

# INFINITESIMAL EINSTEIN DEFORMATIONS OF NEARLY KÄHLER METRICS

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ABSTRACT. It is well known that every 6-dimensional strictly nearly Kähler manifold  $(M, g, J)$  is Einstein with positive scalar curvature  $\text{scal} > 0$ . Moreover, one can show that the space  $E$  of co-closed primitive  $(1, 1)$ -forms on  $M$  is stable under the Laplace operator  $\Delta$ . Let  $E(\lambda)$  denote the  $\lambda$ -eigenspace of the restriction of  $\Delta$  to  $E$ . If  $M$  is compact, and has normalized scalar curvature  $\text{scal} = 30$ , we prove that the moduli space of infinitesimal Einstein deformations of the nearly Kähler metric  $g$  is naturally isomorphic to the direct sum  $E(2) \oplus E(6) \oplus E(12)$ . From [5], the last summand is itself isomorphic with the moduli space of infinitesimal nearly Kähler deformations.

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## 1. INTRODUCTION

Nearly Kähler manifolds, introduced by Alfred Gray in the 70s in the framework of weak holonomy, are defined as almost Hermitian manifolds  $(M, g, J)$  which are not far from being Kähler in the sense that the covariant derivative of  $J$  with respect to the Levi-Civita connection of  $g$  is totally skew-symmetric.

The class of nearly Kähler manifolds is clearly stable under Riemannian products. Using the generalization by Richard Cleyton and Andrew Swann of the Berger-Simons holonomy theorem to the case of connections with torsion [3], Paul-Andi Nagy showed in [7] that every nearly Kähler manifold is locally a Riemannian product of Kähler manifolds, 3-symmetric spaces, twistor spaces over positive quaternion-Kähler manifolds and 6-dimensional nearly Kähler manifolds. This result shows, in particular, that genuine nearly Kähler geometry only occurs in dimension 6. It turns out that in this dimension, strict (*i.e.* non-Kähler) nearly Kähler manifolds have several other remarkable features: They carry a real Killing spinor – so they are in particular Einstein manifolds with positive scalar curvature – and they have a  $SU_3$  structure whose intrinsic torsion is parallel with respect to the minimal connection (cf. [3]). A strict nearly Kähler structure on a compact 6-dimensional manifold with normalized scalar curvature  $\text{scal} = 30$  is called a Gray structure.

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In [5] we have studied the moduli space  $\mathcal{G}$  of infinitesimal deformations of Gray structures on compact 6-dimensional manifolds, and showed that this space is isomorphic to the space  $E(12)$ , where  $E(\lambda)$  denotes the intersection of the  $\lambda$ -eigenspace of the Laplace operator and the space of co-closed primitive  $(1, 1)$ -forms.

In the present paper we consider the related problem of describing the moduli space  $\mathcal{E}$  of Einstein deformations of a Gray structure. Since every Gray structure is in particular Einstein, one has *a priori*  $\mathcal{E} \supset \mathcal{G}$ . Our main result (Theorem 5.1) gives a canonical isomorphism between  $\mathcal{E}$  and the direct sum  $E(2) \oplus E(6) \oplus E(12)$ .

The main idea is the following. The space of infinitesimal Einstein deformation on every compact manifold consists of trace-free symmetric bilinear tensors in a certain eigenspace of a second order elliptic operator called the Lichnerowicz Laplacian  $\Delta_L$ . On a 6-dimensional nearly Kähler manifold, one can decompose every infinitesimal Einstein deformation  $H$  (viewed as symmetric endomorphism) into its parts  $h$  and  $S$  commuting resp. anti-commuting with  $J$ . Under the  $SU_3$  representation, the space of symmetric endomorphisms commuting with  $J$  is isomorphic to the space of  $(1, 1)$ -forms and that of symmetric endomorphisms anti-commuting with  $J$  is isomorphic to the space of primitive  $(2, 1) + (1, 2)$ -forms and one may interpret the eigenvalue equation for  $\Delta_L$  in terms of the forms  $\varphi$  and  $\sigma$  corresponding to  $h$  and  $S$ . The problem is that  $\Delta_L$  does not commute with the isomorphisms above, because  $J$  is not parallel with respect to the Levi-Civita connection. It is thus natural to introduce a modified Lichnerowicz operator  $\bar{\Delta}_L$ , corresponding to the canonical Hermitian connection, better adapted to the nearly Kähler setting. It turns out that the eigenvalue equation for  $\Delta_L$  translates, *via*  $\bar{\Delta}_L$ , into a differential system for  $\varphi$  and  $\sigma$  involving the usual form Laplacian, which eventually yields the claimed result.

## 2. PRELIMINARIES

**2.1. Notation.** In this section we introduce our objects of study and derive several lemmas which will be needed later. Here and henceforth,  $(M^{2m}, g, J)$  will denote an almost Hermitian manifold with tangent bundle  $TM$ , cotangent bundle  $T^*M$  and tensor bundle  $\mathcal{T}M$ . We denote as usual by  $\Lambda^{(p,q)+(q,p)}M$  the projection of the complex bundle  $\Lambda^{(p,q)}M$  onto the real bundle  $\Lambda^{p+q}M$ . The bundle of  $g$ -symmetric endomorphisms  $\text{Sym}M$  splits in a direct sum  $\text{Sym}M = \text{Sym}^+M \oplus \text{Sym}^-M$ , of symmetric endomorphisms commuting resp. anti-commuting with  $J$ . The trace of every element in  $\text{Sym}^-M$  is automatically 0, and  $\text{Sym}^+M$  decomposes further  $\text{Sym}^+M = \text{Sym}_0^+M \oplus \langle \text{id} \rangle$  into its trace-free part and multiples of the identity.

**2.2. Nearly Kähler manifolds.** An almost Hermitian manifold  $(M^{2m}, g, J)$  is called *nearly Kähler* if

$$(\nabla_X J)(X) = 0, \quad \forall X \in TM, \quad (1)$$

where  $\nabla$  denotes the Levi-Civita connection of  $g$ . The canonical Hermitian connection  $\bar{\nabla}$ , defined by

$$\bar{\nabla}_X Y := \nabla_X Y - \frac{1}{2}J(\nabla_X J)Y, \quad \forall X \in TM, Y \in \chi(M), \quad (2)$$

is a  $U_m$  connection on  $M$  (*i.e.*  $\bar{\nabla}g = 0$  and  $\bar{\nabla}J = 0$ ) with torsion  $\bar{T}_X Y = -J(\nabla_X J)Y$ . A fundamental observation, which – although not explicitly stated – goes back to Gray, is the fact that  $\bar{\nabla}\bar{T} = 0$  on every nearly Kähler manifold (see [1]).

