ALMOST COMPLEX STRUCTURES ON QUATERNION-KÄHLER MANIFOLDS AND INNER SYMMETRIC SPACES

PAUL GAUDUCHON, ANDREI MOROIANU, UWE SEMMELMANN

ABSTRACT. We prove that compact quaternionic-Kähler manifolds of positive scalar curvature admit no almost complex structure, even in the weak sense, except for the complex Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{n+2})$. We also prove that irreducible inner symmetric spaces M^{4n} of compact type are not weakly complex, except for spheres and Hermitian symmetric spaces.

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1. Introduction

It is a well-known fact that the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ have no almost complex structure. The proof goes back to F. Hirzebruch in 1953 for $n \geq 4$ (cf. [12]). The non-existence of almost-complex structure on $\mathbb{H}\mathrm{P}^1 = \mathrm{S}^4$ had been established a few years earlier by Ch. Ehresmann [10] and H. Hopf [15]. According to Hirzebruch's lecture at the 1958 ICM [13], J. Milnor had in the meantime settled the remaining cases $n = 2$ and 3, but his proof has remained unpublished. Later on, W.S. Massey [20] gave an original proof of the non-existence of almost-complex structure on $\mathbb{H}\mathrm{P}^n$, for any n, based on the explicit calculation of the ring $\mathrm{K}(X)$ and of the Chern character $\mathrm{ch}(\mathrm{T}X)$ for $X = \mathbb{H}\mathrm{P}^n$.

Quaternionic projective spaces are particular examples of *quaternion-Kähler manifolds.* These, we recall, are (oriented) $4n$-dimensional Riemannian manifolds, whose holonomy is contained in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$, if $n > 1$, or, if $n = 1$, (oriented) Einstein, self-dual 4-dimensional Riemannian manifolds. In all dimensions $4n$, $n \geq 1$, quaternion-Kähler manifolds are Einstein and are called *of positive type* if their scalar curvature is positive. In this paper, we only consider quaternion-Kähler of positive type and we implicitly assume that they are complete, hence compact.

For $n \geq 2$, the above definition of quaternion-Kähler manifolds is equivalent to the existence of locally defined almost complex structures I, J, K, satisfying the quaternion relations and spanning a global rank 3 sub-bundle $\mathrm{Q} \subset \mathrm{End}(\mathrm{T}M)$, which is preserved by the Levi-Civita connection. Almost complex structures on M which are sections of Q are called *compatible.* In [2], it is shown that quaternion-Kähler manifolds of positive type admit no *compatible* almost complex structure. In particular the natural complex structure of the complex Grassmannians

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$\mathrm{Gr}_2(\mathbb{C}^{n+2})$, which constitute a well-known class of quaternion-Kähler manifolds of positive type (cf. below), is not compatible.

The first main result of this paper is:

Theorem 1.1. *Let M^{4n}, $n \geq 2$, be a compact quaternion-Kähler manifold of positive type, which is not isometric to the complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{n+2})$. Then M^{4n} has no weak almost complex structure, in the sense that the tangent bundle $\mathrm{T}M$ is not stably isomorphic to a complex vector bundle.*

Notice that the assumption $n \geq 2$ is necessary, since $\mathbb{H}\mathrm{P}^1 = \mathrm{S}^4$ is weakly complex but not almost complex.

At the moment, the only known quaternion-Kähler manifolds of positive type are the so-called (symmetric) *Wolf spaces*, namely [28]:

(i) the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}$,
(ii) the Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{n+2}) = \frac{\mathrm{U}(n+2)}{\mathrm{U}(n)\times\mathrm{U}(2)}$ of complex 2-planes in $\mathbb{C}^{n+2}$,
(iii) the real Grassmannians $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{n+4}) = \frac{\mathrm{SO}(n+4)}{\mathrm{SO}(n)\times\mathrm{SO}(4)}$, of oriented real 4-planes in $\mathbb{R}^{n+4}$, and
(iv) the five exceptional spaces $\frac{\mathrm{G}_2}{\mathrm{SO}(4)}$, $\frac{\mathrm{F}_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}$, $\frac{\mathrm{E}_6}{\mathrm{SU}(6)\mathrm{Sp}(1)}$, $\frac{\mathrm{E}_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)}$, $\frac{\mathrm{E}_8}{\mathrm{E}_7\mathrm{Sp}(1)}$, in dimensions $4n$ with $n = 2, 7, 10, 16$ and 28 respectively.

According to Theorem 1.1, none of them admits a (weakly) complex structure except for the complex Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{n+2})$. Note however that this was already known for $\mathbb{H}\mathrm{P}^n$, as mentioned above, and also for most real oriented Grassmannians $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{n+4})$. Indeed, the non-existence of (weakly) complex structures on a large class of real Grassmannians, including all $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{n+4})$ except for $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^8)$ and $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{10})$, was shown in [24] by P. Sankaran and in [27] by Z.-Z. Tang.

