The Weitzenböck Machine

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Abstract

Weitzenböck formulas are an important tool in relating local differential geometry to global topological properties by means of the so–called Bochner method. In this article we give a unified treatment of the construction of all possible Weitzenböck formulas for all irreducible, non–symmetric holonomy groups. The resulting classification is two–fold, we construct explicitly a basis of the space of Weitzenböck formulas on the one hand and characterize Weitzenböck formulas as eigenvectors for an explicitly known matrix on the other. Both classifications allow us to find tailor–suit Weitzenböck formulas for applications like eigenvalue estimates or Betti number estimates.

Contents

1 Introduction

Weitzenböck formulas are an important tool for linking differential geometry and topology of compact Riemannian manifolds. They feature prominently in the Bochner method, where they are used to prove the vanishing of Betti numbers under suitable curvature assumptions or the non–existence of metrics of positive scalar curvature on spin manifolds with non–vanishing $\hat{A}$–genus. Moreover they are used to proof eigenvalue estimates for Laplace and Dirac type operators. In these applications one tries to find a (positive) linear combination of hermitean squares D^*D of first order differential operators D, which sums to an expression in the curvature only. In this approach one need only consider special first order differential operators D known as generalized gradients or Stein–Weiss operators, which are defined as projections of a covariant derivative ∇. Examples for generalized gradients include the exterior derivative d and its adjoint d^* and the Dirac and twistor operator in spin geometry.

In this article we present two different classifications of all possible linear combinations of hermitean squares D^*D of generalized gradients D, which sum to pure curvature expressions, if the underlying connection is the Levi–Civita connection ∇ of a Riemannian manifold M of reduced holonomy $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$ or the exceptional holonomies $\mathbf{G}_2$ and $\mathbf{Spin}(7)$. Both classifications are interesting in their own right, the first describes a recursive procedure to calculate a generating set of Weitzenböck formulas, the second classification provides a simple means to decide, whether a given linear combination of hermitean squares of Stein–Weiss operators is actually a pure curvature expression.

In order to describe the setup of the article in more detail we recall that every representation $G \longrightarrow \mathrm{Aut}\,V$ of the holonomy group G of a Riemannian manifold (M, g) on a complex vector space V defines a complex vector bundle VM on M with a covariant derivative induced from the Levi–Civita connection, in particular the complexified holonomy representation T of G defines the complexified tangent bundle TM. The generalized gradients on VM are the parallel first order differential operators T_ε defined as the projection of $\nabla : VM \longrightarrow TM \otimes VM$ to the parallel subbundles $V_\varepsilon M \subset TM \otimes VM$ arising from a decomposition $T \otimes V = \oplus_\varepsilon V_\varepsilon$ into irreducible subspaces. It will be convenient in this article to call every (finite) linear combination $\sum_\varepsilon c_\varepsilon T_\varepsilon^* T_\varepsilon$ of hermitean squares of generalized gradients a *Weitzenböck formula*.

Our first important observation is that the space $\mathfrak{W}(V)$ of all Weitzenböck formulas on a vector bundle VM can be identified with the vector space $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ and thus is an algebra, which is commutative for irreducible representations V. Moreover it is easy to see that the algebra $\mathfrak{W}(V)$ has a canonical involution, the twist $\tau : \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$, such that a Weitzenböck formula reduces to a pure curvature expression if and only if it is an eigenvector of τ of eigenvalue -1. Of course there are interesting Weitzenböck formulas, which are eigenvectors of τ for the eigenvalue $+1$, perhaps the most prominent example is the connection Laplacian $\nabla^*\nabla$. The classical examples of Weitzenböck formulas like

$$\Delta \;=\; d\,d^* \;+\; d^*\,d \;=\; \nabla^*\nabla \;+\; q(R)$$

the original Weitzenböck formula or the Lichnerowicz–Weitzenböck formula

$$D^2 \;=\; \nabla^*\nabla \;+\; \frac{\kappa}{4}$$

reduce in this setting to the statements that $\Delta - \nabla^*\nabla$ and $D^2 - \nabla^*\nabla$ respectively are eigenvectors of τ for the eigenvalue -1 and thus pure curvature expressions.

Starting with the connection Laplacian $\nabla^*\nabla$, corresponding to $\mathbf{1} \in \mathrm{End}_{\mathfrak{g}}(T \otimes V)$, we will describe a recursion procedure to construct a basis of the space $\mathfrak{W}(V_\lambda)$ of Weitzenböck formulas on an irreducible vector bundle $V_\lambda M$ on M such that the base vectors are eigenvectors of τ with alternating eigenvalues ± 1. Interestingly this recursive procedure makes essential use of a second fundamental Weitzenböck formula $B \in \mathfrak{W}(V)$, the so–called *conformal weight operator*, which was considered for the first time in the work of Paul Gauduchon on conformal geometry [G91]. The details of this recursion procedure and the first initial elements are discussed for the holonomies $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$ only, because the discussion of Weitzenböck formulas in the Kähler holonomies $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ is better done differently, whereas the hyperkähler holonomies $\mathbf{Sp}(1)\,\mathbf{Sp}(n)$ and $\mathbf{Sp}(n)$ will be discussed in more detail in a forthcoming paper.

Eventually we obtain a sequence of B–polynomials $p_i(B)$ such that $p_{2i}(B)$ is in the $(+1)$– and $p_{2i-1}(B)$ is in the (-1)–eigenspace of τ. If b_ε are the B-eigenvalues on $V_\varepsilon \subset T \otimes V$, then the coefficient of $T_\varepsilon^* T_\varepsilon$ in the Weitzenböck formula corresponding to $p_i(B)$ is given by $p_i(b_\varepsilon)$. An interesting feature appears for holonomy $\mathbf{G}_2$ and $\mathbf{Spin}(7)$. Here we have the decomposition $\mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}\, V) \cong \mathrm{Hom}_{g}(\mathfrak{g}, \mathrm{End}\, V) \oplus \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^\perp, \mathrm{End}\, V)$ and because of the holonomy reduction any Weitzenböck formula in the second summand has a zero curvature term.

Finally we would like to mention that the problem of finding all possible Weitzenböck formulas is also considered in the work of Y. Homma (e.g. in [H06]). He gives a solution in the case of Riemannian, Kählerian and HyperKählerian manifolds. Even if there are some similarities in the results, it seems fair to say that our method is completely different. In particular we describe an recursive procedure for obtaining the coefficients of Weitzenböck formulas. The main difference is of course that we give a unified approach including the case of exceptional holonomies.

2 The Holonomy Representation

For the rest of this article we will essentially restrict to irreducible non–symmetric holonomy algebras $\mathfrak{g}$. Most of the statements easily generalize to holonomy algebras $\mathfrak{g}$ with no symmetric irreducible factor in their local de Rham–decomposition, which could be called properly non–symmetric holonomy algebras. Some of the concepts introduced are certainly interesting for symmetric holonomy algebras as well, in particular the central idea used to find the matrix of the twist through the Recursion Formula 4.1. Turning to irreducible non–symmetric holonomy algebras $\mathfrak{g}$ leaves us with seven different cases

	algebra $\mathfrak{g}_{\mathbb{R}}$	holonomy representation $T_{\mathbb{R}}$	T
general Riemannian	$\mathfrak{so}_n$	defining representation $\mathbb{R}^n$	T
Kähler	$\mathfrak{u}_n \cong i\mathbb{R} \oplus \mathfrak{su}_n$	defining representation $\mathbb{C}^n$	$\bar{E} \oplus E$
Calabi–Yau	$\mathfrak{su}_n$	defining representation $\mathbb{C}^n$	$\bar{E} \oplus E$
quaternionic Kähler	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	representation $\mathbb{H}^1 \otimes_{\mathbb{H}} \mathbb{H}^n$	$H \otimes E$
hyper–Kähler	$\mathfrak{sp}(n)$	defining representation $\mathbb{H}^n$	$\mathbb{C}^2 \otimes E$
exceptional $\mathbf{G}_2$	$\mathfrak{g}_2$	standard representation $\mathbb{R}^7$	[7]
exceptional $\mathbf{Spin}(7)$	$\mathfrak{spin}_7$	spinor representation $\mathbb{R}^8$	[8]

(2.1)

according to a theorem of Berger, where T denotes the complexified holonomy representation $T := T_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the $\mathbb{C}$–bilinear extension $\langle\, ,\ \rangle$ of the scalar product. For simplicity we will work with the complexified holonomy representation T and the complexified holonomy algebra $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ throughout as well as with irreducible complex representations V_λ of $\mathfrak{g}$ of highest weight λ. Notations like E or H in the table above fix nomenclature for particularly important representations in special holonomy, say E and $\bar{E}$ refer to the spaces of $(1,0)$– and $(0,1)$–vectors in T in the Kähler and Calabi–Yau case, while [7] and [8] are the standard 7–dimensional representation of $\mathbf{G}_2$ and 8–dimensional spinor representation of $\mathbf{Spin}(7)$ respectively. In passing we note that the complexified holonomy representation T is not isotypical in the Kähler and the Calabi–Yau case and this is precisely the reason why these two cases differ significantly from the rest.