We denote as usual the Kähler form of  $M$  by  $\omega := g(J, \cdot)$ . The tensor  $\psi^+ := \nabla\omega$  is totally skew-symmetric by (1). Moreover, since  $J^2 = -\text{id}$ , it is easy to check that  $\psi^+(X, JY, JZ) = -\psi^+(X, Y, Z)$ . In other words,  $\psi^+$  is a form of type  $(3, 0) + (0, 3)$ . Let us now assume that the dimension of  $M$  is  $2m = 6$  and that the nearly Kähler structure is strict, *i.e.*  $(M, g, J)$  is not Kähler. The form  $\psi^+$  can be seen as the real part of a  $\bar{\nabla}$ -parallel complex volume form on  $M$ , so  $M$  carries an  $SU_3$  structure whose minimal connection (cf. [3]) is exactly  $\bar{\nabla}$ .

Let  $A \in \Lambda^1 M \otimes \text{End}M$  denote the tensor  $A_X := J(\nabla_X J) = -\psi_{JX}^+$ . We will sometimes identify the endomorphism  $A_X$  with the corresponding form in  $\Lambda^{(2,0)+(0,2)}M$ , *e.g.* in formula (8) below. By definition, we have  $\nabla_X = \bar{\nabla}_X + \frac{1}{2}A_X$  on  $TM$ . In fact this relation can be extended on the whole tensor bundle, provided we use the right extension for  $A_X$ .

**2.3. The induced action.** On a manifold  $M$ , every endomorphism  $A$  of  $TM$  extends as derivation to the tensor bundle  $\mathcal{T}M$ . In fact if we identify  $\text{End}(T_x M)$  with  $\mathfrak{gl}(n, \mathbb{R})$  this is precisely the Lie algebra action on the defining representation of  $\mathcal{T}M$ . We denote by  $A_\star$  this induced action. For example, we have

$$A_\star \tau = -\tau \circ A, \quad A_\star f = A \circ f - f \circ A, \quad \text{for } \tau \in T^*M, f \in \text{End}M.$$

If  $(M, g, J)$  is almost Hermitian and  $f \in \text{Sym}^+ M$ , let  $\tilde{f}$  denote the associated  $(1, 1)$ -form  $\tilde{f} := g(Jf, \cdot)$ , so in particular  $\tilde{\text{id}} = \omega$ , where  $\omega$  denotes the Kähler form of  $M$ . We compute, for later use:

$$(f_\star \omega)(X, Y) = -\omega(fX, Y) - \omega(X, fY) = -g(JfX, Y) - g(JX, fY) = -2\tilde{f}(X, Y). \quad (3)$$

A similar calculation shows that

$$S_\star \omega = 0, \quad \forall S \in \text{Sym}^- M. \quad (4)$$

Notice that the map  $\text{End}M \rightarrow \text{End}(\mathcal{T}M)$ ,  $A \mapsto A_\star$  is a Lie algebra morphism, *i.e.*

$$[A, B]_\star = [A_\star, B_\star], \quad \forall A, B \in \text{End}M,$$

which can be expressed as

$$A_\star(B_\star T) = (A_\star B)_\star T + B_\star(A_\star T), \quad \forall A, B \in \text{End}M, T \in \mathcal{T}M. \quad (5)$$

A convenient way of writing the induced action of  $A \in \text{End}M$  on a  $p$ -form  $u$  is

$$A_\star u = -A^*(e_i) \wedge e_i \lrcorner u,$$

where  $A^*$  is the adjoint of  $A$  and  $\{e_i\}$  is a local orthonormal basis of  $TM$ . Here, as well as in the remaining part of this paper, we adopt the Einstein convention of summation on the repeated subscripts.

Notice that by (2), the extensions of  $\nabla$  and  $\bar{\nabla}$  to the tensor bundle  $\mathcal{T}M$  are related by

$$\bar{\nabla}_X = \nabla_X - \frac{1}{2}(A_X)_\star. \quad (6)$$

**2.4. Algebraic results on nearly Kähler manifolds.** Assume that  $(M^6, g, J)$  is a strict nearly Kähler manifold and that the metric on  $M$  is normalized such that  $\text{scal} = 30$ . From [4, Theorem 5.2] it follows that for every unit vector  $X$ , the endomorphism  $\nabla_X J$  (which vanishes on the 2-plane spanned by  $X$  and  $JX$ ) defines a complex structure on the orthogonal complement of that 2-plane. Then the same holds for  $A_X$  (because  $A_X = -\nabla_{JX} J$ ).

The exterior bundle  $\Lambda^2 M$  decomposes into irreducible  $SU_3$  components as follows:

$$\Lambda^2 M \simeq \Lambda^{(2,0)+(0,2)} M \oplus \Lambda_0^{(1,1)} M \oplus \mathbb{R}\omega.$$

The map  $X \mapsto X \lrcorner \psi^+$  identifies the first summand with  $TM$ , and  $h \mapsto g(Jh, \cdot)$  defines an isomorphism between  $\text{Sym}_0^+ M$  and the second summand.

Similarly, one can decompose  $\Lambda^3 M$  into irreducible  $SU_3$  components

$$\Lambda^3 M \simeq \Lambda^{(3,0)+(0,3)} M \oplus \Lambda_0^{(2,1)+(1,2)} M \oplus \Lambda^1 M \wedge \omega.$$

The first summand is a rank 2 trivial bundle spanned by  $\psi^+$  and its Hodge dual  $*\psi^+$ , and the isomorphism  $S \mapsto S_\star \psi^+$  identifies  $\text{Sym}^- M$  with the second summand.

If  $\{e_i\}$  denotes a local orthonormal basis of  $TM$ , it is straightforward to check the following formulas:

$$A_{e_i} A_{e_i}(X) = -4X, \quad \forall X \in TM. \quad (7)$$

$$A_{e_i} \wedge A_{e_i} = 2\omega^2. \quad (8)$$

**Lemma 2.1.** (i) For every  $X \in TM$ , with corresponding 1-form  $X^\flat$  one has

$$A_{X_\star} \psi^+ = -2X^\flat \wedge \omega. \quad (9)$$

(ii) If  $S$  is a section of  $\text{Sym}^- M$ , then the following formula holds for every  $X \in TM$ :

$$A_{X_\star}(S_\star \psi^+) = 2(SX)^\flat \wedge \omega. \quad (10)$$

*Proof.* (i) An easy computation shows:

$$\begin{aligned} A_{X_\star} \psi^+ &= A_X e_i \wedge e_i \lrcorner \psi^+ = A_X J e_i \wedge J e_i \lrcorner \psi^+ = A_{J e_i} X \wedge A_{e_i} \\ &= A_{e_i}(JX) \wedge A_{e_i} = \frac{1}{2} JX \lrcorner (A_{e_i} \wedge A_{e_i}) \stackrel{(8)}{=} -2X \wedge \omega. \end{aligned}$$