Wolf spaces are (irreducible, simply-connected) *inner* symmetric spaces of compact type, i.e. are symmetric spaces of the form G/H, where G, H are connected compact Lie groups of *equal rank*. Apart from the Wolf spaces, the class of simply connected irreducible inner symmetric spaces of compact type includes, cf. e.g. [11], [4], [7]:

(i) the class of (irreducible) Hermitian symmetric spaces of compact type ;
(ii) the even-dimensional spheres $\mathrm{S}^{2n} = \frac{\mathrm{SO}(2n+1)}{\mathrm{SO}(2n)}$, $n > 1$;
(iv) the even-dimensional oriented real Grassmannians $\widetilde{\mathrm{Gr}}_{2p}(\mathbb{R}^{n+2p}) = \frac{\mathrm{SO}(n+2p)}{\mathrm{SO}(n)\times\mathrm{SO}(2p)}$, $n > 1$;
(v) the quaternionic Grassmannians $\mathrm{Gr}_k(\mathbb{H}^{k+n}) = \frac{\mathrm{Sp}(n+k)}{\mathrm{Sp}(n)\,\mathrm{Sp}(k)}$, $n, k > 1$;
(iii) the Cayley projective plane $\frac{\mathrm{F}_4}{\mathrm{Spin}(9)}$;
(vi) the two exceptional inner symmetric spaces $\frac{\mathrm{E}_7}{\mathrm{SU}(8)/\mathbb{Z}_2}$ and $\frac{\mathrm{E}_8}{\mathrm{Spin}(16)/\mathbb{Z}_2}$.

Notice that all spaces in this list are even-dimensional, cf. Section 3. In the second part of this paper, we show that the techniques introduced in the proof of our main Theorem 1.1 can be used to establish a similar non-existence theorem for inner symmetric spaces of compact type. More precisely, we have:

Theorem 1.2. *A $4n$-dimensional simply connected irreducible inner symmetric space of compact type is weakly complex if and only if it is a sphere or a Hermitian symmetric space.*

Recall that, for any n, the sphere S^n is stably parallelizable, hence weakly complex, whereas Hermitian symmetric spaces are complex manifolds in a natural way.

Our method uses a unified argument based on the index calculation of a twisted Dirac operator, via Theorem 3.1 and Proposition 3.2 below. On the other hand, our approach is ineffective for *non-inner* symmetric spaces, due to the fact, established by R. Bott [6], that the index of any homogeneous differential operator vanishes on any non-inner symmetric space of compact type. For reasons which will be explained in Section 3, cf. in particular Remarks 3.3, 3.5, 3.7, it is also ineffective for inner symmetric spaces of dimension $4n+2$. In the above list, these are: (i) Hermitian symmetric spaces of odd complex dimension; (ii) oriented real Grassmannians $\widetilde{\mathrm{Gr}}_{2p}(\mathbb{R}^{2p+q})$, whith p and q both odd; (iii) the exceptional symmetric space $\frac{\mathrm{E}_7}{\mathrm{SU}(8)/\mathbb{Z}_2}$ (which is of dimension 70). The non-existence of weakly complex structure for all oriented real Grassmannians, except for $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^8)$, $\widetilde{\mathrm{Gr}}_6(\mathbb{R}^{12})$ and $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{10})$, in particular for all oriented real Grassmannians of dimensions $4n+2$, was established, by different methods, by P. Sankaran in [24] and by Z.-Z. Tang in [27]. Together with these results, our Theorem 1.2 then covers all simply connected irreducible inner symmetric spaces, except for the exceptional symmetric space $\frac{\mathrm{E}_7}{\mathrm{SU}(8)/\mathbb{Z}_2}$, for which, as far as we know, the existence of a (weak) almost complex structure has remained an open question. Notice that the non-existence of (weak) almost complex structure on quaternionic Grassmannians was previously established, by a different approach, by W. C. Hsiang and R. H. Szczarba in [16]. Note also that A. Borel and F. Hirzebruch [5] have shown that the tangent bundle of the Cayley projective plane has no almost complex structure, but their proof does not exclude the possibility for that bundle to being weakly complex.

The irreducibility assumption in Theorem 1.2 can easily be dropped. Indeed, the de Rham decomposition Theorem implies that any simply connected inner symmetric space can be written as a product of irreducible inner symmetric spaces. Using the fact that a product $M \times N$ is weakly complex if and only if both factors are weakly complex — the restriction of the tangent bundle of a product to each factor being stably isomorphic to the tangent bundle of that factor — and by taking into account the above observations, we thus obtain the following generalization of Theorem 1.2:

Theorem 1.3. *An irreducible component of a simply connected inner symmetric space of compact type admitting a weak almost complex structure is isomorphic to an even-dimensional sphere, or to a Hermitian symmetric space or (conceivably) to the exceptional symmetric space* $\frac{\mathrm{E}_7}{\mathrm{SU}(8)/\mathbb{Z}_2}$.

Our method gives however no information concerning the existence of genuine almost complex structures on products of even-dimensional spheres and Hermitian symmetric spaces (in contrast, the product $\mathrm{S}^{2p+1} \times \mathrm{S}^{2q+1}$ of two odd-dimensional spheres admits *integrable* almost complex structures [9]). Theorem 1.3 can be viewed as a topological version of the well-known fact that an inner symmetric space of compact type which admits an *integrable* almost complex structure *compatible with the invariant metric* has to be Hermitian symmetric [7], [8].

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2. Proof of Theorem 1.1

For notation and basic properties of quaternion-Kähler manifolds we refer to [22] and [23]. Let (M, g) be a $4n$-dimensional quaternion-Kähler manifold of positive type, $n \geq 1$. Since the holonomy group of M is contained in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, the standard representations of $\mathrm{Sp}(n)$ on $\mathbb{C}^{2n}$ and of $\mathrm{Sp}(1)$ on $\mathbb{C}^2$ give rise to locally defined complex vector bundles, denoted by E and H respectively. These are globally defined only on the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ ([22, Theorem 6.3]). However, tensor products of any *even* number of copies of H and E are globally defined complex bundles over any quaternion-Kähler manifold M.