In order to understand Weitzenböck formulas or parallel second order differential operators it is a good idea to start with parallel first order differential operators usually called generalized gradients or Stein–Weiss operators. Their representation theoretic background is the decomposition of tensor products $T \otimes V$ of the holonomy representation T with an arbitrary complex representation V. The general case immediately reduces to studying irreducible representations $V = V_\lambda$ of highest weight λ. In this section we will see that the isotypical components of $T \otimes V_\lambda$ are always irreducible for a properly non–symmetric holonomy algebra $\mathfrak{g}$ and isomorphic to irreducible representations $V_{\lambda+\varepsilon}$ of highest weight $\lambda + \varepsilon$ for some weight ε of the holonomy representation T. Thus the decomposition of $T \otimes V_\lambda$ is completely described by the subset of relevant weights ε:

Definition 2.1 (Relevant Weights)
A weight ε of the holonomy representation T is called relevant for an irreducible representation V_λ of highest weight λ if the irreducible representation $V_{\lambda+\varepsilon}$ of highest weight $\lambda + \varepsilon$ occurs in the tensor product $T \otimes V_\lambda$. We will write $\varepsilon \subset \lambda$ for a relevant weight ε for a given irreducible representation V_λ.

Lemma 2.2 (Characterization of Relevant Weights)
Consider the holonomy representation T of an irreducible non–symmetric holonomy algebra $\mathfrak{g}$ and an irreducible representation V_λ of highest weight λ. The decomposition of the tensor product $T \otimes V_\lambda$ is multiplicity free in the sense that all irreducible subspaces are pairwise non–isomorphic. The complete decomposition of $T \otimes V_\lambda$ is thus the sum

$$T \otimes V_\lambda \;\cong\; \bigoplus_{\varepsilon \subset \lambda} V_{\lambda+\varepsilon}$$

over all relevant weights ε. A weight $\varepsilon \neq 0$ is relevant if and only if $\lambda + \varepsilon$ is dominant. The zero weight $\varepsilon = 0$ only occurs for the holonomy algebras $\mathfrak{so}_n$ with n odd and $\mathfrak{g}_2$, it is relevant if $\lambda - \lambda_\Sigma$ or $\lambda - \lambda_T$ respectively is still dominant, where λ_Σ and λ_T are the highest weights of the spinor representation of $\mathfrak{so}_n$ and the standard representation of $\mathfrak{g}_2$.

Proof: The proof is essentially an exercise in Weyl's character formula $\square$

A particular consequence of Lemma 2.2 is that for sufficiently complicated representations V_λ all weights ε of the holonomy representation T are relevant. With this motivation we will call a highest weight λ *generic* if $\lambda + \varepsilon$ is dominant for all weights ε of the holonomy representation T. A simple consideration shows that λ is generic if and only if $\lambda - \rho$ is dominant, where ρ is the half sum of positive roots or equivalently the sum of fundamental weights, unless we consider odd–dimensional generic holonomy $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{g} = \mathfrak{g}_2$. In the latter holonomies the generic weights λ must have $\lambda - \rho - \lambda_\Sigma$ or $\lambda - \rho - \lambda_T$ dominant respectively. In any case the number of relevant weights for the representation V_λ

$$N(G, \lambda) \quad := \quad \sharp\{\ \varepsilon \mid \varepsilon \text{ is relevant for } \lambda\ \} \quad \leq \quad \dim T$$

is bounded above by dim T with equality if and only if λ is generic. In particular there are at most dim T summands in the decomposition of $T \otimes V_\lambda$ into irreducibles, exactly one copy of $V_{\lambda+\varepsilon}$ for every relevant weight ε.

On the other hand the number $N(G, \lambda)$ of irreducible summands in the decomposition of $T \otimes V_\lambda$ agrees with the dimension of the algebra $\mathrm{End}_{\mathfrak{g}}(\, T \otimes V_\lambda\,)$ of $\mathfrak{g}$–invariant endomorphisms of $T \otimes V_\lambda$, because all isotypical components are irreducible by Lemma 2.2. In the next section we will study the identification $\mathrm{End}_{\mathfrak{g}}(\, T \otimes V_\lambda\,) = \mathrm{Hom}_{\mathfrak{g}}(\, T \otimes T, \mathrm{End}\, V_\lambda\,)$ extensively, which allows us to break up $\mathrm{End}_{\mathfrak{g}}(\, T \otimes V_\lambda\,)$ into interesting subspaces called Weitzenböck classes, whose dimension can be calculated in the following way:

Lemma 2.3 (Dimension of Weitzenböck Classes)
Let us call the space $W^{\mathfrak{t}} := \mathrm{Hom}_{\mathfrak{t}}(\mathbb{R}, W) \subset W$ of elements of a G–representation W invariant under a fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ the zero weight space of W. The dimension of the zero weight space provides an upper bound

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\, W, \mathrm{End}\, V_\lambda\,) \quad \leq \quad \dim W^{\mathfrak{t}}$$

for the dimension of the space $\mathrm{Hom}_{\mathfrak{g}}(\, W, \mathrm{End}\, V_\lambda\,)$ for an irreducible representation V_λ. For sufficiently dominant highest weight λ in dependence on W this upper bound is sharp.

The lemma follows again from the Weyl character formula, but it is also an elementary consequence of Kostant's theorem 8.3 formulated below. We will mainly use Lemma 2.3 for the subspaces W_α occuring in the decomposition $T \otimes T = \oplus W_\alpha$ into irreducibles. In the case of the holonomy algebras $\mathfrak{so}_n$, $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ we have the decomposition $T \otimes T = \mathbb{C} \oplus \mathrm{Sym}^2_0 T \oplus \mathfrak{g} \oplus \mathfrak{g}^\perp$ and the following dimensions of the zero weight spaces:

	$\dim T$	$\dim [\,\mathbb{C}\,]^{\mathfrak{t}}$	$\dim [\,\mathrm{Sym}^2_0 T\,]^{\mathfrak{t}}$	$\dim [\,\mathfrak{g}\,]^{\mathfrak{t}}$	$\dim [\,\mathfrak{g}^\perp\,]^{\mathfrak{t}}$
$\mathfrak{so}_n$	n	1	$\lfloor \frac{n-1}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	—
$\mathfrak{g}_2$	7	1	2	3	1
$\mathfrak{spin}_7$	8	1	3	3	1

(2.2)

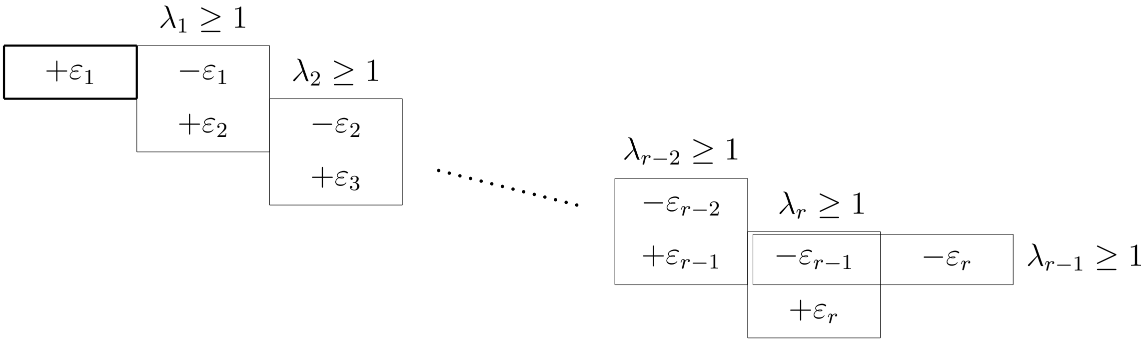
Note in particular that the dimensions of the zero weight spaces sum up to $\dim T$.

Although complete the decision criterion given in Lemma 2.2 is not particularly straightforward in general. At the end of this section we want to give a graphic interpretation of this decision criterion for all irreducible holonomy groups in order to simplify the task of finding the relevant weights. For a fixed holonomy algebra $\mathfrak{g}$ the information necessary in this graphic algorithm is encoded in a single diagram featuring the weights of the holonomy representation T and labeled boxes. A weight ε is relevant for an irreducible representation V_λ if and only if the highest weight $\lambda = \lambda_1\omega_1 + \ldots + \lambda_r\omega_r$ of V_λ, written as a linear combination of fundamental weights $\omega_1, \ldots, \omega_r$, satisfies all inequalities labeling the boxes containing ε. The notation introduced for the weights of the holonomy representation T and the fundamental weights will be used throughout this article.