(ii) The symmetric endomorphism  $A_{X_\star} S = A_X \circ S - S \circ A_X$  commutes with  $J$  and is trace-free. Consequently, by Schur's Lemma (cf. [5] for a more detailed argument)

$$(A_{X_\star} S)_\star \psi^+ = 0. \quad (11)$$

Notice that if  $X^\flat$  is the 1-form corresponding to  $X$  (which we usually identify with  $X$ ), then  $f_\star X^\flat = -(fX)^\flat$  for every symmetric endomorphism  $f$ . We then compute:

$$\begin{aligned} A_{X_\star}(S_\star\psi^+) &\stackrel{(5)}{=} (A_{X_\star}S)_\star\psi^+ + S_\star(A_{X_\star}\psi^+) \stackrel{(9),(11)}{=} -2S_\star(X^\flat \wedge \omega) \\ &= 2(SX)^\flat \wedge \omega - 2X \wedge (S_\star\omega) \stackrel{(4)}{=} 2(SX)^\flat \wedge \omega. \end{aligned}$$

□

### 3. THE CURVATURE OPERATOR

Let  $(M^n, g)$  be a Riemannian manifold. The curvature operator  $\mathcal{R} : \Lambda^2 M \rightarrow \Lambda^2 M$  is defined by the equation  $g(\mathcal{R}(X \wedge Y), U \wedge V) = g(R_{X,Y}V, U)$ , for any vector fields  $X, Y, U, V$  on  $M$ , identified with the corresponding 1-forms *via* the metric. In a local orthonormal frame  $\{e_i\}$  it can be written as

$$\mathcal{R}(e_i \wedge e_j) = \frac{1}{2}R_{ijkl}e_l \wedge e_k = -\frac{1}{2}e_k \wedge R_{e_i, e_j}e_k. \quad (12)$$

Using the identification of 2-vectors and (skew-symmetric) endomorphisms given by  $(X \wedge Y)(Z) := g(X, Z)Y - g(Y, Z)X$ , formula (12) yields  $\mathcal{R}(X \wedge Y)(Z) = -R_{X,Y}Z$ . Notice that a manifold with curvature operator  $\mathcal{R} = \text{cid}$  has Ricci curvature  $c(n-1)$  and in particular the curvature operator of the sphere is a positive multiple of the identity.

Let  $EM$  be the vector bundle associated to the bundle of orthonormal frames *via* some representation  $\pi : SO(n) \rightarrow \text{Aut}(E)$ . Every orthogonal automorphism  $f$  of  $TM$  defines in a canonical way an automorphism of  $EM$ , denoted, by a slight abuse of notation,  $\pi(f)$ . The differential of  $\pi$  maps skew-symmetric endomorphisms of  $TM$  (or equivalently elements of  $\Lambda^2 M$ ) to endomorphisms of  $EM$ . The Levi-Civita connection of  $M$  induces a connection on  $EM$  whose curvature  $R^E$  satisfies  $R^E(X, Y) = \pi_\star(R(X, Y)) = -\pi_\star(\mathcal{R}(X \wedge Y))$ . Notice that  $\pi_\star(A)$  is exactly  $A_\star$  in the notation of Section 2.

We now define the curvature endomorphism  $q(R) \in \text{End}(EM)$  as

$$q(R) := -\frac{1}{2}(e_i \wedge e_j)_\star \mathcal{R}(e_i \wedge e_j)_\star. \quad (13)$$

For example the curvature endomorphism  $q(R)$  on the form bundle  $EM = \Lambda^p M$  satisfies

$$q(R) = (e_j \wedge e_i \lrcorner) \circ (R_{e_i, e_j}e_k \wedge e_k \lrcorner). \quad (14)$$

In particular we have  $q(R) = \text{Ric}$  on 1-forms, and  $q(R) = -\text{Ric}_\star - 2\mathcal{R}$  on 2-forms.

It is easy to check that the action of  $q(R)$  is compatible with the identification of  $\Lambda^2 M$  with the space of skew-symmetric endomorphisms (and, more generally, with all  $SO_n$  equivariant isomorphisms):

**Lemma 3.1.** *Let  $\varphi \in \Lambda^2 M$  be a 2-form with associated skew-symmetric endomorphism  $A$ , i.e.  $\varphi(Y, Z) = g(AY, Z)$  for any vector fields  $Y, Z$ . Then*

$$(q(R)\varphi)(Y, Z) = g((q(R)A)Y, Z).$$

We now return to the case of a 6-dimensional strict nearly Kähler manifold  $(M^6, g, J)$  with scalar curvature  $\text{scal} = 30$ .

Let  $R$  be the curvature of the Levi-Civita connection and let  $\bar{R}$  be the curvature of the canonical Hermitian connection  $\bar{\nabla}$ . The following relation between  $R$  and  $\bar{R}$  is implicitly contained in [4].

**Lemma 3.2.** *For any tangent vectors  $W, X, Y, Z$  one has*

$$\begin{aligned} R_{WXYZ} &= \bar{R}_{WXYZ} - \frac{1}{4}g(Y, W)g(X, Z) + \frac{1}{4}g(X, Y)g(Z, W) \\ &\quad + \frac{3}{4}g(Y, JW)g(JX, Z) - \frac{3}{4}g(Y, JX)g(JW, Z) - \frac{1}{2}g(X, JW)g(JY, Z). \end{aligned}$$

*Proof.* The stated formula follows using equation (3.1) and the polarization of equation (5.1) from [4]. Note that there is a different sign convention for the curvature tensors in [4].  $\square$

The Ricci curvature of  $\bar{R}$  satisfies  $\overline{\text{Ric}} = 4g$ . This follows from the formula above and the fact that  $(M^6, g, J)$  is Einstein with  $\text{Ric} = 5g$ .