It is well known that the complexified tangent bundle of M is given as $\mathrm{T}M^{\mathbb{C}} = \mathrm{E} \otimes \mathrm{H}$. Recall that a quaternion-Kähler manifold M^{4n} of positive type is spin if and only if either $M^{4n} = \mathbb{H}\mathrm{P}^n$, or the quaternionic dimension n is even ([22, Proposition 2.3]). If this holds, the spinor bundle ΣM decomposes as the direct sum of $\mathrm{R}^{p,q} := \mathrm{Sym}^p\mathrm{H} \otimes \Lambda^q_0\mathrm{E}$ over all positive integers p, q with $p + q = n$, cf. e.g. [18, Proposition 2.1]. Here $\Lambda^q_0\mathrm{E}$ denotes the sub-bundle of $\Lambda^q\mathrm{E}$ defined as the kernel of the contraction with the symplectic form of E. In particular, the twisted spin bundles $\Sigma^\pm M \otimes \mathrm{R}^{p,q}$ are globally defined whenever $p + q + n$ is even. We then denote by $\mathrm{D}_{\mathrm{R}^{p,q}}$ be the (twisted) Dirac operator defined on sections of $\Sigma^+ M \otimes \mathrm{R}^{p,q}$ and by $\mathrm{ind}(\mathrm{D}_{\mathrm{R}^{p,q}})$ the index of $\mathrm{D}_{\mathrm{R}^{p,q}}$.

Our argument crucially relies on the following result of C. LeBrun and S. Salamon [19, Theorem 5.1] (cf. also [25]):

$$\mathrm{ind}(\mathrm{D}_{\mathrm{R}^{p,q}}) = \begin{cases} 0 & \text{for} \quad p + q < n \\ (-1)^q\,(\mathrm{b}_{2q}(M) + \mathrm{b}_{2q-2}(M)) & \text{for} \quad p + q = n\ , \end{cases} \tag{1}$$

where $\mathrm{b}_i(M)$ denote the Betti numbers of M. Consider the twist bundle $V = \mathrm{Sym}^{n-2}\mathrm{H} \otimes \mathrm{T}M^{\mathbb{C}}$ (it is here that the assumption $n \geq 2$ is needed). The Clebsch-Gordan decomposition yields

$$V = (\mathrm{Sym}^{n-1}\mathrm{H} \otimes \mathrm{E}) \,\oplus\, (\mathrm{Sym}^{n-3}\mathrm{H} \otimes \mathrm{E})\ .$$

The bundle $\Sigma M \otimes V$ is globally defined for all quaternionic dimensions n and we can therefore compute the index $\mathrm{ind}(\mathrm{D}_V)$ of the corresponding twisted Dirac operator by using (1). We thus obtain

$$\mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-2}\mathrm{H} \otimes \mathrm{T}M^{\mathbb{C}}}) = \mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-1}\mathrm{H} \otimes \mathrm{E}}) + \mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-3}\mathrm{H} \otimes \mathrm{E}}) = -(\mathrm{b}_2(M) + \mathrm{b}_0(M))\ . \tag{2}$$

A key fact, cf. [19, Corollary 4.3], is that $\mathrm{b}_2(M) = 0$ for all compact quaternion-Kähler manifold M of positive type other than the complex Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{n+2})$, whereas $\mathrm{b}_2(M) = 1$ if $M = \mathrm{Gr}_2(\mathbb{C}^{n+2})$, which, as already observed, has a natural complex structure.

We now assume that M is different from $\mathrm{Gr}_2(\mathbb{C}^{n+2})$, so that $\mathrm{b}_2(M) = 0$. The above index calculation then reads

$$\mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-2}\mathrm{H}\otimes \mathrm{T}M^{\mathbb{C}}}) \;=\; -1\;. \tag{3}$$

Assume, for a contradiction, that M carries an almost complex structure. Then the tangent bundle $\mathrm{T}M$ is a complex vector bundle and its complexification splits into the sum of two complex sub-bundles $\mathrm{T}M^{\mathbb{C}} = \theta \oplus \theta^*$. For the components of the Chern character we have $\mathrm{ch}_i(\theta^*) = (-1)^i \mathrm{ch}_i(\theta)$. On the other hand, $\mathrm{ch}(\mathrm{Sym}^{n-2}\mathrm{H})$ and $\hat{\mathrm{A}}(\mathrm{T}M)$ have non-zero components only in degree $4k$. Indeed, $\hat{\mathrm{A}}(\mathrm{T}M)$ is a polynomial in the Pontryagin classes of M and $\mathrm{Sym}^{n-2}\mathrm{H}$ is a self-dual locally defined complex bundle. The Atiyah-Singer formula for twisted Dirac operators (cf. [3]) then yields

$$\begin{aligned}\mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-2}\mathrm{H}\otimes \mathrm{T}M^{\mathbb{C}}}) =& \mathrm{ch}(\mathrm{Sym}^{n-2}\mathrm{H})\mathrm{ch}(\mathrm{T}M^{\mathbb{C}})\hat{\mathrm{A}}(\mathrm{T}M)[M]\\ =& 2\,\mathrm{ch}(\mathrm{Sym}^{n-2}\mathrm{H})\mathrm{ch}(\theta)\hat{\mathrm{A}}(\mathrm{T}M)[M]\;.\end{aligned} \tag{4}$$

Notice that $\mathrm{ch}(\mathrm{Sym}^{n-2}\mathrm{H})$ is well-defined in $H^*(M,\mathbb{Q})$, even if n is odd.

Now, $\mathrm{ch}(\mathrm{Sym}^{n-2}\mathrm{H})\mathrm{ch}(\theta)\hat{\mathrm{A}}(\mathrm{T}M)[M]$ is the index of the twisted Dirac operator $\mathrm{D}_{\mathrm{Sym}^{n-2}\mathrm{H}\otimes\theta}$ on the (globally defined) bundle $\Sigma M \otimes \mathrm{Sym}^{n-2}\mathrm{H} \otimes \theta$ and thus has to be an integer. This implies that that $\mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-2}\mathrm{H}\otimes \mathrm{T}M^{\mathbb{C}}})$ is *even*, hence contradicts (3).