To begin with let us consider even dimensional Riemannian geometry with generic holonomy $\mathfrak{g} = \mathfrak{so}_{2r}$, $r \geq 2$. In this case the holonomy representation T is the defining representation, whose weights $\pm\varepsilon_1, \pm\varepsilon_2, \ldots, \pm\varepsilon_r$ form an orthonormal basis for a suitable scalar product $\langle\, ,\, \rangle$ on the dual $\mathfrak{t}^*$ of the maximal torus. The ordering of weights can be chosen in such a way that the fundamental weights $\omega_1, \ldots, \omega_r$ are given by:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
\omega_1 & = & \varepsilon_1 & \pm\varepsilon_1 & = & \pm\,\omega_1 \\
\omega_2 & = & \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = & \pm(\omega_2 - \omega_1) \\
\vdots & & \vdots & \vdots & & \vdots \\
\omega_{r-2} & = & \varepsilon_1 + \ldots + \varepsilon_{r-2} & \pm\varepsilon_{r-2} & = & \pm(\omega_{r-2} - \omega_{r-3}) \\
\omega_{r-1} & = & \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{r-1} + \varepsilon_r) & \pm\varepsilon_{r-1} & = & \pm(\omega_{r-1} + \omega_r - \omega_{r-2}) \\
\omega_r & = & \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{r-1} - \varepsilon_r) & \pm\varepsilon_r & = & \pm(\omega_{r-1} - \omega_r)
\end{array}$$

Every dominant integral weight of $\mathfrak{so}_{2r}$ can be written $\lambda = \lambda_1\omega_1 + \ldots + \lambda_r\omega_r$ with natural numbers $\lambda_1, \ldots, \lambda_r \geq 0$ and the criterion of Lemma 2.2 becomes:

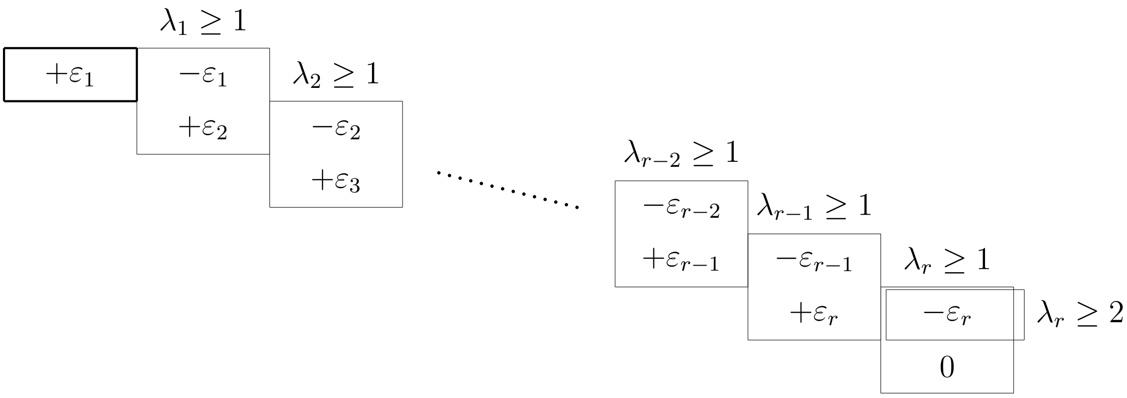


A weight ε of the holonomy representation T of $\mathfrak{so}_{2r}$ is relevant for the irreducible representation V_λ if and only if λ satisfies all the conditions labeling the boxes containing ε. Say the weights $-\varepsilon_1$ and $+\varepsilon_2$ are relevant for all irreducible representations V_λ with $\lambda_1 \geq 1$, whereas $-\varepsilon_{r-1}$ is relevant for V_λ if and only if $\lambda_{r-1} \geq 1$ *and* $\lambda_r \geq 1$.

Odd dimensional Riemannian geometry $\mathfrak{g} = \mathfrak{so}_{2r+1}$, $r \geq 1$, with generic holonomy is of course closely related to $\mathfrak{g} = \mathfrak{so}_{2r}$. The weights $\pm\varepsilon_1, \pm\varepsilon_2, \ldots, \pm\varepsilon_r$ of the defining holonomy representation T besides the zero weight form an orthonormal basis for a suitable scalar product $\langle\, ,\, \rangle$ on the dual $\mathfrak{t}^*$ of the maximal torus. With a suitable choice of ordering of weights the fundamental weights $\omega_1, \ldots, \omega_r$ and the weights of T relate via:

$$
\begin{array}{lcl@{\qquad\qquad}lcl}
\omega_1 & = & \varepsilon_1 & \pm\varepsilon_1 & = & \pm\,\omega_1 \\
\omega_2 & = & \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = & \pm(\omega_2 - \omega_1) \\
\vdots & & \vdots & \vdots & & \vdots \\
\omega_{r-1} & = & \varepsilon_1 + \ldots + \varepsilon_{r-1} & \pm\varepsilon_{r-2} & = & \pm(\omega_{r-1} - \omega_{r-2}) \\
\omega_r & = & \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{r-1} + \varepsilon_r) & \pm\varepsilon_r & = & \pm(2\omega_r - \omega_{r-1})
\end{array}
$$

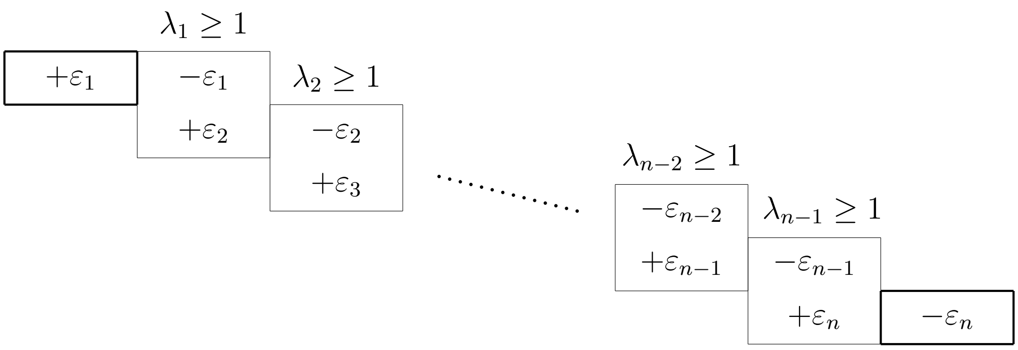
Writing a dominant integral weight $\lambda = \lambda_1\omega_1 + \ldots + \lambda_r\omega_r$ as a linear combination of fundamental weights with integers $\lambda_1, \ldots, \lambda_r \geq 0$ the criterion of Lemma 2.2 becomes:



Turning from the Riemannian case to the Kähler case $\mathfrak{g} = \mathfrak{u}_n$ we observe that the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_n$ of the defining standard representation $T = E \oplus \bar{E}$ form an orthonormal basis for an invariant scalar product on the dual $\mathfrak{t}^*$ of a maximal torus $\mathfrak{t} \subset \mathfrak{u}_n$, but they become linearly dependent when projected to the dual of a maximal torus of the ideal $\mathfrak{su}_n \subset \mathfrak{u}_n$. In any case the fundamental weights and the weights of T relate as

$$
\begin{array}{lcl@{\qquad\qquad}lcl}
\omega_1 & = & \varepsilon_1 & \pm\varepsilon_1 & = & \pm\,\omega_1 \\
\omega_2 & = & \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = & \pm(\omega_2 - \omega_1) \\
\vdots & & \vdots & \vdots & & \vdots \\
\omega_n & = & \varepsilon_1 + \ldots + \varepsilon_n & \pm\varepsilon_n & = & \pm(\omega_n - \omega_{n-1})
\end{array}
$$

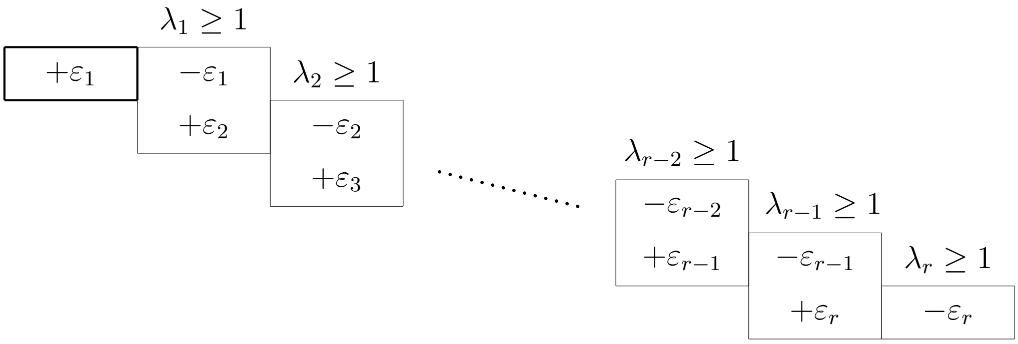
and the criterion of Lemma 2.2 becomes:



The quaternionic Kähler and hyperkähler cases are more complicated, because the condition of being relevant has to be checked for both ideals $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ of $\mathfrak{g}$. For a single ideal however the condition becomes simple again. The weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_n$ of E are again orthonormal for a suitable scalar product $\langle\, ,\, \rangle$ on the dual $\mathfrak{t}^*$ of a maximal torus $\mathfrak{t} \subset \mathfrak{sp}(r)$ for $r = 1$ or $r = n$ and relate to the fundamental weights by the formulas:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\omega_1 & = & \varepsilon_1 & \pm\varepsilon_1 & = & \pm\,\omega_1 \\
\omega_2 & = & \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = & \pm(\omega_2 - \omega_1) \\
\vdots & & \vdots & \vdots & & \vdots \\
\omega_r & = & \varepsilon_1 + \ldots + \varepsilon_r & \pm\varepsilon_r & = & \pm(\omega_r - \omega_{r-1})
\end{array}$$

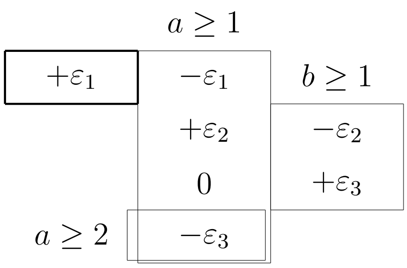
The graphical interpretation of Lemma 2.2 is given by the diagram:



Finally we consider the two exceptional cases $\mathfrak{g}_2$ and $\mathfrak{spin}_7$. Recall that the group $\mathbf{G}_2$ is the group of automorphisms of the octonions $\mathbb{O}$ as an algebra over $\mathbb{R}$. In this sense the holonomy representation $T_{\mathbb{R}}$ is the defining representation $\operatorname{Im}\mathbb{O}$ of $\mathfrak{g}_2$ with complexification $T = [7]$. There are too many weights of the holonomy representation to be orthonormal for any scalar product on the dual $\mathfrak{t}^*$ of a fixed maximal torus $\mathfrak{t} \subset \mathfrak{g}_2$, but at least we can choose an ordering of weights for $\mathfrak{t}^*$ so that the weights of T become totally ordered $+\varepsilon_1 > +\varepsilon_2 > +\varepsilon_3 > 0 > -\varepsilon_3 > -\varepsilon_2 > -\varepsilon_1$. In this notation we have:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\omega_1 & = & \varepsilon_1 & \pm\varepsilon_1 & = & \pm\,\omega_1 \\
\omega_2 & = & \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = & \pm(\omega_2 - \omega_1) \\
 & & & \pm\varepsilon_3 & = & \mp(\omega_2 - 2\omega_1)
\end{array}$$

The scalar product of choice on $\mathfrak{t}^*$ is specified by $\langle\, \varepsilon_1, \varepsilon_1 \,\rangle = 1 = \langle\, \varepsilon_2, \varepsilon_2 \,\rangle$ and $\langle\, \varepsilon_1, \varepsilon_2 \,\rangle = \frac{1}{2}$. Writing a dominant integral weight as $\lambda = a\omega_1 + b\omega_2$, $a, b \geq 0$, we read Lemma 2.2 as:



The holonomy representation of the holonomy algebra $\mathfrak{g} = \mathfrak{spin}_7$ is the 8–dimensional spinor representation $T = [8]$. It is convenient to write the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_4$ of T

and the fundamental weights ω_1, ω_2 and ω_3 in terms of the weights $\pm\eta_1$, $\pm\eta_2$, $\pm\eta_3$, 0 of the 7–dimensional defining representation of $\mathfrak{spin}_7$, which form an orthonormal basis for a suitable scalar product on the dual $\mathfrak{t}^*$ of the maximal torus. With this proviso the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_4$ of T and the fundamental weights ω_1, ω_2 and ω_3 can be written as:

$$\begin{array}{rcl}
\omega_1 &=& \eta_1\\
\omega_2 &=& \eta_1+\eta_2\\
\omega_3 &=& \frac{1}{2}(\eta_1+\eta_2+\eta_3)
\end{array}
\qquad
\begin{array}{rclcl}
\pm\varepsilon_1 &=& \pm\frac{1}{2}(\eta_1+\eta_2+\eta_3) &=& \pm\,\omega_3\\
\pm\varepsilon_2 &=& \pm\frac{1}{2}(\eta_1+\eta_2-\eta_3) &=& \pm(\omega_2-\omega_3)\\
\pm\varepsilon_3 &=& \pm\frac{1}{2}(\eta_1-\eta_2+\eta_3) &=& \pm(\omega_3-\omega_2+\omega_1)\\
\pm\varepsilon_4 &=& \pm\frac{1}{2}(\eta_1-\eta_2-\eta_3) &=& \pm(\omega_1-\omega_3)
\end{array}$$

and Lemma 2.2 for a dominant integral weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ translates into:

$$\begin{array}{cccc}
 & & c \geq 1 & \\
 & \boxed{+\varepsilon_1} & -\varepsilon_1 & b \geq 1\\
 & & +\varepsilon_2 & -\varepsilon_2\\
a \geq 1 & -\varepsilon_4 & -\varepsilon_3 & +\varepsilon_3\\
 & & +\varepsilon_4 &
\end{array}$$

3 The Space $\mathfrak{W}(V)$ of Weitzenböck Formulas

In this section we will define twistor operators, Weitzenböck formulas and the space of Weitzenböck formula with its different realizations. Then we will introduce the conformal weight operator, which in many cases generates all possible Weitzenböck formulas. Finally we define the classifying endomorphism and study the corresponding eigenspace decomposition.

3.1 Weitzenböck Formulas

We consider parallel second order differential operators P on sections of a vector bundle VM over a Riemannian manifold M with special holonomy G. By definition these are differential operators, which up to first order differential operators can always be written as the composition

$$\Gamma(VM) \xrightarrow{\nabla^2} \Gamma(T^*M\otimes T^*M\otimes VM) \xrightarrow{\cong} \Gamma(TM\otimes TM\otimes VM) \xrightarrow{F} \Gamma(VM)$$

where F is a parallel section of the vector bundle $\mathrm{Hom}\,(TM\otimes TM\otimes VM, VM)$ corresponding to a G–equivariant homomorphism $F \in \mathrm{Hom}_G(T\otimes T\otimes V, V)$. A particularly important example is the connection Laplacian $\nabla^*\nabla$ which arises from the linear map $a\otimes b\otimes\psi \longmapsto -\langle\, a,\, b\,\rangle\psi$. Note that we are only considering reduced holonomy groups G, which are connected by definition, so that G–equivariance is equivalent to $\mathfrak{g}$–equivariance. Taking advantage of this fact we describe other parallel differential operators by means of the following identifications of spaces of invariant homomorphisms:

$$\mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T\otimes V,\,V\,) \;=\; \mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T,\,\mathrm{End}\,V\,) \;=\; \mathrm{End}_{\mathfrak{g}}(\,T\otimes V\,)$$

Of course the identification $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T\otimes V,V) = \mathrm{Hom}_{\mathfrak{g}}(T\otimes T,\mathrm{End}\,V)$ is the usual tensor shuffling $F(a\otimes b\otimes v) = F_{a\otimes b}v$ for all $a, b \in T$ and $v \in V$. The second identification $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T\otimes V,V) = \mathrm{End}_{\mathfrak{g}}(T\otimes V)$ depends on the existence of a $\mathfrak{g}$–invariant scalar product on T or the musical isomorphism $T \cong T^*$ via a summation

$$F(\,b\otimes v\,) \;=\; \sum_\mu t_\mu\otimes F(t_\mu\otimes b\otimes v) \qquad F(a\otimes b\otimes v) \;=\; (\langle\, a,\,\cdot\,\rangle\,\lrcorner\,\otimes\mathrm{id})\,F(b\otimes v)$$

over an orthonormal basis $\{t_\mu\}$. Under this identification the identity of $T\otimes V$ becomes the homomorphism $a\otimes b\otimes\psi\longmapsto\langle\, a,\, b\,\rangle\psi$ corresponding to the connection Laplacian $-\nabla^*\nabla$. The composition of endomorphisms turns $\mathrm{End}_{\mathfrak{g}}(\,T\otimes V\,)$ and thus $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T,\,\mathrm{End}\,V)$ into an algebra, for $F,\,\tilde{F}\in\mathrm{Hom}_{\mathfrak{g}}(T\otimes T,\mathrm{End}\,V)$ the resulting algebra structure reads:

$$(\,F\circ\tilde{F}\,)_{a\otimes b} \;=\; \sum_\mu F_{a\otimes t_\mu}\,\circ\,\tilde{F}_{t_\mu\otimes b} \tag{3.3}$$

Last but not least we note that the invariance condition for $F\in\mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T,\mathrm{End}\,V\,)$ is equivalent to the identity $[\,X,\,F_{a\otimes b}\,] = F_{Xa\otimes b}+F_{a\otimes Xb}$ for all $X\in\mathfrak{g}$ and $a,\,b\in T$.

