4_{27}$. This time we obtain

$$\begin{aligned}
3\,\pi_3 \circ j_3(\gamma) \;&=\; \pi_{\Lambda^4_{27}} \Big(\sum e_i \wedge (e_i \lrcorner\, \gamma)_{27} \Big) \\
&=\; \pi_{\Lambda^4_{27}} \Big(\sum e_i \wedge [\, e_i \lrcorner\, \gamma \;-\; \langle\, e_i \lrcorner\, \gamma, \, \omega \,\rangle\, \tfrac{\omega}{7} \;-\; \tfrac{1}{12}\, p\, P\,(e_i \lrcorner\, \gamma)\,] \Big) \\
&=\; 4\,\gamma \;-\; \tfrac{1}{12} \sum e_i \wedge p\, P\,(e_i \lrcorner\, \gamma) \;.
\end{aligned}$$

In this case the sum turns out to be $12\,\gamma$, which finishes the proof of the lemma. □

Lemma 6.2.13 *Let* X *be a vector,* $\beta \in \Lambda^3_{27}$, $\alpha \in \Lambda^2_{14}$ *and* $\gamma \in \Lambda^4_{27}$, *then*

$$\sum e_i \lrcorner\, p\, P(e_i \wedge X) \;=\; 6\, X \;, \qquad \sum e_i \lrcorner\, p\, P(e_i \wedge \alpha) \;=\; 6\,\alpha \;,$$

$$\sum e_i \wedge p\, P(e_i \lrcorner\, \beta) \;=\; 2\,\beta \;, \qquad \sum e_i \wedge p\, P(e_i \lrcorner\, \gamma) \;=\; 12\,\gamma \;.$$

Proof. All the four sums considered in the lemma define G_2–equivariant maps on irreducible G_2–representations. Hence, by Schur's Lemma, they have to be a multiple of the identity. We start with the map on Λ^1 assuming it to be $c\,\mathrm{id}$ for some constant c we obtain

$$\begin{aligned}
c\,|X|^2 \;&=\; \sum \langle\, e_i \lrcorner\, p\, P(e_i \wedge X), \, X \,\rangle \;=\; \sum \langle\, P(e_i \wedge X), \, P(e_i \wedge X) \,\rangle \\
&=\; \sum |\, e_i \wedge X \,|^2 \;=\; 6\,|\, X \,|^2 \;.
\end{aligned}$$

Note that by the definition of a vector cross product P is an isometry on decomposable 2-vectors. Next, we consider the sum for $\alpha \in \Lambda^2_{14}$. Here we compute

$$
\begin{aligned}
c\,|\alpha|^2 &= \sum \langle\, e_i \lrcorner\, p\, P(e_i \wedge \alpha),\, \alpha \,\rangle \;=\; \sum \langle\, P(e_i \wedge \alpha),\, P(e_i \wedge \alpha)\,\rangle \\
&= -\sum \langle\, p(e_i \lrcorner\, \alpha),\, P(e_i \wedge \alpha)\,\rangle \;=\; -\sum \langle\, e_i \lrcorner\, \alpha,\, P^2(e_i \wedge \alpha)\,\rangle \\
&= 3 \sum \langle\, e_i \lrcorner\, \alpha,\, e_i \lrcorner\, \alpha \,\rangle \;=\; 6\,|\,\alpha\,|^2 \,.
\end{aligned}
$$

In this computation we used the formulas of Lemma 6.2.6. The easiest way to check the last two equations is to compute them for one special form $\beta \in \Lambda^3_{27}$ resp. $\gamma \in \Lambda^4_{27}$. We take $\beta = e_0 \wedge e_1 \wedge e_2 - e_2 \wedge e_4 \wedge e_6$. To see whether β is indeed an element of Λ^3_{27} we only have to check $\langle\, \beta,\, \omega\,\rangle = 0 = \beta \wedge \omega$, which is easily done. As above we compute

$$
\begin{aligned}
c\,|\,\beta\,|^2 &= \sum |\, P(e_i \lrcorner\, \beta)\,|^2 \\
&= |\, P(e_1 \wedge e_2)\,|^2 \;+\; |\, P(e_0 \wedge e_2)\,|^2 \;+\; |\, P(e_0 \wedge e_1 \,-\, e_4 \wedge e_6)\,|^2 \\
&\qquad\qquad\qquad\qquad + |\, P(e_2 \wedge e_6)\,|^2 \;+\; |\, P(e_2 \wedge e_4)\,|^2 \\
&= 4 \;=\; 2\,|\,\beta\,|^2 \,.
\end{aligned}
$$

Again, we used that P is an isometry on decomposable 2-vectors and that $P(e_0 \wedge e_1 - e_4 \wedge e_6) = 0$, which can be checked from the explicit formula for ω. Finally, we do the same for $\gamma := *\beta = e_3 \wedge e_4 \wedge e_5 \wedge e_6 - e_0 \wedge e_1 \wedge e_3 \wedge e_5 \in \Lambda^4_{27}$ and obtain

$$
\begin{aligned}
c\,|\,\gamma\,|^2 \;=\; \sum |\, P(e_i \lrcorner\, \gamma)\,|^2 \;&=\; |\, P(e_1 \wedge e_3 \wedge e_5)\,|^2 \;+\; |\, P(e_0 \wedge e_3 \wedge e_5)\,|^2 \\
&+ |\, P(e_4 \wedge e_5 \wedge e_6) \,-\, P(e_0 \wedge e_1 \wedge e_5)\,|^2 \;+\; |\, P(e_3 \wedge e_5 \wedge e_6)\,|^2 \\
&+ |\, P(e_3 \wedge e_4 \wedge e_6) \,+\, P(e_0 \wedge e_1 \wedge e_3)\,|^2 \;+\; |\, P(e_3 \wedge e_4 \wedge e_5)\,|^2 \,.
\end{aligned}
$$

From the explicit formula for ω we immediately conclude that $P(e_3 \wedge e_4 \wedge e_6) = 0$ and

$P(e_0 \wedge e_1 \wedge e_3) = 0$. Moreover, we find

$$\begin{aligned}
P(e_1 \wedge e_3 \wedge e_5) &= P(e_1 \wedge e_3) \wedge e_5 + P(e_3 \wedge e_5) \wedge e_1 + P(e_5 \wedge e_1) \wedge e_3 \\
&= e_0 \wedge e_5 + e_2 \wedge e_1 - e_6 \wedge e_3 , \\
P(e_0 \wedge e_3 \wedge e_5) &= P(e_0 \wedge e_3) \wedge e_5 + P(e_3 \wedge e_5) \wedge e_0 + P(e_5 \wedge e_0) \wedge e_3 \\
&= -e_1 \wedge e_5 + e_2 \wedge e_0 + e_4 \wedge e_3 , \\
P(e_4 \wedge e_5 \wedge e_6) &= P(e_4 \wedge e_5) \wedge e_6 + P(e_5 \wedge e_6) \wedge e_4 + P(e_6 \wedge e_4) \wedge e_5 \\
&= e_0 \wedge e_6 + e_1 \wedge e_4 - e_3 \wedge e_5 , \\
P(e_0 \wedge e_1 \wedge e_5) &= P(e_0 \wedge e_1) \wedge e_5 + P(e_1 \wedge e_5) \wedge e_0 + P(e_5 \wedge e_0) \wedge e_1 \\
&= e_3 \wedge e_5 + e_6 \wedge e_0 + e_4 \wedge e_1 , \\
P(e_3 \wedge e_5 \wedge e_6) &= P(e_3 \wedge e_5) \wedge e_6 + P(e_5 \wedge e_6) \wedge e_3 + P(e_6 \wedge e_3) \wedge e_5 \\
&= e_2 \wedge e_6 + e_1 \wedge e_3 + e_4 \wedge e_5 , \\
P(e_3 \wedge e_4 \wedge e_5) &= P(e_3 \wedge e_4) \wedge e_5 + P(e_4 \wedge e_5) \wedge e_3 + P(e_5 \wedge e_3) \wedge e_4 \\
&= e_6 \wedge e_5 + e_0 \wedge e_3 - e_2 \wedge e_4 .
\end{aligned}$$

The first conclusion from this explicit calculation is $P(e_4 \wedge e_5 \wedge e_6) = -P(e_0 \wedge e_1 \wedge e_5)$. Hence, the norm of the corresponding summand is 12 and the norm of each of the remaining 4 summands is 3. Since the norm of γ is 2 we obtain $c = 12$. □

For the sake of completeness we also give the formula for the embedding j_1 of Λ^2_7 into the tensor product $\Lambda^1 \otimes \Lambda^3_{27}$. Let $\alpha \in \Lambda^2_7$ then we define

$$j_1(\alpha) = -\tfrac{7}{18} \sum e_i \otimes (e_i \wedge \alpha)_{27} .$$

As above we obtain

Lemma 6.2.14 *The map j_1 is a right inverse for the projection π_1 and if α is any 2-form in Λ^2_7, then*

$$|\, j_1(\alpha)\,|^2 = \tfrac{7}{18}\,|\,\alpha\,|^2 .$$

Proof. It is again sufficient to prove that $\pi_1 \circ j_1 = \mathrm{id}_{\Lambda^2_7}$ holds. Let $\alpha \in \Lambda^2_7$, then

$$-\tfrac{18}{7}\,\pi_1 \circ j_1\,(\alpha) = \pi_{\Lambda^2_7}\Big(\sum e_i \lrcorner\, [e_i \wedge \alpha]_{27}\Big) .$$