If the manifold is assumed to be weakly complex then there exists a trivial real vector bundle ε such that $\mathrm{T}M \oplus \varepsilon$ is a complex vector bundle. By replacing $V = \mathrm{Sym}^{n-2}\mathrm{H} \otimes \mathrm{T}M^{\mathbb{C}}$ with $V = \mathrm{Sym}^{n-2}\mathrm{H} \otimes (\mathrm{T}M \oplus \varepsilon)^{\mathbb{C}}$ in the above argument, this remains unchanged, as the extra term

$$\mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-2}\mathrm{H}\otimes\varepsilon^{\mathbb{C}}}) = \mathrm{rk}(\varepsilon)\,\mathrm{ind}(\mathrm{D}_{\mathrm{Sym}^{n-2}\mathrm{H}})$$

in (2) is zero, again because of (1). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

We first establish a general formula, of separate interest, for the index of a family of homogeneous twisted Dirac operators defined on inner symmetric spaces.

Let $M = G/K$ be an irreducible inner symmetric space of compact type, where G denotes a (connected) compact simple Lie group and K a connected closed subgroup of G. Notice that the condition implies that M is even-dimensional. We fix a common maximal torus $T \subset K \subset G$ and we endow the dual Lie algebra $\mathfrak{t}^*$ with a suitable positive definite scalar product $\langle\cdot,\cdot\rangle$, proportional to the one induced by the opposite of the Killing form of G. We denote by $\rho^{\mathfrak{k}}$ and $\rho^{\mathfrak{g}}$ the half-sum of the positive roots of K and G respectively.

The isotropy representation $K \to \mathrm{SO}(\mathfrak{m})$ induces a group homomorphism $\tilde{K} \to \mathrm{Spin}(\mathfrak{m})$ and thus a representation of $\tilde{K}$ on the spin modules $\Sigma^{\pm}_{\mathfrak{m}}$, where $\tilde{K}$ stands for K itself or a two-fold covering of K. Let V_μ be a complex representation of $\tilde{K}$ with highest weight $\mu \in \mathfrak{t}^*$. We assume that the induced representation of $\tilde{K}$ on $V_\mu \otimes \Sigma^{\pm}_{\mathfrak{m}}$ descends to a representation of K. We then denote by $\Sigma^{\pm}_{\mu} M := G \times_K (V_\mu \otimes \Sigma^{\pm}_{\mathfrak{m}})$ the corresponding *twisted spin bundles* and by D_{V_μ} the *twisted Dirac operator* acting on sections of $\Sigma^{\pm}_{\mu} M := G \times_K (V_\mu \otimes \Sigma^{\pm}_{\mathfrak{m}})$.

Theorem 3.1. *Let $w \in W_{\mathfrak{g}}$ be a Weyl group element for which $w(\mu+\rho^\kappa)-\rho^{\mathfrak{g}}$ is $\mathfrak{g}$-dominant. Then the index of the twisted Dirac operator $\mathrm{D}_{V_\mu} : C^\infty(\Sigma^+_\mu M) \to C^\infty(\Sigma^-_\mu M)$ is given by the formula*

$$\mathrm{ind}(\mathrm{D}_{V_\mu}) \;=\; \prod_{\alpha\in\mathcal{R}^+} \frac{\langle \mu+\rho^\kappa, \alpha\rangle}{\langle \rho^{\mathfrak{g}}, \alpha\rangle} =: \mathrm{i}(\mu), \tag{5}$$

where the product goes over all positive roots of G. If such a w does not exist the index is zero.

Proof. The generalized Bott-Borel-Weil theorem (cf. [17], Theorem 4.5.1) states that the kernel of D_{V_μ} is an irreducible G-representation of highest weight $w(\mu+\rho^\kappa)-\rho^{\mathfrak{g}}$, where w is as in the assumption of the theorem. Note that w is unique as soon as it exists. If such a w does not exist, the kernel is zero. It follows that the index of D_{V_μ} is given as $(-1)^{l(w)} \dim V_{w(\mu+\rho^\kappa)-\rho^{\mathfrak{g}}}$, or zero if such a w does not exist (cf. [17], Corollary 4.5.2). Here $l(w)$ is the length of the Weyl group element w, defined as the number of positive roots α such that $w(\alpha)$ is a negative root. It is also the smallest integer k for which w can be written as the product of k reflections in simple roots. Thus the length of w is the same as the length of $w^{-1}=w^t$. The Weyl dimension formula implies

$$\dim(V_{w(\mu+\rho^\kappa)-\rho^{\mathfrak{g}}}) = \prod_{\alpha\in\mathcal{R}^+} \frac{\langle w(\mu+\rho^\kappa), \alpha\rangle}{\langle \rho^{\mathfrak{g}}, \alpha\rangle} = \prod_{\alpha\in\mathcal{R}^+} \langle \rho^{\mathfrak{g}}, \alpha\rangle^{-1} \prod_{\alpha\in\mathcal{R}^+} \langle \mu+\rho^\kappa, w^t(\alpha)\rangle \tag{6}$$

If we replace in the set $\{w^t(\alpha) |\, \alpha\in\mathcal{R}^+\}$ the $l(w)$ roots which are mapped by w^t to negative roots by their negative, we obtain again the set $\mathcal{R}^+$ of positive roots. Hence

$$\prod_{\alpha\in\mathcal{R}^+} \langle \mu+\rho^\kappa, w^t(\alpha)\rangle = (-1)^{l(w)} \prod_{\alpha\in\mathcal{R}^+} \langle \mu+\rho^\kappa, \alpha\rangle.$$

Substituting this into formula (6) and using $\mathrm{ind}(\mathrm{D}_{V_\mu}) = (-1)^{l(w)} \dim V_{w(\mu+\rho^\kappa)-\rho^{\mathfrak{g}}}$ completes the proof of the theorem. ☐

The following criterion, extracted from the proof of Theorem 1.1 in Section 2, provides a general obstruction for the tangent bundle of a compact manifold to being weakly complex.