Replacing  $R$  by  $\bar{R}$  in formula (14) yields a curvature endomorphism  $q(\bar{R})$ . It is easy to check that the curvature operator with respect to  $\bar{\nabla}$ , denoted by  $\bar{\mathcal{R}}$ , is a section of  $\text{Sym}(\Lambda_0^{(1,1)}M)$  so we can express  $q(\bar{R}) = -\sum \alpha_{i\star} \bar{\mathcal{R}}(\alpha_i)_\star$  for any orthonormal basis  $\alpha_i$  of  $\Lambda_0^{(1,1)}M$ . Since  $\Lambda_0^{(1,1)}\mathbb{R}^6 \simeq \mathfrak{su}_3$ , we see that  $q(\bar{R})$  preserves all tensor bundles associated to  $\text{SU}_3$  representations. Moreover, a straightforward computation using the fact that  $\alpha_\star J = 0$  and  $\alpha_\star \psi^+ = 0$  for every  $\alpha \in \mathfrak{su}_3$  yields:

$$q((q(\bar{R})h)J, \cdot) = q(\bar{R})\varphi, \quad \text{and} \quad q(\bar{R})(S_\star \psi^+) = (q(\bar{R})S)_\star \psi^+, \quad (15)$$

for every sections  $h \in \text{Sym}^+M$  and  $S \in \text{Sym}^-M$ , where  $\varphi$  denotes the  $(1,1)$ -form defined by  $\varphi = g(hJ, \cdot)$ .

The following lemma describes the difference  $q(R) - q(\bar{R})$ . It is an immediate consequence of the curvature formula in Lemma 3.2. We will denote with  $\text{Cas} = \frac{1}{2}(e_i \wedge e_j)_\star(e_i \wedge e_j)_\star$  the Casimir operator of  $\mathfrak{so}(n)$  acting on the representation  $E$  and at the same time the corresponding endomorphism of  $EM$ .

**Lemma 3.3.** *The difference  $q(R) - q(\bar{R}) \in \text{End}(EM)$  is given as*

$$q(R) - q(\bar{R}) = -\frac{1}{4}\text{Cas} + \frac{3}{8}(e_i \wedge e_j)_\star(Je_i \wedge Je_j)_\star - \frac{1}{8}(e_i \wedge Je_i)_\star(e_k \wedge Je_k)_\star$$

In the remaining part of this section we will apply Lemma 3.3 in order to compute  $q(R) - q(\bar{R})$  on certain spaces of endomorphisms and forms.

On the bundle  $\text{End}M$  we define projections  $\text{pr}_\pm$  by  $\text{pr}_\pm(H) = \frac{1}{2}(H \mp JHJ)$ . Then  $H = \text{pr}_+(H) + \text{pr}_-(H)$  is the decomposition of the endomorphism  $H$  in a part commuting

resp. anti-commuting with  $J$ . Using this notation we find for the first sum in the above equation

$$(e_i \wedge e_j)_*(Je_i \wedge Je_j)_* = \begin{cases} -2\text{id} & \text{on } TM \\ -8\text{pr}_- & \text{on } \text{Sym}M \\ -8\text{pr}_+ & \text{on } \Lambda_0^2 M \end{cases}$$

Indeed for any tangent vector  $v \in TM$  we have

$$(e_i \wedge e_j)_*(Je_i \wedge Je_j)_*v = (e_i \wedge e_j)_*(g(Je_i, v)Je_j - g(Je_j, v)Je_i) = -2(Jv \wedge e_j)_*Je_j = -2v.$$

We now recall that for every skew-symmetric endomorphism  $A \in \mathfrak{so}(TM) \cong \Lambda^2 M$  and for every  $H \in \text{End}M$  we have  $A_*H = [A, H]$ . Hence, we obtain

$$\begin{aligned} (e_i \wedge e_j)_*(Je_i \wedge Je_j)_*H &= [(e_i \wedge e_j)_*, [(Je_i \wedge Je_j)_*, H]] \\ &= (e_i \wedge e_j)_*(Je_i \wedge Je_j)_*H + H(e_i \wedge e_j)_*(Je_i \wedge Je_j)_* \\ &\quad - 2(e_i \wedge e_j)_*H(Je_i \wedge Je_j)_* \\ &= -4H - 2(e_i \wedge e_j)_*H(Je_i \wedge Je_j)_*. \end{aligned}$$

It remains to compute the endomorphism  $B := (e_i \wedge e_j)_*H(Je_i \wedge Je_j)_*$ . Applying it to a vector  $Y$  and taking the scalar product with a vector  $Z$  we find

$$\begin{aligned} g(BY, Z) &= -g(H(Je_i \wedge Je_j)_*Y, (e_i \wedge e_j)_*Z) \\ &= -g(H[g(Je_i, Y)Je_j - g(Je_j, Y)Je_i], [g(e_i, Z)e_j - g(e_j, Z)e_i]) \\ &= -2(g(JZ, Y)g(HJe_j, e_j) + g(HJZ, JY)). \end{aligned}$$

Hence the sum  $B = (e_i \wedge e_j)_*H(Je_i \wedge Je_j)_*$  equals  $2JHJ$  if the endomorphism  $H$  is symmetric and  $-2JHJ$  if  $H$  is skew-symmetric and  $\text{tr}(HJ) = 0$ .

Next we compute the second sum of Lemma 3.3 on tangent vectors and endomorphisms. Since  $2J = (e_i \wedge Je_i)_*$  we immediately obtain

$$(e_i \wedge Je_i)_*(e_j \wedge Je_j)_* = \begin{cases} -4\text{id} & \text{on } TM \\ -16\text{pr}_- & \text{on } \text{End}M \end{cases}$$

We next determine the two sums of Lemma 3.3 on the space of 3-forms. Recall the type decomposition

$$\Lambda^3 M = \Lambda^{(3,0)+(0,3)} M \oplus \Lambda^{(2,1)+(1,2)} M,$$

which coincides with the eigenspace decomposition of  $(J_*)^2$ , with eigenvalue  $-9$  on the first and eigenvalue  $-1$  on the second summand. For any 3-form  $\alpha$  we define a new 3-form  $\hat{\alpha}$  by the formula  $\hat{\alpha}(X, Y, Z) = \alpha(JX, JY, Z) + \alpha(JX, Y, JZ) + \alpha(X, JY, JZ)$ . Then  $(J_*)^2\alpha = -3\alpha + 2\hat{\alpha}$  and the two components of  $\Lambda^3 M$  may also be characterized by

$$\alpha \in \Lambda^{(3,0)+(0,3)} M \quad \text{if and only if} \quad \hat{\alpha} = -3\alpha,$$

and similarly

$$\alpha \in \Lambda^{(2,1)+(1,2)} M \quad \text{if and only if} \quad \hat{\alpha} = \alpha.$$

Let  $\text{pr}_{3,0}$  and  $\text{pr}_{2,1}$  denote the projections onto the two summands of  $\Lambda^3 M$ . Then for  $\alpha \in \Lambda^3 M$

$$\text{pr}_{3,0}(\alpha) = \frac{1}{4}(\alpha - \hat{\alpha}) \quad \text{and} \quad \text{pr}_{2,1}(\alpha) = \frac{1}{4}(3\alpha + \hat{\alpha}).$$

For any 3-form  $\alpha$  we have  $(e_i \wedge J e_i)_*(e_k \wedge J e_k)_*\alpha = 4(J_*)^2\alpha = -12\alpha + 8\hat{\alpha}$ , which gives the second sum of Lemma 3.3 and a simple calculation yields  $(e_i \wedge e_j)_*(J e_i \wedge J e_j)_*\alpha = -6\alpha - 4\hat{\alpha}$  for the first sum.