Assuming that $V = V_\lambda$ is irreducible of highest weight λ we know from Lemma 2.2 that the isotypical components of $T\otimes V_\lambda$ are irreducible for non–symmetric holonomy groups. The algebra $\mathrm{End}_{\mathfrak{g}}(\,T\otimes V_\lambda\,)$ is thus commutative and spanned by the pairwise orthogonal idempotents pr_ε projecting onto the irreducible subspaces $V_{\lambda+\varepsilon}$ of $T\otimes V_\lambda$. In order to describe the corresponding second order differential operators we introduce first order differential operators T_ε known as Stein–Weiss operators or generalized gradients by:

$$T_\varepsilon:\quad \Gamma(\,V_\lambda M\,)\;\longrightarrow\;\Gamma(\,V_{\lambda+\varepsilon}M\,),\qquad \psi\;\longmapsto\;\mathrm{pr}_\varepsilon(\nabla\psi)\;.$$

A typical example of a Stein–Weiss operator is the twistor operator of spin geometry, which projects the covariant derivative of a spinor onto the kernel of the Clifford multiplication. Straightforward calculations show that the second order differential operator associated to the idempotent pr_ε is the composition of T_ε with its formal abjoint operator $T_\varepsilon^*:\Gamma(V_{\lambda+\varepsilon}M)\longrightarrow\Gamma(V_\lambda M)$ in the sense $\mathrm{pr}_\varepsilon(\nabla^2) = -T_\varepsilon^*T_\varepsilon$ compare [S06]. In consequence we can write the second order differential operator $F(\nabla^2)$ associated to $F\in\mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T,V_\lambda\,)$ as a linear combination of the squares of Stein–Weiss operators:

$$F(\,\nabla^2\,) \;=\; -\sum_\varepsilon f_\varepsilon\,T_\varepsilon^*T_\varepsilon\;. \tag{3.4}$$

In fact with $\mathrm{End}_{\mathfrak{g}}(\,T\otimes V_\lambda\,) = \mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T,\,\mathrm{End}\,V_\lambda\,)$ being spanned by the idempotents pr_ε every $F\in\mathrm{End}_{\mathfrak{g}}(\,T\otimes V_\lambda\,)$ expands as $F = \sum_\varepsilon f_\varepsilon\,\mathrm{pr}_\varepsilon$ with coefficients f_ε determined by $F|_{V_{\lambda+\varepsilon}} = f_\varepsilon\,\mathrm{id}$. A particular instance of equation (3.4) is the identity $\nabla^*\nabla = \sum_\varepsilon T_\varepsilon^*T_\varepsilon$ associated to the expansion $\mathrm{id}_{T\otimes V_\lambda} = \sum_\varepsilon\mathrm{pr}_\varepsilon$. Motivated by this and other well–known identities of second order differential operators of the form (3.4) we will in general call all elements $F\in\mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T\otimes V,\,V\,) = \mathrm{End}_{\mathfrak{g}}(T\otimes V)$ Weitzenböck formulas:

Definition 3.1 (Space of Weitzenböck Formulas on VM)
The Weitzenböck formulas on a vector bundle VM correspond bijectively to vectors in:

$$\mathfrak{W}(\,V\,)\;:=\;\mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T\otimes V,\,V\,)\;=\;\mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T,\,\mathrm{End}\,V\,)\;=\;\mathrm{End}_{\mathfrak{g}}(\,T\otimes V\,)\,.$$

Of course we are mainly interested in Weitzenböck formulas inducing differential operators of zeroth order or equivalently "pure curvature terms". Clearly a Weitzenböck formula $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V)$ skew–symmetric in its two T–arguments will induce a pure curvature term $F(\nabla^2)$, because we can then reshuffle the summation in the calculation:

$$F(\nabla^2 v) \;=\; \tfrac{1}{2}\sum_{\mu\nu} F(t_\mu \otimes t_\nu \otimes (\nabla^2_{t_\mu,t_\nu} - \nabla^2_{t_\nu,t_\mu})v) \;=\; \tfrac{1}{2}\sum_{\mu\nu} F_{t_\mu\otimes t_\nu}\, R_{t_\mu,t_\nu} v\;. \qquad (3.5)$$

Here and in the following we will denote with $\{t_\nu\}$ an orthonormal basis of T and also a local orthonormal basis of the tangent bundle. Conversely the principal symbol of the differential operator $F(\nabla^2)$ is easily computed to be $\sigma_{F(\nabla^2)}(\xi)v = F_{\xi^\flat\otimes\xi^\flat}v$ for every cotangent vector ξ and every $v \in VM$. Hence the principal symbol vanishes identically exactly for the skew–symmetric Weitzenböck formulas. Weitzenböck formulas F leading to a pure curvature term $F(\nabla^2)$ are thus completely characterized by being eigenvectors of eigenvalue -1 for the involution

$$\tau: \qquad \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V), \qquad F \longmapsto \tau(F)$$

defined in the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V)$ as precomposition with the twist $\tau : T\otimes T\otimes V \longrightarrow T\otimes T\otimes V$, $a\otimes b\otimes v \longmapsto b\otimes a\otimes v$. In other words a Weitzenböck formula F will reduce to a pure curvature term if and only if $\tau(F) := F \circ \tau = -F$.

Considering the space of Weitzenböck formulas $\mathfrak{W}(V)$ as the algebra $\mathrm{End}_{\mathfrak{g}}(T\otimes V)$ we can introduce additional structures on it: the unit $\mathbf{1} := \mathrm{id}_{T\otimes V} \in \mathfrak{W}(V)$, the scalar product

$$\langle F, \tilde{F} \rangle \;:=\; \frac{1}{\dim V}\,\mathrm{tr}_{T\otimes V}(F\tilde{F}) \qquad\qquad F,\, \tilde{F} \in \mathfrak{W}(V)$$

satisfying $\langle FG, \tilde{F} \rangle = \langle F, G\tilde{F} \rangle$ and the trace $\mathrm{tr}\, F := \langle F, \mathbf{1} \rangle$. Clearly the trace of the unit is given by $\mathrm{tr}\,\mathbf{1} = \dim T$. The definition of the trace can be rewritten in the form

$$\mathrm{tr}\, F \;=\; \frac{1}{\dim V}\,\mathrm{tr}_V\Big(v \longmapsto \sum_\mu F_{t_\mu\otimes t_\mu} v \Big)$$

so that the trace is invariant under the twist τ. A slightly more complicated argument using (3.3) shows that the scalar product is invariant under the twist, too. In particular the eigenspaces for τ for the eigenvalues ± 1 are orthogonal and all eigenvectors in the (-1)–eigenspace of τ have vanishing trace. Fom the definition of the trace we obtain that the trace of an element $F = \sum f_\varepsilon \mathrm{pr}_\varepsilon$ of $\mathfrak{W}(V_\lambda)$ in the irreducible case is given by

$$\mathrm{tr}\, F \;=\; \sum_\varepsilon f_\varepsilon\, \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda} \qquad (3.6)$$

in particular the idempotents pr_ε form an orthogonal basis of $\mathfrak{W}(V_\lambda)$:

$$\langle \mathrm{pr}_\varepsilon, \mathrm{pr}_{\tilde{\varepsilon}} \rangle \;=\; \delta_{\varepsilon\tilde{\varepsilon}}\,\frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}$$

A different way to interprete the trace is to note that for every Weitzenböck formula $F \in \mathfrak{W}(V)$ considered as an equivariant homomorphism $F : T\otimes T \longrightarrow \mathrm{End}\, V$ the trace endomorphism $\sum_\mu F_{t_\mu\otimes t_\mu} \in \mathrm{End}_{\mathfrak{g}} V$ is invariant under the action of $\mathfrak{g}$. For an irreducible representation V_λ it is thus the multiple $\sum_\mu F_{t_\mu\otimes t_\mu} = (\mathrm{tr}\, F)\,\mathrm{id}_{V_\lambda}$ of id_{V_λ} by Schur's Lemma.

3.2 The conformal weight operator

In order to study the fine structure of the algebra $\mathfrak{W}(V) = \mathrm{End}_{\,\mathfrak{g}}(T \otimes V)$ of Weitzenböck formulas it is convenient to introduce the *conformal weight operator* $B \in \mathfrak{W}(V)$ of the holonomy algebra $\mathfrak{g}$ and its variations $B^{\mathfrak{h}} \in \mathfrak{W}(V)$ associated to the non–trivial ideals $\mathfrak{h} \subset \mathfrak{g}$ of $\mathfrak{g}$. All these conformal weight operators commute and the commutative subalgebra of $\mathfrak{W}(V)$ generated by them in the irreducible case $V = V_\lambda$ is actually all of $\mathfrak{W}(V)$ for generic highest weight λ. In this subsection we work out some direct consequences of the description of Weitzenböck formulas as polynomials in the conformal weight operators.