The sum defines an equivariant map, hence it equals $c\,\alpha$ for some constant c. To determine this constant we take the scalar product with $\alpha = p(\xi)$ and obtain

$$\begin{aligned}
c\,|\,\alpha\,|^2 &= \sum \langle\, e_i \lrcorner\, [e_i \wedge \alpha]_{27},\, \alpha\,\rangle \\
&= \sum \langle\, e_i \wedge \alpha \;-\; \langle e_i \wedge \alpha, \omega\rangle\, \tfrac{1}{7}\,\omega \;-\; \tfrac{1}{12}\, p\, P\,(e_i \wedge \alpha),\, e_i \wedge \alpha\,\rangle \\
&= 5\,|\,\alpha\,|^2 \;-\; \tfrac{1}{7} \sum \langle p(\xi), p(e_i)\rangle^2 \;-\; \tfrac{1}{12} \sum |\, P(e_i \wedge p(\xi))\,|^2 \\
&= 5\,|\,\alpha\,|^2 \;-\; \tfrac{9}{7} \sum \langle \xi, e_i\rangle^2 \;-\; \tfrac{1}{3} \sum |\, p(P(e_i \wedge \xi))\,|^2 \\
&= 5\,|\,\alpha\,|^2 \;-\; \tfrac{3}{7} \sum |\, p(\xi)\,|^2 \;-\; \sum |\, e_i \wedge \xi\,|^2 \\
&= 5\,|\,\alpha\,|^2 \;-\; \tfrac{3}{7} \sum |\,\alpha\,|^2 \;-\; 2\,|\,p(\xi)\,|^2 \\
&= \tfrac{1}{7}\,(35 - 3 + 14)\,|\,\alpha\,|^2 \;=\; \tfrac{18}{7}\,|\,\alpha\,|^2\;.
\end{aligned}$$

Here we used several times the equation $|p(X)|^2 = 3|X|^2$ and that $p \circ P = \mathrm{id}_{\Lambda^1}$. Finally, we still use the following lemma which is not difficult to prove.

Lemma 6.2.15 *Let X and Y be any vectors, then*

$$P(X \wedge p(Y)) \;=\; 2\,(\,(X \wedge Y)\lrcorner *\omega \;-\; X \wedge Y\,) \;=\; -\,2\,p\,(P(X \wedge Y))\;.$$

Chapter 7

Further results

7.1 Conformal Killing forms on Einstein manifolds

In this section we consider conformal Killing forms on Einstein manifold. The starting point is $\delta R^{\pm} = 0$, which is true on Einstein manifolds or more generally on manifolds with parallel Ricci tensor. The following three corollaries are direct consequences of this fact and Proposition 4.4.5 resp. Proposition 4.4.12.

Corollary 7.1.1 *Let (M^n, g) be an Einstein manifold with scalar curvature s and let ψ be a conformal Killing p-form, then*

$$\nabla_X(\Delta\psi) = \tfrac{1}{p}\, X \lrcorner \left(\tfrac{s}{n}\, d\psi + \tfrac{p-1}{p+1}\, 2q(R)\, d\psi \right) - \tfrac{1}{n-p}\, X \wedge \left(\tfrac{s}{n}\, d^*\psi + \tfrac{n-p-1}{n-p+1}\, 2q(R)\, d^*\psi \right) + 2q(\nabla_X R)\psi \ .$$

Corollary 7.1.2 *Let (M^n, g) be an Einstein manifold with scalar curvature s and let ψ be a conformal Killing p-form, then*

$$d(2q(R)\,\psi) = \tfrac{s}{n}\, d\psi + \tfrac{p-1}{p+1}\, 2q(R)\, d\psi \ , \qquad (7.1.1)$$

$$d^*(2q(R)\,\psi) = \tfrac{s}{n}\, d^*\psi + \tfrac{n-p-1}{n-p+1}\, 2q(R)\, d^*\psi \ . \qquad (7.1.2)$$

Corollary 7.1.3 *Let (M, g) be an Einstein manifold of scalar curvature s and let ψ be a conformal Killing form with $\Delta\psi = \lambda\psi$. Then*

$$\lambda\, d\psi = \left(\tfrac{p+1}{np}\, s + \tfrac{p-1}{p}\, 2q(R) \right) d\psi \qquad \textit{and} \qquad \lambda\, d^*\psi = \left(\tfrac{n-p+1}{n(n-p)}\, s + \tfrac{n-p-1}{n-p}\, 2q(R) \right) d^*\psi \ .$$

Proof. Because of $\Delta\psi = \lambda\psi$ we have that $\Delta\psi$ is again a conformal Killing form, i.e. $T(\Delta\psi) = 0$ and we can apply Proposition 4.4.6 to conclude $2q(\nabla_X R)\psi = 0$. Substituting this into Corollary 7.1.1 implies

$$\lambda \nabla_X \psi \;=\; \tfrac{1}{p}\, X \lrcorner \left(\tfrac{s}{n}\, d\psi \,+\, \tfrac{p-1}{p+1}\, 2q(R)\, d\psi \right) \,-\, \tfrac{1}{n-p}\, X \wedge \left(\tfrac{s}{n}\, d^*\psi \,+\, \tfrac{n-p-1}{n-p+1}\, 2q(R)\, d^*\psi \right) .$$

Using the formulas $d\psi = \sum e_i \wedge \nabla_{e_i}\psi$ and $d^*\psi = -\sum e_i \lrcorner \nabla_{e_i}\psi$ we conclude

$$\lambda\, d\psi \;=\; \left(\tfrac{p+1}{np}\, s \,+\, \tfrac{p-1}{p}\, 2q(R) \right) d\psi \qquad \text{and} \qquad \lambda\, d^*\psi \;=\; \left(\tfrac{n-p+1}{n(n-p)}\, s \,+\, \tfrac{n-p-1}{n-p}\, 2q(R) \right) d^*\psi$$

which finishes the proof of the corollary. □

Recall from Corollary 4.4.7 that e.g. on locally symmetric spaces any conformal Killing form can be decomposed into eigenforms of the Laplace operator which are again conformal Killing forms.

7.2 Conformal Killing 2–forms

In this section we will first discuss the consequences of the curvature condition (4.2.1) for conformal Killing 2–forms and in particular for Killing 2–forms. Moreover, we consider the question whether for a conformal Killing 2-form ψ the vector field $\xi := (d^*\psi)^\flat$ is a Killing vector field. We will start with a property of Killing 2-forms, which directly follows from the definition. Nevertheless, we will prove it using the curvature condition (4.2.4). The statement is

Proposition 7.2.1 *Let $(M,\, g)$ be a Riemannian manifold with a Killing 2–form u, then for any vector fields X, Y:*

$$(R(X,\, Y)\, u)(A,\, B) \,+\, (R(A,\, B)\, u)(X,\, Y) \;=\; 0\ . \tag{7.2.3}$$

In particular, a Killing 2-form u satisfies the equation $\mathrm{Ric} \circ \hat{u} = \hat{u} \circ \mathrm{Ric}$, where $\hat{u}$ is the skew-symmetric endomorphism associated with the 2-form u.

Proof. We consider $R(\cdot,\cdot)\, u$ as a section of $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$. But then the equation (7.2.3) is equivalent to $R(\cdot,\cdot)\, u \in \Lambda^2(\Lambda^2 T^*M)$. On the other hand we have the decomposition (4.2.3) of the tensor product. Written on the level of representations it is

$$\Lambda^2 V^* \otimes \Lambda^2 V^* \;\cong\; \left(\Lambda^2 V^* \oplus \Lambda^{3,1} V^*\right) \,\oplus\, \left(\Lambda^{1,1} V^* \oplus \Lambda^4 V^* \oplus \Lambda^0 V^* \oplus \Lambda^{2,2} V^*\right)\ .$$

As a specialty of the case $p = 2$ it turns out, that the first bracket is isomorphic to $\Lambda^2(\Lambda^2 V^*)$, whereas the second bracket is isomorphic to $\mathrm{Sym}(\Lambda^2 V^*)$. Since u is a Killing form it follows from Corollary 4.2.3 that the projection of $R(\cdot,\cdot)\, u$ onto the summands $\Lambda^{2,2} TM^*$ and $\Lambda^{1,1} TM^*$ has to vanish, i.e. $R(\cdot,\cdot)\, u \in \Lambda^2(\Lambda^2 T^*M)$.

To prove the second statement we first write equation (7.2.3) in the following form: $\langle R(X,Y)\,u, A\wedge B\rangle + \langle R(A,B)\,u, X\wedge Y\rangle = 0$. If we set $A = Y = e_i$ and sum over an ortho-normal basis $\{e_i\}$ we immediately obtain: $\langle u, \mathrm{Ric}\,(X)\wedge B - X\wedge \mathrm{Ric}\,(B)\rangle = 0$ which then translates into $\mathrm{Ric}\circ\hat{u} = \hat{u}\circ\mathrm{Ric}$. □

After considering Killing 2–forms we will now derive a more general formula for conformal Killing 2–forms. But we start with considering arbitrary 2–forms. Let u be a 2-form, then we denote with $\hat{u}$ the associated skew-symmetric endomorphism, i.e. $g(\,\hat{u}\,(X), Y) := u(X,Y)$ which can be extended to a skew-symmetric endomorphism of $\Lambda^2 TM$. Denoting this extension again with $\hat{u}$ we have

$$\hat{u}\,(X\wedge Y) \;=\; \hat{u}\,(X)\wedge Y \;+\; X\wedge\;\hat{u}\,(Y)\;.$$

In the next lemma we collect several curvature properties for arbitrary 2-forms, which will then be used to reformulate the curvature condition for conformal Killing 2-forms given in Proposition 4.4.6.

