DEFORMATIONS OF NEARLY PARALLEL G_2-STRUCTURES

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ABSTRACT. We study the infinitesimal deformations of a proper nearly parallel G_2-structure and prove that they are characterized by a certain first order differential equation. In particular we show that the space of infinitesimal deformations modulo the group of diffeomorphisms is isomorphic to a subspace of co-closed Λ^3_{27}-eigenforms of the Laplace operator for the eigenvalue $8\mathrm{scal}\,/21$. We give a similar description for the space of infinitesimal Einstein deformations of a fixed nearly parallel G_2-structure. Moreover we show that there are no deformations on the squashed S^7 and on $\mathrm{SO}(5)/\mathrm{SO}(3)$, but that there are infinitesimal deformations on the Aloff-Wallach manifold $N(1,1) = \mathrm{SU}(3)/U(1)$.

1. INTRODUCTION

A nearly parallel G_2-structure on a 7-dimensional manifold M is given by a 3-form σ of special algebraic type satisfying the differential equation $*d\sigma = \tau_0\sigma$ for some constant τ_0. Such a manifold has a structure group contained in the exceptional Lie group $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and, in particular, a Riemannian metric g induced by σ. It can be shown that nearly parallel G_2-manifolds are irreducible and Einstein with scalar curvature $\mathrm{scal} = \frac{21}{8}\tau_0^2$. Moreover, the existence of such a structure is equivalent to the existence of a spin structure with a Killing spinor.

Another equivalent description of nearly parallel G_2-structures is in terms of the metric cone $(\hat{M}, \hat{g})$, which has to have holonomy contained in $\mathrm{Spin}(7)$, considered as subgroup of $\mathrm{SO}(8)$. The metric cone is the manifold $\hat{M} = M \times \mathbb{R}_+$ with the warped product metric $\hat{g} = r^2 g \oplus dr^2$. If (M^7, g) is simply connected and not isometric to the standard sphere, then there are three possible cases: the holonomy of $(\hat{M}, \hat{g})$ is contained in $Sp(2)$, equivalently, (M^7, g) is a 3-Sasakian manifold, the holonomy can be $\mathrm{SU}(4)$, equivalently, (M^7, g) is an Einstein-Sasaki manifold, or the holonomy is precisely $\mathrm{Spin}(7)$, in which case we call the G_2-structure *proper*. We recall that these three cases correspond to the existence of a 3-, 2- resp. 1-dimensional space of Killing spinors. Proper nearly parallel G_2-structures are also characterized by the vanishing of the Lie derivative $L_\xi\sigma$ for any Killing vector field ξ.

In this article we shall mainly consider the case of proper nearly parallel G_2-manifolds. In [12] it is shown that any 7-dimensional 3-Sasakian manifold admits a second nearly parallel G_2-structure which is proper. The corresponding Einstein metric belongs to the metrics of the canonical variation of the 3-Sasakian Einstein metric. Applying this construction to

the homogeneous 3-Sasakian spaces S^7 and $N(1,1)$ one obtains homogeneous proper nearly parallel G_2-structures: the squashed 7-sphere and the second Einstein metric on $N(1,1)$. The Aloff-Wallach spaces $N(k,l)$ for $(k,l) \neq (1,1)$ also have exactly two nearly parallel G_2-structures, both of which are proper. A further example is the isotropy irreducible space $SO(5)/SO(3)$. In fact, due to the classification [12] these are the only homogeneous nearly parallel G_2-manifolds.

As a last remarkable property of nearly parallel G_2-manifolds we mention the existence of a metric connection $\bar{\nabla}$ with totally skew-symmetric torsion. The so-called *canonical connection* $\bar{\nabla}$ is defined as $\bar{\nabla} = \nabla - \frac{\tau_0}{12}\sigma$ and has holonomy contained in the group $G_2 \subset SO(7)$. Nearly parallel G_2-manifolds appear as one of two exceptional cases in a classification of metric connections with parallel torsion due to Cleyton and Swann [7]. The other exceptional case is the class of 6-dimensional nearly Kähler manifolds, which turns out to be in various ways rather similar to nearly parallel G_2-manifolds. The defining condition is the existence of a nearly parallel almost complex structure J, i.e., J satisfying $(\nabla_X J)(X) = 0$ for any vector field X. Nearly Kähler manifolds in dimension 6 are also Einstein manifolds admitting a Killing spinor. Moreover, the metric cone has holonomy contained in G_2.

In this article we shall show that nearly parallel G_2-manifolds are also in another respect very similar to nearly Kähler manifolds: the description of infinitesimal deformations. In [16] the space of infinitesimal nearly Kähler deformations is identified with the space of primitive co-closed $(1,1)$-eigenforms of the Laplace operator for the eigenvalue $2\text{scal}/5$, [19] contains a similar description of the space of infinitesimal Einstein deformations. This space turns out to be the sum of three such eigenspaces. Finally, in [18] it is shown that infinitesimal deformations for the known homogenous examples only exist in the case of the flag manifold $SU(3)/T^2$. For all three results we shall obtain a counterpart on nearly parallel G_2-manifolds.

We start with the equations of R. Bryant (cf. Proposition 3.1 and [6]) describing the infinitesimal deformation of an arbitrary G_2-structure. They give equations for the tangent vector on a curve of G_2-structures. Specializing to the case of nearly parallel G_2-structures and staying transversal to the action of the diffeomorphism group, we obtain that the space of such deformations is a direct sum of two spaces, D_1 and D_3, consisting of 1-forms and 3-forms respectively. As shown in Section 4, the space D_1 parametrizes Einstein-Sasakian structures compatible with the given nearly parallel G_2-structure. The more interesting space is D_3 which consists of the solutions ϕ in $\Lambda^3_{27} T^* M$ of the differential equation $*d\phi = -\tau_0\phi$. In particular, infinitesimal deformations $\phi \in D_3$ are co-closed and eigenforms of the Hodge-Laplace operator for the eigenvalue $\tau_0^2 = 8\text{scal}/21$. But more important for the computation in examples is that they are also eigenforms for the eigenvalue $\frac{5\tau_0^2}{6}$ of the G_2-Laplace operator $\bar{\Delta}$ introduced in Section 5. In Section 6 we describe the space of infinitesimal Einstein deformations of the metric of a nearly parallel G_2-structure. In addition to D_3 one obtains two other spaces of sections of $\Lambda^3_{27} T^* M$ which are characterized by similar equations. In the last section we compute the infinitesimal Einstein deformations of the normal homogeneous examples: the isotropy irreducible space $SO(5)/SO(3)$, the squashed 7-sphere and the second

Einstein metric on the Aloff-Wallach space $N(1,1)$. We show that there exist no Einstein deformations and, in particular, no deformations of the nearly parallel G_2-structure in the first two cases, while in the third the space of infinitesimal Einstein deformations coincides with the space of infinitesimal nearly parallel G_2-deformations and is 8-dimensional. We do not know whether these infinitesimal deformations integrate to real Einstein deformations.

2. Preliminaries

Let $e_1, \ldots, e_7$ denote the standard basis of $\mathbb{R}^7$ and $e^1, \ldots, e^7$ its dual basis. On $\mathbb{R}^7$ we fix the canonical scalar product $\langle \cdot, \cdot \rangle$ and the standard orientation. We shall write $e^{i_1 \cdots i_k}$ for the wedge product $e^{i_1} \wedge \ldots \wedge e^{i_k} \in \Lambda^k(\mathbb{R}^7)^*$ and define the *fundamental 3-form* as

$$(2.1) \qquad \sigma = e^{123} + e^{145} + e^{246} + e^{347} - e^{167} + e^{257} - e^{356}.$$

The exceptional group G_2 is defined as the subgroup of $\mathrm{GL}(7, \mathbb{R})$ that fixes the 3-form σ, i.e., $G_2 = \{ g \in \mathrm{GL}(7, \mathbb{R}) \mid g^* \sigma = \sigma \}$. The group G_2 is a 14-dimensional compact, connected, simple Lie group, which acts irreducibly on $\mathrm{T} := \mathbb{R}^7$ and preserves the metric, the orientation and the Hodge dual of σ, i.e. the 4-form

$$(2.2) \qquad *\sigma = e^{4567} + e^{2367} + e^{1357} + e^{1256} - e^{2345} + e^{1346} - e^{1247}.$$

The irreducible representations of G_2 can be indexed by their highest weights, which are pairs of non-negative integers (p, q) if written as linear combinations of the two fundamental weights. The corresponding representation will be denoted by $V_{p,q}$. In this paper we will in particular be interested in the following four irreducible G_2-representations: the trivial representation $V_{0,0} = \mathbb{R}$, the standard representation $V_{1,0} = \mathrm{T} := \mathbb{R}^7$, the adjoint representation $V_{0,1} = \mathfrak{g}_2$ and the representation on traceless symmetric 2-forms $V_{2,0} = S_0^2 \mathrm{T}^*$. Among the irreducible representations these are uniquely determined by their dimensions 1, 7, 14 and 27 respectively. Therefore we shall use the dimensions as lower indices when we decompose the space of k-forms $\Lambda^k \mathrm{T}^*$ into irreducible components. In other words, Λ_r^k will denote the r-dimensional irreducible subspace of $\Lambda^k \mathrm{T}^*$. With this notation we have

$$(2.3) \qquad \Lambda^2 = \Lambda^2 \mathrm{T}^* = \Lambda_7^2 \oplus \Lambda_{14}^2, \qquad \Lambda^3 = \Lambda^3 \mathrm{T}^* = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3,$$

with an isomorphic decomposition for $\Lambda^4 \mathrm{T}^* \cong \Lambda^3 \mathrm{T}^*$ and $\Lambda^5 \mathrm{T}^* \cong \Lambda^2 \mathrm{T}^*$ obtained with the help of the Hodge $*$-operator. The one-dimensional spaces in Λ^3 resp. Λ^4 are spanned by σ resp. $*\sigma$. The space Λ_{14}^2 is isomorphic to the Lie algebra of G_2 and the other subspaces can be characterized by

$$\Lambda_7^2 = \{ X \lrcorner \sigma \in \Lambda^2 \mid X \in \mathrm{T} \} \cong \mathrm{T}, \quad \Lambda_7^3 = \{ X \lrcorner *\sigma \in \Lambda^3 \mid X \in \mathrm{T} \} \cong \mathrm{T},$$

$$\Lambda_{27}^3 = \{ \alpha \in \Lambda^3 \mid \alpha \wedge \sigma = 0 = \alpha \wedge *\sigma \} \cong V_{2,0}.$$

In the sequel we shall use the following G_2-equivariant isomorphisms, which were introduced by Bryant in [6]: $\mathbf{i} : S_0^2 \mathrm{T}^* \to \Lambda_{27}^3$ and $\mathbf{j} : \Lambda_{27}^3 \to S_0^2 \mathrm{T}^*$, where $\mathbf{i}$ is the restriction to $S_0^2 \mathrm{T}^* \subset S^2 \mathrm{T}^*$ of the map $S^2 \mathrm{T}^* \to \Lambda^3 \mathrm{T}^*$, defined on decomposable elements by

$$\alpha \odot \beta \mapsto \alpha \wedge (\beta \lrcorner \sigma) + \beta \wedge (\alpha \lrcorner \sigma),$$

while $\mathbf{j}$ is given by

$$\mathbf{j}(\gamma)(X,Y) = *((X \lrcorner \sigma) \wedge (Y \lrcorner \sigma) \wedge \gamma).$$

Note that $\mathbf{j} = -8\mathbf{i}^{-1}$. With the help of $\mathbf{i}$ one can obtain explicit elements of Λ^3_{27}, e.g.

(2.4) $\mathbf{i}(e^1 \odot e^2) = e^{146} + e^{157} + e^{245} - e^{267}.$

Because of $\mathrm{T}^* \otimes \mathrm{T}^* = S^2\mathrm{T}^* \oplus \Lambda^2\mathrm{T}^*$ we have the following decomposition:

(2.5) $V_{1,0} \otimes V_{1,0} \cong \mathbb{R} \oplus V_{2,0} \oplus V_{1,0} \oplus V_{0,1}.$

Later we shall also need the decompositions

(2.6) $V_{1,0} \otimes V_{2,0} \cong V_{1,0} \oplus V_{2,0} \oplus V_{0,1} \oplus V_{1,1} \oplus V_{3,0},$

(2.7) $V_{1,0} \otimes V_{0,1} \cong V_{1,0} \oplus V_{2,0} \oplus V_{1,1}.$

The group G_2 can also be defined as the stabilizer of the vector cross product P, given by

(2.8) $\sigma(X,Y,Z) = \langle P(X,Y), Z \rangle,$

where X, Y, Z are any vectors in T. Recall from [8] that a 2-fold vector cross product P is a bilinear map $P : \mathrm{T} \times \mathrm{T} \to \mathrm{T}$ satisfying for all $X, Y \in \mathrm{T}$ the equations

(2.9) $\langle P(X,Y), X \rangle = \langle P(X,Y), Y \rangle = 0 \quad \text{and} \quad \|P(X,Y)\|^2 = \|X\|^2\|Y\|^2 - \langle X, Y \rangle^2.$

In particular, it follows from the second equation of (2.9) that P is skew-symmetric. Thus we can consider P as a linear map $P : \Lambda^2\mathrm{T} \to \mathrm{T}$ and write $P(X \wedge Y) = P(X,Y)$. In this notation the second equation of (2.9) reads: $\|P(X \wedge Y)\|^2 = \|X \wedge Y\|^2$. We also refer to [8] for the following relations satisfied by a general 2-fold vector cross product:

Lemma 2.1. *For $X, Y, Z \in \mathrm{T}$ we have*

    (1) $\langle P(X,Y), Z \rangle = \langle X, P(Y,Z) \rangle,$
    (2) $P(X, P(X,Y)) = -\|X\|^2 Y + \langle X, Y \rangle X,$
    (3) $2P(P(X,Y),Z) = P(P(Y,Z),X) + P(P(Z,X),Y) + 3\langle X,Z \rangle Y - 3\langle Y,Z \rangle X.$

From now on we will usually identify vectors and 1-forms via the metric and denote with $\{e_i\}$, $i = 1, \ldots, 7$ an orthonormal basis of T. For later use we still note

Lemma 2.2. *Let X and Y be any vectors in T. Then the following equations hold*

(2.10) $(X \lrcorner \sigma) \wedge \sigma = -2X \wedge *\sigma,$

(2.11) $(X \lrcorner \sigma) \wedge *\sigma = 3 * X,$

(2.12) $\sum_i (e_i \lrcorner X \lrcorner \sigma) \lrcorner (e_i \wedge \sigma) = 3X \lrcorner *\sigma,$

(2.13) $\sum_i (e_i \lrcorner X \lrcorner \sigma) \wedge (e_i \lrcorner \sigma) = 3X \lrcorner *\sigma,$

(2.14) $(X \lrcorner Y \lrcorner \sigma) \lrcorner \sigma + X \lrcorner Y \lrcorner *\sigma = -X \wedge Y,$

(2.15) $P(X \lrcorner \sigma) = 3X.$

The GL$_7$-orbit of σ in $\Lambda^3 T^*$ is an open set by dimensional reasons. As usual it is denoted with Λ^3_+. Forms in Λ^3_+ are called *stable* or *definite*.

Let M be a 7-dimensional manifold. The union of the subspaces $\Lambda^3_+ T^*_x M$, $x \in M$, of stable forms defines an open subbundle $\Lambda^3_+ T^* M \subset \Lambda^3 T^* M$. There is a one-to-one correspondence between G$_2$-structures on M, i.e. reductions of the structure group of M to the group G$_2$, and the space of sections of $\Lambda^3_+ T^* M$, which we will denote with $\Omega^3_+(M)$. The defining 3-form $\sigma \in \Omega^3_+(M)$ determines a Riemannian metric g and an orientation of M via the relation

(2.16) $$ -6g(X,Y) * 1 = X \lrcorner \sigma \wedge Y \lrcorner \sigma \wedge \sigma $$

($*1$ denotes the volume form). Let ∇ be the Levi-Civita connection of g. Then the covariant derivative $\nabla \sigma$ is a section of the bundle $T^* \otimes \mathfrak{g}_2^{\perp}$, where $\mathfrak{g}_2^{\perp} \cong T^*$ is the orthogonal complement of $\mathfrak{g}_2$ in $\Lambda^2 T^*$ and we identify bundles with the G$_2$-representation defining it. It follows from (2.5) that this bundle decomposes as $\mathbb{R} \oplus V_{2,0} \oplus V_{1,0} \oplus V_{0,1}$ and thus the covariant derivative of σ has four components. Accordingly, one has the 16 Fernandez-Gray classes of G$_2$-structures, with the four basic classes W_1, W_2, W_3, W_4 corresponding to the four irreducible summands.

In this article we shall consider the class W_1 of so called *nearly parallel* (or *weak*) G$_2$-structures, i.e. G$_2$-structures induced by a non-parallel 3-form $\sigma \in \Omega^3_+(M)$, such that $\nabla \sigma$ is a section of the 1-dimensional subbundle defined by the trivial G$_2$-representation. Nearly parallel G$_2$-structures can be described by several equivalent conditions in terms of σ.

Proposition 2.3. *Let M be a 7-dimensional manifold with a G$_2$-structure defined by a 3-form $\sigma \in \Omega^3_+(M)$. Then the following conditions are equivalent*

(1) *The 3-form σ defines a nearly parallel G$_2$-structure.*
(2) *The 3-form σ is a Killing 3-form, i.e. $\nabla \sigma = \frac{1}{4} d\sigma$.*
(3) *There exists a $\tau_0 \in \mathbb{R} \setminus \{0\}$ with $\nabla \sigma = \frac{\tau_0}{4} * \sigma$.*
(4) *There exists a $\tau_0 \in \mathbb{R} \setminus \{0\}$ with $\nabla_X(*\sigma) = -\frac{\tau_0}{4} X \wedge \sigma$ for all vector fields X.*
(5) *There exists a $\tau_0 \in \mathbb{R} \setminus \{0\}$ with $d\sigma = \tau_0 * \sigma$.*
(6) *$X \lrcorner \nabla_X \sigma = 0$ holds for all vector fields X.*

Proof: The equivalence of (3) and (4) is obvious, while the equivalence of (1), (2), (3) and (6) has been proved in [8]. The only point not mentioned there is that τ_0 is constant. This fact is also known (see e.g. [12]) and can be proven as follows. Since (5) is an obvious consequence of (3), we can differentiate it to obtain $d\tau_0 \wedge *\sigma = 0$, which implies $d\tau_0 = 0$. Finally, that (5) implies the remaining conditions was proved in [12]. This is the only point where one uses that τ_0 is different from zero. $\qquad \square$

Let P be the associated vector cross product, defined in (2.8). Then the condition (6) of the proposition above is equivalent to $(\nabla_X P)(X,Y) = 0$ for any vector fields X, Y, i.e., to P being *nearly parallel* [13]. Further straightforward consequences of Proposition 2.3 in the case of nearly parallel G$_2$-manifolds are: $d^* \sigma = 0$ and $\Delta \sigma = \tau_0^2 \sigma$, where here and in the following $\Delta = dd^* + d^*d$ denotes the Hodge-de Rham Laplacian. Moreover it follows that σ is a special Killing 3-form, i.e. the additional equation $\nabla_X d\sigma = -\frac{1}{4}\tau_0^2 X^* \wedge \sigma$ is satisfied for all vector fields X (cf. [20]).