Proposition 3.2. *Let (M^{4n}, g) be a compact Riemannian manifold carrying a locally defined complex vector bundle E such that the following conditions hold:*

(a) *E is self-dual, i.e. E is isomorphic to its dual bundle E^*.*
(b) *$E\otimes\Sigma M$ is globally defined, where ΣM denotes the (locally defined) spin bundle of M.*
(c) *The index of the twisted Dirac operator $\mathrm{D}_{E\otimes \mathrm{T}M^{\mathbb{C}}}$ is odd.*

Then the tangent bundle of M is not weakly complex and in particular M cannot carry an almost complex structure.

Proof. The Atiyah-Singer index formula for twisted Dirac operators (cf. [3]) reads

$$\mathrm{ind}(\mathrm{D}_V) = \mathrm{ch}(V)\hat{\mathrm{A}}(\mathrm{T}M)[M] \tag{7}$$

for every complex bundle V such that $V\otimes\Sigma M$ is globally defined.

Assume that $\mathrm{T}M$ is weakly complex, i.e. there exists a trivial real vector bundle ε (of even rank) such that $\mathrm{T}M \oplus \varepsilon$ is a complex vector bundle. Then its complexification splits into the sum of two complex bundles $(\mathrm{T}M \oplus \varepsilon)^{\mathbb{C}} = \theta \oplus \theta^*$.

Since $\hat{\mathrm{A}}(\mathrm{T}M)$ is a polynomial in the Pontryagin classes of M, it has non-zero components only in degree $4k$. Condition (a) shows that the Chern character $\mathrm{ch}(E)$ has the same property. Moreover, the components of the Chern characters of θ and θ^* satisfy $\mathrm{ch}_i(\theta^*) = (-1)^i \mathrm{ch}_i(\theta)$. Applying (7) to $V = E \otimes \theta$ and $V = E \otimes \theta^*$ yields

$$\mathrm{ind}(\mathrm{D}_{E\otimes\theta}) = \mathrm{ch}(E)\mathrm{ch}(\theta)\hat{\mathrm{A}}(\mathrm{T}M)[M] = \mathrm{ch}(E)\mathrm{ch}(\theta^*)\hat{\mathrm{A}}(\mathrm{T}M)[M] = \mathrm{ind}(\mathrm{D}_{E\otimes\theta^*}). \tag{8}$$

Using this equation, the condition that $\mathrm{rk}(\varepsilon)$ is even and assumption (b), we infer

$$\begin{aligned} \mathrm{ind}(\mathrm{D}_{E\otimes \mathrm{T}M^{\mathbb{C}}}) &\equiv \mathrm{ind}(\mathrm{D}_{E\otimes \mathrm{T}M^{\mathbb{C}}}) + \mathrm{rk}(\varepsilon)\mathrm{ind}(\mathrm{D}_E) \mod 2 \\ &\equiv \mathrm{ind}(\mathrm{D}_{E\otimes(\mathrm{T}M^{\mathbb{C}}\oplus\varepsilon^{\mathbb{C}})}) \mod 2 \\ &\equiv \mathrm{ind}(\mathrm{D}_{E\otimes\theta}) + \mathrm{ind}(\mathrm{D}_{E\otimes\theta^*}) \mod 2 \\ &\equiv 2\mathrm{ind}(\mathrm{D}_{E\otimes\theta}) \equiv 0 \mod 2, \end{aligned}$$

contradicting (c). This proves the proposition. □

Remark 3.3. Notice that in dimension $4n+2$, there exists no local bundle E satisfying conditions (a)–(c). Indeed, if E is self-dual, the Chern character of $E \otimes \mathrm{T}M^{\mathbb{C}}$ has non-zero components only in degree $4k$. Formula (7) applied to $V = E \otimes \mathrm{T}M^{\mathbb{C}}$ then shows that the index of the twisted Dirac operator $\mathrm{D}_{E\otimes \mathrm{T}M^{\mathbb{C}}}$ vanishes.

We now check that the criterion given by Proposition 3.2 applies to all $4n$-dimensional (simply connected) irreducible inner symmetric spaces of compact type, using as main tool the formula (5) in Theorem 3.1. We focus on those cases which were not fully covered by previous works, namely the oriented real Grassmannians $\widetilde{\mathrm{Gr}}_{2p}(\mathbb{R}^{2p+q})$ with either p or q even (of dimension $2pq$), the exceptional inner symmetric space $\mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ (of dimension 128), and the Cayley projective plane $\mathrm{F}_4/\mathrm{Spin}(9)$ (of dimension 16). The quaternionic Grassmannians can be handled with quite similar methods.

3.1. **The oriented real Grassmannians $\widetilde{\mathrm{Gr}}_{2p}(\mathbb{R}^{2p+q})$.**

3.1.1. *Case I: $q = 2q'$ is even.* Since for $p = 1$ or $q' = 1$ the Grassmannian of oriented 2-planes is a Hermitian symmetric space, we assume $p, q' \geq 2$. The symmetric space $M = G/K := \mathrm{SO}(2p+2q')/\mathrm{SO}(2p) \times \mathrm{SO}(2q')$ is spin (cf. [14]). Let H and H' denote the tautological bundles over M, associated to the standard representations of $\mathrm{SO}(2p) \times \mathrm{SO}(2q')$ on $\mathbb{R}^{2p}$ and $\mathbb{R}^{2q'}$ respectively. It is well-known that $\mathrm{T}M$ is isomorphic to $H \otimes H'$ (cf. [4], p. 312).