Lemma 7.2.2 *Let (M^n, g) be a Riemannian manifold and u an arbitrary 2-form with associated skew-symmetric endomorphism $\hat{u}$. Further let $\mathcal{R}$ denote the Riemannian curvature operator and* Ric *the Ricci curvature extended to an endomorphism of $\Lambda^2 TM$, then*

(1) $R(X,Y)\,u = [\,\hat{u}\circ\mathcal{R}\,]\,(X\wedge Y)$,

(2) $[\,Y\wedge X\lrcorner \;-\; X\wedge Y\lrcorner\,]\,2q(R)\,u \;=\; -[\,\mathrm{Ric}\circ\hat{u} \;+\; \hat{u}\circ\mathrm{Ric} \;-\; 2\,\mathcal{R}\,(u)\,]\,(X\wedge Y)$,

(3) $[\,X\lrcorner\,R^+(Y) \;-\; Y\lrcorner\,R^+(X)\,]\,u \;=\; -[\,\hat{u}\circ\mathcal{R} \;+\; \mathcal{R}\circ\hat{u}\,]\,(X\wedge Y)$,

(4) $[\,X\wedge R^-(Y) \;-\; Y\wedge R^-(X)\,]\,u \;=\; [\,\mathcal{R}\,(u) \;-\; \hat{u}\circ\mathrm{Ric}\,]\,(X\wedge Y)$,

(5) $2\,R^-(X)\,u \;+\; X\lrcorner\,2q(R)\,u \;=\; [\mathrm{Ric}\circ\hat{u} \;-\; \hat{u}\circ\mathrm{Ric}\,]\,X$.

Proof. Starting with the definition of the 2-form $R(X,Y)u$ we will first show that for any vector fields X, Y, A, B:

$$(R(X,Y)u)(A,B) \;=\; g(R_{X,Y}(\,\hat{u}\,(A)) - \hat{u}\,(R_{X,Y}A),\, B)\;,$$

i.e. considered as an endomorphism we have $R(X,Y)u = R_{X,Y} \circ \hat{u} - \hat{u} \circ R_{X,Y}$. The proof of this formula is straightforward

$$
\begin{aligned}
(R(X,Y)u)(A,B) &= \sum (R_{X,Y}(e_i) \wedge e_i \lrcorner u)(A,B) \\
&= \sum g(R_{X,Y}(e_i), A)\, g(\hat{u}(e_i), B) - g(R_{X,Y}(e_i), B)\, g(\hat{u}(e_i), A) \\
&= -g(\hat{u}(R_{X,Y}A), B) + g(R_{X,Y}(\hat{u}(A)), B) \\
&= -g(\mathcal{R}(X \wedge Y), A \wedge \hat{u}(B)) - g(\mathcal{R}(X \wedge Y), \hat{u}(A) \wedge B) \\
&= -g(\mathcal{R}(X \wedge Y), \hat{u}(A \wedge B)) \\
&= g([\hat{u} \circ \mathcal{R}](X \wedge Y), A \wedge B) \; .
\end{aligned}
$$

In addition this proves the first equation of the lemma. To prove the second equation we use Lemma B.0.6 to replace $2q(R)\,u$ with $\mathrm{Ric}\,(u) - 2\,\mathcal{R}\,(u)$. Considering the 2-form $\mathrm{Ric}\,(u)$ as a skew-symmetric endomorphism implies $\mathrm{Ric}\,(u)(X) = \mathrm{Ric}\,(\hat{u}\,(X)) + \hat{u}\,(\mathrm{Ric}\,(X))$, thus

$$
\begin{aligned}
[\,Y \wedge X \lrcorner \;-\; X \wedge Y \lrcorner\,]\, 2q(R)\, u \;=\;& Y \wedge (\mathrm{Ric}\,(\hat{u}\,(X)) + \hat{u}\,(\mathrm{Ric}\,(X)) - 2\,(\mathcal{R}\,u)(X)) \\
& - X \wedge (\mathrm{Ric}\,(\hat{u}\,(Y)) + \hat{u}\,(\mathrm{Ric}\,(Y)) - 2\,(\mathcal{R}\,u)(Y))
\end{aligned}
$$

and the second equation of the lemma follows if we consider the 2-form $\mathcal{R}\,(u)$ as an skew-symmetric endomorphism and extend it to $\Lambda^2 TM$ by $\mathcal{R}\,(u)(X \wedge Y) := (\mathcal{R}\,u)X \wedge Y + X \wedge (\mathcal{R}\,u)Y$. To prove the third equation we first note

$$
X \lrcorner\, R^-(Y)u \;=\; X \lrcorner \sum e_i \wedge R_{Y,\,e_i}u \;=\; -R(X,Y)u \;-\; \sum e_i \wedge X \lrcorner\, R_{Y,\,e_i}u \; .
$$

Hence, we have to compute the following expression

$$
\begin{aligned}
\sum & ([e_i \wedge X \lrcorner R_{Y,e_i} \ - \ e_i \wedge Y \lrcorner R_{X,e_i}]\, u)\ (A,\, B) \\
& = \ \sum (g(e_i, A)\, (R_{Y,e_i} u)\, (X,\, B) \ - \ g(e_i, B)\, (R_{Y,e_i} u)\, (X,\, A)) \\
& \qquad - \sum (g(e_i, A)\, (R_{X,e_i} u)\, (Y,\, B) \ - \ g(e_i, B)\, (R_{X,e_i} u)\, (Y,\, A)) \\
& = \ (R_{Y,A}\, u)\, (X,\, B) \ - \ (R_{Y,B}\, u)\, (X,\, A) \ - \ (R_{X,A}\, u)\, (Y,\, B) \ + \ (R_{X,B}\, u)\, (Y,\, A) \\
& = \ g(R_{Y,A}(\,\hat{u}\,(X)) \ - \ \hat{u}\,(R_{Y,A}X),\, B) \ - \ g(R_{Y,B}(\,\hat{u}\,(X)) \ - \ \hat{u}\,(R_{Y,B}X),\, A) \\
& \qquad - g(R_{X,A}(\,\hat{u}\,(Y)) \ - \ \hat{u}\,(R_{X,A}Y),\, B) \ + \ g(R_{X,B}(\,\hat{u}\,(Y)) \ - \ \hat{u}\,(R_{X,B}Y),\, A) \\
& = \ g(\,\hat{u}\,(X),\, -R_{Y,A}B \ + \ R_{Y,B}A) \ + \ g(\,\hat{u}\,(Y),\, R_{X,A}B \ - \ R_{X,B}A) \\
& \qquad + g(\,\hat{u}\,(A),\, -R_{Y,B}X \ + \ R_{X,B}Y) \ + \ g(\,\hat{u}\,(B),\, R_{Y,A}X \ - \ R_{X,A}Y) \\
& = \ g(\,\hat{u}\,(X),\, R_{A,B}Y) \ - \ g(\,\hat{u}\,(Y),\, R_{A,B}X) \ + \ g(\,\hat{u}\,(A),\, R_{X,Y}B) \\
& \qquad\qquad\qquad\qquad - g(\,\hat{u}\,(B),\, R_{X,Y}A,) \\
& = \ -g(\,\mathcal{R}\,(A \wedge B),\, Y \wedge \hat{u}\,(X) \ - \ X \wedge \hat{u}\,(Y)) \\
& \qquad\qquad\qquad\qquad - g(\,\mathcal{R}\,(X \wedge Y),\, B \wedge \hat{u}\,(A) \ - \ A \wedge \hat{u}\,(B)) \\
& = \ g(\,\mathcal{R}\,(A \wedge B),\, \hat{u}\,(X \wedge Y)) \ + \ g(\,\mathcal{R}\,(X \wedge Y),\, \hat{u}\,(A \wedge B)) \\
& = \ g(\,[\,\mathcal{R} \circ \hat{u} \ - \ \hat{u} \circ \mathcal{R}\,](X \wedge Y),\, A \wedge B)\ .
\end{aligned}
$$

Using this calculation and the first equation of the lemma we find

$$
\begin{aligned}
[\,X \lrcorner R^{+}(Y) \ - \ Y \lrcorner R^{+}(X)\,]\, u \ &= \ -2\, R(X,Y)u \ - \ [\,\mathcal{R} \circ \hat{u} \ - \ \hat{u} \circ \mathcal{R}\,](X \wedge Y) \\
&= \ -[\,\hat{u} \circ \mathcal{R} \ + \ \mathcal{R} \circ \hat{u}\,](X \wedge Y)\ .
\end{aligned}
$$

This finishes the proof of equation (3). To prove equation (4) we first apply the Bianchi identity to obtain

$$
\begin{aligned}
\sum R_{Y,e_i}(\,\hat{u}\,(e_i)) \ &= \ -\sum R_{e_i,\hat{u}\,(e_i)}Y \ - \ \sum R_{\hat{u}\,(e_i),Y}\, e_i \\
&= \ 2\,(\,\mathcal{R}\, u)\, Y \ + \ \sum R_{Y,\hat{u}\,(e_i)}\, e_i \\
&= \ 2\,(\,\mathcal{R}\,\hat{u}\,)Y \ - \ \sum R_{Y,e_i}\,\hat{u}\,(e_i)\ .
\end{aligned}
$$

Hence,

$$
\sum R_{Y,e_i}(\,\hat{u}\,(e_i)) \ = \ (\,\mathcal{R}\, u)Y\ .
$$

We will use this formula in the next calculation. Starting from the definition of $R^-(Y)u$ we find

$$R^-(Y)\,u \;=\; \sum e_i \lrcorner R_{Y,e_i} u \;=\; \sum R_{Y,e_i}(\,\hat{u}\,(e_i)) \;-\; \hat{u}\,(\mathrm{Ric}\,(Y))$$
$$=\; (\,\mathcal{R}\,u)Y \;-\; (\,\hat{u}\circ\mathrm{Ric}\,)Y\;.$$

Then, the proof of equation (4) immediately follows:

$$[\,X\wedge R^-(Y)\;-\;Y\wedge R^-(X)\,]\,u \;=\; X\wedge(\,(\,\mathcal{R}\,\hat{u}\,)Y\;-\;(\,\hat{u}\circ\mathrm{Ric}\,)Y\,)$$
$$-\,Y\wedge(\,(\,\mathcal{R}\,\hat{u}\,)X\;-\;(\,\hat{u}\circ\mathrm{Ric}\,)X\,)$$
$$=\;[\,\mathcal{R}\,(u)\;-\;\hat{u}\circ\mathrm{Ric}\,]\,(X\wedge Y)\;.$$