The canonical connection $\bar{\nabla}$ of a G_2-structure is the unique G_2-connection whose torsion is equal to the intrinsic torsion of the G_2-structure. In the nearly parallel case it has totally skew-symmetric and parallel torsion and is explicitly given by

$$(2.17) \qquad g(\bar{\nabla}_X Y, Z) \;=\; g(\nabla_X Y, Z) \;-\; \tfrac{\tau_0}{12}\,\sigma(X, Y, Z)$$

or, equivalently, by

$$(2.18) \qquad \bar{\nabla}_X = \nabla_X - \tfrac{\tau_0}{12} P_X,$$

where the endomorphism P_X is defined by $P_X Y := P(X, Y)$.

Remark 2.4. The fact that P is G_2-invariant allows the following important application, which we shall use several times in this article. Let V be an irreducible G_2-representation contained in some tensor space and VM be the corresponding associated bundle. Then the endomorphism P_X extends to an endomorphism of VM and we may consider the G_2-equivariant map $V \to \mathrm{T}^* \otimes V$, defined by $\varphi \mapsto \sum_i e^i \otimes P_{e_i}\varphi$, which we again denote by P. By (2.18) we have

$$(\bar{\nabla} - \nabla)\varphi = -\tfrac{\tau_0}{12} P\varphi$$

for any section φ of VM. Let U be an irreducible component in $\mathrm{T}^* \otimes V$. Suppose first that U is not isomorphic to V as a G_2-representation. Then there exists no non-zero G_2-equivariant map from V to U and therefore the UM-part $(P\varphi)_{UM}$ of $P\varphi$ vanishes, which implies $(\nabla\varphi)_{UM} = (\bar{\nabla}\varphi)_{UM}$. On the other hand, if U is isomorphic to V, then $U = i(V)$, where $i : V \to \mathrm{T}^* \otimes V$ is some G_2-equivariant embedding. Let $\pi : \mathrm{T}^* \otimes V \to U$ be the projection. Then $\pi \circ P : V \to U$ is also G_2-equivariant and therefore by Schur's lemma $\pi \circ P = ci$ for some constant c. Thus $(\nabla\varphi)_{i(VM)} = (\bar{\nabla}\varphi)_{i(VM)} + \tfrac{c\tau_0}{12} i(\varphi)$. Finally, since $\bar{\nabla}\varphi$ and $P\varphi$ are sections of $\mathrm{T}^*M \otimes VM$, the same is true for $\nabla\varphi$, despite the fact that ∇ is not a G_2-connection.

Remark 2.5. Our choice of the orientation induced by a stable 3-form σ is the opposite of the choice of Bryant in [6]. As a consequence our $*$, $\mathbf{j}$, τ_0 and f_1 from the next section differ from those in [6] by a sign.

3. Deformations of G_2-structures

In this section we will consider a smooth curve σ_t of nearly parallel G_2-structures and describe its tangent vector $\dot{\sigma}$ in $t = 0$. Here and in the sequel the dot denotes the time derivative at $t = 0$. As a starting point we use the following result of R. Bryant [6] for curves of arbitrary G_2-structures (cf. also [14]).

Proposition 3.1. *Let (M^7, g) be a Riemannian manifold with a family $\sigma_t \in \Omega^3_+(M)$ of G_2-structures. Let g_t be the family of metrics and $*_t$ the Hodge star operator associated with σ_t. Then there exist three time-dependent differential forms $f_0 \in \Omega^0(M)$, $f_1 \in \Omega^1(M)$ and $f_3 \in \Omega^3_{27}(M)$ that satisfy the equations*

(1) $\dot{\sigma} \;=\; 3 f_0\,\sigma \;+\; *(f_1 \wedge \sigma) \;+\; f_3,$

(2) $\dot{g} \;=\; 2 f_0\,g \;-\; \tfrac{1}{2}\,\mathbf{j}(f_3),$

(3) $\dot{*\sigma} = 4 f_0 * \sigma + f_1 \wedge \sigma - *f_3,$

(4) $\dot{*1} = 7 f_0 * 1.$

Our aim is to study deformations of a given nearly parallel G$_2$-structure σ on a compact manifold M by nearly parallel G$_2$-structures σ_t. We will only be interested in deformations of the nearly parallel G$_2$-structures modulo the action of the group $\mathbb{R}^* \times \mathrm{Diff}(M)$, given by

$$(\lambda, \varphi) \cdot \sigma = \lambda^3 \varphi(\sigma) = \lambda^3 (\varphi^{-1})^* \sigma \circ \varphi.$$

If σ induces the metric g, the Hodge dual $*\sigma$ and the volume form $*1$, then $\widetilde{\sigma} = \lambda^3 \sigma$ induces

(3.19) $$\widetilde{g} = \lambda^2 g, \quad \widetilde{*\widetilde{\sigma}} = \lambda^4 * \sigma, \quad \widetilde{*}1 = \lambda^7 * 1, \quad \widetilde{\tau}_0 = \frac{1}{\lambda}\tau_0.$$

Therefore we can always assume that the volume of M with respect to g is normalized. Moreover, we can apply the Ebin's Slice Theorem and assume that g_t is a curve in the slice through g. A nearly parallel G$_2$-structure is Einstein with scalar curvature

(3.20) $$\mathrm{scal}_g = \tfrac{21}{8}\tau_0^2.$$

Thus $\dot{g}$ is an infinitesimal Einstein deformation of g and by the theorem of Berger-Ebin (see [3], Chapter 12) we have

(3.21) $$\mathrm{tr}\, \dot{g} = 0, \qquad \delta\, \dot{g} = 0, \qquad \Delta_L \dot{g} = \tfrac{2\mathrm{scal}}{7}\dot{g} = \tfrac{3}{4}\tau_0^2 \dot{g},$$

where Δ_L is the Lichnerowicz Laplacian (see [3] or Section 6 below). Since $\mathrm{tr}\, \dot{g} = 14 f_0$, it immediately follows that f_0 vanishes and the equations of Proposition 3.1 may be rewritten as

(3.22) $$\dot{\sigma} = *(f_1 \wedge \sigma) + f_3, \quad \dot{g} = -\tfrac{1}{2}\mathbf{j}(f_3), \quad \dot{*\sigma} = f_1 \wedge \sigma - *f_3, \quad \dot{*1} = 0,$$

while equations (3.21) become

(3.23) $$\delta\, \mathbf{j}(f_3) = 0, \qquad \Delta_L \mathbf{j}(f_3) = \tfrac{3}{4}\tau_0^2\, \mathbf{j}(f_3).$$

The fact that σ_t is a family of nearly parallel G$_2$-structures means by definition that

(3.24) $$d\sigma_t = \tau_0(t) * \sigma_t$$

for some function $\tau_0(t)$. However, g_t is a family of Einstein metrics and therefore scal_{g_t} is constant as function in t, as follows from Corollary 2.12 of [3]. This, together with (3.20), implies that the function τ_0 is constant too. Thus, differentiating (3.24) with respect to t, we obtain the linearized equation $d\dot{\sigma} = \tau_0 \dot{*\sigma}$, which by (3.22) yields

(3.25) $$d * (f_1 \wedge \sigma) + df_3 = \tau_0(f_1 \wedge \sigma - *f_3).$$

The discussion above motivates the following definition.

Definition 3.2. An *infinitesimal (nearly parallel) deformation* of a compact nearly parallel G$_2$-manifold (M, σ) is a section (f_1, f_3) of the bundle $\Lambda^1 \mathrm{T}^* M \oplus \Lambda^3_{27} \mathrm{T}^* M$, which satisfies the equations from (3.23) and (3.25).

The rest of this section is devoted to deriving a more explicit description of the space of infinitesimal deformations of nearly parallel G_2-structures. In a first step we obtain information about the 3-form component f_3 of the infinitesimal deformation.

Lemma 3.3. *The covariant derivatives of f_3 with respect to $\bar{\nabla}$ and ∇ have no component in $T \subset T^* \otimes \Lambda^3_{27}$, i.e., it holds $(\bar{\nabla} f_3)_T = (\nabla f_3)_T = 0$. In particular, the differential and codifferential of f_3 satisfy the equations:*

$$(df_3)_{\Lambda^4_1} = 0, \qquad (df_3)_{\Lambda^4_7} = 0, \qquad (d^* f_3)_{\Lambda^2_7} = 0, \qquad (d * f_3)_{\Lambda^5_7} = 0.$$

Proof: The divergence δ is defined as the composition of the covariant derivative ∇ and the equivariant contraction $c : T^* \otimes S^2_0 T^* \to T^*$. It follows from Remark 2.4 and the decomposition (2.6) that $(\nabla \mathbf{j}(f_3))_T = (\bar{\nabla}\mathbf{j}(f_3))_T$ and $(\nabla f_3)_T = (\bar{\nabla} f_3)_T$. This implies

$$(3.26) \qquad \delta\mathbf{j}(f_3) = -c\nabla\mathbf{j}(f_3) = -c(\nabla\mathbf{j}(f_3))_T = -c(\bar{\nabla}\mathbf{j}(f_3))_T = -c \circ (1 \otimes \mathbf{j})(\bar{\nabla} f_3)_T,$$

where the last equality follows from $\mathbf{j}$ being G_2-equivariant and $\bar{\nabla}$-parallel. Since $c \circ (1 \otimes \mathbf{j})$ is non-zero on $T \subset T^* \otimes \Lambda^3_{27}$ (which can be checked on an explicit element), we finally obtain $\delta\mathbf{j}(f_3) = 0$ if and only if $(\bar{\nabla}\mathbf{j}(f_3))_T = (\bar{\nabla} f_3)_T = 0$.

By the definition of the differential we have $df_3 = \varepsilon\nabla f_3$, with the G_2-equivariant wedging map $\varepsilon : T^* \otimes \Lambda^3 T^* \to \Lambda^4 T^*$. Again by (2.6) and Remark 2.4 we see that ∇f_3 has no components in bundles associated with the trivial representation $\mathbb{R}$. Thus df_3 has no components in Λ^4_1, which proves the first equation for df_3. The second follows from $(df_3)_{\Lambda^4_7} = \varepsilon(\nabla f_3)_T = 0$ and the remaining two are proved in a similar way. $\qquad\square$

In the next step we will derive information about the 1-form part of infinitesimal deformations.

Proposition 3.4. *For the 1-form f_1 the following holds:*

(1) $\bar{\nabla} f_1 = -\frac{1}{3}\tau_0\, f_1 \lrcorner \sigma$.
(2) $\nabla f_1 = -\frac{1}{4}\tau_0\, f_1 \lrcorner \sigma$.
(3) f_1 *is a Killing 1-form and* $df_1 = -\frac{1}{2}\tau_0\, f_1 \lrcorner \sigma$. *In particular,* $d^* f_1 = 0$ *and* $(df_1)_{\Lambda^2_{14}} = 0$.
(4) $d(f_1 \wedge *\sigma) = -\frac{3}{2}\tau_0 * f_1$, *in particular* $d^*(f_1 \lrcorner \sigma) = -\frac{3}{2}\tau_0 f_1$.
(5) $\Delta f_1 = \frac{3}{4}\tau_0^2 f_1$.
(6) $d^*(f_1 \wedge \sigma) = -\tau_0 f_1 \lrcorner * \sigma$. *In particular* $d * (f_1 \wedge \sigma) = \tau_0 f_1 \wedge \sigma$.
(7) f_1 *has constant length.*

Proof: Statements (2) and (3) are obviously equivalent and (1) and (2) are equivalent because (2.17) implies $\bar{\nabla} f_1 = \nabla f_1 - \frac{\tau_0}{12} f_1 \lrcorner \sigma$. The remaining properties are consequences of each of the first three. To prove (6) we use $d^* f_1 = 0$, $d^*\sigma = 0$, (3) of Proposition 2.3 and (2.13) to obtain

$$d^*(f_1 \wedge \sigma) = \sum \nabla_{e_i} f_1 \wedge (e_i \lrcorner \sigma) - \nabla_{f_1^\sharp}\sigma = -\frac{1}{4}\tau_0 \sum (e_i \lrcorner f_1 \lrcorner \sigma) \wedge (e_i \lrcorner \sigma) - \frac{1}{4}\tau_0 f_1 \lrcorner * \sigma = -\tau_0 f_1 \lrcorner * \sigma.$$

Property (7) follows from

$$d|f_1|^2(X) = 2\langle \nabla_X f_1, f_1 \rangle = -\frac{1}{2}\tau_0\langle X \lrcorner f_1 \lrcorner \sigma, f_1 \rangle = -\frac{1}{2}\tau_0 f_1 \lrcorner X \lrcorner f_1 \lrcorner \sigma = 0.$$

For the proof of (3) we start with computing the Λ_7^2 part of df_1. Since τ_0 is a non-zero constant we may take the differential of equation (3.25) to obtain $d(f_1 \wedge \sigma) = d * f_3$. Now, let β be the 1-form, defined by $(df_1)_{\Lambda_7^2} = \beta \lrcorner \sigma$. Then the vanishing of the Λ_7^5-part of $d * f_3$ and equation (2.10) imply

$$0 = (d(f_1 \wedge \sigma))_{\Lambda_7^5} = (df_1)_{\Lambda_7^2} \wedge \sigma - \tau_0 f_1 \wedge *\sigma = (\beta \lrcorner \sigma) \wedge \sigma - \tau_0 f_1 \wedge *\sigma = (-2\beta - \tau_0 f_1) \wedge *\sigma.$$

Since the wedge product with $*\sigma$ defines an injective map on 1-forms, we find $\beta = -\frac{1}{2}\tau_0 f_1$. Hence $(df_1)_{\Lambda_7^2} = -\frac{1}{2}\tau_0 f_1 \lrcorner \sigma$. The Λ_7^2-part of ∇f_1 with respect to the decomposition (2.5) is $\frac{1}{2}(df_1)_{\Lambda_7^2}$ and we obtain $(\nabla f_1)_{\Lambda_7^2} = -\frac{1}{4}\tau_0 f_1 \lrcorner \sigma$.

We continue with proving (4). Since the fundamental 3-form σ is coclosed, it follows $d * \sigma = 0$. Moreover, the wedge product with $*\sigma$ defines an equivariant map $\Lambda^2 \to \Lambda^6 \cong \Lambda^1$, which by Schur's Lemma vanishes on Λ_{14}^2. Hence, using (2.11) we obtain

$$d(f_1 \wedge *\sigma) = df_1 \wedge *\sigma = (df_1)_{\Lambda_7^2} \wedge *\sigma = -\frac{1}{2}\tau_0(f_1 \lrcorner \sigma) \wedge *\sigma = -\frac{3}{2}\tau_0 * f_1.$$

As an immediate consequence we obtain in addition $d^*(f_1 \lrcorner \sigma) = -\frac{3}{2}\tau_0 f_1$. Since $\tau_0 \neq 0$, this implies $d^* f_1 = 0$.

Next we want to show that the Λ_7^3-part of $d(f_1 \lrcorner \sigma)$ vanishes. Using Proposition 2.3 we compute

$$\begin{aligned} d(f_1 \lrcorner \sigma) &= \sum_i e^i \wedge \nabla_{e_i}(f_1 \lrcorner \sigma) = \sum_i e^i \wedge (\nabla_{e_i} f_1 \lrcorner \sigma + f_1 \lrcorner \nabla_{e_i} \sigma) \\ &= -\sum_i \nabla_{e_i} f_1 \lrcorner (e_i \wedge \sigma) + (d^* f_1)\sigma - f_1 \lrcorner d\sigma + \nabla_{f_1^\sharp} \sigma \\ &= -\Phi(\nabla f_1) - \frac{3}{4}\tau_0 f_1 \lrcorner * \sigma, \end{aligned}$$

where $\Phi : \Lambda^1 \otimes \Lambda^1 \to \Lambda^3$ denotes the map $\Phi(\gamma) = \sum_i (e_i \lrcorner \gamma) \lrcorner (e^i \wedge \sigma)$. Since Φ is obviously G_2-invariant, we obtain

$$(d(f_1 \lrcorner \sigma))_{\Lambda_7^3} = -(\Phi(\nabla f_1))_{\Lambda_7^3} - \frac{3}{4}\tau_0 f_1 \lrcorner *\sigma = -\Phi((\nabla f_1)_{\Lambda_7^2}) - \frac{3}{4}\tau_0 f_1 \lrcorner *\sigma = \frac{1}{4}\tau_0 \Phi(f_1 \lrcorner \sigma) - \frac{3}{4}\tau_0 f_1 \lrcorner *\sigma.$$

Because (2.12) implies $\Phi(f_1 \lrcorner \sigma) = 3 f_1 \lrcorner * \sigma$, we finally obtain

(3.27) $$\qquad\qquad\qquad\qquad (d(f_1 \lrcorner \sigma))_{\Lambda_7^3} = 0.$$

Now we shall prove the vanishing of the Λ_{14}^2-part of df_1 using the compactness of M. From our equation for $(df_1)_{\Lambda_7^2}$ we conclude

$$0 = d^2 f_1 = d((df_1)_{\Lambda_7^2} + (df_1)_{\Lambda_{14}^2}) = -\frac{1}{2}\tau_0 d(f_1 \lrcorner \sigma) + d(df_1)_{\Lambda_{14}^2}.$$

From here it follows with (3.27) that $(d(df_1)_{\Lambda_{14}^2})_{\Lambda_7^3} = 0$. By definition the differential d is the composition of the invariant wedging map $\varepsilon : T^* \otimes \Lambda^2 \to \Lambda^3$ and the covariant derivative ∇. By Remark 2.4 $\nabla \gamma$ is a section of $T^* M \otimes \Lambda_{14}^2 M$ for any section γ of $\Lambda_{14}^2 M$. Since by (2.7) there is only one component isomorphic to T in $T^* \otimes \Lambda_{14}^2$, we obtain for $\gamma := (df_1)_{\Lambda_{14}^2}$:

$$0 = (d\gamma)_{\Lambda_7^3} = \pi_{\Lambda_7^3} \circ \varepsilon \nabla \gamma = \pi_{\Lambda_7^3} \circ \varepsilon (\nabla \gamma)_\mathrm{T}.$$

Because $\pi_{\Lambda_7^3} \circ \varepsilon$ is different from zero on $\mathrm{T} \subset T^* \otimes \Lambda_{14}^2$, as one checks on an explicit element, this yields $(\nabla \gamma)_\mathrm{T} = 0$.