The root system of G consists of the vectors $\pm e_i \pm e_j$, $1 \leq i < j \leq p+q'$. We choose as fundamental Weyl chamber the one containing the vector $\sum_{i=1}^{p+q'}(p+q'-i)e_i$. The positive roots are then $e_i \pm e_j$, $1 \leq i < j \leq p+q'$, and their half-sum is

$$\rho^{\mathfrak{g}} = \sum_{i=1}^{p+q'}(p+q'-i)e_i.$$

The root system of K is the direct sum of the root systems of $\mathrm{SO}(2p)$ and $\mathrm{SO}(2q')$ and thus

$$\rho^{\mathfrak{k}} = \sum_{i=1}^{p}(p-i)e_i + \sum_{j=1}^{q'}(q'-j)e_{p+j}.$$

Let E be the complex vector bundle over M associated to the complex representation of K with highest weight

$$\mu = (q'-1)\sum_{i=1}^{p} e_i + (p-2)e_{p+1}. \tag{9}$$

In other words, E is the Cartan component in the tensor product of the $(q'-1)$-th symmetric power of $\Lambda^p H^{\mathbb{C}}$ and the $(p-2)$-th symmetric power of $(H')^{\mathbb{C}}$. It clearly satisfies (a) and (b) in Proposition 3.2, and we claim that it also satisfies (c).

To see this, we need to compute the decomposition in irreducible summands of $E \otimes \mathrm{T}M^{\mathbb{C}}$. This is given by the following standard facts:

Lemma 3.4. *The tensor product of the complex irreducible* $\mathrm{Spin}(2r)$ *representations with highest weights* e_1 *and* $k(e_1+\ldots+e_r)$ *is the direct sum of representations with highest weights* $(k+1)e_1 + k(e_2+\ldots+e_r)$ *and* $k(e_1+\ldots+e_{r-1}) + (k-1)e_r$.

The tensor product of the complex irreducible $\mathrm{Spin}(2r)$ *representations with highest weights* e_1 *and* ke_1 *is the direct sum of representations with highest weights* $(k+1)e_1$, ke_1+e_2 *and* $(k-1)e_1$.

Proof. The statements of the lemma follow from a routine decomposition of tensor products. However, in this special case, where one of the factors is the standard representation of $\mathrm{Spin}(2r)$, the decomposition can directly be read off the table given in [26] p. 511. ☐

The complexified bundles $H^{\mathbb{C}}$ and $(H')^{\mathbb{C}}$ are associated to the irreducible $\mathrm{SO}(2p)\times\mathrm{SO}(2q')$ representations with highest weights e_1 and e_{p+1}. (This is where the hypothesis $p, q' \geq 2$ is used: The complexification of the standard representation of $\mathrm{SO}(2)$ on $\mathbb{R}^2$ is reducible!) Lemma 3.4 shows that $E \otimes \mathrm{T}M^{\mathbb{C}}$ is associated to the direct sum of representations with highest weights

$$\mu_1 = q'e_1 + (q'-1)\sum_{i=2}^{p} e_i + (p-1)e_{p+1}, \qquad \mu_2 = (q'-1)\sum_{i=1}^{p-1} e_i + (q'-2)e_p + (p-1)e_{p+1},$$

$$\mu_3 = q'e_1 + (q'-1)\sum_{i=2}^{p} e_i + (p-2)e_{p+1} + e_{p+2}, \quad \mu_4 = (q'-1)\sum_{i=1}^{p-1} e_i + (q'-2)e_p + (p-2)e_{p+1} + e_{p+2},$$

$$\mu_5 = q'e_1 + (q'-1)\sum_{i=2}^{p} e_i + (p-3)e_{p+1}, \qquad \mu_6 = (q'-1)\sum_{i=1}^{p-1} e_i + (q'-2)e_p + (p-3)e_{p+1}.$$

It is clear that the coordinates of $\mu_1 + \rho^{\mathfrak{k}}$ are a permutation of the coordinates of $\rho^{\mathfrak{g}}$, so $i(\mu_1) = \pm 1$. Moreover, $i(\mu_i) = 0$ for $2 \leq i \leq 6$ since $\mu_i + \rho^{\mathfrak{k}}$ has two equal coordinates in each case. By Theorem 3.1, condition (c) in Proposition 3.2 is satisfied, hence the tangent bundle of M is not weakly complex.

3.1.2. *Case II: $p = 2p'$ is even.* We can assume that q is odd since the case where q is even is included in the previous one. The manifold $M = \mathrm{SO}(2p+q)/\mathrm{SO}(2p) \times \mathrm{SO}(q)$ is not spin (cf. [14]). Nevertheless, the tensor product of the (locally defined) spin bundle of M and any odd tensor power of the locally defined spin bundle of H is globally defined. For $q \geq 3$ we take E as the locally defined complex vector bundle over M associated to the complex representation of $\mathrm{Spin}(2p) \times \mathrm{Spin}(q)$ with highest weight

$$\mu = (\tfrac{q}{2} - 1) \sum_{i=1}^{p} e_i + (p-2)e_{p+1} \tag{10}$$

(incidentally this is exactly the same formula as (9)). Like before, Theorem 3.1 shows that the index of the Dirac operator twisted with $E \otimes \mathrm{T}M^{\mathbb{C}}$ is ± 1. Moreover, E is self-dual, being associated to the Cartan component of the tensor product of self-dual representations: The complexification of the standard representation of $\mathrm{SO}(q)$ and the spin representation of $\mathrm{Spin}(4p')$. Proposition 3.2 thus shows that M is not weakly complex.