The proof of the last equation is already contained in the above calculations. Indeed,

$$2\,R^-(X)\,u\;+\;X\lrcorner\,2q(R)\,u \;=\; 2\,(\,\mathcal{R}\,u)X\;-\;2\,(\,\hat{u}\circ\mathrm{Ric}\,)X\;+\;\mathrm{Ric}\,(u)X\;-\;2(\,\mathcal{R}\,u)X$$
$$=\;[\mathrm{Ric}\circ\hat{u}\;-\;\hat{u}\circ\mathrm{Ric}\,]\,X\;.$$

This proves equation (5) and finishes the proof of the lemma. □

As a first application of the lemma we write equation (7.2.3) in a different form:

$$(R(X,\,Y)\,u\,)(A,\,B)\;+\;(R(A,\,B)\,u\,)(X,\,Y)$$
$$=\;g([\,\hat{u}\circ\mathcal{R}\,](X\wedge Y),\,A\wedge B)\;+\;g([\,\hat{u}\circ\mathcal{R}\,](A\wedge B),\,X\wedge Y$$
$$=\;g([\,\hat{u}\circ\mathcal{R}\;-\;\mathcal{R}\circ u](X\wedge Y),\,A\wedge B)\;.$$

Hence, equation (7.2.3) is equivalent to the statement that for a Killing 2-form u the associated endomorphism $\hat{u}$ commutes with the curvature operator. The corresponding result in the general case of conformal Killing 2-forms is given in

Proposition 7.2.3 *Let $(M^n,\,g)$ be a Riemannian manifold with a conformal Killing 2-form u. If $\hat{u}$ denotes the skew-symmetric endomorphism corresponding to u and also its extension to a skew-symmetric endomorphism of Λ^2TM, then $\hat{u}$ commutes with the Weyl curvature W, considered a symmetric endomorphism of Λ^2TM.*

Proof. For the proof we have to express the summands appearing in the curvature condition of Proposition 4.2.1 in the case $p = 2$. This is done using Lemma 7.2.2. We obtain

$$[\,\hat{u}\circ\mathcal{R}\,]\,(X\wedge Y) \;=\; -\,\tfrac{1}{2(n-2)}\,[\,\mathrm{Ric}\circ\hat{u}\;+\;\hat{u}\circ\mathrm{Ric}\;-\;2\,\mathcal{R}\,(u)\,]\,(X\wedge Y)$$
$$+\,\tfrac{1}{2}\,[\,\hat{u}\circ\mathcal{R}\;+\;\mathcal{R}\circ\hat{u}\,]\,(X\wedge Y)$$
$$-\,\tfrac{1}{n-2}\,[\,\mathcal{R}\,(u)\;-\;\hat{u}\circ\mathrm{Ric}\,]\,(X\wedge Y)\;.$$

This simplifies to

$$\mathrm{Ric} \circ \hat{u} \;+\; \hat{u} \circ \mathrm{Ric} \;-\; 2\,\mathcal{R}(u) \;=\; (n-2)\,[\,\mathcal{R} \circ \hat{u} \;-\; \hat{u} \circ \mathcal{R}\,] \;+\; 2\,[\,\hat{u} \circ \mathrm{Ric} \;-\; \mathcal{R}(u)]$$

and then to

$$\mathrm{Ric} \circ \hat{u} \;-\; \hat{u} \circ \mathrm{Ric} \;=\; (n-2)\,(\,\mathcal{R} \circ \hat{u} \;-\; \hat{u} \circ \mathcal{R}\,)\;. \tag{7.2.4}$$

It follows from Lemma B.0.6 that the curvature operator $\mathcal{R}$ is given as $\mathcal{R} = c\,\mathrm{id} + \frac{1}{n-2}\mathrm{Ric} + W$, for some constant c. Substituting this into the above equation completes the proof of the proposition. □

Another application of Lemma 7.2.2 answers the question under which circumstances for a conformal Killing 2-form u the associated vector field $\xi := (d^*u)^\sharp$ is Killing. One condition for this is given in

Proposition 7.2.4 *Let $(M,\, g)$ be a Riemannian manifold and let u be a conformal Killing 2–form with associated vector field $\xi := (d^*u)^\sharp$. Then ξ is a Killing vector field if and only if $\mathrm{Ric} \circ \hat{u} = \hat{u} \circ \mathrm{Ric}$. In particular, ξ is a Killing vector field if $(M,\, g)$ is an Einstein manifold.*

Proof. Let u be a conformal Killing 2–form and assume that u is not a Killing form. Then Corollary 4.4.9 gives a sufficient and necessary condition for d^*u to be a Killing 1-form. We have that d^*u is a (coclosed) conformal Killing 1-form if and only if $2R^-(X)\,u + X \lrcorner\; 2q(R)\,u \;=\; 0$. But due to the last equation of Lemma 7.2.2 this is equivalent to $\mathrm{Ric} \circ \hat{u} = \hat{u} \circ \mathrm{Ric}$. □

There is one immediate corollary which we apply in the case of manifolds with holonomy G_2 resp. Spin_7 and more generally for compact Ricci-flat manifolds. Indeed, these are Einstein manifolds of vanishing scalar curvature and hence a special case of the following

Corollary 7.2.5 *Let $(M,\, g)$ be a compact Einstein manifold of scalar curvature $s \leq 0$. Then any conformal Killing 2–form on M is either a Killing 2–form or it defines a non-trivial parallel vector field.*

Proof. Let u be a conformal Killing 2-form. Assume that u is not a Killing form, i.e. u is not coclosed. Then Proposition 7.2.4 yields a non-trivial Killing vector field ξ. Killing vector fields satisfy the equation $\Delta\xi = 2\,\mathrm{Ric}\,(\xi)$. Since $\mathrm{Ric} \leq 0$ and Δ is positive operator it follows $\Delta\xi = 0$, which on a compact manifold is equivalent to $d\xi = 0 = d^*\xi$. But obviously any closed Killing form has to be parallel. □

7.3 Conformal Killing forms on Sasakian manifolds

For the sake of completes we cite in this section what is known for Sasakian manifolds. Again, it is mainly due to the work of S. Yamaguchi.

Theorem 7.3.1 ([Y72a]) *Let (M, g) be a complete Sasakian manifold, then*

1. *any horizontal conformal Killing form of odd degree is Killing, and*
2. *any conformal Killing form of even degree has a unique decomposition into the sum of a Killing form and a $*$–Killing form.*

7.4 Integrability of the Killing equation

There are several results stating that manifolds which have in some sense many conformal Killing forms have to be conformally flat or have to be spaces of constant curvature. The following result is due to T. Kashiwada.

Theorem 7.4.1 ([Ka68]) *Let (M^n, g) be a Riemannian manifold such that for any point $x \in M$ and any p–form $\psi_0 \in \Lambda^p(T_x^*M)$, with $2 \leq p \leq n-2$, there exists (locally) a conformal Killing p–form ψ with $\psi(x) = \psi_0$, then the manifold is conformally flat.*

Under the assumption that there are many Killing forms one obtains an even stronger restrictions for the underlying manifold. Here we cite the following theorem of S. Tachibana and T. Kashiwada.

Theorem 7.4.2 ([KaTa69]) *Let (M^n, g) be a Riemannian manifold such that for any point $x \in M$ and any p–form $\psi_0 \in \Lambda^p(T_x^*M)$, with $2 \leq p \leq n-2$, there exists (locally) a Killing p–form ψ with $\psi(x) = \psi_0$, then the manifold has constant curvature.*

Appendix A

Linear Algebra

In this section we collect several elementary formulas which we use repeatedly. Let $(V, \langle\cdot,\cdot\rangle)$ be an n–dimensional Euclidean vector space and let $L : V \to V$ be a linear map. Then we can extend L as a derivation to a linear map of $\Lambda^p V$, where we usually identify V with its dual space V^*. On decomposable p-vectors the extension, which we again denote by L, is the given as

$$L\,(X_1 \wedge \ldots \wedge X_p) \;=\; L(X_1) \wedge X_2 \wedge \ldots \wedge X_p \;+\; \ldots \;+\; X_1 \wedge \ldots \wedge X_{p-1} \wedge L(X_p)\ .$$

It is often useful to have a general formula for the extension of L in terms of an ortho-normal basis $\{e_i\}$ of V. We have

$$L \;=\; \sum L(e_i) \,\wedge\, e_i \lrcorner\ .$$

It is obvious that the identity map on V extends to the map $p\,\mathrm{id}_{\Lambda^p V}$, i.e. we obtain

Lemma A.0.3 *Let V, be an n–dimensional Euclidean vector space with an ortho-normal basis $\{e_i\}$, then for any p–form ψ:*

$$\sum e_i^* \,\wedge\, e_i \lrcorner\, \psi \;=\; p\,\psi \qquad \textit{and} \qquad \sum e_i \lrcorner\, e_i^* \,\wedge\, \psi = (n-p)\,\psi\ .$$

One important example is the Riemannian curvature on p-forms. It is easy to see that this is exactly the curvature endomorphism $R_{X,Y}$ extended as a derivation to an endomorphism of p-forms, i.e. for a p-form ψ and any vector fields X, Y we have:

$$R(X,Y)\,\psi \;=\; \sum (R_{X,Y} e_i \,\wedge\, e_i \lrcorner\,)\,\psi\ .$$

This corresponds to the standard representation of the Lie algebra $\mathfrak{so}(n)$ on the space of p-forms.

Also useful is the extension of linear maps to maps on 2-forms. With any 2-form ψ we can associate a skew-symmetric map $\widehat{\psi}$, which is defined by $\psi(X,Y) = \langle \widehat{\psi}(X), Y \rangle$. Usually we will identify a 2-form with its associated skew-symmetric endomorphism and use the same notation.