We may use a similar argument for the codifferential d^*, which is the composition of the invariant contraction map $c : \mathrm{T}^* \otimes \Lambda^2 \to \Lambda^1$ and the covariant derivative. Hence we have

$$d^* \gamma = -c \nabla \gamma = -c \, (\nabla \gamma)_{\mathrm{T}} = 0.$$

Then the L^2-scalar product of $d^* \gamma$ and f_1 yields

$$0 = (d^* \gamma, f_1) = (\gamma, df_1) = \|\gamma\|^2.$$

Thus it follows that $\gamma = 0$, i.e. $(df_1)_{\Lambda_{14}^2} = 0$, and that df_1 is indeed a section of $\Lambda_7^2 \mathrm{T}^* M$ with $df_1 = (df_1)_{\Lambda_7^2} = -\frac{1}{2} \tau_0 f_1 \lrcorner \sigma$.

We already know that f_1 is coclosed and thus

$$\Delta f_1 = d^* df_1 = -\frac{1}{2} \tau_0 d^* (f_1 \lrcorner \sigma) = \frac{3}{4} \tau_0^2 f_1,$$

which proves (5). Using the fact that the manifold (M^7, g) is Einstein with scal $= \frac{21}{8} \tau_0^2$, we obtain $\Delta f_1 = 2\mathrm{Ric}(f_1)$. By the well-known characterization of Killing vector fields on compact manifolds this implies that f_1 is Killing. □

Finally we combine Proposition 3.4 with the initial equations to obtain a characterization of infinitesimal deformations of nearly parallel G_2-structures.

Theorem 3.5. *The space of infinitesimal deformations of a compact nearly parallel G_2-manifold (M, σ) is the direct sum of the finite-dimensional spaces*

$$D_1 := \{f_1 \in \Omega^1(M) \mid \nabla f_1 = -\tfrac{1}{4} \tau_0 \, f_1 \lrcorner \sigma\} \quad and \quad D_3 := \{f_3 \in \Omega_{27}^3(M) \mid *df_3 = -\tau_0 f_3\}.$$

In particular, f_1 and f_3 are co-closed eigenforms of the Laplace operator for the eigenvalues $\frac{3}{4} \tau_0^2$ and τ_0^2 respectively.

Proof: It remains to prove the equations for f_3. For this we substitute the expression for $d * (f_1 \wedge \sigma)$ of Proposition 3.4 back into equation (3.25) and obtain $df_3 = -\tau_0 * f_3$. Since $\tau_0 \neq 0$ this immediately implies that f_3 is coclosed. Then the Laplace operator is computed as $\Delta f_3 = d^* df_3 = (*d)^2 f_3 = \tau_0^2 f_3$. It follows that an infinitesimal deformation lies in the direct sum of the spaces D_1 and D_3. They are finite-dimensional since they are contained in certain eigenspaces of the Laplace operator.

Conversely, by Proposition 3.4 $\nabla f_1 = -\frac{1}{4} \tau_0 \, f_1 \lrcorner \sigma$ and $*df_3 = -\tau_0 f_3$ imply (3.25). Further, $df_3 = -\tau_0 * f_3$ yields $(df_3)_{\Lambda_7^4} = 0$, i.e., $\pi_{\Lambda_7^4} \circ \varepsilon (\nabla f_3)_{\mathrm{T}} = 0$, where $\mathrm{T} \subset \mathrm{T}^* \otimes \Lambda_{27}^3$ is the component isomorphic to $V_{1,0}$. Since $\pi_{\Lambda_7^4} \circ \varepsilon$ is non-zero on T (which can be checked on an explicit element), it follows that $(\nabla f_3)_{\mathrm{T}} = 0$. Thus, by Remark 2.4 also $(\bar{\nabla} f_3)_{\mathrm{T}} = 0$ and therefore by (3.26) $\delta \mathbf{j}(f_3) = -c(1 \otimes \mathbf{j})(\bar{\nabla} f_3)_{\mathrm{T}} = 0$. It remains to show that $*df_3 = -\tau_0 f_3$ implies $\Delta_L \mathbf{j}(f_3) = \frac{3}{4} \tau_0^2 \mathbf{j}(f_3)$, which will be done in Section 6. □

4. G_2-DEFORMATIONS AND SASAKIAN STRUCTURES

In this section we will investigate the relation between nearly parallel G_2-manifolds with a non-trivial space D_1 in Theorem 3.5 and Sasakian structures. The first result in this direction is the following.

Proposition 4.1. *Let (M, g, σ) be a compact nearly parallel G_2-manifold normalized so that $\tau_0 = 4$. Then:*

    (1) *If $\dim D_1 \geq 1$, then (M, g) is a Sasakian-Einstein manifold.*
    (2) *If $\dim D_1 \geq 2$, then (M, g) is a 3-Sasakian manifold.*

Proof: The assumption about the normalization is not a restriction because of (3.19). Let $0 \neq f_1 \in D_1$, then Proposition 3.4 shows that f_1 is a Killing 1-form of constant length and we can assume $|f_1| = 1$. Thus, to prove that f_1 is the contact form of a Sasakian structure it remains (see [4]) to verify the curvature condition

$$(\nabla^2_{X,Y} f_1)(Z) \; = \; f_1(Y)g(X, Z) \; - \; f_1(Z)g(X, Y) \; = \; (f_1 \wedge X)(Y, Z)$$

However, taking the covariant derivative of the defining equation of D_1 immediately implies:

$$\begin{aligned}
(\nabla^2_{X,Y} f_1)(Z) \;\; &= -\tfrac{1}{4}\,\tau_0\,(\nabla_X (f_1 \lrcorner \sigma))(Y, Z) \; = \; -(\nabla_X f_1 \lrcorner \sigma)(Y, Z) \; - \; (f_1 \lrcorner \nabla_X \sigma)(Y, Z) \\
&= \;\; \tfrac{1}{4}\,\tau_0\,((X \lrcorner f_1 \lrcorner \sigma) \lrcorner \sigma)(Y, Z) - \tfrac{1}{4}\,\tau_0\,(f_1 \lrcorner X \lrcorner * \sigma)(Y, Z) \; = \; (f_1 \wedge X)(Y, Z)
\end{aligned}$$

where we also used (2.14). Since g is known to be Einstein, we obtain the first statement.

If $\dim D_1 \geq 2$, then (M, g) has two Sasakian structures, whose contact forms are linearly independent. This implies the second statement (see [4], Lemma 8.1.17). $\hfill\square$

Recall that a G_2-structure on a 7-dimensional manifold M defines a canonical spin structure on M. The G_2-structure is furthermore nearly parallel if and only if the associated spin structure admits real Killing spinors [12]. In this case the nearly parallel G_2-structures inducing the given metric and spin structure are in bijective correspondence with the projectivization of the space of Killing spinors in the real spinor bundle [12]. The complex spinor bundle is the complexification of the real spinor bundle and the space of real Killing spinors is the complexification of the space of Killing spinors in the real spinor bundle, so both spaces have the same dimension over the respective field. After a suitable normalization of the metric (which in our case amounts to ensuring that $\tau_0 = 4$) this dimension is also equal to the dimension of the space of parallel spinors on the metric cone $\hat{M}$ of M for the spin structure induced by the one on M. This is a result of Bär [2] in the simply connected case and holds also in general, as explained by Wang in [21]. Hence, as noticed in [2], if M is compact, then either the restricted holonomy group of $\hat{M}$ is one of $\mathrm{Spin}(7)$, $\mathrm{SU}(4)$, $\mathrm{Sp}(2)$, or $\hat{M}$ is flat. In the latter case M is a quotient of the standard sphere S^7. According to a result of Friedrich [9], all nearly parallel G_2-structures on S^7 which induce the standard metric are conjugated under the action of the isometry group. Thus neither S^7 nor its quotients admit G_2-deformations. Therefore from now on we shall exclude from our considerations the case of nearly parallel G_2-manifolds with constant curvature. Under this assumption the compact nearly parallel G_2-manifolds split into the following three different types.

Type 1. The space of real Killing spinors is 1-dimensional. Then there is only one 3-form inducing the given metric, orientation and spin structure. We call such nearly parallel G_2-structures *proper*. Notice that our definition of a proper nearly parallel G_2-structure is slightly different from those in [12] and [4]. In [12] one assumes additionally that the manifold

is simply connected, while the definition in [4] requires that the cone has holonomy equal to Spin(7). For simply connected manifolds the three definitions are equivalent.

Type 2. The space of real Killing spinors is 2-dimensional. Then the given metric and orientation are induced by a Sasaki-Einstein structure but not by a 3-Sasakian structure. In terms of the cone $\hat{M}$ this is equivalent to saying that the holonomy group of $\hat{M}$ is contained in SU(4) but not in Sp(2). Indeed, the subgroup of Spin(8) which acts as identity on a 2-dimensional subspace of one of the half-spin representations is Spin(6) $\cong$ SU(4). In this case the 3-forms inducing the given metric, orientation and spin structure are parametrized by $\mathbb{R}P^1$.

Type 3. The space of real Killing spinors is 3-dimensional. The given metric and orientation are induced by a 3-Sasakian structure. In terms of the cone $\hat{M}$ this is equivalent to saying that the holonomy group of $\hat{M}$ is equal to Sp(2). In this case the 3-forms inducing the given metric and orientation are parametrized by $\mathbb{R}P^2$.

Now we shall describe the nearly parallel G_2-structures of types 2 and 3 without reference to Killing spinors. Recall that the cone of a Riemannian manifold (M, g) is $(\hat{M}, \hat{g})$, where $\hat{M} := \mathbb{R}_+ \times M$, $\hat{g} := dr^2 + r^2 g$ and r is the natural coordinate on $\mathbb{R}_+$. As shown in [2], if we normalize the nearly parallel G_2-structure so that $\tau_0 = 4$, then $\sigma = \partial_r \lrcorner \varphi|_{r=1}$, where φ is a parallel (and also stable) 4-form on the cone.

Suppose first that the holonomy group of the cone is equal to SU(4) (which implies that M is Sasaki-Einstein but not 3-Sasakian). Then the space of parallel 4-forms on $\hat{M}$ is spanned by $\Omega_{\hat{I}} \wedge \Omega_{\hat{I}}$, $\operatorname{Re}\Psi_{\hat{I}}$, $\operatorname{Im}\Psi_{\hat{I}}$. Here $\Omega_{\hat{I}}$ is the Kähler form and $\Psi_{\hat{I}}$ the complex volume form of the SU(4)-structure. Thus

$$\sigma = \partial_r \lrcorner \left(\frac{1}{2} c_0 \Omega_{\hat{I}} \wedge \Omega_{\hat{I}} + c_1 \operatorname{Re}\Psi_{\hat{I}} + c_2 \operatorname{Im}\Psi_{\hat{I}} \right) \Bigg|_{r=1}.$$

Equivalently, one can write this as

$$\sigma = c_0 \eta \wedge \Omega + c_1 \operatorname{Re}\Psi + c_2 \operatorname{Im}\Psi,$$

where η is the contact form of the Sasaki-Einstein structure on M, $\Omega = \nabla \eta$ is the horizontal Kähler form and Ψ is the horizontal complex volume form. Now a straightforward computation using (2.16) shows that σ induces the given metric and orientation if and only if $c_0 = -1$ and $c_1^2 + c_2^2 = 1$. Hence we have the following explicit S^1-family of nearly parallel G_2-structures:

$$(4.28) \qquad \sigma_t = -\eta \wedge \Omega + \cos t \operatorname{Re}\Psi + \sin t \operatorname{Im}\Psi.$$

In particular, each σ_t is of type 2.

Now suppose that the holonomy group of the cone is Sp(2) (i.e., M is a 3-Sasakian manifold). Then the space of parallel 4-forms is spanned by

$$\Omega_{\hat{I}_1} \wedge \Omega_{\hat{I}_1}, \quad \Omega_{\hat{I}_2} \wedge \Omega_{\hat{I}_2}, \quad \Omega_{\hat{I}_3} \wedge \Omega_{\hat{I}_3}, \quad \Omega_{\hat{I}_1} \wedge \Omega_{\hat{I}_2}, \quad \Omega_{\hat{I}_2} \wedge \Omega_{\hat{I}_3}, \quad \Omega_{\hat{I}_3} \wedge \Omega_{\hat{I}_1}.$$

Here $\Omega_{\hat{I}_1}$, $\Omega_{\hat{I}_2}$, $\Omega_{\hat{I}_3}$ are the Kähler forms of the hyper-Kähler structure $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_3$ on the cone (we use the convention $\hat{I}_1\hat{I}_2 = -\hat{I}_3$). Thus

$$\sigma = \partial_r \lrcorner \left(\frac{1}{2} \sum_\lambda s_{\lambda\lambda} \Omega_{\hat{I}_\lambda} \wedge \Omega_{\hat{I}_\lambda} + \sum_{\lambda<\mu} s_{\lambda\mu} \Omega_{\hat{I}_\lambda} \wedge \Omega_{\hat{I}_\mu} \right)\Bigg|_{r=1}.$$

Equivalently, this can be written as

$$(4.29) \qquad \sigma = \sum_{\lambda,\mu=1}^{3} s_{\lambda\mu} \eta_\lambda \wedge \Omega_\mu$$

with $s_{\lambda\mu} = s_{\mu\lambda}$, where η_1, η_2, η_3 are the contact forms of the 3-Sasakian structure on M and $\Omega_\lambda = \nabla\eta_\lambda$ are the corresponding Kähler forms. Again a straightforward computation using (2.16) shows that σ given by (4.29) induces the given metric and orientation if and only if the matrix $S = (s_{\lambda\mu})$ is in SO(3) and $\operatorname{tr} S = -1$. The condition $s_{\lambda\mu} = s_{\mu\lambda}$ means furthermore that σ is nearly parallel if and only if S is symmetric. An orthogonal matrix is symmetric if and only if its eigenvalues are real and the condition $\operatorname{tr} S = -1$ implies that they are $1, -1, -1$. But an orthogonal matrix with eigenvalues $1, -1, -1$ is completely determined by its 1-eigenspace. Thus we obtain that the nearly parallel G₂-structures are parametrized by $\mathbb{R}P^2$ (in particular, they are of type 3). We shall identify $\mathbb{R}^3$ with $\operatorname{span}\{\eta_1, \eta_2, \eta_3\}$. Then $\eta \in \operatorname{span}\{\eta_1, \eta_2, \eta_3\}$ is the contact form of a Saskai-Einstein structure if and only if η lies on the unit sphere S^2. Let $S(\eta) = (s_{\lambda\mu}(\eta))$ denote the orthogonal matrix with eigenvalues $1, -1, -1$ whose 1-eigenspace is spanned by η. Then the nearly parallel G₂-structures are

$$\left\{ \sigma_{S(\eta)} = \sum_{\lambda,\mu} s_{\lambda\mu}(\eta) \eta_\lambda \wedge \Omega_\mu \mid \eta \in S^2 \right\}$$

(notice that $S(\eta) = S(-\eta)$).

Fixing an η, we can again write the S^1-family $\sigma_{\eta,t}$ from the SU(4)-case. Inside the $\mathbb{R}P^2$-family it is identified by

$$\{\sigma_{\eta,t}\} = \{\sigma_{S(\eta')} \mid \eta' \in S^2, \ \eta' \perp \eta\}.$$

This follows from the fact that $\Psi_{\hat{I}_1} = \frac{1}{2}(\Omega_{\hat{I}_2} - i\Omega_{\hat{I}_3}) \wedge (\Omega_{\hat{I}_2} - i\Omega_{\hat{I}_3})$, i.e.,

$$\operatorname{Re}\Psi_1 = \eta_2 \wedge \Omega_2 - \eta_3 \wedge \Omega_3, \quad \operatorname{Im}\Psi_1 = -\eta_2 \wedge \Omega_3 - \eta_3 \wedge \Omega_2.$$

Finally, let the holonomy group $Hol(\hat{M})$ of the cone lie strictly between SU(4) and Sp(2). Then the restricted holonomy group is Sp(2) and therefore $Hol(\hat{M}) \subset \mathrm{Sp}(2)\mathrm{Sp}(1)$ as the normalizer of Sp(2) in $O(8)$ is Sp(2)Sp(1). Now the fact that $Hol(\hat{M})$ preserves a complex structure implies $Hol(\hat{M}) \subset \mathrm{Sp}(2)\mathrm{U}(1)$. Finally, $Hol(\hat{M})$ preserves a complex volume form, so the $U(1)$ part of $Hol(\hat{M})$ is contained in

$$\{a \in U(1) \mid a^4 = 1\} = \{1, i, -1, -i\} = \mathbb{Z}_4.$$

Since $\mathrm{Sp}(2)\mathbb{Z}_2 = \mathrm{Sp}(2)$, it remains $Hol(\hat{M}) = \mathrm{Sp}(2)\mathbb{Z}_4$. Now we have to find which Sp(2)-invariant 4-forms are also Sp(2)$\mathbb{Z}_4$-invariant. Notice that the action of $i \in \mathbb{Z}_4$ is in fact the

complex structure $\hat{I} = \hat{I}_1$. Since $\hat{I}_1$ acts on $\Omega_{\hat{I}_1}$ as the identity and on $\Omega_{\hat{I}_2}$ and $\Omega_{\hat{I}_3}$ as minus identity, the space of $\mathrm{Sp}(2)\mathbb{Z}_4$-invariant 4-forms is 4-dimensional and is spanned by

$$\Omega_{\hat{I}_1} \wedge \Omega_{\hat{I}_1}, \quad \Omega_{\hat{I}_2} \wedge \Omega_{\hat{I}_2}, \quad \Omega_{\hat{I}_3} \wedge \Omega_{\hat{I}_3}, \quad \Omega_{\hat{I}_2} \wedge \Omega_{\hat{I}_3}.$$

Now the results of the $\mathrm{Sp}(2)$-case imply that σ is given by (4.29) with $s_{12} = s_{21} = s_{13} = s_{31} = 0$. Thus either $s_{11} = 1$ and

$$\sigma = \sigma_{S(\eta_1)} = \eta_1 \wedge \Omega_1 - \eta_2 \wedge \Omega_2 - \eta_3 \wedge \Omega_3$$

or $s_{11} = -1$ and $\sigma = \sigma_{S(\eta')}$ for some η' orthogonal to $\eta = \eta_1$, i.e.,

$$\sigma = \sigma_{\eta,t} = -\eta \wedge \Omega + \cos t \operatorname{Re} \Psi + \sin t \operatorname{Im} \Psi.$$

Thus in this case we have nearly parallel G_2-structures of different types sharing the same metric: $\sigma_{S(\eta_1)}$ is of type 1, while $\sigma_{\eta,t}$ are of type 2.