Recall that the scalar product $\langle\, ,\, \rangle$ on T induces a scalar product on all exterior powers $\Lambda^k T$ of T via Gram's determinant. Using this scalar product on $\Lambda^2 T$ we can identify the adjoint representation $\mathfrak{so}\, T$ of $\mathbf{SO}(n)$ with $\Lambda^2 T$ through $\langle\, X,\, a \wedge b\,\rangle \;=\; \langle\, Xa,\, b\,\rangle$ and hence think of the holonomy algebra $\mathfrak{g} \;\subset\; \mathfrak{so}\, T$ as a subspace of the euclidian vector space $\Lambda^2 T$:

Definition 3.2 (Conformal Weight Operator)
Consider an ideal $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ in the real holonomy algebra. Its complexification $\mathfrak{h} \;:=\; \mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is an ideal in $\mathfrak{g}$ and a regular subspace $\mathfrak{h} \subset \mathfrak{g} \subset \Lambda^2 T$ in $\Lambda^2 T$ with associated orthogonal projection $\mathrm{pr}_{\mathfrak{h}} : \Lambda^2 T \longrightarrow \mathfrak{h}$. The conformal weight operator $B^{\mathfrak{h}} \;\in\; \mathfrak{W}(V)$ is defined by

$$B^{\mathfrak{h}}_{a \otimes b}\, v \;\;:=\;\; \mathrm{pr}_{\mathfrak{h}}(a \wedge b)\, v$$

in the interpretation of Weitzenböck formulas as linear maps $T \otimes T \longrightarrow \mathrm{End}\, V$. Under the identification $\mathrm{Hom}_{\,\mathfrak{g}}(T \otimes T, \mathrm{End}\, V) \;=\; \mathrm{End}_{\,\mathfrak{g}}(T \otimes V)$ discussed above the conformal weight operator $B^{\mathfrak{h}}$ becomes the following sum over an orthonormal basis $\{t_\mu\}$ of T:

$$B^{\mathfrak{h}}(\, b \otimes v\,) \;\;=\;\; \sum_\mu t_\mu \otimes \mathrm{pr}_{\mathfrak{h}}(t_\mu \wedge b)\, v$$

The notation $B \;:=\; B^{\mathfrak{g}}$ will be used for the conformal weight operator of the algebra $\mathfrak{g}$.

Most of the irreducible non–symmetric holonomy algebras $\mathfrak{g}$ are simple and hence there is only one weight operator B (c.f. table (2.1)). The exceptions are Kähler geometry $\mathfrak{g}_{\mathbb{R}} \;=\; i\mathbb{R} \oplus \mathfrak{su}_n$ with a one–dimensional center in dimension $2n$ and two commuting weight operators $B^{i\mathbb{R}}$ and $B^{\mathfrak{su}}$ and quaternionic Kähler geometry $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ in dimension $4n$, $n \geq 2$ with two commuting weight operators B^H and B^E.

Lemma 3.3 (Fegan's Lemma [F76])
The conformal weight operator $B^{\mathfrak{h}} \in \mathfrak{W}(V)$ of an ideal $\mathfrak{h} \subset \mathfrak{g} \subset \Lambda^2 T$ can be written

$$B^{\mathfrak{h}} \;\;=\;\; -\sum_\alpha X_\alpha \otimes X_\alpha \;\;\in\;\; \mathrm{End}_{\,\mathfrak{g}}(\, T \otimes V\,)$$

where $\{X_\alpha\}$ is an orthonormal basis of $\mathfrak{h}$ for the scalar product on $\Lambda^2 T$ induced from T.

Proof: Let $\{t_\mu\}$ and $\{X_\alpha\}$ be orthonormal bases of T and $\mathfrak{h}$ respectively. Using the characterization $\langle\, X,\, a \wedge b\,\rangle \;=\; \langle\, Xa,\, b\,\rangle$ of the identification $\mathfrak{so}\, T \;=\; \Lambda^2 T$ we find:

$$\begin{aligned} B^{\mathfrak{h}}(\, b \otimes v\,) \;&=\; \sum_\mu t_\mu \otimes \mathrm{pr}_{\mathfrak{h}}(\, t_\mu \wedge b\,)\, v \\ &=\; \sum_{\mu\alpha} t_\mu \otimes \langle\, X_\alpha,\, t_\mu \wedge b\,\rangle X_\alpha\, v \;\;=\;\; -\sum_\alpha X_\alpha b \otimes X_\alpha\, v\,. \end{aligned} \qquad \square$$

A particularly nice consequence of Fegan's Lemma is that the conformal weight operators $B^{\mathfrak{h}}$ and $B^{\tilde{\mathfrak{h}}}$ associated to two ideals $\mathfrak{h},\,\tilde{\mathfrak{h}}\subset\mathfrak{g}$ always commute. In fact two disjoint ideals $\mathfrak{h}\cap\tilde{\mathfrak{h}} = \{0\}$ of $\mathfrak{g}$ commute by definition, the general case follows easily. Hence the algebra structure on $\mathfrak{W}(V)$ allows us to use the evaluation homomorphism

$$\Phi:\qquad \mathbb{C}[\ \{\ B^{\mathfrak{h}}\ |\ \mathfrak{h}\text{ irreducible ideal of }\mathfrak{g}\ \}\ ]\ \longrightarrow\ \mathfrak{W}(\ V\ )\tag{3.7}$$

from the polynomial algebra on abstract symbols $\{B^{\mathfrak{h}}\}$ to the algebra $\mathrm{End}_{\mathfrak{g}}(T\otimes V)$ for studying the fine structure of the space $\mathfrak{W}(V)$ of Weitzenböck formulas.

In order to turn Fegan's Lemma into an effective formula for the eigenvalues of the conformal weight operator $B^{\mathfrak{h}}$ of an ideal $\mathfrak{h}\subset\mathfrak{g}$ we need to calculate the Casimir operator in the normalization given by the scalar product on Λ^2T. Recall that the Casimir operator is defined as a sum $\mathrm{Cas} := \sum_\alpha X_\alpha^2 \in \mathcal{U}\mathfrak{h}$ over an orthonormal basis $\{X_\alpha\}$ of $\mathfrak{h}$ and is thus determined only up to a constant. Usually it is much more convenient to calculate the Casimir Cas with respect to a scalar product of choice and later normalize it to the Casimir Cas^{Λ^2} with respect to the invariant scalar product induced on $\mathfrak{h}\subset\Lambda^2T$.

For a given irreducible ideal $\mathfrak{h}\subset\mathfrak{g}$ in an irreducible holonomy algebra $\mathfrak{g}$ the Casimir operator Cas for $\mathfrak{h}$ is now real, symmetric and $\mathfrak{g}$–invariant. Although the holonomy representation T of $\mathfrak{g}$ may not be irreducible itself, it is the complexification of the irreducible real representation $T_{\mathbb{R}}$ so that we can still conclude that Cas acts as the scalar multiple $\mathrm{Cas}_T\,\mathrm{id}$ of the identity on T. The Casimir eigenvalue $\mathrm{Cas}^{\Lambda^2}_{V_\lambda}$ of the properly normalized Casimir operator on a general irreducible representation V_λ of $\mathfrak{g}$ of highest weight λ can then be calculated from the Casimir Cas using

$$\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\quad=\quad -\,2\,\frac{\dim\mathfrak{h}}{\dim T}\,\frac{\mathrm{Cas}_{V_\lambda}}{\mathrm{Cas}_T}\,,\tag{3.8}$$

where the ambiguity in the choice of normalization cancels out in the quotient $\frac{\mathrm{Cas}_{V_\lambda}}{\mathrm{Cas}_T}$. In fact the normalization (3.8) is readily checked for the holonomy representation $V = T$

$$\mathrm{tr}_T\,\mathrm{Cas}^{\Lambda^2}\quad=\quad\dim T\cdot\mathrm{Cas}^{\Lambda^2}_T\quad=\quad\sum_\alpha\mathrm{tr}_T\,X_\alpha^2\quad=\quad-\,2\,\dim\mathfrak{h}\,,$$

because the scalar product induced from T on Λ^2T satisfies $\langle\,X,\,Y\,\rangle\;=\;-\frac{1}{2}\mathrm{tr}_T\,XY$.