Finally we may substitute the Casimir eigenvalues and our explicit expressions into the formula of Lemma 3.3. Recall that in the normalization with  $\{e_i \wedge e_j\}$  as orthonormal basis of  $\Lambda^2 M \cong \mathfrak{so}(TM)$ , the Casimir operator acts as  $-p(n-p)\text{id}$  on  $\Lambda^p M$ , and as  $-2n\text{id}$  on  $\text{Sym} M$ . Hence we obtain for  $n = 6$ :

**Proposition 3.4.**

$$q(R) - q(\bar{R}) = \begin{cases} \text{id} & \text{on } TM \\ 3\text{pr}_+ + 2\text{pr}_- & \text{on } \text{Sym} M \\ -\text{pr}_+ + 4\text{pr}_- & \text{on } \Lambda_0^2 M \\ -\text{pr}_{2,1} + 9\text{pr}_{3,0} & \text{on } \Lambda^3 M \end{cases}$$

**Corollary 3.5.**

$$q(R) - q(\bar{R}) = \begin{cases} 3\text{id} & \text{on } \text{Sym}^+ M \\ 2\text{id} & \text{on } \text{Sym}^- M \\ -\text{id} & \text{on } \Lambda_0^{(1,1)} M \\ -\text{id} & \text{on } \Lambda^{(2,1)+(1,2)} M \end{cases}$$

#### 4. COMPARING ROUGH LAPLACIANS

Let  $(M, g, J)$  be a strict nearly Kähler manifold with Levi-Civita connection  $\nabla$  and canonical Hermitian connection  $\bar{\nabla}$ . In this section we compare the actions of the rough Laplacians  $\nabla^* \nabla$  and  $\bar{\nabla}^* \bar{\nabla}$  on several tensor bundles.

We will perform all calculations below at some fixed point  $x \in M$  using a local orthonormal frame  $\{e_i\}$  which is  $\nabla$ -parallel at  $x$ . On any tensor bundle on  $M$  we can write  $\nabla^* \nabla = -\nabla_{e_i} \nabla_{e_i}$  and because  $\bar{\nabla}_{e_i} e_i = \nabla_{e_i} e_i - \frac{1}{2} J(\nabla_{e_i} J) e_i = 0$ , we also have  $\bar{\nabla}^* \bar{\nabla} = -\bar{\nabla}_{e_i} \bar{\nabla}_{e_i}$ . We are interested in the operator  $P := \nabla^* \nabla - \bar{\nabla}^* \bar{\nabla}$ . Using (6) and the fact that the tensor  $A := J \nabla J$  is  $\bar{\nabla}$ -parallel, we have

$$\begin{aligned} P &= -\nabla_{e_i} \nabla_{e_i} + \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} = -(\bar{\nabla}_{e_i} + \frac{1}{2} A_{e_i \star})(\bar{\nabla}_{e_i} + \frac{1}{2} A_{e_i \star}) + \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} \\ &= -\frac{1}{4} A_{e_i \star} A_{e_i \star} - A_{e_i \star} \bar{\nabla}_{e_i}. \end{aligned}$$

We now compute the action of the two operators occurring in the previous formula on several tensor bundles which are of interest in the deformation problem.



**Lemma 4.1.** *Let  $\varphi$ ,  $\sigma$ ,  $h$  and  $S$  be sections of  $\Lambda_0^{(1,1)}M$ ,  $\Lambda_0^{(2,1)+(1,2)}M$ ,  $\text{Sym}^+M$  and  $\text{Sym}^-M$  respectively. Then*

$$A_{e_i\star}(A_{e_i\star}\varphi) = -4\varphi. \quad (16)$$

$$A_{e_i\star}(A_{e_i\star}\sigma) = -4\sigma. \quad (17)$$

$$A_{e_i\star}(A_{e_i\star}h) = -12h. \quad (18)$$

$$A_{e_i\star}(A_{e_i\star}S) = -8S. \quad (19)$$

*Proof.* Let  $X \in TM$  be a tangent vector. Since  $A_X$  and  $\varphi$  are 2-forms of type  $(2, 0) + (0, 2)$  and  $(1, 1)$  respectively, we get

$$A_X(e_k, e_j)\varphi(e_k, e_j) = 2\langle A_X, \varphi \rangle = 0, \quad \forall X \in TM. \quad (20)$$

We then compute

$$\begin{aligned} A_{e_i\star}(A_{e_i\star}\varphi) &= A_{e_i}e_j \wedge e_j \lrcorner (A_{e_i}e_k \wedge e_k \lrcorner \varphi) \\ &= A_{e_i}e_j A_{e_i}(e_k, e_j) \wedge (e_k \lrcorner \varphi) - A_{e_i}e_j \wedge A_{e_i}e_k \varphi(e_k, e_j) \\ &\stackrel{(20)}{=} A_{e_i}^2(e_k) \wedge (e_k \lrcorner \varphi) - e_j \lrcorner (A_{e_i} \wedge A_{e_i}e_k \varphi(e_k, e_j)) \\ &\stackrel{(7)}{=} -4e_k \wedge (e_k \lrcorner \varphi) - \frac{1}{2}e_j \lrcorner e_k \lrcorner (A_i \wedge A_i \varphi(e_k, e_j)) \\ &\stackrel{(8)}{=} -8\varphi - e_j \lrcorner e_k \lrcorner (\omega^2 \varphi(e_k, e_j)) = -8\varphi - 2e_j \lrcorner (Je_k \wedge \omega \varphi(e_k, e_j)) \\ &= -8\varphi - 2\omega \varphi(e_k, Je_k) + 2Je_k \wedge Je_j \varphi(e_k, e_j) \\ &= -8\varphi + 2e_k \wedge e_j \varphi(e_k, e_j) = -8\varphi + 4\varphi = -4\varphi. \end{aligned}$$

In order to prove (17), we express  $\sigma$  as  $\sigma = S_\star \psi^+$  for some section  $S$  of  $\text{Sym}^-M$ . Using (10) we obtain

$$\begin{aligned} A_{e_i\star}(A_{e_i\star}\sigma) &= 2A_{e_i\star}(Se_i \wedge \omega) = 2Se_i \wedge (A_{e_i\star}\omega) = 2Se_i \wedge J\psi_{e_i}^+ e_j \wedge Je_j \\ &= -2Se_i \wedge \psi_{e_i}^+ Je_j \wedge Je_j = 4Se_i \wedge \psi_{e_i}^+ = -4S_\star \psi^+ = -4\sigma. \end{aligned}$$