Now we can prove the main result of this section.

Theorem 4.2. *Let (M, σ) be a compact nearly parallel G_2-manifold which is normalized so that $\tau_0 = 4$ and is not a space of constant curvature. Then:*

(1) *(M, σ) is of type 1 if and only if $\dim D_1 = 0$.*
(2) *(M, σ) is of type 2 if and only if $\dim D_1 = 1$.*
(3) *(M, σ) is of type 3 if and only if $\dim D_1 = 2$.*

Proof: Suppose that (M, σ) is of type 2. Then the holonomy of the cone is $\mathrm{SU}(4)$ or $\mathrm{Sp}(2)\mathbb{Z}_4 \subset \mathrm{SU}(4)$ and the consideration above show that σ is σ_t from (4.28) for some t. Again by Proposition 4.1 we have $\dim D_1 \leq 1$. By definition, the contact form η of the Sasakian structure satisfies $\nabla \eta = \Omega$. On the other hand, $\eta \lrcorner \sigma_t = -\Omega$ and therefore $\nabla \eta = -\frac{1}{4}\tau_0 \eta \lrcorner \sigma_t$. Thus $\eta \in D_1$, $D_1 = \operatorname{span}\{\eta\}$ and $\dim D_1 = 1$.

Let (M, σ) be of type 3. Then M is 3-Sasakian and the holonomy of the cone is $\mathrm{Sp}(2)$, so $\sigma = \sigma_{S(\eta)}$ for some $\eta \in S^2$. We shall show that D_1 is the orthogonal complement of η in $\operatorname{span}\{\eta_1, \eta_2, \eta_3\}$. Without loss of generality we can assume that $\eta = \eta_1$ (otherwise we shall change the orthonormal frame η_1, η_2, η_3). Then $\sigma \in \{\sigma_{\eta_2,t}\}$ and $\sigma \in \{\sigma_{\eta_3,t}\}$, so as above $\eta_2, \eta_3 \in D_1$ and therefore $\dim D_1 \geq 2$. By Proposition 4.1 every element of D_1 induces a Sasakian structure on (M, g) and by Lemma 8.1.17 in [4] it lies in $\operatorname{span}\{\eta_1, \eta_2, \eta_3\}$. Thus, if we assume that $\dim D_1 \geq 3$, we must have $D_1 = \operatorname{span}\{\eta_1, \eta_2, \eta_3\}$. But $\nabla \eta_1 = \Omega_1$, while $S(\eta_1)$ is the diagonal matrix with diagonal elements $1, -1, -1$ and

$$-\frac{1}{4}\tau_0 \eta_1 \lrcorner \sigma_{S(\eta_1)} = -\eta_1 \lrcorner (\eta_1 \wedge \Omega_1 - \eta_2 \wedge \Omega_2 - \eta_3 \wedge \Omega_3) = -\Omega_1 - \eta_2 \wedge \eta_1 \lrcorner \Omega_2 - \eta_3 \wedge \eta_1 \lrcorner \Omega_3$$

$$= -\Omega_1 - 2\eta_2 \wedge \eta_3 \neq \Omega_1.$$

Hence $\eta_1 \notin D_1$ and we have $D_1 = \operatorname{span}\{\eta_2, \eta_3\} = \eta_1^\perp$ and $\dim D_1 = 2$. This proves (3) since the reverse implication follows from Proposition 4.1.

Suppose now that $\dim D_1 = 1$. Then, by Proposition 4.1, (M, g) is Sasaki-Einstein but not 3-Sasakian. Thus, to prove the reverse implication of (2) we only have to show that the case $\mathrm{Hol}(\hat{M}) = \mathrm{Sp}(2)\mathbb{Z}_4$ with $\sigma = \sigma_{S(\eta_1)}$ is impossible. Indeed, the proof of Proposition 4.1 yields

that the contact form η_1 must be an infinitesimal deformation of $\sigma_{S(\eta_1)}$ but in the type 3 case above we saw that this is not true. This completes the proof of (2).

Finally, (1) follows from Proposition 4.1, (2) and (3). □

Remark 4.3. A nearly parallel G$_2$-structure of type 2 is a part of a whole curve σ_t of such structures. It is easy to see that $\frac{d\sigma_t}{dt} = *(\eta \wedge \sigma_t)$ (η is the contact form of the Sasaki-Einstein structure). Let ξ be the vector field dual to η. Since all σ_t have $\tau_0 = 4$ and $D_1 = \mathrm{span}\{\eta\}$, we obtain from Proposition 3.4 and Proposition 2.3 that

$$L_\xi \sigma_t = d(\xi \lrcorner \sigma_t) + \xi \lrcorner d\sigma_t = -\frac{1}{2} d^2\eta + 4\xi \lrcorner * \sigma_t = -4 * (\eta \wedge \sigma_t).$$

Let φ_s be the flow of ξ. Since ξ is Killing, φ_s preserves the metric and thus also η and $*$. Now the fact $L_\xi \sigma_t = - \left.\frac{d\varphi_s(\sigma_t)}{ds}\right|_{s=0}$ and the above equations imply

$$\frac{d\varphi_s(\sigma_t)}{ds} = 4 * (\eta \wedge \varphi_s(\sigma_t)) = \frac{d\sigma_{t+4s}}{ds}.$$

This and $\varphi_0(\sigma_t) = \sigma_t$ show that $\varphi_s(\sigma_t) = \sigma_{t+4s}$ for all s. Thus the flow of ξ acts transitively on the family $\{\sigma_t\}$ and so the members of this family are equivalent G$_2$-structures.

In a similar way, if the type is 3, one can generate the whole D_1 through curves in the $\mathbb{R}P^2$-family $\{\sigma_{S(\eta)}\}$. But this family consists of equivalent G$_2$-structures since a 3-Sasakian manifold admits an isometric SO(3) or Sp(1) action which is transitive on the oriented orthonormal frames (η_1, η_2, η_3) and therefore transitive also on the family $\{\sigma_{S(\eta)}\}$.

Thus, whatever the type of the nearly parallel G$_2$-structure, the "interesting" infinitesimal deformations are in the space D_3.

Remark 4.4. We have seen above that if the holonomy group of the cone $\hat{M}$ is Sp(2)$\mathbb{Z}_4$, then M has nearly parallel G$_2$-structures of different type sharing the same metric and orientation. This is possible because they induce different spin structures on M and therefore also on $\hat{M}$. Indeed, Sp(2)$\mathbb{Z}_4$ has two different embeddings in Spin(8). The first one, i_1, is the restriction on Sp(2)$\mathbb{Z}_4$ of the embedding of SU(4) in Spin(8). The second, i_2, is equal to i_1 on the identity component of Sp(2)$\mathbb{Z}_4$ and to $-i_1$ on the other component, i.e.,

$$i_2([a, 1]) = i_1([a, 1]), \quad i_2([a, i]) = -i_1([a, i]) \quad \text{for } a \in \mathrm{Sp}(2).$$

Let $E \cong \mathbb{C}^4$ be the standard representation of Sp(2). Then the spin representation, restricted to Sp(2), is isomorphic to $\sum_{p=0}^4 \Lambda^p E$. The action of $i_1(\mathrm{Sp}(2)\mathbb{Z}_4)$ is given by

$$i_1([a, z])\alpha = z^p a\alpha \quad \text{for } \alpha \in \Lambda^p E$$

and the space of invariant spinors is 2-dimensional: $\Lambda^0 E \oplus \Lambda^4 E$. On the other hand, the action of $i_2(\mathrm{Sp}(2)\mathbb{Z}_4)$ is

$$i_2([a, 1])\alpha = a\alpha, \quad i_2([a, i])\alpha = -i^p a\alpha \quad \text{for } \alpha \in \Lambda^p E$$

and the space of invariant spinors is 1-dimensional: $\mathbb{C}\sigma_E \subset \Lambda^2 E = \mathbb{C}\sigma_E \oplus \Lambda_0^2 E$, where σ_E is the Sp(2)-invariant symplectic form. Thus an 8-dimensional manifold with holonomy group Sp(2)$\mathbb{Z}_4$ is equipped with two canonical spin structures, one of which carries $N = 2$ and the

other $N = 1$ parallel spinors. Similarly, a 7-dimensional manifold whose cone has holonomy group $\mathrm{Sp}(2)\mathbb{Z}_4$ has two spin structures, with $N = 2$ and $N = 1$ real Killing spinors respectively. This adds to the results in [21], where in part 2b of Theorem 4.1 $N = 1$ is given as the only possibility, while the group $\mathrm{Sp}(2)\mathbb{Z}_4$ is completely missing in part 3 of Corollary 5.2. Notice that the existence of 7-dimensional manifolds with cones having holonomy group $\mathrm{Sp}(2)\mathbb{Z}_4$ has been proved in [17].

5. The G_2-Laplace operator

In Section 3 we have seen that infinitesimal deformations of nearly parallel G_2-manifolds give rise to coclosed eigenforms of the Hodge-de Rham Laplacian acting on sections of $\Lambda^3_{27}\mathrm{T}^*M$. By the classical Weitzenböck formula the Hodge-de Rham Laplacian is written as

$$(5.30) \qquad \Delta = d^*d + dd^* = \nabla^*\nabla + q(R),$$

where $q(R)$ is an endomorphism of the form bundle, which is linear in the curvature R and satisfies $q(R) = \mathrm{Ric}$ on the space of 1-forms. We will define the operator $q(R)$ in the following more general setting. Let (M, g) be an n-dimensional Riemannian manifold. For a representation V of $\mathrm{O}(n)$ let VM denote the corresponding associated vector bundle. We denote the action of $\alpha \in \Lambda^2\mathrm{T}^* \cong \mathfrak{so}(n)$ on V by α_* (here T denotes the standard representation $\mathbb{R}^n$ of $\mathrm{O}(n)$) and in a similar way the action of $\alpha \in \Lambda^2\mathrm{T}^*_xM$ on V_xM, $x \in M$. The endomorphism $q(A) \in \mathrm{End}\,(VM)$ is defined for any $A \in \Lambda^2\mathrm{T}^*M \otimes \mathrm{End}\,(VM)$ by

$$(5.31) \qquad q(A) = \sum_{i<j} (e_i \wedge e_j)_* A(e_i \wedge e_j),$$

where $\{e_i\}$ is a local orthonormal frame of $\mathrm{T}M$. Notice that in this definition $\{e_i \wedge e_j \mid i < j\}$ could be replaced by any other orthonormal basis of $\Lambda^2\mathrm{T}M$. The curvature R of the Levi-Civita connection ∇ or, more generally, the curvature $\bar{R}$ of any metric connection $\bar{\nabla}$ on (M, g) defines a section of $\Lambda^2\mathrm{T}^*M \otimes \mathrm{End}\,(VM)$, thus the endomorphisms $q(R)$ and $q(\bar{R})$ are well defined. We denote by $\bar{\Delta}$ the Laplace type operator

$$(5.32) \qquad \bar{\Delta} := \bar{\nabla}^*\bar{\nabla} + q(\bar{R}).$$

The operator $\Delta_L := \nabla^*\nabla + q(R)$ for the Levi-Civita connection ∇ and a subrepresentation $V \subset \otimes^p \mathrm{T}$ is also called *Lichnerowicz Laplacian* (cf. [3], Chapter 1 I). Because of (5.30) it coincides on differential forms with the Hodge deRham Laplacian Δ.

Now let us return to the case of nearly parallel G_2-manifolds. We will call the operator $\bar{\Delta}$, defined with the canonical connection $\bar{\nabla}$, *the G_2-Laplace operator*. In order to compute the spectrum of the Lichnerowicz Laplacian Δ_L on naturally reductive spaces, it turns out to be convenient to express Δ_L through $\bar{\Delta}$. Thus, our next aim will be to compute the difference $\bar{\Delta} - \Delta_L$, which we do by calculating the differences $\bar{\nabla}^*\bar{\nabla} - \nabla^*\nabla$ and $q(\bar{R}) - q(R)$ separately.

A direct calculation using (2.18) and the third equation of Lemma 2.1 gives

$$\bar{R}_{X,Y}Z - R_{X,Y}Z = (\tfrac{\tau_0}{12})^2 \left[2P(P(X,Y),Z) + P(P(Y,Z),X) + P(P(Z,X),Y)\right]$$
$$= (\tfrac{\tau_0}{12})^2 \left[4P(P(X,Y),Z) - 3g(X,Z)Y + 3g(Y,Z)X\right].$$

Thus $\bar{R}(X \wedge Y) - R(X \wedge Y) = (\tfrac{\tau_0}{12})^2 [4P_{P(X \wedge Y)} - 3(X \wedge Y)_*]$. Substituting this equation into the definition of the curvature endomorphisms $q(R)$ and $q(\bar{R})$, we obtain

$$(5.33) \qquad q(\bar{R}) - q(R) = -3(\tfrac{\tau_0}{12})^2 \operatorname{Cas}^{\mathfrak{so}(7)} + 4(\tfrac{\tau_0}{12})^2 S$$

where $\operatorname{Cas}^{\mathfrak{so}(n)}$ is the $\mathfrak{so}(n)$-Casimir operator $\sum_{i<j}(e_i \wedge e_j)_*(e_i \wedge e_j)_*$ and S is defined as

$$(5.34) \qquad S = \sum_{i<j}(e_i \wedge e_j)_* P_{P(e_i \wedge e_j)}.$$

Since $P : \Lambda^2 T \cong \Lambda^2_7 \oplus \Lambda^2_{14} \to T$ is a G_2-equivariant map, $P|_{\Lambda^2_{14}} = 0$ and we may replace in the sum in (5.34) the orthonormal basis $\{e_i \wedge e_j \mid i < j\}$ of $\Lambda^2 T$ with the orthonormal basis $\{f_i = \frac{1}{\sqrt{3}} e_i \lrcorner \sigma \mid i = 1, \ldots, 7\}$ of Λ^2_7. Because obviously $f_{i*} = \frac{1}{\sqrt{3}}(e_i \lrcorner \sigma)_* = \frac{1}{\sqrt{3}} P_{e_i}$ and, by (2.15), $P(f_i) = \sqrt{3}\, e_i$, we obtain $S = \sum f_{i*} P_{P(f_i)} = \sum P_{e_i} P_{e_i}$. For the difference $\bar{\nabla}^* \bar{\nabla} - \nabla^* \nabla$ of the two rough Laplacians we derive directly from (2.18)

$$\bar{\nabla}^* \bar{\nabla} - \nabla^* \nabla = \sum \left(\tfrac{\tau_0}{6} P_{e_i} \bar{\nabla}_{e_i} + (\tfrac{\tau_0}{12})^2 P_{e_i} P_{e_i}\right).$$

Summarizing these calculation we obtain an expression for the difference of $\bar{\Delta}$ and Δ_L.

Proposition 5.1. *The difference of the Laplace type operators $\bar{\Delta}$ and Δ_L on a nearly parallel G_2-manifold is given by*

$$(5.35) \qquad \bar{\Delta} - \Delta_L = \tfrac{\tau_0}{6} \sum P_{e_i} \bar{\nabla}_{e_i} - 3(\tfrac{\tau_0}{12})^2 \operatorname{Cas}^{\mathfrak{so}(7)} + 5(\tfrac{\tau_0}{12})^2 \sum P_{e_i} P_{e_i}.$$

We shall apply this result for the space Λ^3_{27}. Recalling that the $\mathfrak{so}(n)$-Casimir operator acts as $-p(n-p)\mathrm{id}$ on the space of p-forms, we obtain $\operatorname{Cas}^{\mathfrak{so}(7)} \gamma = -12\gamma$ for $\gamma \in \Lambda^3_{27}$. A straightforward computation on an explicit element (e.g. the element from (2.4)) shows that the G_2-equivariant map $\sum P_{e_i} P_{e_i}$ acts as $-8\mathrm{id}$ on Λ^3_{27}. Thus it remains to compute $\sum P_{e_i} \bar{\nabla}_{e_i}$. The map $\sum P_{e_i} \circ e_i \lrcorner : T^* \otimes \Lambda^3_{27} \to \Lambda^3$ is G_2-equivariant. Hence, because of (2.3) and (2.6), it can be non-zero only on the components of $T^* \otimes \Lambda^3_{27}$ which are isomorphic to $V_{1,0} \cong T$ and $V_{2,0} \cong \Lambda^3_{27}$. A straightforward computation on explicit elements shows that $\sum P_{e_i} \circ e_i \lrcorner$ is $-3 * \circ \varepsilon$ on the component $T \subset T^* \otimes \Lambda^3_{27}$ and $* \circ \varepsilon$ on the component $\Lambda^3_{27} \subset T^* \otimes \Lambda^3_{27}$. Here again $\varepsilon : T^* \otimes \Lambda^3 \to \Lambda^4$ denotes the wedging map. Thus

$$\sum P_{e_i} \bar{\nabla}_{e_i} \gamma = \sum P_{e_i} \circ e_i \lrcorner \bar{\nabla}\gamma = -3 * \varepsilon(\bar{\nabla}\gamma)_T + *\varepsilon(\bar{\nabla}\gamma)_{\Lambda^3_{27}} = -3 * (\bar{d}\gamma)_{\Lambda^4_7} + *(\bar{d}\gamma)_{\Lambda^4_{27}},$$

where $\bar{d} := \varepsilon \circ \bar{\nabla}$. Now, by (2.18) we have $\bar{d} = d - \tfrac{\tau_0}{12} \sum e_i \wedge P_{e_i}$. Again a simple computation on an element of Λ^3_{27} shows that $\sum e_i \wedge P_{e_i} = -2*$ on Λ^3_{27} and we eventually obtain

Lemma 5.2. *On sections of $\Lambda^3_{27} T^* M$ it holds that $\bar{d} = d + \tfrac{\tau_0}{6} *$. In particular, we have*

$$(\bar{d}\gamma)_{\Lambda^4_7} = (d\gamma)_{\Lambda^4_7}, \quad (\bar{d}\gamma)_{\Lambda^4_{27}} = (d\gamma)_{\Lambda^4_{27}} + \tfrac{\tau_0}{6} * \gamma, \quad (d\gamma)_{\Lambda^4_1} = (\bar{d}\gamma)_{\Lambda^4_1} = 0 \quad \text{for } \gamma \in \Omega^3_{27}(M).$$

Using this lemma we obtain $\sum P_{e_i} \bar{\nabla}_{e_i} \gamma = -3 * (d\gamma)_{\Lambda_7^4} + *(d\gamma)_{\Lambda_{27}^4} + \frac{\tau_0}{6} \gamma$, which finally enables us to compute the difference $\bar{\Delta} - \Delta$ on $\Omega_{27}^3(M)$. Combining the formulas above we find

Proposition 5.3. *Let (M^7, g, σ) be a nearly parallel* G_2*-manifold and let* γ *be a 3-form in* $\Omega_{27}^3(M)$. *Then*

$$\bar{\Delta}\gamma = \Delta\gamma - \tfrac{\tau_0}{2} * (d\gamma)_{\Lambda_7^4} + \tfrac{\tau_0}{6} * (d\gamma)_{\Lambda_{27}^4}.$$

In particular, Δ *and* $\bar{\Delta}$ *coincide on closed forms in* $\Omega_{27}^3(M)$. *Moreover, if* γ *is a 3-form in* $\Omega_{27}^3(M)$ *with* $(d^*\gamma)_{\Lambda_7^2} = 0$, *then* $\bar{\Delta}\gamma = \Delta\gamma + \frac{\tau_0}{6} * d\gamma$.