Corollary 3.4 (Explicit Formula for Conformal Weights)
Consider the tensor product $T\otimes V_\lambda = \oplus_{\varepsilon\subset\lambda}V_{\lambda+\varepsilon}$ of the holonomy representation T with the irreducible representation V_λ of highest weight λ. For an ideal $\mathfrak{h}\subset\mathfrak{g}$ let ε_{max} be the highest weight of T and ρ be the half sum of positive weights of $\mathfrak{h}$ in the dual $\mathfrak{t}^*$ of a maximal torus $\mathfrak{t}$. With respect to the basis $\{\mathrm{pr}_\varepsilon\}$ of idempotents the conformal weight operator $B^{\mathfrak{h}}$ of the ideal $\mathfrak{h}$ can be expanded $B^{\mathfrak{h}} = \sum_{\varepsilon\subset\lambda}b^{\mathfrak{h}}_\varepsilon\,\mathrm{pr}_\varepsilon$ with conformal weights

$$b^{\mathfrak{h}}_\varepsilon\quad=\quad 2\,\frac{\dim\mathfrak{h}}{\dim\,T}\,\frac{\langle\,\lambda+\rho,\,\varepsilon\,\rangle\;-\;\langle\,\rho,\,\varepsilon_{max}\,\rangle\;+\;\frac{1}{2}\,(|\varepsilon|^2-|\varepsilon_{max}|^2)}{\langle\,\varepsilon_{max}+2\rho,\,\varepsilon_{max}\,\rangle}\,,$$

where $\langle\,\cdot,\,\cdot\,\rangle$ is an arbitrary scalar product on $\mathfrak{t}^*$ invariant under the Weyl group of $\mathfrak{h}$.

Proof: According to Lemma 3.3 the conformal weight operator can be written as a difference $B^{\mathfrak{h}} = -\frac{1}{2}\,(\mathrm{Cas}^{\Lambda^2} - \mathrm{Cas}^{\Lambda^2} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{Cas}^{\Lambda^2})$ of properly normalized Casimir operators. In particular its restriction to the irreducible summand $V_{\lambda+\varepsilon} \subset T \otimes V_\lambda$ acts by multiplication with $b^{\mathfrak{h}}_\varepsilon := -\frac{1}{2}(\mathrm{Cas}^{\Lambda^2}_{V_{\lambda+\varepsilon}} - \mathrm{Cas}^{\Lambda^2}_T - \mathrm{Cas}^{\Lambda^2}_{V_\lambda})$. The conformal weights $b^{\mathfrak{h}}_\varepsilon$ can thus be calculated using Freudenthal's formula $\mathrm{Cas}_{V_\lambda} = \langle\, \lambda + 2\rho,\, \lambda\, \rangle$ for the Casimir eigenvalues of irreducible representations V_λ and the normalization (3.8). □

It is clear from the definition that the conformal weight operator $B^{\mathfrak{h}} \in \mathfrak{W}(V)$ of an ideal $\mathfrak{h} \subset \mathfrak{g}$ of the holonomy algebra $\mathfrak{g}$ is in the (-1)–eigenspace of the involution τ and thus induces a pure curvature term $B^{\mathfrak{h}}(\,\nabla^2\,)$ on every vector bundle VM associated to the holonomy reduction of M. Explicitly we can describe this curvature term using an orthonormal basis $\{X_\alpha\}$ of the ideal $\mathfrak{h}$ for the scalar product induced on $\mathfrak{h} \subset \Lambda^2 T$. Namely the curvature operator $R : \Lambda^2 TM \longrightarrow \mathfrak{g}M \subset \Lambda^2 TM$, $a \wedge b \longmapsto R_{a,b}$, associated to the curvature tensor R of M allows us to write down a well–defined global section

$$q^{\mathfrak{h}}(R) \;\; := \;\; \sum_\alpha X_\alpha\, R(X_\alpha) \;\; \in \;\; \Gamma(\,\mathcal{U}^{\,\leq 2}\mathfrak{g}M\,) \tag{3.9}$$

of the universal enveloping algebra bundle associated to the holonomy reduction. Fixing a representation $G \longrightarrow \mathrm{Aut}(V)$ of the holonomy group the section $q^{\mathfrak{h}}(R)$ in turn induces an endomorphism on the vector bundle VM associated to V and the holonomy reduction.

A particularly important example of Weitzenböck formulas is the classical formula of Weitzenböck for the Laplace operator $\Delta = d^* d \,+\, d\, d^*$ acting on differential forms, i.e.

$$\Delta \;\; = \;\; \nabla^* \nabla \; + \; q(R) \tag{3.10}$$

The curvature term in this formula is precisely the curvature endomorphism for the full holonomy algebra $\mathfrak{g}$, in particular $q(R) = \mathrm{Ric}$ on the bundle of 1–forms on M. We recall that the curvature term in the Weitzenböck formula (3.10) is known to be the Casimir operator of the holonomy algebra $\mathfrak{g}$ on a symmetric space $M = \tilde{G}/G$, more precisely for every ideal $\mathfrak{h} \subset \mathfrak{g}$ the curvature term $q^{\mathfrak{h}}(R)$ acts as the Casimir operator of the ideal $\mathfrak{h}$ on every homogeneous vector bundle VM over a symmetric space M.

A minor subtlety in the definition of the curvature terms $q^{\mathfrak{h}}(R)$ should not pass unnoticed, in difference to the conformal weight operators $B^{\mathfrak{h}}$ the curvature terms $q^{\mathfrak{h}}(R)$ associated to two ideals $\mathfrak{h}, \tilde{\mathfrak{h}} \subset \mathfrak{g}$ do not in general commute. The problem is that the curvature operator $R : \Lambda^2 TM \longrightarrow \mathfrak{g}M$ does not necessarily map the parallel subbundle $\mathfrak{h}M \subset \Lambda^2 TM$ associated to an ideal $\mathfrak{h} \subset \mathfrak{g}$ to itself due to the presence of "mixed terms" in the curvature tensor R. In other words the section $q^{\mathfrak{h}}(R)$ is not in general a section of $\mathcal{U}^{\,\leq 2}\mathfrak{h}M \subset \mathcal{U}^{\,\leq 2}\mathfrak{g}M$, so it is of no use that disjoint ideals centralize each other.

Interestingly the problem of mixed terms is absent for symmetric holonomies as well as for the holonomies $\mathfrak{so}_n$, $\mathfrak{g}_2$, $\mathfrak{spin}_7$ and the quaternionic holonomies $\mathfrak{sp}(n)$ and $\mathfrak{sp}(1)\,\mathfrak{sp}(n)$. Mixed terms may however spoil commutativity of the curvature endomorphisms $q^{\mathfrak{h}}(R)$ on a Kähler manifold M, in fact the curvature term associated to the center $i\mathbb{R} \subset \mathfrak{u}_n$ reads

$$q^{i\mathbb{R}}(\,R\,) \;\; = \;\; -\frac{1}{n}\, I\,(\,I\mathrm{Ric}\,) \;\; \in \;\; \Gamma(\mathcal{U}^{\,\leq 2}\mathfrak{u}M\,)\;,$$

according to equation (5.29), where I is the parallel complex structure and $I\mathrm{Ric}$ is the composition of I with the symmetric Ricci endomorphism of TM thought of as a section of the holonomy bundle $\mathfrak{u}M$. In consequence $q^{i\mathbb{R}}(R)$ is a section of $\mathcal{U}^{\leq 2}\, i\mathbb{R}M$ if and only if M is Kähler–Einstein, otherwise it will not in general commute with $q^{\mathfrak{su}}(R)$. The central curvature term features prominently in the Bochner identity for Kähler manifolds.

Lemma 3.5 $$B^{\mathfrak{h}}(\,\nabla^2\,) \;=\; q^{\mathfrak{h}}(\,R\,)$$

Proof: Expanding the second covariant derivative $\nabla^2\psi = \sum t_\mu \otimes t_\nu \otimes \nabla^2_{t_\mu,t_\nu}\psi$ of the section ψ with an orthonormal basis $\{t_\mu\}$ of T and using the same resummation as in the derivation of equation (3.5) we find for an orthonormal basis $\{X_\alpha\}$ of the ideal $\mathfrak{h}$:

$$\begin{aligned} B^{\mathfrak{h}}(\,\nabla^2\psi\,) &= \frac{1}{2}\sum_{\mu\nu} \mathrm{pr}_{\mathfrak{h}}(t_\mu\wedge t_\nu)\, R^V_{t_\mu,t_\nu}\,\psi \\ &= \frac{1}{2}\sum_{\alpha\mu\nu} \langle\, t_\mu\wedge t_\nu,\, X_\alpha\,\rangle\, X_\alpha\, R^V_{t_\mu,t_\nu}\,\psi \;=\; q^{\mathfrak{h}}(\,R\,) \end{aligned} \qquad \square$$

On the other hand Corollary 3.4 tells us how to write the conformal weight operator $B^{\mathfrak{h}}$ in terms of the basis $\{\mathrm{pr}_\varepsilon\}$ of projections onto the irreducible summands $V_{\lambda+\varepsilon}\subset T\otimes V_\lambda$. Using the identification of $B^{\mathfrak{h}}(\nabla^2)$ with the universal curvature terms $q^{\mathfrak{h}}(\,R\,)$ proved above we obtain some prime examples of Weitzenböck formulas:

Proposition 3.6 (Universal Weitzenböck Formula)
Consider a Riemannian manifold M of dimension n with holonomy group $G\subset \mathbf{SO}(n)$ and the vector bundle $V_\lambda M$ over M associated to the holonomy reduction of M and the irreducible representation V_λ of G of highest weight λ. In terms of the Stein–Weiss operators $T_\varepsilon : \Gamma(V_\lambda M)\longrightarrow \Gamma(V_{\lambda+\varepsilon}M)$ arising from the decomposition $T\otimes V_\lambda = \oplus_{\varepsilon\subset\lambda} V_{\lambda+\varepsilon}$ the action of the curvature endomorphisms $q^{\mathfrak{h}}(R)$ can be written

$$q^{\mathfrak{h}}(R) \;=\; -\sum_{\varepsilon\subset\lambda} b^{\mathfrak{h}}_\varepsilon\, T^*_\varepsilon\, T_\varepsilon\;,$$

where the $b^{\mathfrak{h}}_\varepsilon$ are the eigenvalues of the conformal weight operator $B^{\mathfrak{h}}\in \mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda)$.