Now, for every endomorphism  $H$  of  $TM$ , we have

$$A_{e_i\star}(A_{e_i\star}H) = A_{e_i}^2 H + HA_{e_i}^2 - 2A_{e_i}HA_{e_i} \stackrel{(7)}{=} -8H - 2A_{e_i}HA_{e_i}. \quad (21)$$

If  $h \in \text{Sym}^+M$ , let  $\varphi(\cdot, \cdot) = g(Jh\cdot, \cdot)$  be its associated  $(1, 1)$ -form. By (16) we have for every tangent vectors  $X, Y$

$$\begin{aligned} -4\varphi(X, Y) &= A_{e_i\star}(A_{e_i\star}\varphi)(X, Y) \\ &= \varphi(A_{e_i}^2 X, Y) + \varphi(X, A_{e_i}^2 Y) + 2\varphi(A_{e_i}X, A_{e_i}Y) \\ &\stackrel{(7)}{=} -8\varphi(X, Y) + 2g(hJA_{e_i}X, A_{e_i}Y) \\ &= -8\varphi(X, Y) + 2g(JA_{e_i}hA_{e_i}X, Y), \end{aligned}$$

whence  $A_{e_i}hA_{e_i} = 2h$ . This, together with (21), yields (18). If  $S \in \text{Sym}^-M$ , using the fact that  $A_{JX} = A_X \circ J = -J \circ A_X$  for every  $X$ , we can write

$$A_{e_i}SA_{e_i} = A_{Je_i}SA_{Je_i} = -A_{e_i}JSJA_{e_i} = -A_{e_i}SA_{e_i},$$

which together with (21) yields (19).  $\square$

**Lemma 4.2.** *The following relations hold:*

$$e_i \lrcorner (A_{e_i \star} \varphi) = 0, \quad \forall \varphi \in \Lambda_0^{(1,1)} M. \quad (22)$$

$$(A_{e_i \star} h)(e_i) = 0, \quad \forall h \in \text{Sym}^+ M. \quad (23)$$

$$e_i \wedge (A_{e_i \star} \varphi) = 0, \quad \forall \varphi \in \Lambda_0^{(1,1)} M. \quad (24)$$

$$e_i \lrcorner (A_{i \star} (S \star \psi^+)) = 0, \quad \forall S \in \text{Sym}^- M. \quad (25)$$

*Proof.* Simple application of the Schur Lemma, taking into account the decomposition of the exterior bundles into irreducible components with respect to the  $\text{SU}_3$  action.  $\square$

**Lemma 4.3.** *Let  $\varphi$  and  $S$  be sections of  $\Lambda_0^{(1,1)} M$  and  $\text{Sym}^- M$  respectively. If  $h$  and  $\sigma$  are defined as usual by  $g(Jh, \cdot) := \varphi(\cdot, \cdot)$  and  $\sigma := S \star \psi^+$ , then*

$$A_{e_i \star} \bar{\nabla}_{e_i} \varphi = -(J\delta\varphi) \lrcorner \psi^+. \quad (26)$$

$$A_{e_i \star} \bar{\nabla}_{e_i} \sigma = -2\delta S \wedge \omega. \quad (27)$$

$$(A_{e_i \star} \bar{\nabla}_{e_i} h) \star \psi^+ = -2\delta h \wedge \omega - 4d\varphi. \quad (28)$$

$$A_{e_i \star} \bar{\nabla}_{e_i} S = (\delta S \lrcorner \psi^+ + \delta(S \star \psi^+)) \circ J. \quad (29)$$

Here  $\delta$  denotes the co-differential on exterior forms and the divergence operator whenever applied to symmetric endomorphisms.

*Proof.* Since  $A_{e_i}$  anti-commutes with  $J$  and  $\bar{\nabla}_{e_i} \varphi$  is of type  $(1, 1)$ , it follows that  $A_{e_i \star} \bar{\nabla}_{e_i} \varphi$  is a form of type  $(2, 0) + (0, 2)$ , so there exists a vector field  $\alpha$  such that  $A_{e_i \star} \bar{\nabla}_{e_i} \varphi = \alpha \lrcorner \psi^+$ . In order to find  $\alpha$ , we use the relation  $(\alpha \lrcorner \psi^+) \wedge \psi^+ = \alpha \wedge \omega^2$  (see [5]) and compute:

$$\begin{aligned} \alpha \wedge \omega^2 &= (\alpha \lrcorner \psi^+) \wedge \psi^+ = (A_{e_i \star} \bar{\nabla}_{e_i} \varphi) \wedge \psi^+ \\ &= A_{e_i \star} ((\bar{\nabla}_{e_i} \varphi) \wedge \psi^+) - \bar{\nabla}_{e_i} \varphi \wedge (A_{e_i \star} \psi^+) \stackrel{(9)}{=} 2\bar{\nabla}_{e_i} \varphi \wedge e_i \wedge \omega \\ &= -\bar{\nabla}_{e_i} \varphi \wedge (J e_i \lrcorner \omega^2) = -J e_i \lrcorner (\bar{\nabla}_{e_i} \varphi \wedge \omega^2) + (J e_i \lrcorner \bar{\nabla}_{e_i} \varphi) \wedge \omega^2 \\ &= J(e_i \lrcorner \bar{\nabla}_{e_i} \varphi) \wedge \omega^2 \stackrel{(22)}{=} J(e_i \lrcorner \nabla_{e_i} \varphi) \wedge \omega^2 = -J\delta\varphi \wedge \omega^2, \end{aligned}$$

so  $\alpha = -J\delta\varphi$ , thus proving (26). Using the fact that  $\psi^+$  is  $\bar{\nabla}$ -parallel, we get:

$$\begin{aligned} A_{e_i \star} \bar{\nabla}_{e_i} \sigma &= A_{e_i \star} (\bar{\nabla}_{e_i} S \star \psi^+) \stackrel{(5)}{=} (A_{e_i \star} \bar{\nabla}_{e_i} S) \star \psi^+ + \bar{\nabla}_{e_i} S \star (A_{e_i \star} \psi^+) \\ &\stackrel{(9),(11)}{=} -2\bar{\nabla}_{e_i} S \star (e_i \wedge \omega) \stackrel{(4)}{=} 2(\bar{\nabla}_{e_i} S) e_i \wedge \omega = -2\delta S \wedge \omega. \end{aligned}$$