Proof: It only remains to prove the last statement: Recall from the proof of Lemma 3.3, that for $\gamma \in \Omega_{27}^3(M)$ the condition $(d^*\gamma)_{\Lambda_7^2} = 0$ is equivalent to $(d\gamma)_{\Lambda_7^4} = 0$ and thus, by Lemma 5.2, also to $(d\gamma)_{\Lambda_{27}^4} = d\gamma$. Substituting this into the equation for $\bar{\Delta}$ implies the last statement.   □

6. Infinitesimal Einstein deformations

Nearly parallel G_2-structures induce Einstein metrics and thus infinitesimal deformations of such structures are related to infinitesimal Einstein deformations. In this section we shall consider the space of infinitesimal Einstein deformations of a given nearly parallel G_2-metric and realize it as a direct sum of certain spaces of 3-forms in $\Omega_{27}^3(M)$.

Let g be an Einstein metric with Ric $= Eg$. From [3], Theorem 12.30, the space of infinitesimal Einstein deformations of g is isomorphic to the set of trace-free symmetric bilinear forms h on TM with $\delta h = 0$ and $\Delta_L h = 2Eh$, where $\Delta_L = \nabla^*\nabla + q(R)$ is the so-called Lichnerowicz Laplacian (see the previous section). Note that for a nearly parallel G_2-metric the eigenvalue can be written as $2E = \frac{2\mathrm{scal}}{7} = \frac{3\tau_0^2}{4}$.

As a G_2-representation the space $S_0^2 T^*$ is isomorphic to $\Lambda_{27}^3 T$. We shall now use the explicit identification $\mathbf{i}$ in order to identify infinitesimal Einstein deformations with certain eigenforms of the Laplacian on forms in $\Lambda_{27}^3 T$. To do this we still need an analogue of Proposition 5.3.

We apply the results of Proposition 5.1 to the space $S_0^2 T^*$. It is well known that the $\mathfrak{so}(n)$-Casimir operator acts on $S_0^2 T^*$ as $-2n\mathrm{Id}$, i.e., as $-14\mathrm{Id}$ in our case. Moreover it is clear that similarly $\sum P_{e_i} P_{e_i}$, as a G_2-equivariant map, acts as a multiple of the identity. An explicit calculation, e.g. on the element $e^1 \odot e^2$, shows that $\sum P_{e_i} P_{e_i} = -14\mathrm{Id}$.

It remains to determine $\sum P_{e_i} \bar{\nabla}_{e_i} h$, i.e., $Q(\bar{\nabla} h)$, where $Q : T^* \otimes S_0^2 T^* \to S_0^2 T^*$ is the G_2-equivariant map defined as $Q = \sum P_{e_i} \circ e_{i\lrcorner}$. The map Q is different from zero only on the component of $T^* \otimes S_0^2 T^*$, which is isomorphic to $S_0^2 T^*$. Let $i_2 : S_0^2 T^* \to T^* \otimes S_0^2 T^*$ be the embedding given as $i_2(h) = (1 \otimes \pi_0) \circ C(g \otimes h)$, where g is the metric, $C : T^{*\otimes 4} \to T^{*\otimes 3}$ is defined by $C(a \otimes b \otimes c \otimes d) = a \otimes P(b, c) \otimes d$ and $\pi_0 : T^* \otimes T^* \to S_0^2 T^*$ denotes the standard projection. Moreover, let $\pi_2 : T^* \otimes S_0^2 T^* \to S_0^2 T^*$ be the projection "inverse" to i_2, i.e. $\pi_2 \circ i_2 = \mathrm{id}$ and π_2 vanishes on the components of $T^* \otimes S_0^2 T^*$ that are not isomorphic to $S_0^2 T^*$. Then an explicit calculation, e.g. on $e_1 \odot e_2$, shows that $Q \circ i_2 = -7\mathrm{id}$ and thus

$Q = -7\pi_2$. Substituting this and the results for $\mathrm{Cas}^{\,\mathfrak{so}(7)}$ and $\sum P_{e_i} P_{e_i}$ into equation (5.35), we obtain

$$(6.36) \qquad (\bar{\Delta} - \Delta_L)h \;=\; -\tfrac{7\tau_0}{6}\,\pi_2(\bar{\nabla}h) \;-\; \tfrac{7\tau_0^2}{36}\,h.$$

Since $S_0^2 \mathrm{T}^*$ and Λ_{27}^3 are isomorphic representations of G$_2$ and $\bar{\nabla}$ is a G$_2$-connection, the bundles $S_0^2 \mathrm{T}^* M$ and $\Lambda_{27}^3 M$ share the same G$_2$-Laplace operator $\bar{\Delta}$, i.e., with the G$_2$-equivariant isomorphism $\mathbf{i} : S_0^2 \mathrm{T}^* \to \Lambda_{27}^3$ we have $\mathbf{i} \circ \bar{\Delta} \circ \mathbf{i}^{-1} = \bar{\Delta}$. Hence, to compute $\mathbf{i} \circ \Delta_L \circ \mathbf{i}^{-1}$ we need to compute $\mathbf{i} \circ \pi_2 \circ \bar{\nabla} \circ \mathbf{i}^{-1}$. An easy calculation shows that $\mathbf{i} = \pi_1 \circ (1 \otimes \mathbf{i}) \circ i_2$, where $\pi_1 : \mathrm{T}^* \otimes \Lambda_{27}^3 \to \Lambda_{27}^3$ is defined as $\pi_1(\alpha \otimes \gamma) = \tfrac{2}{7} * (\alpha \wedge \gamma)_{\Lambda_{27}^4}$. The map $i_2 \circ \pi_2$ is the projection on the component isomorphic to $S_0^2 \mathrm{T}^*$ in $\mathrm{T}^* \otimes S_0^2 \mathrm{T}^*$ and since $\pi_1 \circ (1 \otimes \mathbf{i})$ is invariant with values in $S_0^2 \mathrm{T}^*$, it vanishes on all other components of $\mathrm{T}^* \otimes S_0^2 \mathrm{T}^*$. Hence $\pi_1 \circ (1 \otimes \mathbf{i}) = \mathbf{i} \circ \pi_2$ and we obtain from Lemma 5.2

$$(6.37) \quad \mathbf{i} \circ \pi_2(\bar{\nabla}h) = \pi_1 \circ (1 \otimes \mathbf{i})\bar{\nabla}h = \pi_1 \bar{\nabla}(\mathbf{i}(h)) = \tfrac{2}{7} * (\bar{d}\,\mathbf{i}(h))_{\Lambda_{27}^4} = \tfrac{2}{7} * (d\,\mathbf{i}(h))_{\Lambda_{27}^4} + \tfrac{\tau_0}{21}\mathbf{i}(h).$$

Let h be an infinitesimal Einstein deformation and let $\gamma \in \Omega_{27}^3(M)$ denote the 3-form $\mathbf{i}(h)$. Then the condition $\delta h = 0$ translates into $(d^*\gamma)_{\Lambda_7^2} = 0$ or, equivalently, to $d\gamma = (d\gamma)_{\Lambda_{27}^4}$. Indeed, by Remark 2.4 we have that $(\nabla h)_{\mathrm{T}} = (\bar{\nabla}h)_{\mathrm{T}}$ and $(\nabla \gamma)_{\mathrm{T}} = (\bar{\nabla}\gamma)_{\mathrm{T}}$. Thus $\delta h = 0$ is equivalent to $(\nabla h)_{\mathrm{T}} = 0$ and also to $(\bar{\nabla}h)_{\mathrm{T}} = 0$. But since $\mathbf{i}$ is an G$_2$-equivariant map, $(\bar{\nabla}h)_{\mathrm{T}} = 0$ if and only if $(\bar{\nabla}\gamma)_{\mathrm{T}} = 0$, i.e., $(\nabla \gamma)_{\mathrm{T}} = 0$. However this is equivalent to $(d^*\gamma)_{\Lambda_7^2} = 0$ and also to $(d\gamma)_{\Lambda_7^4} = 0$. Then by Lemma 5.2 $(d\gamma)_{\Lambda_7^4} = 0$ can be written as $d\gamma = (d\gamma)_{\Lambda_{27}^4}$.

Finally, we apply $\mathbf{i}$ to (6.36), use Proposition 5.3 and substitute (6.37) to obtain

Proposition 6.1. *For each $\gamma \in \Omega_{27}^3(M)$ the following equation is satisfied:*

$$(6.38) \qquad \mathbf{i}\,\Delta_L\,\mathbf{i}^{-1}(\gamma) \;=\; \Delta\gamma \;-\; \tfrac{\tau_0}{2} * (d\gamma)_{\Lambda_7^4} \;+\; \tfrac{\tau_0}{2} * (d\gamma)_{\Lambda_{27}^4} \;+\; \tfrac{\tau_0^2}{4}\gamma.$$

With this formula we are able to translate the conditions for infinitesimal Einstein deformations into equivalent conditions for 3-forms in $\Omega_{27}^3(M)$: The traceless symmetric bilinear form h is an infinitesimal Einstein deformation if and only if $\gamma = \mathbf{i}(h)$ is a section of $\Lambda_{27}^3 \mathrm{T}^* M$ with $(d\gamma)_{\Lambda_7^4} = 0$ (or, equivalently, $(d^*\gamma)_{\Lambda_7^2} = 0$), satisfying the equation

$$(6.39) \qquad \Delta\gamma \;+\; \tfrac{\tau_0}{2} * d\gamma \;-\; \tfrac{\tau_0^2}{2}\gamma \;=\; 0.$$

We want to decompose the solution space of this equation into eigenspaces of the operator $*d$. This is possible since $*d$ is a symmetric operator, commuting with the operator on the left hand side of equation (6.39) and preserving the condition $(d^*\gamma)_{\Lambda_7^2} = 0$. Indeed, $*d\alpha$ is coclosed for any differential form α. Moreover, the solution space is finite dimensional because it is the kernel of an elliptic operator. Assume that $*d\gamma = \lambda\gamma$ with $\lambda \neq 0$. Then γ is coclosed (in particular, $(d^*\gamma)_{\Lambda_7^2} = 0$ and $(d\gamma)_{\Lambda_7^4} = 0$) and (6.39) yields the quadratic equation

$$(6.40) \qquad \lambda^2 \;+\; \tfrac{\tau_0}{2}\lambda \;-\; \tfrac{\tau_0^2}{2} \;=\; 0$$

with the solutions $\lambda = -\tau_0$ and $\lambda = \tfrac{\tau_0}{2}$. In the case $\lambda = 0$ we obtain $d\gamma = 0$ and $dd^*\gamma = \tfrac{\tau_0^2}{2}\gamma$. Moreover a solution γ of the last equation is automatically closed and thus $(d\gamma)_{\Lambda_7^4} = 0$ as well as $(d^*\gamma)_{\Lambda_7^2} = 0$. Summarizing we have

Theorem 6.2. *Let (M, σ, g) be a compact nearly parallel G_2-manifold. Then the space of infinitesimal Einstein deformations of g is isomorphic to the direct sum of the spaces*

$$\{\gamma \in \Omega^3_{27} \mid *d\gamma = -\tau_0\gamma\}, \quad \{\gamma \in \Omega^3_{27} \mid *d\gamma = \tfrac{\tau_0}{2}\gamma\}, \quad \{\gamma \in \Omega^3_{27} \mid dd^*\gamma = \tfrac{\tau_0^2}{2}\gamma\}.$$

Notice that the first space is the space D_3 from Theorem 3.5. Thus any element $f_3 \in D_3$ satisfies $\mathbf{i}\,\Delta_L\mathbf{i}^{-1}\,(f_3) = \frac{3\tau_0^2}{4}\,f_3$, or, equivalently, $\Delta_L\,\mathbf{j}(f_3) = \frac{3\tau_0^2}{4}\,\mathbf{j}(f_3)$, which finishes the proof of Theorem 3.5.

In order to check in the examples whether or not infinitesimal Einstein deformations exist it will be convenient to embed these three spaces into eigenspaces of the operator $\bar{\Delta}$ acting on sections of $\Lambda^3_{27}T^*M$. Let γ be a 3-form as above with $*d\gamma = \lambda\gamma$ for $\lambda \neq 0$. Then γ is coclosed and $\Delta\gamma = \lambda^2\gamma$. Thus Proposition 5.3 implies $\bar{\Delta}\gamma = (\lambda^2 + \frac{\tau_0}{6}\lambda)\gamma$. In the case $dd^*\gamma = \frac{\tau_0^2}{2}\gamma$ it follows that γ is closed and we obtain $\bar{\Delta}\gamma = \Delta\gamma = dd^*\gamma = \frac{\tau_0^2}{2}\gamma$. This proves

Lemma 6.3. *The three summands of Theorem 6.2 are contained in the eigenspaces of $\bar{\Delta}$ acting on $\Omega^3_{27}(M)$ for the eigenvalues $\frac{5\tau_0^2}{6}$, $\frac{\tau_0^2}{3}$ and $\frac{\tau_0^2}{2}$ respectively.*

7. NATURALLY REDUCTIVE SPACES

In this section we will make some general remarks which will help us to compute the infinitesimal Einstein deformations of nearly parallel G_2-manifolds that are naturally reductive homogenous spaces, i.e. reductive spaces where the torsion of the canonical homogenous connection can be considered as a 3-form.

Lemma 7.1. *Let G/H be a 7-dimensional oriented naturally reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Suppose that at the initial point o the torsion of the canonical homogeneous connection $\hat{\nabla}$ is $\hat{T}_o = -\frac{\tau_0}{6}\sigma_o$ with $\tau_0 \neq 0$ and that σ_o is stable and induces the given metric and orientation on $\mathfrak{m}$. Then σ_o defines by translations a G-invariant 3-form σ and thus a G_2-structure on G/H compatible with the given metric and orientation. This G_2-structure is nearly parallel and its canonical connection is $\bar{\nabla} = \hat{\nabla}$. In particular, $d\sigma = \tau_0 * \sigma$.*

Moreover if G/H is standard up to a scaling factor c^2, i.e., $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form B of $\mathfrak{g}$ and the metric is induced by the restriction of $-c^2B$ to $\mathfrak{m}$, then the scalar curvature is $\mathrm{scal} = \frac{63}{20c^2}$ and $\tau_0^2 = \frac{6}{5c^2}$.

Proof: Since $\sigma_o = -\frac{6}{\tau_0}\hat{T}_o$ is an H-invariant 3-form on $\mathfrak{m}$, $\sigma = -\frac{6}{\tau_0}\hat{T}$ is a G-invariant 3-form on G/H. In particular, σ is parallel with respect to the canonical homogeneous connection

$$\hat{\nabla} = \nabla + \tfrac{1}{2}\hat{T} = \nabla - \tfrac{\tau_0}{12}\sigma.$$

For $X \in \mathbb{R}^7$ we have the identity $P_X\sigma = 3X \lrcorner *\sigma$ (which follows from (2.12) and (2.13) or by an explicit computation for some $X \neq 0$). Thus

$$\nabla_X\sigma = \hat{\nabla}_X\sigma + \tfrac{\tau_0}{12}P_X\sigma = \tfrac{\tau_0}{4}X \lrcorner *\sigma$$

and Proposition 2.3 implies that the G$_2$-structure is nearly parallel with $d\sigma = \tau_0 * \sigma$. Moreover $\hat{\nabla}$ coincides with the canonical connection $\bar{\nabla}$ of the G$_2$-structure because of (2.18).

Suppose now that G/H is standard (up to a scaling factor c^2). Obviously, it is enough to prove the statement about the scalar curvature when $c = 1$. Recall that $\hat{T}_o(X, Y) = -[X, Y]_{\mathfrak{m}}$ for $X, Y \in \mathfrak{m}$. Then, considering again $\hat{T}_o$ as a 3-form and using (7.39) in [3], we obtain

$$\mathrm{scal} \;=\; -\tfrac{6}{4}\,|\hat{T}_o|^2 + \tfrac{7}{2} \;=\; -\tfrac{\tau_0^2}{24}\,|\sigma_o|^2 + \tfrac{7}{2}.$$

Since σ_o induces the metric on $\mathfrak{m}$ we have $|\sigma_o|^2 = 7$ and, using (3.20) to replace scal, the equation above yields $\tau_0^2 = \tfrac{6}{5}$ and therefore $\mathrm{scal} = \tfrac{21}{8}\tau_0^2 = \tfrac{63}{20}$. $\qquad\square$

In view of this lemma and the results of the previous section it will be useful to have an algebraic description of some differential operators on naturally reductive spaces.