The argument does not apply for $q = 1$ since μ is no longer a highest weight in that case. This is of course compatible with the fact that the sphere $S^{2p} = \mathrm{SO}(2p+1)/\mathrm{SO}(2p)$ is weakly complex, having stably trivial tangent bundle.

Remark 3.5. The above construction can also be carried out verbatim on the remaining even-dimensional oriented real Grassmannians $\widetilde{\mathrm{Gr}}_{2p}(\mathbb{R}^{2p+q})$, when p and q are both odd, by choosing for E the locally defined complex vector bundle associated to the complex representation of $\mathrm{Spin}(2p) \times \mathrm{Spin}(q)$ with highest weight given by (10). In other words, E is the Cartan component in the tensor product of the $(q-2)$-th symmetric power of the spin representation Σ_{2p}^{+} and the $(p-2)$-th symmetric power of $(H')^{\mathbb{C}}$. The same argument shows that the index of the corresponding twisted Dirac operator is ± 1. However, the bundle E is no longer self-dual if p and q are both odd, since the spin representation of Σ_{2p}^{+} – and thus its $(q-2)$-th symmetric power – is not self-dual in this case.

3.2. **The exceptional symmetric space $M = \mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$.** The group $\mathbb{Z}_2$ acting on $\mathrm{Spin}(16)$ is generated by the volume element $v := e_1 \ldots e_{16} \in \mathrm{Spin}(16)$ (cf. [1]). The positive half-spin representation factors through v and induces a representation of $\mathrm{Spin}(16)/\mathbb{Z}_2$ on Σ_{16}^{+} whose associated bundle is just $\mathrm{T}M^{\mathbb{C}}$. Since v maps to $-\mathrm{id} \in \mathrm{SO}(16)$, the representation of $\mathrm{Spin}(16)$ on $\mathbb{R}^{16}$ defined by the spin covering $\xi : \mathrm{Spin}(16) \to \mathrm{SO}(16)$ induces a locally defined real vector bundle H on M. Of course, all even tensor products of H are globally defined vector bundles on M. Moreover, the manifold M is spin (cf. [14]). Conditions (a) and (b) in Proposition 3.2 are thus satisfied for $E = \mathrm{Sym}_0^{2k} H^{\mathbb{C}}$, i.e. the Cartan summand of the $2k$-th tensor power of $H^{\mathbb{C}}$, associated to the representation of $\mathrm{Spin}(16)$ with highest weight $(2k, 0, 0, 0, 0, 0, 0, 0) \in \mathfrak{t}^* \simeq \mathbb{R}^8$.

We will use Theorem 3.1 in order to compute the index of the Dirac operator on M twisted with $\mathrm{T}M^{\mathbb{C}} \otimes \mathrm{Sym}_0^{2k} H^{\mathbb{C}}$. Since $\mathrm{T}M^{\mathbb{C}}$ is associated to the positive half-spin representation, whose highest weight is $\frac{1}{2}(1,1,1,1,1,1,1,1)$, we need to decompose $\mathrm{Sym}_0^{2k}H^{\mathbb{C}} \otimes \Sigma_{16}^+$ into irreducible components.

Lemma 3.6. $\mathrm{Sym}_0^{2k}H^{\mathbb{C}} \otimes \Sigma_{16}^+$ *is the direct sum of the* $\mathrm{Spin}(16)$*-representations with highest weights* $\frac{1}{2}(4k+1,1,1,1,1,1,1,1)$ *and* $\frac{1}{2}(4k-1,1,1,1,1,1,1,-1)$.

Proof. Again the decomposition follows from a standard calculation, where in this case the result can also be found in [21], p. 303. □

The root system $\mathcal{R}(\mathrm{E}_8)$ is the disjoint union of the root system of $\mathrm{Spin}(16)$ and the weights of the half-spin representation Σ_{16}^+. It thus consists of the vectors $\pm e_i \pm e_j$, $1 \le i < j \le 8$ and

$$\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i, \qquad \varepsilon_i = \pm 1,\ \varepsilon_1 \cdots \varepsilon_8 = 1.$$

With respect to the fundamental Weyl chamber containing the vector $(23,6,5,4,3,2,1,0)$, the set of positive roots of $\mathrm{Spin}(16)$ is $\mathcal{R}^+(\mathrm{Spin}(16)) = \{e_i \pm e_j \mid 1 \le i < j \le 8\}$, and the set of positive roots of E_8 is

$$\mathcal{R}^+(\mathrm{E}_8) = \mathcal{R}^+(\mathrm{Spin}(16)) \cup \left\{ \frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i \mid \varepsilon_1 = 1,\ \varepsilon_i = \pm 1,\ \varepsilon_1 \cdots \varepsilon_8 = 1 \right\}.$$

The half-sums of the positive roots of $K = \mathrm{Spin}(16)/\mathbb{Z}_2$ and $G = \mathrm{E}_8$ are thus given by

$$\rho^{\mathfrak{k}} = (7,6,5,4,3,2,1,0) \quad \text{and} \quad \rho^{\mathfrak{g}} = (23,6,5,4,3,2,1,0).$$