Lemma A.0.4 *Let V be an n-dimensional Euclidean vector space with a skew-symmetric map A and a symmetric map B. Then A resp. B extend to a skew-symmetric resp. a symmetric map of $\Lambda^2 V$. Moreover, for any 2-form ψ and for any vector X the extensions of A resp. B satisfy*

$$A(\psi)\,X \;=\; A(\psi(X)) + \psi(A(X)) \qquad \textit{and} \qquad B(\psi)\,X \;=\; B(\psi(X)) - \psi(B(X))\ .$$

Proof. We give the proof for a symmetric endomorphism A. Of course it is a completely elementary calculation. For any vectors X, Y we have

$$\begin{aligned} A(\psi)\,(X,Y) &= \langle\, A(\psi), X \wedge Y \rangle \;=\; \langle\, A(\psi)X, Y \rangle \;=\; -\,\langle X, A(\psi) Y \rangle \\ &= \langle\, \psi, A(X \wedge Y) \rangle \;=\; \langle\, \psi, A(X) \wedge Y \;+\; X \wedge A(Y) \rangle \\ &= \langle\, \psi(A(X)), Y \rangle \;+\; \langle \psi(X), A(Y) \rangle \\ &= \langle\, \psi(A(X)) \;+\; A(\psi(X)), Y \rangle \\ &= -\,\langle\, X, A(\psi(Y)) \;+\; \psi(A(Y)) \rangle\ . \quad \square \end{aligned}$$

We will use this lemma for the symmetric endomorphism $A = \mathrm{Ric}$ and the skew-symmetric endomorphism $B = R_{X,Y}$.

Appendix B

The curvature endomorphism

In Chapter 1 we defined the symmetric endomorphism $2q(R) : \Lambda^p(T^*M) \to \Lambda^p(T^*M)$, which appears as the curvature term in the classical Weitzenböck formula for the Laplacian on p-forms: $\Delta = d^*d + dd^* = \nabla^*\nabla + 2q(R)$. Recall that it was defined with respect to a local ortho-normal basis $\{e_i\}$ as

$$2q(R) \;=\; \sum e_j^* \wedge e_i \lrcorner R_{e_i,e_j}.$$

The aim of this section is to describe the endomorphism $2q(R)$ in more detail. We will give several elementary properties and prove that it is indeed the curvature term in the Weitzenböck formula for the Laplace operator on forms. At the end of this section we present a more general definition of $2q(R)$ as an endomorphism of an arbitrary bundle associated to a representation of the holonomy group. In particular, we will show that it depends only on the representation defining the bundle.

It is well-known that $2q(R)$ acts as the Ricci tensor on 1-forms and as $\mathrm{Ric} - 2\mathcal{R}$ on 2-forms. Here Ric denotes the Ricci tensor extended as a derivation and $\mathcal{R}$ is the Riemannian curvature operator, defined by $g(\mathcal{R}(X \wedge Y), Z \wedge U) = -g(R(X,Y)Z, U)$. Using the curvature operator it is possible to write the action of $2q(R)$ in a slightly different form, which also gives a simple proof for the special expressions of $2q(R)$ on 1- resp. 2-forms. We have

Lemma B.0.5 *Let $\{\omega_i\}$ be any local ortho-normal basis of $\Lambda^2(T^*M)$. Then*

$$2q(R) \;=\; \mathrm{Ric} \;-\; 2\sum \mathcal{R}(\omega_i) \wedge \omega_i \lrcorner \;.$$

In particular, $2q(R)$ acts as Ric on 1-forms and as $\mathrm{Ric} - 2\mathcal{R}$ on 2-forms.

Proof. We rewrite the definition of the endomorphism $2q(R)$ using the standard notation for the components of the curvature tensor.

$$\begin{aligned} 2q(R) &= \sum e_j \wedge e_i \lrcorner R_{i,j} \;=\; \sum e_j \wedge e_i \lrcorner (R_{i,j} e_k \wedge e_k \lrcorner) \\ &= \sum R_{ijki}\, e_j \wedge e_k \lrcorner \;-\; \sum R_{ijkl}\, e_j \wedge e_l \wedge e_i \lrcorner\, e_k \lrcorner \\ &= \mathrm{Ric} \;+\; \sum R_{ijlk}\, e_j \wedge e_l \wedge e_i \lrcorner\, e_k \lrcorner \\ &= \mathrm{Ric} \;-\; \tfrac{1}{2} \sum R_{jlik}\, e_j \wedge e_l \wedge e_i \lrcorner\, e_k \lrcorner \\ &= \mathrm{Ric} \;-\; 2 \sum_{j<l,i<k} \mathcal{R}(e_j \wedge e_l) \wedge (e_i \wedge e_k) \lrcorner \;. \end{aligned}$$

Here we also applied the Bianchi identity and a renaming of indices. In the final sum we can replace the 2-vectors $e_j \wedge e_l$ by any other ortho-normal basis $\{\omega_i\}$ of $\Lambda^2 TM$. □

The following Lemma contains a well-known expression for the Riemannian curvature operator $\mathcal{R}$. However, we give it in a slightly different form involving the Ricci curvature.

Lemma B.0.6 *Let (M^n, g) be a Riemannian manifold. Then the Riemannian curvature operator $\mathcal{R}$ is given as*

$$\mathcal{R} \;=\; -\,\tfrac{s}{(n-1)(n-2)}\,\mathrm{id}_{\Lambda^2} \;+\; \tfrac{1}{n-2}\,\mathrm{Ric} \;+\; W,$$

where s denotes the scalar curvature, W the Weyl curvature and Ric *is the Ricci curvature considered as symmetric endomorphisms of $\Lambda^2 T^* M$.*

Proof. Recall that the Kulkarni-Nomizu product $h \triangle k$ between two symmetric bilinear forms h and k is the symmetric endomorphism of $\Lambda^2 TM$ defined by

$$\begin{aligned} &(h\triangle k)(X,Y,Z,U) \\ &\quad= \; h(X,Z)\,k(Y,U) \;+\; h(Y,U)\,k(X,Z) \;-\; h(X,U)\,k(Y,Z) \;-\; h(Y,Z)\,k(X,Y)\;. \end{aligned}$$

If g is the Riemannian metric than it is easy to check that $g \triangle g = 2\,\mathrm{id}$. Using this notation we have the well-known description of the curvature operator $\mathcal{R}$ of an n-dimensional Riemannian manifold (M, g):

$$\mathcal{R} \;=\; \tfrac{s}{2n(n-1)}\, g\triangle g \;+\; \tfrac{1}{n-2}\,\mathrm{Ric}_0 \triangle g \;+\; W \;,$$

where s is the scalar curvature, W the Weyl curvature and Ric_0 denotes the trace-free Ricci tensor, i.e. $\mathrm{Ric}_0 = \mathrm{Ric} - \frac{s}{n}\mathrm{id}$. Finally, we note that the symmetric map $\mathrm{Ric} \triangle g$ is just the Ricci tensor extended to $\Lambda^2 TM$ as a derivation. Substituting this into the equation for $\mathcal{R}$ finishes the proof of the lemma. □

On spaces of constant curvature the endomorphism $2q(R)$ is in its simplest form. Indeed, let (M, g) be an n-dimensional space of constant curvature c, i.e. a Riemannian manifold where $R(X,Y)Z = c\,(g(Y,Z)X - g(X,Z)Y)$ for any vector fields X, Y, Z. Then the curvature operator satisfies $\mathcal{R} = c\,\mathrm{id}_{\Lambda^2}$. Moreover, one has $\mathrm{Ric} = c\,(n-1)g$ and thus scalar curvature $s = c\,n(n-1)$. For the endomorphism $2q(R)$ we find

Lemma B.0.7 *Let $(M^n,\, g)$ be a space of constant curvature c, then for any p–form ψ:*

$$2q(R)\,\psi \;=\; c\,p(n-p)\,\psi\ .$$

Surprisingly it turns out that this property is in some sense characteristic for spaces of constant curvature. Indeed, we have the following

Proposition B.0.8 *A Riemannian manifold $(M^n,\, g)$ has constant sectional curvature if and only if there is a p, with $1 < p < n-1$, such that $q(R)$ acts as a scalar multiple of the identity on $\Gamma(\Lambda^p T^*M)$.*

Proof. We will give only a sketch of the proof. First of all it is not difficult to derive a formula for $2q(R)$ starting from the decomposition of Lemma B.0.6 and in particular using Lemma B.0.5. It follows for the action of $2q(R)$ on p-forms

$$2q(R) \;=\; s\,\tfrac{p(n-p)}{n(n-1)}\,\mathrm{id}_{\Lambda^p} \;+\; \tfrac{n-2p}{n-2}\,\mathrm{Ric}_0 \;+\; 2q(W)\ . \tag{B.0.1}$$

Note that the summand with the trace-free Ricci tensor vanishes if $p = n$. Next, we can easily prove that for a symmetric endomorphism h one has

$$2q(h\triangle g) \;=\; (n-2p)\,h \;+\; \mathrm{tr}(h)\,p\,\mathrm{id}_{\Lambda^p}\ ,$$

where h also denotes the extension of h to a map of Λ^p. As special cases we obtain $2q(g\triangle g) = 2p(n-p)\,\mathrm{id}_{\Lambda^p}$ and the formula $2q(\mathrm{Ric}\triangle g) = (n-2p)\,\mathrm{Ric} \;+\; s\,p\,\mathrm{id}_{\Lambda^p}$. Then we have to consider the map

$$\begin{aligned} \mathrm{Curv}(V) &\;\longrightarrow\; \mathrm{End}(\Lambda^p V^*) \\ R &\;\longmapsto\; q(R)\ , \end{aligned}$$

where $\mathrm{Curv}(V)$ denotes the space of algebraic curvature tensors of V. To prove the lemma we have to determine the preimage of id_{Λ^p}. But since two elements in the preimage differ by an element of the kernel of this map it suffices to show that the map is injective. Obviously the map is $\mathrm{SO}(V)$-equivariant and the kernel has to be an $\mathrm{SO}(V)$–invariant subspace of $\mathrm{Curv}(V)$. Hence, it is enough to check the statement on the three irreducible summands. We have already shown that the map is injective on the summands $\mathbb{R}$ and $\mathrm{Sym}^2_0(V^*)$ corresponding to curvature operators of the form $g\triangle g$ resp. $h\triangle g$ for a symmetric and trace free endomorphism h. But it is also not difficult to show that the map is injective on the third summand, the space of Weyl tensors. $\square$

The following two lemmas give further useful properties of the curvature endomorphism $q(R)$, which can easily be proved.