Let $M = G/H$ be a reductive homogeneous space and ρ be a representation of H on a vector space V. Denote by $E := G \times_\rho V$ the associated vector bundle over M. If a G-invariant metric is fixed on M, then the canonical homogeneous connection $\hat{\nabla}$ is a metric connection and, as explained in Section 5, we can define the Laplace type operator $\hat{\Delta}_\rho = \hat{\nabla}^* \hat{\nabla} + q(\hat{R})$ acting on sections of E. With the same proof as for Lemma 5.2 in [19] we have

Lemma 7.2. *Let G be a compact semi-simple Lie group, $H \subset G$ a compact subgroup and let $M = G/H$ be standard (up to a scaling factor c^2). Then the endomorphism $q(\hat{R})$ acts fibrewise on E as $-\frac{1}{c^2}\mathrm{Cas}_\rho^H$ and the operator $\hat{\Delta}_\rho$ acts on $\Gamma(E)$, considered as a G-representation via the left-regular representation l, as $-\frac{1}{c^2}\mathrm{Cas}_l^G$, where the Casimir operator $\mathrm{Cas}_{V_\gamma}^G$ of a G-representation V_γ is defined with respect to the Killing form of G.*

Lemma 7.2 can be used to compute the spectrum of $\hat{\Delta}_\rho$. We recall that the Peter-Weyl theorem and the Frobenius reciprocity yield the following decomposition of the left-regular representation of G into irreducible summands:

$$(7.41) \qquad\qquad \Gamma(E) \;\cong\; \overline{\bigoplus} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, V),$$

where the sum is taken over the set of (non-isomorphic) irreducible G-representations V_γ, labeled by their highest weight γ. The Casimir operator acts on V_γ as a certain multiple of the identity, which can be computed explicitly by the Freudenthal formula. Hence the eigenspace of $\hat{\Delta}_\rho$ for the eigenvalue λ is isomorphic as a G-representation to the direct sum of the spaces $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, V)$ for which $\mathrm{Cas}_{V_\gamma}^G = -c^2\lambda$.

Corollary 7.3. *Let G/H be standard (up to a scaling factor c^2), satisfying the assumptions of Lemma 7.1. Then the eigenspaces of the G$_2$-Laplace operator $\bar{\Delta}$ on $\Omega_{27}^3(M)$ for the eigenvalues $\frac{5\tau_0^2}{6}$, $\frac{\tau_0^2}{3}$, $\frac{\tau_0^2}{2}$ are isomorphic as G-representations to the direct sum of spaces $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Lambda_{27}^3\mathfrak{m}^*)$, on which the Casimir operator $\mathrm{Cas}_{V_\gamma}^G$ acts as -1, $-\frac{2}{5}$, $-\frac{3}{5}$.*

In the examples below we have to solve equations of the form $\bar{d}\varphi + c * \varphi = 0$ for 3-forms φ on naturally reductive spaces $M = G/H$. Using the explicit embedding of $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, V)$ of (7.41) into $\Gamma(E)$ we will translate this into an algebraic equation.

As above let $E := G \times_\rho V$ be the vector bundle over $M = G/H$ associated to a representation $\rho : H \to \mathrm{Aut}(V)$. The space of H-equivariant functions from G to V, i.e., functions $f : G \to V$ with $f \circ R_h = \rho(h^{-1}) \circ f$ for all $h \in H$, can be identified with the space of the sections of E. Indeed, the section φ corresponding to the function f is given by $\varphi(\pi(a)) = a(f(a))$. Here $\pi : G \to G/H$ denotes the projection, $\pi(a) = aH$, and $a \in G$ is considered as a linear isomorphism from V to the fibre $E_{\pi(a)}$, defined on $v \in V$ as $a(v) := [a, v] \in E_{\pi(a)}$. Since G acts from the left on the space of H-equivariant functions from G to V by $a \cdot f := L_{a^{-1}}^* f = f \circ L_{a^{-1}}$, we obtain a left action of G on $\Gamma(E)$.

Let U be an irreducible G-representation. Then $U \otimes \mathrm{Hom}_H(U, V)$ embeds into $\Gamma(E)$ by

$$U \otimes \mathrm{Hom}_H(U, V) \ni \alpha \otimes A \mapsto f_\alpha^A, \quad \text{where} \quad f_\alpha^A : G \to V, \quad f_\alpha^A(a) = A(a^{-1}\alpha).$$

In particular, fixing $A \in \mathrm{Hom}_H(U, V)$ one obtains a G-equivariant homomorphism $U \to \Gamma(E)$, given by $U \ni \alpha \mapsto f_\alpha^A$. The meaning of (7.41) is that each G-equivariant homomorphism $U \to \Gamma(E)$ is obtained in this way. In other words, a subspace of $\Gamma(E)$ is isomorphic as a G-representation to U if and only if it coincides with the space $\{f_\alpha^A : \alpha \in U\}$ for some $A \in \mathrm{Hom}_H(U, V)$, $A \neq 0$.

Let $M = G/H$ be reductive with $\mathrm{Ad}\,(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $E = \Lambda^s T^* M$, i.e., the vector bundle associated to the H-representation $V = \Lambda^s \mathfrak{m}^*$. Then a straightforward computation shows that in this case $a \cdot \varphi = L_{a^{-1}}^* \varphi$ for $a \in G$ and $\varphi \in \Gamma(E)$. This means that if φ corresponds to the function f, then $L_a^* \varphi$ corresponds to the function $L_a^* f$. Let $\hat{\nabla}$ be the canonical homogeneous connection and consider the operator $\hat{d} = \varepsilon \circ \hat{\nabla} : \Gamma(\Lambda^s T^* M) \to \Gamma(\Lambda^{s+1} T^* M)$. Since $\hat{\nabla}$ is translation invariant, we have

$$(\hat{d}\varphi)_{\pi(a)} = L_{a^{-1}}^*((\hat{d}L_a^*\varphi)_{\pi(e)}).$$

For $(\hat{d}L_a^*\varphi)_{\pi(e)}$ we obtain the equation

$$(\hat{d}L_a^*\varphi)_{\pi(e)}(X_1, \ldots, X_{s+1}) = \sum_{i=1}^{s+1} (-1)^{i-1} dL_a^* f(X_i)(X_1, \ldots, \hat{X}_i, \ldots, X_{s+1})$$

for $X_1, \ldots, X_{s+1} \in \mathfrak{m} \cong T_{\pi(e)} M$. Let φ correspond to f_α^A. Then

$$dL_a^* f_\alpha^A(X) = (dA(b^{-1}a^{-1}\alpha))_{b=e}(X) = A((d(b^{-1}))_{b=e}(X) \cdot a^{-1}\alpha) = -A(X \cdot a^{-1}\alpha),$$

where $X \cdot \alpha$ denotes the action of $X \in \mathfrak{g}$ on $\alpha \in U$. Thus

$$(\hat{d}L_a^*\varphi)_{\pi(e)}(X_1, \ldots, X_{s+1}) = \sum_{i=1}^{s+1} (-1)^i A(X_i \cdot a^{-1}\alpha)(X_1, \ldots, \hat{X}_i, \ldots, X_{s+1})$$

for $X_1, \ldots, X_{s+1} \in \mathfrak{m} \cong T_{\pi(e)} M$. In a similar way one obtains

$$(dL_a^*\varphi)_{\pi(e)}(X_1, \ldots, X_{s+1}) = \sum_{i=1}^{s+1} (-1)^i A(X_i \cdot a^{-1}\alpha)(X_1, \ldots, \hat{X}_i, \ldots, X_{s+1})$$

$$+ \sum_{1 \leq i < j \leq s+1} (-1)^{i+j} A(a^{-1}\alpha)([X_i, X_j]_{\mathfrak{m}}, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{s+1})$$

for $X_1, \ldots, X_{s+1} \in \mathfrak{m} \cong T_{\pi(e)}M$. From these formulas one can compute $(\hat{d}\varphi)_{\pi(a)}$ and $(d\varphi)_{\pi(a)}$ for any $a \in G$.

Next we fix a G-invariant metric and an orientation on M. Then $*\varphi = L_{a^{-1}}^*((*L_a^*\varphi)_{\pi(e)})$. Therefore, if we would like to solve the G-invariant equation $\hat{d}\varphi + c*\varphi = 0$, for a certain constant c, it is enough to solve $(\hat{d}L_a^*\varphi)_{\pi(e)} + c*(L_a^*\varphi)_{\pi(e)} = 0$ for all $a \in G$. In fact, we shall be interested in subspaces of solutions of this equation, which are isomorphic to a given irreducible G-representation U. Thus we have to find $A \in \mathrm{Hom}_H(U, V)$ so that

$$\sum_{1 \le i_1 < \cdots < i_{s+1} \le n} \sum_{j=1}^{s+1} (-1)^i A(e_{i_j} \cdot a^{-1}\alpha)(e_{i_1}, \ldots, \hat{e}_{i_j}, \ldots, e_{i_{s+1}}) e^{i_1 \ldots i_{s+1}} + c * A(a^{-1}\alpha) = 0$$

for all $a \in G$, $\alpha \in U$. Here $e_1, \ldots, e_n$ is a basis of $\mathfrak{m}$. It is clear that it suffices to write $a = e$ in this equation, i.e., we are looking for $A \in \mathrm{Hom}_H(U, V)$ so that

$$(7.42) \qquad \sum_{1 \le i_1 < \cdots < i_{s+1} \le n} \sum_{j=1}^{s+1} (-1)^j A(e_{i_j} \cdot \alpha)(e_{i_1}, \ldots, \hat{e}_{i_j}, \ldots, e_{i_{s+1}}) e^{i_1 \ldots i_{s+1}} + c * A(\alpha) = 0$$

holds for all $\alpha \in U$. Notice that this equation is H-invariant.

8. EXAMPLES

In this section we shall compute the infinitesimal Einstein deformations of three examples of proper nearly parallel G$_2$-structures on standard homogeneous spaces (up to a factor).

The first example is $\mathrm{SO}(5)/\mathrm{SO}(3)$, where the embedding of $\mathrm{SO}(3)$ in $\mathrm{SO}(5)$ is given by the 5-dimensional irreducible representation of $\mathrm{SO}(3)$. This space is isotropy irreducible. In fact, the isotropy representation is the unique 7-dimensional irreducible representation of $\mathrm{SO}(3)$, which also defines an embedding of $\mathrm{SO}(3)$ in G$_2$ and thus a G$_2$-structure on $\mathrm{SO}(5)/\mathrm{SO}(3)$. The G$_2$-structure is proper nearly parallel (cf. [5]).

The other two examples come from 3-Sasakian geometry. Recall that there is a second Einstein metric in the canonical variation of a 3-Sasakian metric. In the 7-dimensional case this metric is induced by a proper nearly parallel G$_2$-structure [12]. In general, for each simply connected compact simple Lie group G there exists exactly one simply connected 3-Sasakian homogeneous manifold of the form G/H and the only other 3-Sasakian homogeneous manifolds are the real projective spaces [4]. The second Einstein metric is also G-homogeneous but not normal (neither is the 3-Sasakian metric). But if one writes the space in the form $\frac{G \times \mathrm{Sp}(1)}{H \times \mathrm{Sp}(1)}$, it becomes normal [1] and in the 7-dimensional case even standard (up to a factor).

The simply connected 7-dimensional homogeneous 3-Sasakian manifolds are the round sphere S^7 and the Aloff-Wallach space $N(1,1)$. The corresponding second Einstein metrics are the standard homogenous metrics (up to a factor) on $\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}$ (the so-called squashed sphere) and on $\frac{\mathrm{SU}(3) \times \mathrm{Sp}(1)}{\mathrm{U}(1) \times \mathrm{Sp}(1)}$. As remarked by B. Wilking in [22], the latter space was overlooked in the Berger classification of normal homogenous spaces of positive sectional curvature. It follows from Equation (7.87b) of [3] that a normal homogenous space has non-negative

sectional curvature. However, if in addition one has a G_2-structure as desribed in Lemma 7.1 the torsion is non-degenerate and the sectional curvature has to be positive. Thus by the Berger classification there are only the examples considered above.

To compute the space of infinitesimal Einstein deformations on our examples $M = G/H$, we shall proceed in the following way. First we determine which H-representation V define the bundle $\Lambda^3_{27} \mathrm{T}^* M$ and then we use Corollary 7.3 to find the irreducible G-representations U appearing in the eigenspaces of $\bar{\Delta}$ for the eigenvalues $\frac{5\tau_0^2}{6}$, $\frac{\tau_0^2}{3}$ and $\frac{\tau_0^2}{2}$ (as given in Lemma 6.3). In all three examples the computation of the Casimir eigenvalues will show that the eigenvalues $\frac{\tau_0^2}{3}$ and $\frac{\tau_0^2}{2}$ do not appear and thus the spaces of infinitesimal Einstein deformations and infinitesimal G_2-deformations coincide. It is interesting to note that in all three cases the non-zero candidates U comming from the eigenvalue $\frac{5\tau_0^2}{6}$ turn out to be exactly the components of the adjoint representation of G. However this is not too surprising since the Casimir eigenvalue of the adjoint representation (with respect to the Killing form) is always -1. If such representations U do exist, we have to solve the equation $d\varphi = -\tau_0 * \varphi$ or equivalently $(\bar{d} + c*)\varphi = 0$ for the constant $c = \frac{5}{6}\tau_0$. By the results of the previous section, this is reduced to finding $A \in \mathrm{Hom}_H(U, \Lambda^3_{27}\mathfrak{m})$ that solves (7.42) with $c = \frac{5}{6}\tau_0$.

For reference below we mention the following facts about Casimir operators. The Casimir operator of the representation $V(k_1, \ldots, k_n)$ of $\mathrm{Sp}(n)$ with highest weight $\gamma = (k_1, \ldots, k_n)$, where $k_1 \geq \cdots \geq k_n \geq 0$ are integers, is given by

$$(8.43) \qquad \mathrm{Cas}^{\mathrm{Sp}(n)}_{V(k_1, \ldots, k_n)} = -\tfrac{1}{4(n+1)} \sum_{i=1}^{n} \left(2(n-i+1)k_i + k_i^2\right)$$

and the Casimir operator of the representation $V(k_1, \ldots, k_n)$ of $\mathrm{SU}(n)$ with highest weight $\gamma = (k_1, \ldots, k_n)$, where $k_1 \geq \cdots \geq k_n$ are integers satisfying $-\frac{n}{2} < k_1 + \cdots + k_n \leq \frac{n}{2}$, by

$$(8.44) \qquad \mathrm{Cas}^{\mathrm{SU}(n)}_{V(k_1, \ldots, k_n)} = -\tfrac{1}{2n} \sum_{i=1}^{n} \left((n+1-2i)k_i + k_i^2\right) + \tfrac{1}{2n^2}\left(\sum_{i=1}^{n} k_i\right)^2.$$

Finally, if V_1 and V_2 are representations of the groups G_1 and G_2 respectively, then

$$(8.45) \qquad \mathrm{Cas}^{G_1 \times G_2}_{V_1 \otimes V_2} = \mathrm{Cas}^{G_1}_{V_1} + \mathrm{Cas}^{G_2}_{V_2}.$$

8.1. The example $\mathrm{SO}(5)/\mathrm{SO}(3)$.

We have the reductive, i.e. $\mathrm{Ad}\,(\mathrm{SO}(3))$-invariant, decomposition $\mathfrak{so}(5) = \mathfrak{so}(3) \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{so}(3)$ with respect to the Killing form of $\mathfrak{so}(5)$. As mentioned above, $\mathfrak{m}$ is the irreducible 7-dimensional representation of $\mathrm{SO}(3)$. The complex irreducible $\mathrm{SO}(3)$-representations can be written as the symmetric powers $S^{2k}E$, where $E = \mathbb{C}^2$ is the standard representation of the double cover $\mathrm{Sp}(1)$ of $\mathrm{SO}(3)$, in particular $\mathfrak{m}^{\mathbb{C}} \cong S^6 E$. It is easy to obtain the following decomposition into irreducible summands

$$\Lambda^3 \mathfrak{m}^{*\mathbb{C}} \cong \Lambda^3 S^6 E = \mathbb{C} \oplus S^4 E \oplus S^6 E \oplus S^8 E \oplus S^{12} E.$$

We see that there is a 1-dimensional space of SO(3)-invariant 3-forms, which implies that on $M = \mathrm{SO}(5)/\mathrm{SO}(3)$ the canonical homogeneous connection coincides with the canonical G₂-connection. Moreover, since $\Lambda^3\mathfrak{m}^* \cong \mathbb{R} \oplus \mathfrak{m} \oplus \Lambda^3_{27}\mathfrak{m}^*$ as a G₂-representation, we obtain

$$(8.46) \qquad \Lambda^3_{27}\mathfrak{m}^{*\mathbb{C}} \cong S^4E \oplus S^8E \oplus S^{12}E.$$

Since Sp(2) double covers SO(5), the two groups have the same Casimir operator. Therefore, by Corollary 7.3 and (8.43), we have to find all pairs of integers (k_1, k_2) with $k_1 \geq k_2 \geq 0$, such that $-\frac{1}{12}(4k_1 + k_1^2 + 2k_2 + k_2^2)$ is equal to one of $-1, -\frac{2}{5}, -\frac{3}{5}$. The only solution is $(k_1, k_2) = (2, 0)$ for the eigenvalue -1. The representation $V(2, 0)$ is the adjoint representation of Sp(2) and it corresponds, of course, to the adjoint representation $\mathfrak{so}(5)$ of SO(5). It remains to compute the dimension of $\mathrm{Hom}_{\mathrm{SO}(3)}(\mathfrak{so}(5)^{\mathbb{C}}, \Lambda^3_{27}\mathfrak{m}^{*\mathbb{C}})$, which turns out to be zero. Indeed, from the reductive decomposition above we have the following decomposition of $\mathfrak{so}(5)^{\mathbb{C}}$ into irreducible SO(3)-representations:

$$\mathfrak{so}(5)^{\mathbb{C}} \cong \mathfrak{so}(3)^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} \cong S^2E \oplus S^6E.$$

Comparing this with (8.46), we see that $\mathfrak{so}(5)^{\mathbb{C}}$ and $\Lambda^3_{27}\mathfrak{m}^{*\mathbb{C}}$ do not have any common components and therefore $\mathrm{Hom}_{\mathrm{SO}(3)}(\mathfrak{so}(5)^{\mathbb{C}}, \Lambda^3_{27}\mathfrak{m}^{*\mathbb{C}}) = 0$. Thus the eigenvalues $\frac{5\tau_0^2}{6}, \frac{\tau_0^2}{3}$ and $\frac{\tau_0^2}{2}$ do not appear in the $\bar{\Delta}$-spectrum on $\Omega^3_{27}(M)$ and so we have proved

Proposition 8.1. *There are no infinitesimal Einstein deformations and, in particular, no infinitesimal G₂-deformations of the nearly parallel G₂-structure on* SO(5)/SO(3).