This proves (27). We next use (23) to write

$$\delta h = -(\nabla_{e_i} h) e_i = -(\bar{\nabla}_{e_i} h) e_i \quad (30)$$

whence

$$\begin{aligned}
 (A_{e_i^*} \bar{\nabla}_{e_i} h)_* \psi^+ &\stackrel{(5)}{=} A_{e_i^*}((\bar{\nabla}_{e_i} h)_* \psi^+) - (\bar{\nabla}_{e_i} h)_*(A_{e_i^*} \psi^+) \\
 &\stackrel{(9)}{=} 2(\bar{\nabla}_{e_i} h)_*(e_i \wedge \omega) = 2(\bar{\nabla}_{e_i} h)e_i \wedge \omega + 2e_i \wedge ((\bar{\nabla}_{e_i} h)_* \omega) \\
 &\stackrel{(3)}{=} -2\delta h \wedge \omega - 4e_i \wedge \bar{\nabla}_{e_i} \varphi = -2\delta h \wedge \omega - 4d\varphi.
 \end{aligned}$$

In order to check (29) we first compute

$$\begin{aligned}
 \delta(S_* \psi^+) &= -e_i \lrcorner \nabla_{e_i}(S_* \psi^+) \stackrel{(25)}{=} -e_i \lrcorner \bar{\nabla}_{e_i}(S_* \psi^+) = e_i \lrcorner \bar{\nabla}_{e_i}(S(e_j) \wedge \psi_{e_j}^+) \\
 &= e_i \lrcorner ((\bar{\nabla}_{e_i} S)e_j \wedge \psi_{e_j}^+) = -g(\delta S, e_j) \psi_{e_j}^+ - (\bar{\nabla}_{e_i} S)e_j \wedge (e_i \lrcorner \psi_{e_j}^+) \\
 &= -\delta S \lrcorner \psi^+ + (e_i \lrcorner \psi_{e_j}^+) \wedge (\bar{\nabla}_{e_i} S)e_j.
 \end{aligned}$$

Let  $B$  denote the endomorphism of  $TM$  corresponding to the 2-form  $\delta(S_* \psi^+) + \delta S \lrcorner \psi^+$ . By the calculation above we get

$$\begin{aligned}
 B(X) &= ((e_i \lrcorner \psi_{e_j}^+) \wedge (\bar{\nabla}_{e_i} S)e_j)(X) = \psi^+(e_j, e_i, X)(\bar{\nabla}_{e_i} S)e_j - (\bar{\nabla}_{e_i} S)(e_j, X)(e_i \lrcorner \psi_{e_j}^+) \\
 &= (\bar{\nabla}_{e_i} S)(\psi_{e_i}^+ X) + \psi_{e_i}^+((\bar{\nabla}_{e_i} S)X) = (\bar{\nabla}_{e_i} S)(A_{e_i} JX) + A_{e_i}(J(\bar{\nabla}_{e_i} S)X) \\
 &= (\bar{\nabla}_{e_i} S)(A_{e_i} JX) - A_{e_i}((\bar{\nabla}_{e_i} S)JX) = -(A_{e_i^*}(\bar{\nabla}_{e_i} S))(JX).
 \end{aligned}$$

Replacing  $X$  by  $JX$  yields (29).  $\square$

From Lemma 4.1 and Lemma 4.3 we infer directly

**Corollary 4.4.** *Let  $\varphi$  and  $S$  be sections of  $\Lambda_0^{(1,1)}M$  and  $\text{Sym}^-M$  respectively. If  $h$  and  $\sigma$  are defined by  $g(Jh, \cdot) := \varphi(\cdot, \cdot)$  and  $\sigma := S_* \psi^+$ , then*

$$(\nabla^* \nabla - \bar{\nabla}^* \bar{\nabla})\varphi = \varphi + (J\delta\varphi) \lrcorner \psi^+. \quad (31)$$

$$(\nabla^* \nabla - \bar{\nabla}^* \bar{\nabla})\sigma = \sigma + 2\delta S \wedge \omega. \quad (32)$$

$$(\nabla^* \nabla - \bar{\nabla}^* \bar{\nabla})h = 3h + s, \quad \text{where } s \in \text{Sym}^-M, \text{ and } s_* \psi^+ = 2\delta h \wedge \omega + 4d\varphi. \quad (33)$$

$$(\nabla^* \nabla - \bar{\nabla}^* \bar{\nabla})S = 2S - (\delta S \lrcorner \psi^+ + \delta\sigma) \circ J. \quad (34)$$

Finally, we obtain the invariance of the space of primitive co-closed  $(1, 1)$ -forms under the Laplace operator:

**Proposition 4.5.** *If  $\varphi$  is a co-closed section of  $\Lambda_0^{(1,1)}M$ , then the same holds for  $\Delta\varphi$ .*

*Proof.* The 2-form  $\Delta\varphi$  is clearly co-closed since  $\varphi$  is co-closed. Using (31), Corollary 3.5 and the classical Weitzenböck formula on 2-forms yield

$$\Delta\varphi = (\nabla^* \nabla + q(R))\varphi = (\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))\varphi.$$

The last term is a section of  $\Lambda_0^{(1,1)}M$  since both  $\bar{\nabla}$  and  $q(\bar{R})$  preserve this space.  $\square$

## 5. THE MODULI SPACE OF EINSTEIN DEFORMATIONS

We now have all the ingredients for the main result of this paper:

**Theorem 5.1.** *Let  $(M^6, g, J)$  be a Gray manifold. Then the moduli space of infinitesimal Einstein deformations of  $g$  is isomorphic to the direct sum of the spaces of primitive co-closed  $(1, 1)$ -eigenforms of the Laplace operator for the eigenvalues 2, 6 and 12.*

*Proof.* Let  $g$  be an Einstein metric with  $\text{Ric} = Eg$ . From [2], Theorem 12.30, the space of infinitesimal Einstein deformations of  $g$  is isomorphic to the set of symmetric trace-free endomorphisms  $H$  of  $TM$  such that  $\delta H = 0$  and such that  $\Delta_L H = 2EH$ , where  $\Delta_L = \nabla^* \nabla + q(R)$  is the so-called Lichnerowicz Laplacian  $\Delta_L$ . Remark that  $q(R) = 2\bar{R} + 2E\text{id}$  in the notation of [2].