Lemma B.0.9 *The endomorphism $q(R)$ commutes with the Hodge star operator, with contraction with parallel forms and with the Laplace operator.*

Lemma B.0.10 *Let X be any vector field, then*

$$\nabla_X (q(R)\,\psi) \quad = \quad q(\nabla_X R)\,\psi \;+\; q(R)\,\nabla_X\,\psi\ .$$

In the remainder of this section we discuss the endomorphism $2q(R)$ in a more general context. Starting from the classical situation, where $2q(R)$ is given as the curvature term in the Weitzenböck formula of the Laplace operator on forms, we will show that there is natural generalization to a curvature endomorphism acting on any bundle which is induced by some representation of the holonomy group. These considerations were the starting point of [SW01] which eventually lead to vanishing results for Betti numbers on compact quaternion Kähler manifolds.

The basic example of a Clifford bundle is the bundle of exterior forms $\Lambda^\bullet T^*M$ equipped with the scalar product induced by the metric on M and Clifford multiplication with tangent vectors

$$\star :\; T_pM \times \Lambda^\bullet T^*_pM \;\longrightarrow\; \Lambda^\bullet T^*_pM, \qquad (X,\,\omega) \;\longmapsto\; X \star \omega$$

defined by $X \star \omega := X^\sharp \wedge \omega - X \lrcorner\, \omega$. The Levi–Civita–connection induces a connection ∇ on $\Lambda^\bullet T^*M$ and an associated second order elliptic differential operator $\nabla^*\nabla := -\sum_i \nabla^2_{e_i,e_i}$ where $\nabla^2_{X,Y} := \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$ and the sum is over a local ortho-normal base $\{e_i\}$. On the other hand we have the exterior differential d and its formal adjoint d^* as natural first order differential operators on $\Lambda^\bullet T^*M$ linked to $\nabla^*\nabla$ by the classical Weitzenböck formula

$$\Delta \quad := \quad (d + d^*)^2 \quad = \quad \nabla^*\nabla \;+\; \frac{1}{2}\sum_{ij} e_i \star e_j \star R_{e_i,e_j} \qquad \text{(B.0.2)}$$

where $R_{X,Y}$ is the curvature endomorphism of $\Lambda^\bullet T^*_pM$. However, the connection on $\Lambda^\bullet T^*M$ is induced by a connection on TM and consequently the curvature endomorphism $R_{X,Y}$ is just the curvature endomorphism of T_pM in a different representation, namely the representation

$$\bullet :\; \mathfrak{so}(T_pM) \times \Lambda^\bullet T^*_pM \;\longrightarrow\; \Lambda^\bullet T^*_pM, \qquad (X,\,\omega) \;\longmapsto\; X \bullet \omega$$

of the Lie algebra $\mathfrak{so}(T_pM)$ of $\mathrm{SO}(T_pM)$ on the exterior algebra induced by its representation on T_pM. The canonical identification of $\mathfrak{so}(T_pM)$ with the bi-vectors characterized by

$$\Lambda^2 T_pM \stackrel{\cong}{\longrightarrow} \mathfrak{so}(T_pM), \qquad \langle\, (X\wedge Y)\bullet A,\, B\,\rangle \;:=\; \langle\, X\wedge Y,\, A\wedge B\,\rangle$$

reads $(X\wedge Y)\bullet A \;:=\; \langle X, A\rangle\, Y \;-\; \langle Y, A\rangle\, X$ and defines a unique bi-vector $R(X\wedge Y)$ via:

$$\langle R(X\wedge Y)\bullet Z, W\rangle \quad:=\quad \langle R_{X,Y}Z, W\rangle \qquad\qquad R(X\wedge Y) \quad=\quad \frac{1}{2}\sum_i e_i\wedge R_{X,Y}e_i$$

Using this identification the representation of $\mathfrak{so}(T_pM)$ on $\Lambda^\bullet T_p^*M$ is given by $(X\wedge Y)\bullet \;=\; Y^*\wedge X\,\lrcorner \;-\; X^*\wedge Y\,\lrcorner$. In particular, the classical Weitzenböck formula becomes

$$\begin{aligned}\Delta \;&=\; \nabla^*\nabla \;+\; \frac{1}{2}\sum_{ij}(e_i^*\wedge e_j^*\wedge - e_i\,\lrcorner\, e_j^*\wedge - e_i^*\wedge e_j\,\lrcorner \;+\; e_i\,\lrcorner\, e_j\,\lrcorner\,)\,R(e_i\wedge e_j)\bullet \\ &=\; \nabla^*\nabla \;+\; \frac{1}{2}\sum_{ij}(e_i\wedge e_j)\bullet R(e_i\wedge e_j)\bullet\end{aligned}$$

here the inhomogeneous terms cancel because of the first Bianchi identity. Hence, the curvature term depends linearly on the curvature tensor:

$$R \quad:=\quad \frac{1}{4}\sum_{ij}(e_i\wedge e_j)\,\cdot\, R(e_i\wedge e_j) \qquad\in\qquad \mathrm{Sym}^2(\Lambda^2 T_pM)\;.$$

It will be convenient to compose the identification $\Lambda^2 T_pM \stackrel{\cong}{\longrightarrow} \mathfrak{so}(T_pM)$ with the quantization map $q:\,\mathrm{Sym}^2\mathfrak{so}(T_pM) \longrightarrow \mathcal{U}\,\mathfrak{so}(T_pM),\ X^2 \longmapsto X^2$, into the universal enveloping algebra of $\mathfrak{so}(T_pM)$ to get an element $q(R)\in\mathcal{U}\,\mathfrak{so}(T_pM)$ with:

$$\Delta \quad=\quad \nabla^*\nabla \;+\; 2\,q(R) \tag{B.0.3}$$

Writing the well known classical Weitzenböck formula (B.0.2) this way we can bring the holonomy group of the underlying manifold into play. Recall that the holonomy group $\mathrm{Hol}_pM \subset \mathbf{O}\,(T_pM)$ is the closure of the group of all parallel transports along piecewise smooth loops in $p\in M$. We will assume throughout that M is connected so that the holonomy groups in different points p and $\tilde{p}$ are conjugated by parallel transport $T_pM \longrightarrow T_{\tilde{p}}M$. Choosing a suitable representative $\mathrm{Hol} \subset \mathbf{O}_n\mathbb{R}$ with $n := \dim M$ of their common conjugacy class acting on the abstract vector space $\mathbb{R}^n$ we can define the holonomy bundle of M:

$$\mathrm{Hol}\,(M) \quad:=\quad \{\, f:\,\mathbb{R}^n \longrightarrow T_pM \,|\; p\in M \text{ and } f \text{ isometry with } f(\mathrm{Hol}\,) \;=\; \mathrm{Hol}_pM\,\}\;.$$

The holonomy bundle is a reduction of the orthonormal frame bundle $\mathbf{O}\,(M)$ to a principal bundle with structure group Hol, which is stable under parallel transport. Consequently the Levi–Civita connection is tangent to $\mathrm{Hol}\,(M)$ and descends to a connection on $\mathrm{Hol}\,(M)$.

The associated fibre bundle $\mathrm{Hol}(M) \times_{\mathrm{Hol}} \mathbf{O}_n\mathbb{R}$ is canonically diffeomorphic to the full orthonormal frame bundle $\mathbf{O}(M)$. This construction provides an explicit foliation of $\mathbf{O}(M)$ into mutually equivalent principal sub-bundles stable under parallel transport. Choosing a leaf different from the distinguished leaf $\mathrm{Hol}(M)$ amounts to choosing a different representative for the conjugacy class of $\mathrm{Hol} \subset \mathbf{O}_n\mathbb{R}$. In particular, every principal sub-bundle of $\mathbf{O}(M)$ stable under parallel transport is a union of leaves and is characterized by a subgroup of $\mathbf{O}_n\mathbb{R}$ containing a representative of the conjugacy class of the holonomy group Hol.