8.2. The example $\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}$.

We denote by $\mathrm{Sp}(1)_u$ and $\mathrm{Sp}(1)_d$ the following embeddings of Sp(1) in $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$:

$$\mathrm{Sp}(1)_u := \{(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1) : a \in \mathrm{Sp}(1)\}, \qquad \mathrm{Sp}(1)_d := \{(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, a) : a \in \mathrm{Sp}(1)\}.$$

In this realization the Lie algebras of $\mathrm{Sp}(1)_u$ and $\mathrm{Sp}(1)_d$ are given as

$$\mathfrak{sp}(1)_u := \{(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, 0) : a \in \mathfrak{sp}(1)\}, \qquad \mathfrak{sp}(1)_d := \{(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, a) : a \in \mathfrak{sp}(1)\}.$$

We consider the homogeneous space $\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d}$ as a normal homogeneous space taking the metric induced by $-\frac{1}{24}B$, where B is the Killing form of $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$. Then we have the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with

$$\mathfrak{h} = \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d, \qquad \mathfrak{m} = \mathfrak{h}^{\perp} = \mathfrak{sp}(1)_o \oplus \mathfrak{m}'.$$

The Lie algebra $\mathfrak{sp}(1)_o$ and the space $\mathfrak{m}'$ are given as

$$\mathfrak{sp}(1)_o := \{(\begin{pmatrix} 0 & 0 \\ 0 & 2a \end{pmatrix}, -3a) : a \in \mathfrak{sp}(1)\}, \qquad \mathfrak{m}' := \{(\begin{pmatrix} 0 & x \\ -\bar{x} & 0 \end{pmatrix}, 0) : x \in \mathbb{H}\}.$$

We define the orientation by means of the following orthonormal frame of $\mathfrak{m}$:

$$e_1 := \frac{1}{\sqrt{5}}(\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}, -3i), \quad e_2 := \frac{1}{\sqrt{5}}(\begin{pmatrix} 0 & 0 \\ 0 & 2j \end{pmatrix}, -3j), \quad e_3 := \frac{1}{\sqrt{5}}(\begin{pmatrix} 0 & 0 \\ 0 & 2k \end{pmatrix}, -3k),$$

$$e_4 := (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0), \quad e_5 := (\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, 0), \quad e_6 := (\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, 0), \quad e_7 := (\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, 0).$$

Then, computing the comutators of these basis elements, we see that at the initial point o the torsion of the canonical homogeneous connection is $\hat{\mathrm{T}}_o = \frac{2}{\sqrt{5}}\sigma_o$, where σ_o is given by (2.1). Hence, by Lemma 7.1 we obtain a nearly parallel G_2-structure on $\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d}$ with $\tau_0 = -\frac{12}{\sqrt{5}}$.

We want to determine infinitesimal Einstein deformations of this structure. Thus by Corollary 7.3 together with (8.45) and (8.43) we are looking for $k_1 \geq k_2 \geq 0$ and $l \geq 0$ such that

$$\mathrm{Cas}^{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}_{V(k_1,k_2) \otimes V(l)} = -\frac{1}{12}\left(4k_1 + k_1^2 + 2k_2 + k_2^2\right) - \frac{1}{8}\left(2l + l^2\right)$$

is equal to one of $-1, -\frac{2}{5}, -\frac{3}{5}$. The only solutions, both for the eigenvalue -1, are $k_1 = 2$, $k_2 = 0$, $l = 0$ and $k_1 = 0$, $k_2 = 0$, $l = 2$. Thus the space of infinitesimal Einstein deformations is equal to the space of infinitesimal G_2-deformations and the only two representations of $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ which could be contained in this space are $V(2,0) \cong \mathfrak{sp}(2)$ and $V(2) \cong \mathfrak{sp}(1)$. Next we have to determine whether these spaces admit H-invariant homomorphisms to $\Lambda^3_{27}\mathfrak{m}^*$.

If the standard representations of $\mathrm{Sp}(1)_u$ and $\mathrm{Sp}(1)_d$ are denoted by E and H respectively, an arbitrary irreducible representation of $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$ can be written as $S^k E S^l H$. (In this and the next subsection we shall omit the tensor product sign and the complexification sign). Then we have the following decompositions into irreducible $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$-representations:

$$\begin{aligned}
\mathfrak{sp}(1)_o &\cong S^2 H, \qquad \mathfrak{m}' \cong EH, \qquad \mathfrak{m} \cong S^2 H \oplus EH, \\
\Lambda^3 \mathfrak{m}^* &\cong S^2 E S^2 H \oplus E S^3 H \oplus 2EH \oplus S^4 H \oplus S^2 H \oplus 2\mathbb{C}, \\
\Lambda^3_{27} \mathfrak{m}^* &\cong S^2 E S^2 H \oplus E S^3 H \oplus EH \oplus S^4 H \oplus \mathbb{C}, \\
V(2,0) &\cong \mathfrak{sp}(2) \cong S^2 E \oplus EH \oplus S^2 H, \qquad V(2) \cong \mathfrak{sp}(1) = S^2 H.
\end{aligned}$$

Since $\Lambda^3_{27}\mathfrak{m}^*$ and $\mathfrak{sp}(1)$ have no common summands, $\mathrm{Hom}_{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d}(\mathfrak{sp}(1), \Lambda^3_{27}\mathfrak{m}^*) = 0$ and therefore the $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-representation $\mathfrak{sp}(1)$ is not contained in $\Omega^3_{27}(M)$.

The only common summand of $\Lambda^3_{27}\mathfrak{m}^*$ and $\mathfrak{sp}(2)$ is EH, so $\mathrm{Hom}_{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d}(\mathfrak{sp}(2), \Lambda^3_{27}\mathfrak{m}^*)$ is 1-dimensional. In order to proceed we have to find an explicit equivariant homomorphism $A : \mathfrak{sp}(2) \to \Lambda^3_{27}\mathfrak{m}^*$ spanning this space. Since $*\sigma_o$ and e^{4567} are the two linear independent $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$-invariant forms in $\Lambda^4\mathfrak{m}^*$, an arbitrary embedding of EH in $\Lambda^3\mathfrak{m}^*$ is given by

$$EH \cong \mathfrak{m}' \ni X \mapsto X \lrcorner (\lambda * \sigma_o + \mu e^{4567}) \in \Lambda^3\mathfrak{m}^*.$$

The image of this map is contained in $\Lambda^3_{27}\mathfrak{m}^*$ if and only if it is orthogonal to the EH in $\Lambda^3_7\mathfrak{m}^*$. Obviously this is equivalent to $\mu = -4\lambda$ and we can take the embedding

$$i : EH \cong \mathfrak{m}' \to \Lambda^3_{27}\mathfrak{m}^*, \quad EH \cong \mathfrak{m}' \ni X \mapsto X \lrcorner (*\sigma_o - 4e^{4567}).$$

Hence $\mathrm{Hom}_{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d}(\mathfrak{sp}(2), \Lambda^3_{27}\mathfrak{m}^*)$ is spanned by the equivariant homomorphism $A := i \circ p$, where $p : \mathfrak{sp}(2) \to EH$ is the orthogonal projection.

Thus $U = \mathfrak{sp}(2)$ is the only $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-representation which remains for the solution space of the equation $*d\varphi = -\tau_0\varphi$, describing the infinitesimal G₂-deformations. As mentioned above, this equation is equivalent to $(\bar{d} + \frac{5}{6}\tau_0*)\varphi = 0$, i.e. in the case at hand to $(\bar{d} - 2\sqrt{5}*)\varphi = 0$. From the results of the last part of Section 7 for $V = \Lambda_{27}^3 \mathfrak{m}^*$ and $U = \mathfrak{sp}(2)$ it follows that equation (7.42) with $c = -2\sqrt{5}$ must be satisfied for the chosen A and all $u \in \mathfrak{sp}(2)$. However this is not the case: take $\alpha := e_4 \in EH \cong \mathfrak{m}' \subset \mathfrak{sp}(2)$. Then

$$
\begin{array}{ll}
e_1 \cdot \alpha = [e_1, e_4] = -\frac{2}{\sqrt{5}}e_5, & i(e_5) = 3e^{467} + e^{137} + e^{126} + e^{234}, \\
e_2 \cdot \alpha = [e_2, e_4] = -\frac{2}{\sqrt{5}}e_6, & i(e_6) = -3e^{457} + e^{237} - e^{125} - e^{134}, \\
e_3 \cdot \alpha = [e_3, e_4] = -\frac{2}{\sqrt{5}}e_7, & i(e_7) = 3e^{456} - e^{236} - e^{135} + e^{124}, \\
e_4 \cdot \alpha = [e_4, e_4] = 0, & i(e_4) = -3e^{567} - e^{235} + e^{136} - e^{127}.
\end{array}
$$

Using these equations one easily sees that the coefficient of e^{1234} in the left-hand side of (7.42) is $\frac{36}{\sqrt{5}} \neq 0$. Hence $\mathfrak{sp}(2)$ is not contained in the space of solutions of $(\bar{d} - 2\sqrt{5}*)\varphi = 0$.

Since the nearly parallel G₂-structure of the squashed sphere is a double covering of the one on $\mathbb{R}P^7$ the same argument applies for the real projective space and we obtain

Proposition 8.2. *There are no infinitesimal Einstein deformations and, in particular, no infinitesimal* G₂*-deformations of the nearly parallel* G₂*-structure on the squashed sphere* $\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}$ *and of the nearly parallel* G₂*-structure on* $\mathbb{R}P^7$ *inducing the second Einstein metric.*

8.3. The example $\frac{\mathrm{SU}(3) \times \mathrm{SU}(2)}{U(1) \times \mathrm{SU}(2)}$.

We denote by $\mathrm{SU}(2)_d$ the following embedding of $\mathrm{SU}(2)$ in $\mathrm{SU}(3) \times \mathrm{SU}(2)$:

$$
\mathrm{SU}(2)_d := \{(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a) : a \in \mathrm{SU}(2)\}.
$$

The group $U(1)$ is realized as a subgroup of $\mathrm{SU}(3) \subset \mathrm{SU}(3) \times \mathrm{SU}(2)$ by the embedding

$$
U(1) = \{(\begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix}, 1) : t \in \mathbb{R}\}.
$$

We consider the homogeneous space $\frac{\mathrm{SU}(3) \times \mathrm{SU}(2)}{U(1) \times \mathrm{SU}(2)_d}$ as a normal homogeneous space taking the metric induced by $-\frac{1}{24}B$, where B is the Killing form of $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$. Then we have the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with

$$
\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(2)_d, \qquad \mathfrak{m} = \mathfrak{h}^\perp = \mathfrak{su}(2)_o \oplus \mathfrak{m}'.
$$

Here

$$
\mathfrak{u}(1) := \mathrm{span}\{C\}, \text{ where } C := (\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0), \text{ and } \mathfrak{su}(2)_d := \{(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a) : a \in \mathfrak{su}(2)\}
$$

are the Lie algebras of $U(1)$ and $\mathrm{SU}(2)_d$ respectively and

$$\mathfrak{su}(2)_o := \{(\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}, -3a) : a \in \mathfrak{su}(2)\}, \qquad \mathfrak{m}' := \{(\begin{pmatrix} 0 & z \\ -\bar{z}^t & 0 \end{pmatrix}, 0) : z \in \mathbb{C}^2\}.$$

Let

$$I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2), \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{su}(2), \quad K := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{su}(2).$$

Then we define the orientation fixing the following orthonormal frame of $\mathfrak{m}$:

$$e_1 := -\frac{1}{\sqrt{5}}(\begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}, -3I), \quad e_2 := -\frac{1}{\sqrt{5}}(\begin{pmatrix} 2J & 0 \\ 0 & 0 \end{pmatrix}, -3J), \quad e_3 := -\frac{1}{\sqrt{5}}(\begin{pmatrix} 2K & 0 \\ 0 & 0 \end{pmatrix}, -3K),$$

$$e_4 := (\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}, 0), \quad e_5 := (\begin{pmatrix} 0 & 0 & \sqrt{2}i \\ 0 & 0 & 0 \\ \sqrt{2}i & 0 & 0 \end{pmatrix}, 0),$$

$$e_6 := (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, 0), \quad e_7 := (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}, 0).$$

Then, as in the previous example we see that $\hat{T}_o = \frac{2}{\sqrt{5}}\sigma_o$, where σ_o is given by (2.1). Hence, by Lemma 7.1 we obtain a nearly parallel G_2-structure on $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{U(1)\times\mathrm{SU}(2)_d}$ with $\tau_0 = -\frac{12}{\sqrt{5}}$.

Again we want to find the infinitesimal Einstein deformations of this structure. By Corollary 7.3 together with (8.45) and (8.44) we are this time looking for integers $k_1 \geq k_2 \geq k_3$ and $l_1 \geq l_2$, satisfying $-\frac{3}{2} < k_1 + k_2 + k_3 \leq \frac{3}{2}$ and $-1 < l_1 + l_2 \leq 1$, such that

$$\mathrm{Cas}^{\mathrm{SU}(3)\times\mathrm{SU}(2)}_{V(k_1,k_2,k_3)\otimes V(l_1,l_2)} = -\frac{1}{9}(3k_1+k_1^2+k_2^2-3k_3+k_3^2-k_1k_2-k_2k_3-k_3k_2)-\frac{1}{8}(2l_1+l_1^2-2l_2+l_2^2-2l_1l_2).$$

is equal to one of -1, $-\frac{2}{5}$, $-\frac{3}{5}$. The only solutions, both for the eigenvalue -1, are

$$k_1 = 1, \ k_2 = 0, \ k_3 = -1, \ l_1 = 0, \ l_2 = 0 \quad \text{and} \quad k_1 = 0, \ k_2 = 0, \ k_3 = 0, \ l_1 = 1, \ l_2 = -1,$$

Thus the space of infinitesimal Einstein deformations is equal to the space of infinitesimal G_2-deformations and the only two representations of $\mathrm{SU}(3) \times \mathrm{SU}(2)$ which could be contained in this space are $V(1,0,-1) \cong \mathfrak{su}(3)$ and $V(1,-1) \cong \mathfrak{su}(2)$. Next we have to determine the H-equivariant homomorphisms of these spaces into $\Lambda^3_{27}\mathfrak{m}^*$.

If the representation of $U(1)$ with weight k is denoted by $F(k)$ and the standard representation of $\mathrm{SU}(2)_d$ by H, then an arbitrary irreducible representation of $U(1) \times \mathrm{SU}(2)_d$ has

the form $F(k)S^l H$. We have the following decompositions into irreducible $U(1) \times \mathrm{SU}(2)_d$-representations:

$$\mathfrak{su}(2)_o \cong S^2 H, \qquad \mathfrak{m}' \cong F(3)H \oplus F(-3)H, \qquad \mathfrak{m} \cong S^2 H \oplus F(3)H \oplus F(-3)H,$$

$$\Lambda^3 \mathfrak{m}^* \cong F(6)S^2 H \oplus F(-6)S^2 H \oplus F(3)S^3 H \oplus F(-3)S^3 H \oplus 2F(3)H \oplus 2F(-3)H$$
$$\oplus S^4 H \oplus 2S^2 H \oplus 2\mathbb{C},$$

$$\Lambda_{27}^3 \mathfrak{m}^* \cong F(6)S^2 H \oplus F(-6)S^2 H \oplus F(3)S^3 H \oplus F(-3)S^3 H \oplus F(3)H \oplus F(-3)H$$
$$\oplus S^4 H \oplus S^2 H \oplus \mathbb{C},$$

$$V(1,0,-1) \cong \mathfrak{su}(3) \cong \mathbb{C} \oplus F(3)H \oplus F(-3)H \oplus S^2 H, \qquad V(1,-1) \cong \mathfrak{su}(2) \cong S^2 H,$$

The only common summand of $\Lambda_{27}^3 \mathfrak{m}^*$ and $\mathfrak{su}(2)$ is $S^2 H$, so $\mathrm{Hom}_{U(1)\times\mathrm{SU}(2)_d}(\mathfrak{su}(2), \Lambda_{27}^3 \mathfrak{m}^*)$ is 1-dimensional. Let

$$q_2 : S^2 H \cong \mathfrak{su}(2) \to \mathfrak{su}(2)_o, \quad S^2 H \cong \mathfrak{su}(2) \ni a \mapsto \left(\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}, -3a \right) \in \mathfrak{su}(2)_o$$

be the identification of $S^2 H$ and $\mathfrak{su}(2)_o$. The $S^2 H$ in $\Lambda^3 \mathfrak{m}^*$ coming from $\Lambda_7^3 \mathfrak{m}^*$ is given by the embedding

$$S^2 H \ni a \mapsto q_2(a) \mapsto q_2(a) \lrcorner * \sigma_0 \in \Lambda_7^3 \mathfrak{m}^*.$$

Since the 2-form $\Omega := e^{45} + e^{67}$ is also $U(1) \times \mathrm{SU}(2)_d$-invariant, another embedding of $S^2 H$ in $\Lambda^3 \mathfrak{m}^*$ is

$$i_2 : S^2 H \to \Lambda^3 \mathfrak{m}^*, \quad S^2 H \ni a \mapsto q_2(a) \mapsto q_2(a)^\flat \wedge \Omega \in \Lambda^3 \mathfrak{m}^*.$$

It is easy to see that $i_2(S^2 H)$ is orthogonal to the $S^2 H$ in $\Lambda_7^3 \mathfrak{m}^*$, so in fact i_2 is the embedding of $S^2 H$ into $\Lambda_{27}^3 \mathfrak{m}^*$. Therefore $\mathrm{Hom}_{U(1)\times\mathrm{SU}(2)_d}(\mathfrak{su}(2), \Lambda_{27}^3 \mathfrak{m}^*)$ is spanned by $A := i_2$.

Now, as in the previous example, it remains to solve the equation $(\bar{d} - 2\sqrt{5}*)\varphi = 0$ by applying the results of the last part of section 7 for $V = \Lambda_{27}^3 \mathfrak{m}^*$ and for the summands $U = \mathfrak{su}(3)$ and $U = \mathfrak{su}(2)$ found above.

If $\mathfrak{su}(2)$ is contained in the space of solutions of $(\bar{d} - 2\sqrt{5}*)\varphi = 0$, then equation (7.42) (with $s = 3$ and $c = -2\sqrt{5}$) must be satisfied for the chosen A and all $\alpha \in \mathfrak{su}(2)$. We shall show this is not the case. Take $\alpha := (0, I) \in S^2 H \cong \mathfrak{su}(2) \subset \mathfrak{su}(3) \oplus \mathfrak{su}(2)$. Then

$$e_2 \cdot \alpha = [e_2, \alpha] = \frac{6}{\sqrt{5}}(0, K), \quad i_2(0, K) = q_2(0, K)^\flat \wedge \Omega = -\sqrt{5}e^3 \wedge (e^{45} + e^{67}) = -\sqrt{5}(e^{345} + e^{367}),$$

$$e_3 \cdot \alpha = [e_3, \alpha] = -\frac{6}{\sqrt{5}}(0, J), \quad i_2(0, J) = q_2(0, J)^\flat \wedge \Omega = -\sqrt{5}e^2 \wedge (e^{45} + e^{67}) = -\sqrt{5}(e^{245} + e^{267}),$$

$$e_4 \cdot \alpha = [e_4, \alpha] = 0, \quad e_5 \cdot \alpha = [e_5, \alpha] = 0,$$

$$i_2(0, I) = q_2(0, I)^\flat \wedge \Omega = -\sqrt{5}e^1 \wedge (e^{45} + e^{67}) = -\sqrt{5}(e^{145} + e^{167}).$$

Using these equations one easily sees that the coefficient of e^{2345} in the left-hand side of (7.42) is $22 \neq 0$. Hence $\mathfrak{su}(2)$ is not contained in the space of solutions of $(\bar{d} - 2\sqrt{5}*)\varphi = 0$.