In our present situation the Einstein constant equals  $E = 5$ , so the space of infinitesimal Einstein deformations of  $g$  is isomorphic to the set of  $H \in \text{Sym}M$  with  $\delta H = 0 = \text{tr}H$  such that

$$(\nabla^* \nabla + q(R))H = 10H. \quad (35)$$

Let  $h := \text{pr}_+ H$  and  $S := \text{pr}_- H$  denote the projections of  $H$  onto  $\text{Sym}^\pm M$ . We define the primitive  $(1, 1)$ -form  $\varphi(\cdot, \cdot) := g(Jh, \cdot)$  and the 3-form  $\sigma := S_* \psi^+$ . The key idea is to express (35) in terms of an exterior differential system for  $\varphi$  and  $\sigma$ . Using Corollary 3.5 and Corollary 4.4, (35) becomes

$$(\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))(h + S) = 10(h + S) - (3h + s) - (2S - (\delta S \lrcorner \psi^+ + \delta \sigma) \circ J) - 3h - 2S,$$

where  $s$  is the section of  $\text{Sym}^- M$  defined in the second part of (33). Since the operator  $(\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))$  preserves the decomposition  $\text{Sym}M = \text{Sym}^+ M \oplus \text{Sym}^- M$ , the previous equation is equivalent to the system

$$\begin{cases} (\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))h = 4h + \delta \sigma \circ J, \\ (\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))S = 6S - s, \\ \delta S = 0. \end{cases} \quad (36)$$

Taking the composition with  $J$  and using (15), the first equation of (36) becomes

$$(\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))\varphi = 4\varphi - \delta \sigma. \quad (37)$$

Similarly, taking the action on  $\psi^+$  and using (15) and the definition of  $s$ , the second equation of (36) becomes

$$(\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))\sigma = 6\sigma - 2\delta h \wedge \omega - 4d\varphi. \quad (38)$$

Notice that  $\delta h = \delta H - \delta S = 0$ , which can also be seen by examining the algebraic types in equation (38). From (30) we get

$$0 = \delta h = -(\bar{\nabla}_{e_i} h)e_i = (\bar{\nabla}_{e_i} J\varphi)e_i = J(\bar{\nabla}_{e_i} \varphi)e_i = -J\delta \varphi,$$

so finally the system (36) is equivalent to

$$\begin{cases} (\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))\varphi = 4\varphi - \delta\sigma, \\ (\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))\sigma = 6\sigma - 4d\varphi, \\ \delta\varphi = 0. \end{cases} \quad (39)$$

Using Corollary 3.5 together with the equations (31) and (32) (keeping in mind that  $\delta S = 0$  and  $\delta\varphi = 0$ ) we get the two identities  $(\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))\varphi = (\nabla^* \nabla + q(R))\varphi$  and  $(\bar{\nabla}^* \bar{\nabla} + q(\bar{R}))\sigma = (\nabla^* \nabla + q(R))\sigma$ . Hence the classical Weitzenböck formula for the Laplace operator on forms implies that (39) is equivalent to

$$\begin{cases} \Delta\varphi = 4\varphi - \delta\sigma, \\ \Delta\sigma = 6\sigma - 4d\varphi, \\ \delta\varphi = 0. \end{cases} \quad (40)$$

**Lemma 5.2.** *Let  $E(\lambda)$  be the  $\lambda$ -eigenspace of  $\Delta$  restricted to the space of co-closed primitive  $(1, 1)$ -forms. Then the space of solutions of the system (40) is isomorphic to the direct sum  $E(2) \oplus E(6) \oplus E(12)$ . The isomorphism can be written explicitly as*

$$(\varphi, \sigma) \xrightarrow{\Psi} (8\varphi + \delta\sigma, *d\sigma, 2\varphi - \delta\sigma) \in E(2) \oplus E(6) \oplus E(12)$$

with inverse

$$(\alpha, \beta, \gamma) \in E(2) \oplus E(6) \oplus E(12) \xrightarrow{\Phi} \left( \frac{\alpha + \gamma}{10}, \frac{3d\alpha - 5 * d\beta - 2d\gamma}{30} \right).$$

*Proof.* The first thing to check is the fact that  $\Phi$  and  $\Psi$  take values in the right spaces.

Let  $(\alpha, \beta, \gamma) \in E(2) \oplus E(6) \oplus E(12)$  and  $(\varphi, \sigma) := \Phi(\alpha, \beta, \gamma)$ . From Lemma 4.4 in [5], the exterior derivative of every co-closed primitive  $(1, 1)$ -form is a primitive  $(2, 1) + (1, 2)$ -form. Thus  $\varphi \in \Omega_0^{(1,1)}M$  and  $\sigma \in \Omega_0^{(2,1)+(1,2)}M$ . A simple calculation shows that  $(\varphi, \sigma)$  satisfy the system (40).

Conversely, let  $(\varphi, \sigma)$  be a solution of (40) and  $(\alpha, \beta, \gamma) := \Psi(\varphi, \sigma)$ . Clearly  $\alpha$ ,  $\beta$  and  $\gamma$  are co-closed. From Proposition 4.5 and the first equation of (40) we see that  $\delta\sigma$  is a section of  $\Lambda_0^{(1,1)}M$ , so the same holds for  $\alpha$  and  $\gamma$ . The fact that  $*d\sigma$  is a section of the same bundle follows from Lemma 4.3 in [5]. A direct check shows that  $\Delta\alpha = 2\alpha$ ,  $\Delta\beta = 6\beta$  and  $\Delta\gamma = 12\gamma$ .

Finally, it is straightforward to check that  $\Phi$  and  $\Psi$  are inverse to each other. This proves the lemma and the theorem.  $\square$

In order to apply this result, one should try to compute the spectrum of the Laplacian on 2-forms on some explicit compact nearly Kähler 6-dimensional manifolds. Besides the sphere  $S^6$  – which has no infinitesimal Einstein deformations (cf. [2]) – the only known examples are 3-symmetric spaces  $\mathbb{C}P^3 = SO_5/U_2$ ,  $F(1, 2) = SU_3/U_1 \times U_1$  and  $SU_2 \times SU_2 = SU_2 \times SU_2 \times SU_2/\Delta$ .

Computations of the Laplace spectrum using the Peter-Weyl theorem show that  $E(2)$  and  $E(6)$  vanish on each of these spaces. Moreover,  $E(12)$  vanishes on  $\mathbb{C}P^3$  and on  $SU_2 \times SU_2$ , and is 8-dimensional on  $F(1, 2)$  (cf. [6]). As a consequence of these facts, we deduce:

- (1) every infinitesimal Einstein deformation of the 6-dimensional 3-symmetric spaces is an infinitesimal Gray deformation (cf. [5]);
- (2) the nearly Kähler structure on  $\mathbb{C}P^3$  and on  $SU_2 \times SU_2$  is rigid;
- (3) there is an 8-dimensional space of infinitesimal deformations of the nearly Kähler structure on  $F(1, 2)$ .

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