With the Levi–Civita connection being tangent to the holonomy bundle $\mathrm{Hol}(M)$ its curvature tensor R takes values in the holonomy algebra $\mathfrak{hol}_pM$ at every point $p \in M$, so that $R \in \mathrm{Sym}^2\mathfrak{hol}_pM \subset \mathrm{Sym}^2\Lambda^2T_pM$ and $q(R) \in \mathcal{U}\,\mathfrak{hol}_pM$. However by definition every point $f \in \mathrm{Hol}(M)$ identifies $\mathfrak{hol}_pM$ with $\mathfrak{hol}$ making $q(R)$ a $\mathcal{U}\,\mathfrak{hol}$–valued function on $\mathrm{Hol}(M)$:

$$q(R) \quad \in \quad C^\infty(\,\mathrm{Hol}(M), \mathcal{U}\,\mathfrak{hol}\,)^{\mathrm{Hol}} \;\cong\; \Gamma(\mathrm{Hol}(M) \times_{\mathrm{Hol}} \mathcal{U}\,\mathfrak{hol})$$

For an arbitrary irreducible complex representation π of Hol the associated vector bundle $\pi(M) := \mathrm{Hol}(M) \times_{\mathrm{Hol}} \pi$ over M is equipped with the connection induced from the Levi–Civita connection. Moreover, there is a canonical second order differential operator defined on sections of $\pi(M)$:

$$\Delta_\pi \quad := \quad \nabla^*\nabla \;+\; 2\,q(R) \tag{B.0.4}$$

It is evident from the Weitzenböck formula (B.0.2) written as in (B.0.3) that the diagram

$$\begin{array}{ccc} \pi(M) & \xrightarrow{\;\Delta_\pi\;} & \pi(M) \\ F\big\downarrow & & \big\downarrow F \\ \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\;\Delta\;} & \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

commutes for any $F \in \mathrm{Hom}_{\mathrm{Hol}}(\pi, \Lambda^\bullet\mathbb{C}^{n*})$ or equivalently for any globally parallel embedding $F : \pi(M) \longrightarrow \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$. Hence the pointwise decomposition of $\Lambda^\bullet T^*_pM \otimes_{\mathbb{R}} \mathbb{C}$ into irreducible complex representations of Hol_pM becomes a global decomposition of any eigenspace of Δ, e.g. we have for its kernel:

$$H^\bullet_{dR}(M, \mathbb{C}) \;=\; \bigoplus_\pi \mathrm{Hom}_{\mathrm{Hol}}(\,\pi,\, \Lambda^\bullet\mathbb{C}^{n*}\,) \otimes \mathrm{Kern}\,\Delta_\pi$$

At this point we want to emphasize the important property of Δ_π and $2q(R)$, that their action on a bundle $\pi(M)$ depends only on the defining representation π and not of the bundles having $\pi(M)$ as a parallel sub-bundle. In particular, to show that the forms in a bundle $\pi(M) \subset \Lambda^\bullet T^*M$ cannot contribute to the cohomology it would suffice to find any bundle E which has $\pi(M)$ as a parallel sub-bundle and where Δ_π is positive. This idea was used in [SW01] to prove the vanishing of odd Betti numbers on compact quaternion Kähler manifolds. In the present situation we have a more elementary application of this idea, i.e. we use it to show that $2q(R)$ acts trivially on every 7-dimensional parallel sub-bundle of the form bundles on a G_2-manifold.

Returning to the general situation we note that same kind of reasoning is possible for the Dirac operator on spinors, assuming the manifold M to be spin and taking Hol_pM to be its spin holonomy group. Ignoring for the moment the Lichnerowicz result that the curvature term reduces to multiplication by the scalar curvature and employing the formula $(X\wedge Y)\bullet := \frac{1}{2}(X\star Y\star + \langle X,Y\rangle)$ for the representation of $\mathfrak{so}(T_pM)$ on the spinor bundle $\mathbf{S}(M)$ we can proceed from (B.0.2) directly to:

$$D^2 \quad = \quad \nabla^*\nabla \;+\; 4\,q(R)\,. \tag{B.0.5}$$

In particular, all eigenspaces of D^2 decompose globally according to the pointwise decomposition of the spinor bundle under the spin holonomy group Hol_pM. From Lichnerowicz's result we already know that $q(R)$ acts by scalar multiplication with $\frac{s}{16}$ on $\mathbf{S}(M)$, where s is the scalar curvature of (M,g). Hence, we can read equation (B.0.5) as

$$D^2\Big|_\pi \quad = \quad \Delta_\pi \;+\; \frac{\kappa}{8}$$

where the restriction to π is a short hand notation for any globally parallel embedding $F:\ \pi(M)\longrightarrow \mathbf{S}(M)$ induced by some non–trivial $F\in\mathrm{Hom}_{\mathrm{Hol}}(\pi,\mathbf{S})$. Written in this way formula (B.0.5) is seen to be a generalization of the Partharasathy formula for the Dirac square D^2 on a symmetric space G/K of compact type. Indeed, in this case the operators Δ_π defined above on sections of $\pi(M)$ all become the Casimir of G.

Lemma B.0.11 *Let $M = G/K$ be a Riemannian symmetric space. Then the endomorphism $2q(R)$ coincides with the Casimir operator C_K of the group K.*

Proof. We will show that on a Riemannian symmetric space G/K the operators $\Delta_\pi := \nabla^*\nabla + 2q(R)$ acts as the Casimir operator C_G of the group G. It is then well-known that $\nabla^*\nabla$ acts as the difference, $C_G - C_K$, of the two Casimir operators, which then proves the lemma. We consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ of the Lie algebra of G and an invariant regular symmetric bilinear form B on $\mathfrak{g}$, which agrees with minus the Riemannian metric $-\langle\,,\rangle$ on $\mathfrak{p}\cong\mathfrak{g}/\mathfrak{k}$. Such a choice can always be made, usually $\mathfrak{g}$ will be semi-simple and B will be taken to be the Killing form of $\mathfrak{g}$ multiplied by -1 on every simple non-compact factor. The isomorphisms $*:\ \mathfrak{g}\longrightarrow\mathfrak{g}^*$ and $\flat:\ \mathfrak{g}^*\longrightarrow\mathfrak{g}$ will always be taken with respect to B. Hence, the definition of the Casimir operator of G with respect to B is given as

$$\mathrm{Cas}_G \quad = \quad \sum_\mu e_\mu\, e_\mu \;+\; \sum_\alpha f_\alpha\, f_\alpha$$

where $\{e_\mu\}$ and $\{f_\alpha\}$ are ortho-normal bases of $\mathfrak{p}$ resp. $\mathfrak{k}$. The first summand on the right hand side is evidently the Laplacian $\nabla^*\nabla$ for the Levi–Civita connection on G/K, because

B agrees with minus the Riemannian metric $-\langle\,,\rangle$ on $\mathfrak{p}$. The second summand is just the Casimir Cas_K of K with respect to the restriction of B to $\mathfrak{k}$.

Since the isomorphisms with respect to the Riemannian metric $\langle\,,\rangle$ are simply the restrictions $-*|_{\mathfrak{p}} : \mathfrak{p} \longrightarrow \mathfrak{p}^*$ and $-\flat|_{\mathfrak{p}^*} : \mathfrak{p}^* \longrightarrow \mathfrak{p}$ we arrive at the following formula for the representation of $\mathfrak{k}$ in $\mathfrak{so}(\mathfrak{p}) \cong \Lambda^2\mathfrak{p}$

$$\mathfrak{k} \longrightarrow \Lambda^2\mathfrak{p}, \qquad K \longmapsto -\frac{1}{2}\sum_\mu e_\mu^\flat \wedge [K, e_\mu]$$

because:

$$\begin{aligned}\Big(-\frac{1}{2}\sum_\mu e_\mu^\flat \wedge [K, e_\mu]\Big) \star X &= \frac{1}{2}\sum_\mu \Big(B(e_\mu^\flat, X)\,[K, e_\mu] - B([K, e_\mu], X)\,e_\mu^\flat\Big)\\ &= [K, X] \quad = \quad K \star X\end{aligned}$$

It is well known that for $X, Y \in \mathfrak{p}$, the curvature $R_{X,Y}$ of the symmetric space G/K acts by $-[X, Y]\star$ on every vector bundle associated to the principal K–bundle G via a representation $\star$ of K. Combined with the definition of the curvature term $q(R)$ this classical result implies that

$$\begin{aligned}2\,q(R) &= \frac{1}{2}\sum_{\mu\nu}(e_\mu^\flat \wedge e_\nu^\flat) \star R_{e_\mu, e_\nu}\\ &= -\frac{1}{2}\sum_{\mu\nu\alpha} B(f_\alpha^\flat, [e_\mu, e_\nu])\,(e_\mu^\flat \wedge e_\nu^\flat) \star K_\alpha \star\\ &= -\frac{1}{2}\sum_{\mu\alpha}(e_\mu^\flat \wedge [f_\alpha^\flat, e_\mu]) \star f_\alpha \star \quad = \quad \sum_\alpha f_\alpha^\flat \star f_\alpha \star\end{aligned}$$

Hence, $2\,q(R)$ coincides with Cas_K on every representation $\star$ of $\mathfrak{so}(\mathfrak{p})$. In particular, Cas_K is the extension of $2\,q(R)$ to arbitrary representations $\star$ of $\mathfrak{k}$ and the differential operator Δ agrees with Cas_G on every vector bundle associated to the principal K–bundle G. $\square$

Appendix C

The commutator rule

Let (M, g) be a Riemannian manifold with holonomy group Hol, further let π be a $\mathrm{Hol}-$ representation and $\pi(M)$ the associated vector bundle defined by π. The Levi-Civita connection induces a connection ∇ on $\pi(M)$ and we also have a natural second order differential operator $\Delta_\pi := \nabla^*\nabla + 2q(R)$ acting on sections of $\pi(M)$ (c.f. Section B). The aim of this section is to prove a general commutator formula between ∇ and Δ_π, which we apply in the case where $\pi(M)$ is any parallel sub-bundle of the form bundle and the operator Δ_π coincides with the usual form Laplacian. As a corollary we obtain the commutator rule of Proposition 4.4.6 between the twistor operator and the Laplace operator. As a first step we have to compute the commutator of $\nabla^*\nabla$ and ∇. For this we recall the definition the third iterated covariant derivative. It is given as

$$\nabla^3_{X,Y,Z} \;=\; \nabla_X\nabla_Y\nabla_Z \;-\; \nabla_X\nabla_{\nabla_Y Z} \;-\; \nabla_{\nabla_X Y}\nabla_Z \;+\; \nabla_{\nabla_{\nabla_X Y}Z} \;-\; \nabla_Y\nabla_{\nabla_X Z} \;+\; \nabla_{\nabla_Y\nabla_X Z} \;.$$

The following lemma provides two important equations for ∇^3, which are known as *Ricci identities* (c.f. [Be]).