There are four common summands of $\Lambda_{27}^3 \mathfrak{m}^*$ and $\mathfrak{su}(3)$: $\mathbb{C}$, $S^2 H$, $F(3)H$, $F(-3)H$. Since they are all different, $\mathrm{Hom}_{U(1)\times\mathrm{SU}(2)_d}(\mathfrak{su}(3), \Lambda_{27}^3 \mathfrak{m}^*)$ is 4-dimensional. Our next goal is to determine a basis A_1, A_2, A_3, A_4 of $\mathrm{Hom}_{U(1)\times\mathrm{SU}(2)_d}(\mathfrak{su}(3), \Lambda_{27}^3 \mathfrak{m}^*)$ corresponding to these spaces.

The $\mathbb{C}$ in $\Lambda^3\mathfrak{m}^*$ coming from $\Lambda_1^3\mathfrak{m}^*$ is spanned by σ_o and the second $\mathbb{C}$ in $\Lambda^3\mathfrak{m}^*$ is spanned by e^{123}. Thus an arbitrary $\mathbb{C}$ in $\Lambda^3\mathfrak{m}^*$ is spanned by $\lambda\sigma_o + \mu e^{123}$. This is orthogonal to σ_o if and only if $\mu = -7\lambda$. Hence the $\mathbb{C}$ in $\Lambda_{27}^3\mathfrak{m}^*$ is spanned by $\sigma_o - 7e^{123}$. On the other hand, $\mathbb{C}$ in $\mathfrak{su}(3)$ is $\mathfrak{u}(1)$ and is spanned by C. Define

$$i_1 : \mathfrak{u}(1) \to \Lambda_{27}^3\mathfrak{m}^*, \quad C \mapsto \sigma_o - 7e^{123}.$$

Then the subspace of $\mathrm{Hom}_{U(1)\times\mathrm{SU}(2)_d}(\mathfrak{su}(3), \Lambda_{27}^3\mathfrak{m}^*)$ which corresponds to $\mathbb{C}$ is spanned by $A_1 := i_1 \circ p_1$, where $p_1 : \mathfrak{su}(3) \to \mathfrak{u}(1)$ is the projection.

Let

$$j_2 : S^2H \cong \mathfrak{su}(2) \to \mathfrak{su}(3), \quad S^2H \cong \mathfrak{su}(2) \ni a \mapsto (\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, 0) \in \mathfrak{su}(3) \subset \mathfrak{su}(3) \oplus \mathfrak{su}(2).$$

Then the subspace of $\mathrm{Hom}_{U(1)\times\mathrm{SU}(2)_d}(\mathfrak{su}(3), \Lambda_{27}^3\mathfrak{m}^*)$ corresponding to S^2H is spanned by $A_2 := i_2 \circ j_2^{-1} \circ p_2$, where $p_2 : \mathfrak{su}(3) \to S^2H$ is the projection and i_2 was defined earlier.

Considered as subspaces of $\mathfrak{m}' \subset \mathfrak{su}(3) \oplus \mathfrak{su}(2)$, $F(3)H$ and $F(-3)H$ are

$$F(3)H \cong \mathrm{span}\{e_4 - ie_5, e_6 - ie_7\}, \quad F(-3)H \cong \mathrm{span}\{e_4 + ie_5, e_6 + ie_7\}.$$

In the same way as for EH in the case of $\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$ we obtain that the embeddings $i_3 : F(3)H \to \Lambda_{27}^3\mathfrak{m}^*$ and $i_4 : F(-3)H \to \Lambda_{27}^3\mathfrak{m}^*$ are given by the restrictions on $F(3)H$ and $F(-3)H$ of the embedding

$$\mathfrak{m}' \ni X \mapsto X \lrcorner (*\sigma_o - 4e^{4567}).$$

Then the subspaces of $\mathrm{Hom}_{U(1)\times\mathrm{SU}(2)_d}(\mathfrak{su}(3), \Lambda_{27}^3\mathfrak{m}^*)$ corresponding to $F(3)H$ and $F(-3)H$ are spanned by $A_3 := i_3 \circ p_3$ and $A_4 := i_4 \circ p_4$, where $p_3 : \mathfrak{su}(3) \to F(3)H$, $p_4 : \mathfrak{su}(3) \to F(-3)H$ are the projections.

Thus we have to find for which $A = c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4$ equation (7.42) (with $s = 3$ and $c = -2\sqrt{5}$) is satisfied for all $\alpha \in \mathfrak{su}(3)$. As this equation is $U(1) \times \mathrm{SU}(2)_d$-invariant, this is equivalent to the requirement that the equation is satisfied for one representative of each of the four summands in $\mathfrak{su}(3)$. We take

$$\begin{aligned} \alpha_1 &:= C \in \mathbb{C} \subset \mathfrak{su}(3), & \alpha_2 &:= j_2(I) \in S^2H \subset \mathfrak{su}(3), \\ \alpha_3 &:= e_4 - ie_5 \in F(3)H \subset \mathfrak{su}(3), & \alpha_4 &:= e_4 + ie_5 \in F(-3)H \subset \mathfrak{su}(3). \end{aligned}$$

Then we have

$$A(C) = c_1 i_1(C) = c_1(\sigma_o - 7e^{123}),$$
$$A(j_2(I)) = c_2 i_2(I) = -\sqrt{5}c_2 e^1 \wedge \Omega = -\sqrt{5}c_2(e^{145} + e^{167}),$$
$$A(j_2(J)) = c_2 i_2(J) = -\sqrt{5}c_2 e^2 \wedge \Omega = -\sqrt{5}c_2(e^{245} + e^{267}),$$
$$A(j_2(K)) = c_2 i_2(K) = -\sqrt{5}c_2 e^3 \wedge \Omega = -\sqrt{5}c_2(e^{345} + e^{367}),$$
$$A(e_4 - ie_5) = c_3 i_3(e_4 - ie_5) = c_3((-3e^{567} - e^{235} + e^{136} - e^{127}) - i(3e^{467} + e^{137} + e^{126} + e^{234})),$$
$$A(e_6 - ie_7) = c_3 i_3(e_6 - ie_7) = c_3((-3e^{457} + e^{237} - e^{125} - e^{134}) - i(3e^{456} - e^{236} - e^{135} + e^{124})),$$
$$A(e_4 + ie_5) = c_4 i_4(e_4 + ie_5) = c_4((-3e^{567} - e^{235} + e^{136} - e^{127}) + i(3e^{467} + e^{137} + e^{126} + e^{234})),$$
$$A(e_6 + ie_7) = c_4 i_4(e_6 + ie_7) = c_4((-3e^{457} + e^{237} - e^{125} - e^{134}) + i(3e^{456} - e^{236} - e^{135} + e^{124})).$$

Since equation (7.42) is invariant with respect to $U(1) \times \mathrm{SU}(2)_d$, its left-hand side for $\alpha = \alpha_1$ lies in $2\mathbb{C} = \mathrm{span}\{*\sigma_o, e^{4567}\} \subset \Lambda^4\mathfrak{m}^*$. Hence, to determine it, it is enough to compute the coefficients of e^{4567} and e^{2367}.

We have

$$e_1 \cdot \alpha_1 = [e_1, C] = 0, \quad e_2 \cdot \alpha_1 = [e_2, C] = 0, \quad e_3 \cdot \alpha_1 = [e_3, C] = 0,$$
$$e_4 \cdot \alpha_1 = [e_4, C] = -3e_5, \quad e_5 \cdot \alpha_1 = [e_5, C] = 3e_4,$$
$$e_6 \cdot \alpha_1 = [e_6, C] = -3e_7, \quad e_7 \cdot \alpha_1 = [e_7, C] = 3e_6.$$

Using these equations we see that the coefficients of e^{4567} and e^{2367} are $-18i(c_3 - c_4) + 12\sqrt{5}c_1$ and $3i(c_3 - c_4) - 2\sqrt{5}c_1$. So the whole left-hand side of (7.42) for $\alpha = \alpha_1$ is

$$(-2\sqrt{5}c_1 + 3i(c_3 - c_4))(*\sigma_0 - 7e^{4567})$$

and this vanishes if and only if $-2\sqrt{5}c_1 + 3i(c_3 - c_4) = 0$.

Since equation (7.42) is $U(1) \times \mathrm{SU}(2)_d$-invariant, its left-hand side for $\alpha \in S^2H$ lies in $2S^2H \subset \Lambda^4\mathfrak{m}^*$. The two embeddings of S^2H in $\Lambda^4\mathfrak{m}^*$ are obtained as the composition of the two embeddings of S^2H in $\Lambda^3\mathfrak{m}^*$ with $*$. Thus the left-hand side of (7.42) for $\alpha = \alpha_2$ lies in

$$\mathrm{span}\{e^{1246} + e^{1347} + e^{1257} - e^{1356}, e^{2367} + e^{2345}\}.$$

Hence, to determine it, it is enough to compute the coefficients of e^{1246} and e^{2367}.

We have

$$e_1 \cdot \alpha_2 = [e_1, j_2(I)] = 0, \quad e_2 \cdot \alpha_2 = [e_2, j_2(I)] = -\frac{4}{\sqrt{5}}j_2(K), \quad e_3 \cdot \alpha_2 = [e_3, j_2(I)] = \frac{4}{\sqrt{5}}j_2(J),$$

$$e_4 \cdot \alpha_2 = [e_4, j_2(I)] = -e_5, \quad e_5 \cdot \alpha_2 = [e_5, j_2(I)] = e_4,$$
$$e_6 \cdot \alpha_2 = [e_6, j_2(I)] = e_7, \quad e_7 \cdot \alpha_2 = [e_7, j_2(I)] = -e_6.$$

With these equations we see that the coefficients of e^{1246} and e^{2367} are $c_3 + c_4$ and $2c_2 - i(c_3 - c_4)$. So the whole left-hand side of (7.42) for $\alpha = \alpha_2$ is

$$(c_3 + c_4)(e^{1246} + e^{1347} + e^{1257} - e^{1356}) + (2c_2 - i(c_3 - c_4))(e^{2367} + e^{2345})$$

and this vanishes if and only if $c_3 + c_4 = 0$ and $2c_2 - i(c_3 - c_4) = 0$.

Again the $U(1) \times \mathrm{SU}(2)_d$-invariance of equation (7.42) implies that its left-hand side for $\alpha \in F(3)H$ lies in $2F(3)H \subset \Lambda^4\mathfrak{m}^*$. The two embeddings of $F(3)H$ in $\Lambda^4\mathfrak{m}^*$ are obtained as the composition of the two embeddings of $F(3)H$ in $\Lambda^3\mathfrak{m}^*$ with $*$. Thus the left-hand side of (7.42) for $\alpha = \alpha_3$ lies in

$$\mathrm{span}\{e^{1234} - ie^{1235}, (e^{1467} - e^{2457} + e^{3456}) - i(e^{2456} + e^{3457} + e^{1567})\}.$$

Hence, to determine it, it is enough to compute the coefficients of e^{1234} and e^{1467}.

We have

$$e_1 \cdot \alpha_3 = [e_1, e_4 - ie_5] = -\frac{2}{\sqrt{5}}i(e_4 - ie_5), \quad e_2 \cdot \alpha_3 = [e_2, e_4 - ie_5] = -\frac{2}{\sqrt{5}}(e_6 - ie_7),$$

$$e_3 \cdot \alpha_3 = [e_3, e_4 - ie_5] = -\frac{2}{\sqrt{5}}i(e_6 - ie_7),$$

$$e_4 \cdot \alpha_3 = [e_4, e_4 - ie_5)] = -2iC - 2ij_2(I), \quad e_5 \cdot \alpha_3 = [e_5, e_4 - ie_5] = -2C - 2j_2(I),$$

$$e_6 \cdot \alpha_3 = [e_6, e_4 - ie_5] = -2j_2(J) - 2ij_2(K), \quad e_7 \cdot \alpha_3 = [e_7, e_4 - ie_5] = i(2j_2(J) + 2ij_2(K)).$$

Then, using these equations we find that the coefficients of e^{1234} and e^{1467} are $12ic_1 + \frac{36}{\sqrt{5}}c_3$ and $2ic_1 + 2i\sqrt{5}c_2 + \frac{16}{\sqrt{5}}c_3$. So the whole left-hand side of (7.42) for $\alpha = \alpha_3$ is

$$(12ic_1 + \frac{36}{\sqrt{5}}c_3)(e^{1234} - ie^{1235})$$

$$+(2ic_1 + 2i\sqrt{5}c_2 + \frac{16}{\sqrt{5}}c_3)((e^{1467} - e^{2457} + e^{3456}) - i(e^{2456} + e^{3457} + e^{1567}))$$

and this vanishes if and only if $12ic_1 + \frac{36}{\sqrt{5}}c_3 = 0$ and $2ic_1 + 2i\sqrt{5}c_2 + \frac{16}{\sqrt{5}}c_3 = 0$.

The computations for $\alpha = \alpha_4 \in F(-3)H$ are similar. In fact, one has to take the results for α_3, change c_3 to c_4, preserve c_1 and c_2 and take the complex conjugate of everything else. So the whole left-hand side of (7.42) for $\alpha = \alpha_4$ is

$$(-12ic_1 + \frac{36}{\sqrt{5}}c_4)(e^{1234} + ie^{1235})$$

$$+(-2ic_1 - 2i\sqrt{5}c_2 + \frac{16}{\sqrt{5}}c_4)((e^{1467} - e^{2457} + e^{3456}) + i(e^{2456} + e^{3457} + e^{1567}))$$

and this vanishes if and only if $-12ic_1 + \frac{36}{\sqrt{5}}c_4 = 0$ and $-2ic_1 - 2i\sqrt{5}c_2 + \frac{16}{\sqrt{5}}c_4 = 0$.

Hence equation (7.42) is satisfied for $A = c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4$ and all $\alpha \in \mathfrak{su}(3)$ if and only if

$$-2\sqrt{5}c_1 + 3i(c_3 - c_4) = 0, \quad c_3 + c_4 = 0, \quad 2c_2 - i(c_3 - c_4) = 0,$$

$$12ic_1 + \frac{36}{\sqrt{5}}c_3 = 0, \quad 2ic_1 + 2i\sqrt{5}c_2 + \frac{16}{\sqrt{5}}c_3 = 0,$$

$$-12ic_1 + \frac{36}{\sqrt{5}}c_4 = 0, \quad -2ic_1 - 2i\sqrt{5}c_2 + \frac{16}{\sqrt{5}}c_4 = 0.$$

The solution of this linear system is 1-dimensional:

$$c_2 = \frac{\sqrt{5}}{3}c_1, \quad c_3 = -\frac{\sqrt{5}}{3}ic_1, \quad c_4 = \frac{\sqrt{5}}{3}ic_1.$$

This means that exactly one copy of $\mathfrak{su}(3)$ is contained in the space of solutions of the equation $(\bar{d} - 2\sqrt{5}*)\varphi = 0$.

Thus we have proved

Proposition 8.3. *The space of infinitesimal Einstein deformations of the proper nearly parallel G_2-structure on $\frac{\mathrm{SU}(3) \times \mathrm{SU}(2)}{U(1) \times \mathrm{SU}(2)}$ coincides with the space of its infinitesimal G_2-deformations. This space is 8-dimensional and is isomorphic to $\mathfrak{su}(3)$ as an $\mathrm{SU}(3) \times \mathrm{SU}(2)$-representation.*

References

[1] B. Alexandrov, *3-Sasakian manifolds and intrinsic torsion*, to appear.

[2] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993), 509–521.

[3] A. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 10. Springer-Verlag, Berlin, 1987.

[4] C. Boyer, K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[5] R. Bryant, *Metrics with exceptional holonomy*, Ann. Math. **126** (1987), 525–576.

[6] R. Bryant, *Some remarks on G_2-structures*, Proceedings of Gökova Geometry-Topology Conference 2005, 75–109, Gökova Geometry/Topology Conference (GGT), Gökova, 2006. arXiv:math/0305124

[7] R. Cleyton, A. Swann, *Einstein metrics via intrinsic or parallel torsion*, Math. Z. **247** (2004), no. 3, 513–528.

[8] M. Fernandez, A. Gray, *Riemannian manifolds with structure group G_2*, Ann. Mat. Pura Appl. (4) **132** (1982), 19–45 (1983).

[9] Th. Friedrich, *Nearly Kähler and nearly parallel G₂-structures on spheres*, Arch. Math. Brno **42** (2006), Supplement, 241–243.

[10] Th. Friedrich, S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), no. 2, 303–336.

[11] Th. Friedrich, I. Kath, *7-dimensional compact Riemannian manifolds with Killing spinors*, Commun. Math. Phys. **133** (1990), 543–561.

[12] Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, *On nearly parallel G_2-structures*, J. Geom. Phys. **23** (1997), no. 3-4, 259–286.

[13] A. Gray, *Vector cross products on manifolds*, Trans. Amer. Math. Soc. **141** (1969), 463–504. Correction, **148** (1970), 69-75.

[14] D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[15] S. Kobayashi, *Transformation groups in differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[16] A. Moroianu, P-A. Nagy, U. Semmelmann, *Deformations of nearly Kähler structures*, Pacific J. Math. **235** (2008), no. 1, 57–72.

[17] A. Moroianu, U. Semmelmann, *Parallel Spinors and holonomy groups*, J. Math. Phys. **41** (2000), 2395-2402.

[18] A. Moroianu, U. Semmelmann, *Infinitesimal Einstein Deformations of Nearly Kähler Metrics*, Pacific J. Math. **235** (2008), 57-72.

[19] A. Moroianu, U. Semmelmann, *The Hermitian Laplace Operator on Nearly Kähler Manifolds*, Commun. Math. Phys. **294** (2010), 251-272.

[20] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Math. Z. **245** (2003), no. 3, 503–527.

[21] M. Wang, *On non-simply connected manifolds with non-trivial parallel spinors*, Ann. Global Anal. Geom. **13** (1995), 31–42.

[22] B. Wilking, *The normal homogeneous space $(\mathrm{SU}(3) \times \mathrm{SO}(3))/\mathrm{U}^{\bullet}(2)$ has positive sectional curvature*, Proc. Amer. Math. Soc. **127** (1999), no. 4, 1191–1194.

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