It is clear that $\mu_1 + \rho^{\mathfrak{k}}$ is orthogonal to the root $\alpha = \frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1)$ for $\mu_1 := \frac{1}{2}(33,1,1,1,1,1,1,1)$, so the integer $\mathrm{i}(\mu)$ defined by (5) vanishes for $\mu = \mu_1$. Moreover, an elementary computation shows that $\mathrm{i}(\mu_2) = -1$ for $\mu_2 := \frac{1}{2}(31,1,1,1,1,1,1,-1)$. By Lemma 3.6, the tensor product $\mathrm{T}M^{\mathbb{C}} \otimes \mathrm{Sym}_0^{16}H^{\mathbb{C}}$ is associated to the direct sum of $\mathrm{Spin}(16)$-representations with highest weights μ_1 and μ_2. Theorem 3.1 thus shows that condition (c) in Proposition 3.2 is satisfied for $E = \mathrm{Sym}_0^{16}H^{\mathbb{C}}$, hence the tangent bundle of M is not weakly complex. Notice that the index of the Dirac operator twisted with $\mathrm{T}M^{\mathbb{C}} \otimes \mathrm{Sym}_0^{k}H^{\mathbb{C}}$ vanishes for every $k < 16$, by Lemma 3.6 and Theorem 3.1 again.

3.3. **The Cayley projective plane $M = \mathrm{F}_4/\mathrm{Spin}(9)$.** The complexified tangent bundle $\mathrm{T}M^{\mathbb{C}}$ is associated to the spin representation on $\Sigma_9 \simeq \mathbb{C}^{16}$ (cf. [4], p. 302). Let H denote the real bundle associated to the representation of $\mathrm{Spin}(9)$ on $\mathbb{R}^9$ defined by the spin covering $\mathrm{Spin}(9) \to \mathrm{SO}(9)$, with highest weight $(1,0,0,0) \in \mathfrak{t}^* \simeq \mathbb{R}^4$. It is well-known that the Cayley projective plane is spin (cf. [14]). Conditions (a) and (b) in Proposition 3.2 are thus satisfied for $E = H^{\mathbb{C}}$.

We will use Theorem 3.1 again in order to compute the index of the Dirac operator on M twisted with $\mathrm{T}M^{\mathbb{C}} \otimes H^{\mathbb{C}}$. Recall that $\Sigma_9 \otimes \mathbb{C}^9 \simeq \Sigma_9 \oplus \Sigma_9^{\frac{3}{2}}$, the two summands having highest weights $\frac{1}{2}(1,1,1,1)$ and $\frac{1}{2}(3,1,1,1)$ respectively.

The root system $\mathcal{R}(\mathrm{F}_4)$ is the disjoint union of the root system of Spin(9) and the weights of the spin representation Σ_9. It thus consists of the vectors $\pm e_i \pm e_j$, $1 \le i < j \le 4$, $\pm e_i$, $1 \le i \le 4$ and

$$\frac{1}{2}\sum_{i=1}^{4} \varepsilon_i e_i, \qquad \varepsilon_i = \pm 1.$$

With respect to the fundamental Weyl chamber containing the vector $(11, 5, 3, 1)$, the set of positive roots of Spin(9) is $\mathcal{R}^+(\mathrm{Spin}(9)) = \{e_i \pm e_j \mid 1 \le i < j \le 4\} \cup \{e_i \mid 1 \le i \le 4\}$, and the set of positive roots of F_4 is

$$\mathcal{R}^+(\mathrm{F}_4) = \mathcal{R}^+(\mathrm{Spin}(9)) \cup \left\{ \frac{1}{2}\sum_{i=1}^{4} \varepsilon_i e_i \mid \varepsilon_1 = 1,\ \varepsilon_i = \pm 1 \right\}.$$

The half-sums of the positive roots of $K = \mathrm{Spin}(9)$ and $G = \mathrm{F}_4$ are thus given by

$$\rho^{\mathfrak{k}} = \tfrac{1}{2}(7, 5, 3, 1) \quad \text{and} \quad \rho^{\mathfrak{g}} = \tfrac{1}{2}(11, 5, 3, 1).$$

It is clear that $\frac{1}{2}(1, 1, 1, 1) + \rho^{\mathfrak{k}}$ is orthogonal to the root $\alpha = \frac{1}{2}(1, -1, -1, 1)$ so the integer $\mathrm{i}(\mu)$ defined by (5) vanishes for $\mu = \frac{1}{2}(1, 1, 1, 1)$. An easy elementary computation shows that $\mathrm{i}(\mu) = -1$ for $\mu := \frac{1}{2}(3, 1, 1, 1)$. Theorem 3.1 thus shows that condition (c) in Proposition 3.2 is satisfied for $E = H^{\mathbb{C}}$, hence the tangent bundle of M is not weakly complex.

Remark 3.7. A similar argument shows that the remaining exceptional symmetric space $M = \mathrm{E}_7/(\mathrm{SU}(8)/\mathbb{Z}_2)$ (which is spin [14] and of dimension 70) also carries a complex vector bundle E satisfying conditions (b) and (c) in Proposition 3.2. More precisely, E is the 10-th symmetric power of the locally defined bundle H on M associated to the standard representation of SU(8) (like in Section 3.2, Theorem 3.1 shows that the index of the twisted Dirac operator vanishes for every lower even symmetric power $\mathrm{Sym}^{2k} H$, $0 \le k \le 4$). Of course, E is not self-dual, cf. Remark 3.3.

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Paul Gauduchon and Andrei Moroianu, CMLS, École Polytechnique, UMR 7640 du CNRS, 91128 Palaiseau, France

E-mail address: pg@math.polytechnique.fr, am@math.polytechnique.fr

Uwe Semmelmann, Mathematisches Institut, Universität zu Köln, Weyertal 86-90 D-50931 Köln, Germany

E-mail address: uwe.semmelmann@math.uni-koeln.de