Lemma C.0.12 *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let X, Y, Z be any vector fields. Then*

$$(1) \qquad \nabla^3_{X,Y,Z} \;-\; \nabla^3_{Y,X,Z} \;=\; R_{X,Y}\nabla_Z \;-\; \nabla_{R_{X,Y}Z} \;,$$

$$(2) \qquad \nabla^3_{X,Y,Z} \;-\; \nabla^3_{X,Z,Y} \;=\; (\nabla_X R)_{Y,Z} \;+\; R_{Y,Z}\nabla_X \;.$$

Proof. The proof is a simple computation starting form the definition of ∇^3.

$$\begin{aligned}
&\nabla^3_{X,Y,Z} \;-\; \nabla^3_{Y,X,Z} \\
&\quad = \nabla_X\nabla_Y\nabla_Z \;-\; \nabla_X\nabla_{\nabla_Y Z} \;-\; \nabla_{\nabla_X Y}\nabla_Z \;+\; \nabla_{\nabla_{\nabla_X Y}Z} \;-\; \nabla_Y\nabla_{\nabla_X Z} \;+\; \nabla_{\nabla_Y\nabla_X Z} \\
&\qquad -\nabla_Y\nabla_X\nabla_Z \;+\; \nabla_Y\nabla_{\nabla_X Z} \;+\; \nabla_{\nabla_Y X}\nabla_Z \;-\; \nabla_{\nabla_{\nabla_Y X}Z} \;+\; \nabla_X\nabla_{\nabla_Y Z} \;-\; \nabla_{\nabla_X\nabla_Y Z} \\
&\quad = R_{X,Y}\nabla_Z \;-\; \nabla_{R_{X,Y}Z} \;.
\end{aligned}$$

And similarly

$$\begin{aligned}
&\nabla^3_{X,Y,Z} \;-\; \nabla^3_{X,Z,Y}\\
&\quad=\; \nabla_X\nabla_Y\nabla_Z \;-\; \nabla_X\nabla_{\nabla_Y Z} \;-\; \nabla_{\nabla_X Y}\nabla_Z \;+\; \nabla_{\nabla_{\nabla_X Y}Z} \;-\; \nabla_Y\nabla_{\nabla_X Z} \;+\; \nabla_{\nabla_Y\nabla_X Z}\\
&\qquad\; -\nabla_X\nabla_Z\nabla_Y \;+\; \nabla_X\nabla_{\nabla_Z Y} \;+\; \nabla_{\nabla_X Z}\nabla_Y \;-\; \nabla_{\nabla_{\nabla_X Z}Y} \;+\; \nabla_Z\nabla_{\nabla_X Y} \;-\; \nabla_{\nabla_Z\nabla_X Y}\\
&\quad=\; \nabla_X R_{Y,Z} \;-\; R_{\nabla_X Y,Z} \;-\; R_{Y,\nabla_X Z}\\
&\quad=\; (\nabla_X R)_{Y,Z} \;+\; R_{Y,Z}\nabla_X \;. \quad \square
\end{aligned}$$

The next step is to calculate $\nabla \circ \nabla^*\nabla$. If $\{e_i\}$ denotes a local ortho-normal basis we obtain

$$\begin{aligned}
\nabla \circ \nabla^*\nabla \;&=\; -\sum e_i \otimes \nabla_{e_i}(\nabla_{e_j}\nabla_{e_j} \;-\; \nabla_{\nabla_{e_j}e_j})\\
&=\; -\sum e_i \otimes \nabla^3_{e_i,e_j,e_j} \;.
\end{aligned}$$

All the other terms in the third covariant derivative cancel each other because of the relation $\sum \nabla_X e_j \otimes e_j = -\sum e_j \otimes \nabla_X e_j$. A slightly more complicated calculation shows

$$\begin{aligned}
\nabla^*\nabla \circ \nabla \;&=\; -\sum \Big[\nabla^2_{e_j,e_j} e_i \otimes \nabla_{e_i} \;+\; 2\,\nabla_{e_j} e_i \otimes \nabla_{e_j}\nabla_{e_i} \;+\; e_i \otimes \nabla^2_{e_j,e_j}\nabla_{e_i}\Big]\\
&=\; -\sum e_i \otimes \Big[\nabla_{\nabla_{e_j}\nabla_{e_j}e_i} \;+\; \nabla_{\nabla_{\nabla_{e_j}e_j}e_i} \;-\; 2\,\nabla_{e_j}\nabla_{\nabla_{e_j}e_i} \;+\; \nabla_{e_j}\nabla_{e_j}\nabla_{e_i} \;-\; \nabla_{\nabla_{e_j}e_j}\nabla_{e_i}\Big]\\
&=\; -\sum e_i \otimes \nabla^3_{e_j,e_j,e_i} \;.
\end{aligned}$$

This yields for the commutator between ∇ and $\nabla^*\nabla$ the following expression

$$\begin{aligned}
[\nabla,\, \nabla^*\nabla] \;&=\; -\sum e_i \otimes \Big[\nabla^3_{e_i,\,e_j,\,e_j} \;-\; \nabla^3_{e_j,\,e_i,\,e_j} \;+\; \nabla^3_{e_j,\,e_i,\,e_j} \;-\; \nabla^3_{e_j,\,e_j,\,e_i}\Big]\\
&=\; -\sum e_i \otimes \Big[R_{e_i,\,e_j}\nabla_{e_j} \;-\; \nabla_{R_{e_i,e_j}e_j} \;+\; (\nabla_{e_j}R)_{e_i,\,e_j} \;+\; R_{e_i,\,e_j}\,\nabla_{e_j}\Big]\\
&=\; \sum e_i \otimes \nabla_{\mathrm{Ric}\,(e_i)} \;-\; 2\sum e_i \otimes R_{e_i,\,e_j}\,\nabla_{e_j} \;-\; \sum e_i \otimes (\nabla_{e_j}R)_{e_i,\,e_j} \;.
\end{aligned}$$

In particular, this shows that $[\nabla,\, \nabla^*\nabla]$ is a differential operator of first order. The next task is to compute the commutator between ∇ and $2q(R)$. Recall that the curvature term $2q(R)$ was defined as $2q(R) = \sum e_k \wedge e_l \lrcorner R_{e_l,e_k}$, which in the special case of 1-forms is just the Ricci endomorphism.

$$\begin{aligned}
[\,\nabla,\, 2q(R)\,] \;&=\; 2\sum e_i \otimes \nabla_{e_i}\, q(R) \;-\; \sum 2q(R)\, e_i \otimes \nabla_{e_i}\\
&\qquad +\; \sum (e_k \,\wedge\, e_l \lrcorner e_i) \otimes R_{e_k,e_l}\nabla_{e_i} \;-\; e_i \otimes 2q(R)\,\nabla_{e_i}\\
&=\; 2\sum e_i \otimes q(\nabla_{e_i}R) \;-\; \sum e_i \otimes \nabla_{\mathrm{Ric}\,(e_i)} \;+\; 2\sum e_k \otimes R_{e_k,e_i}\nabla_{e_i} \;.
\end{aligned}$$

Adding the two formulas for $[\nabla, \nabla^*\nabla]$ resp. $[\nabla, 2q(R)]$ we eventually obtain

$$\begin{aligned}[\nabla, \nabla^*\nabla + 2q(R)] &= 2\sum e_i \otimes q(\nabla_{e_i} R) + \sum e_i \otimes (\nabla_{e_j} R)_{e_j, e_i} \\ &= \sum e_i \otimes \left[2q(\nabla_{e_i} R) + (\nabla_{e_j} R)_{e_j, e_i}\right] .\end{aligned}$$

Note that the second summand vanishes on Einstein manifolds. Indeed, using the notation of Chapter 4.1 we have $(\delta R)_X = -\sum (\nabla_{e_j} R)_{e_j, X}$ and Lemma 4.4.3 shows that this expression has to vanish on manifolds with parallel Ricci tensor. It still remains to derive the commutator of $\nabla^*\nabla + 2q(R)$ and a twistor operator. A *twistor operator* T is defined as $T := \mathrm{pr} \circ \nabla$, where pr denotes the projection onto a sum of irreducible summands in $\Lambda^1 \otimes \pi$. In our case $\mathrm{pr} := \mathrm{pr}_{\Lambda^{p,1}}$ is the projection onto $\Lambda^{p,1} \subset \Lambda^1 \otimes \Lambda^p$. Equivalently (and on the level of bundles) we can say that $\mathrm{pr} : T^*M \otimes \pi(M) \to T^*M \otimes \pi(M)$ is a Hol–equivariant endomorphism, where Hol denotes the holonomy group of the manifold. Let T be any twistor operator, then

$$\begin{aligned}[T, \nabla^*\nabla + 2q(R)] &= [\mathrm{pr} \circ \nabla, \nabla^*\nabla + 2q(R)] \\ &= \mathrm{pr} \circ [\nabla, \nabla^*\nabla + 2q(R)] + [\mathrm{pr}, \nabla^*\nabla + 2q(R)] \circ \nabla \\ &= \sum \mathrm{pr}\left(e_i \otimes \left[2q(\nabla_{e_i} R) + (\nabla_{e_j} R)_{e_j, e_i}\right]\right) .\end{aligned}$$

Here we used that pr commutes with $\nabla^*\nabla$ and $2q(R)$ because of the holonomy invariance. Summarizing our calculations, we have proved the following

Theorem C.0.13 *Let (M, g) be a Riemannian manifold with holonomy group* Hol *and let π be a* Hol*–representation with associated vector bundle $\pi(M)$, which is equipped with the connection ∇ induced from the Levi-Civita connection. Then*

$$[\nabla, \nabla^*\nabla + 2q(R)] = \sum e_i \otimes \left[2q(\nabla_{e_i} R) + (\nabla_{e_j} R)_{e_j, e_i}\right] .$$

Moreover, let $T := \mathrm{pr} \circ \nabla$ be a twistor operator defined by an Hol*–equivariant endomorphism $\mathrm{pr} : T^*M \otimes \pi(M) \to T^*M \otimes \pi(M)$, then*

$$[T, \nabla^*\nabla + 2q(R)] = \sum \mathrm{pr}\left(e_i \otimes \left[2q(\nabla_{e_i} R) + (\nabla_{e_j} R)_{e_j, e_i}\right]\right) .$$

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