As a direct consequence of Proposition 3.6 and the classical Weitzenböck formula (3.10) for the Laplace operator $\Delta = dd^* + d^*d$ on the bundle of differential forms we obtain

$$\Delta \;=\; \sum_{\varepsilon\subset\lambda} (1\,-\,b_\varepsilon)\, T^*_\varepsilon\, T_\varepsilon\;.$$

In the case of Riemannian holonomy $G = \mathbf{SO}(n)$ the universal Weitzenböck formula of Proposition 3.6 was considered in [G91] for the first time. The definition of the conformal weight operator and its expression in terms of the Casimir is taken from the same article. The conformal weight operator B has been used for other purposes as well, see [CGH00] for example. Similar results can be found in [H04].

Considering B as an element of the algebra $\mathfrak{W}(V)$ all powers of B are $\mathfrak{g}$–invariant endomorphisms. In the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}\,V)$ these powers read:

$$B^k_{a\otimes b} \;=\; \sum_{\mu_1,\ldots,\mu_{k-1}} \mathrm{pr}_{\mathfrak{g}}(a\wedge t_{\mu_1})\,\mathrm{pr}_{\mathfrak{g}}(t_{\mu_1}\wedge t_{\mu_2})\,\ldots\,\mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-2}}\wedge t_{\mu_{k-1}})\,\mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-1}}\wedge b)\;. \qquad (3.11)$$

Recall now that in the irreducible case the trace $\sum F_{t_\mu \otimes t_\mu} = (\mathrm{tr}\, F)\mathrm{id}_{V_\lambda}$ of an element $F \in \mathfrak{W}(V_\lambda)$ is a multiple of the identity of V_λ. Evidently the traces of the powers B^k of B correspond to the action of the elements

$$\mathrm{Cas}^{[k]} \;:=\; \sum_{\mu_1,\ldots,\mu_{k-1}} \mathrm{pr}_{\mathfrak{g}}(t_{\mu_0} \wedge t_{\mu_1})\, \mathrm{pr}_{\mathfrak{g}}(t_{\mu_1} \wedge t_{\mu_2}) \,\ldots\, \mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-2}} \wedge t_{\mu_{k-1}})\, \mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-1}} \wedge t_{\mu_0})\,. \tag{3.12}$$

of the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}$ on V. The elements $\mathrm{Cas}^{[k]}$, $k \geq 2$, all belong to the center of the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}$ and are called *higher Casimirs* since $\mathrm{Cas}^{[2]} = -2\mathrm{Cas}^{\Lambda^2}$ (c.f. [CGH00]). A straightforward calculation shows

$$\mathrm{Cas}^{[k]} \;=\; \mathrm{tr}\,(B^k)\,\mathrm{id}_{V_\lambda} \;=\; \left(\sum_\varepsilon b_\varepsilon^k \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda} \right) \mathrm{id}_{V_\lambda}\,, \tag{3.13}$$

for an irreducible representation $V = V_\lambda$, where we use the second equation in (3.6) for computing the trace of $B^k = \sum b_\varepsilon^k \mathrm{pr}_\varepsilon$ explicitly. As an example we consider the equation $\mathrm{Cas}^{[3]} = -\frac{1}{2}\mathrm{Cas}_{\mathfrak{g}}^{\Lambda^2}\mathrm{Cas}^{\Lambda^2}$, which follows from the recursion formula of Corollary 4.2 or by direct calculation. Indeed $B^2 - \frac{1}{4}\mathrm{Cas}_{\mathfrak{g}}^{\Lambda^2} B$ is an eigenvector of the involution τ for the eigenvalue $+1$. Thus it is orthogonal to the eigenvector B for the eigenvalue -1 and so:

$$0 \;=\; \langle\, B^2 - \frac{1}{4}\mathrm{Cas}_{\mathfrak{g}}^{\Lambda^2} B,\, B\,\rangle \;=\; \mathrm{tr}(B^3) - \frac{1}{4}\mathrm{Cas}_{\mathfrak{g}}^{\Lambda^2}\mathrm{tr}(B^2) \;=\; \mathrm{tr}(B^3) + \frac{1}{2}\mathrm{Cas}_{\mathfrak{g}}^{\Lambda^2}\mathrm{Cas}^{\Lambda^2}\,.$$

From a slightly more general point of view the evaluation at the conformal weight operator defines an algebra homomorphism $\Phi :\ \mathbb{C}[B] \longrightarrow \mathrm{End}_{\mathfrak{g}}(T \otimes V)$, whose kernel is generated by the minimal polynomial of B as an endomorphism on $T \otimes V$. With B being diagonalizable its minimal polynomial is the product $\min(B) = \prod_{b \in \{b_\varepsilon\}}(B - b)$ over all *different* conformal weights. In consequence the injective algebra homomorphism

$$\Phi : \qquad \mathbb{C}[B]/\langle \min(\,B\,)\rangle \;\longrightarrow\; \mathrm{End}_{\mathfrak{g}}(\,T \otimes\; V\,)\,.$$

is an isomorphism as soon as all conformal weights are pairwise different. Indeed the dimension of $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ is the number $N(G, \lambda)$ of relevant weights or the number of conformal weights counted with multiplicity, while the number of different conformal weights determines the degree of $\min(B)$ and so the dimension of $\mathbb{C}[B]/\langle \min(B)\rangle$.

In section 4.2 we compute the B–eigenvalues in the cases $\mathfrak{g} = \mathfrak{so}_n, \mathfrak{g}_2$ and $\mathfrak{spin}_7$. It follows that they are pairwise different with the only exceptions of $\mathfrak{g} = \mathfrak{so}_{2r}$ and a representation of highest weight $\lambda = \lambda_1\omega_1 + \ldots + \lambda_r\omega_r$, with $\lambda_r = \lambda_{r-1}$, which is equivalent to $b_{\varepsilon_r} = b_{-\varepsilon_r}$ and $\mathfrak{g} = \mathfrak{spin}_7$ and a representation with highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ with $c = 2a + 1$, which is equivalent to $b_{-\varepsilon_4} = b_{\varepsilon_4}$. In these cases the degree of the minimal polynom is reduced by one and hence the image of Φ has codimension one. We thus have proved the following

Proposition 3.7 (Structure of the Algebra of Weitzenböck Formulas)
Let G be one of the holonomy groups $\mathbf{SO}_n$, $\mathbf{G}_2$ or $\mathbf{Spin}(7)$ of non–symmetric manifolds. If V_λ is irreducible, then Φ is an isomorphism

$$\Phi :\;\; \mathbb{C}[B]/\langle \min(\,B\,)\rangle \;\stackrel{\cong}{\longrightarrow}\; \mathrm{End}_{\mathfrak{g}}(\,T \otimes V_\lambda\,)\,,$$

with the only exception of the cases $G = \mathbf{SO}_{2r}$ and a highest weight λ with $\lambda_{r-1} = \lambda_r$, or $G = \mathbf{Spin}(7)$ and a highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ with $c = 2a + 1$. In both cases the homomorphism Φ is not surjective and its image has codimension one.

3.3 The Classifying Endomorphism

The decomposition of the space $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ of Weitzenböck formulas into the (± 1)–eigenspaces of the involution τ can be written as

$$\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V) \;\cong\; \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}\, V) \,\oplus\, \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Sym}^2 T, \mathrm{End}\, V)\ .$$

However in general we have a further splitting of $T \otimes T$ leading to a further decomposition of the τ–eigenspaces. Our aim is now to introduce an endomorphism K on $\mathfrak{W}(V)$ whose eigenspaces correspond to this finer decomposition.

Definition 3.8 (The Classifying Endomorphism)
The classifying endomorphism $K^{\mathfrak{h}}$ of an ideal $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ of the real holonomy algebra $\mathfrak{g}_{\mathbb{R}}$ is the endomorphism $K^{\mathfrak{h}} : \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$ on the space of Weitzenböck formulas defined in the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ by the formula

$$K^{\mathfrak{h}}(\,F\,)_{a \otimes b}\, v \;\;:=\;\; -\sum_{\alpha} F_{X_\alpha a \otimes X_\alpha b} v$$

where $\{X_\alpha\}$ is an orthonormal basis for the scalar product induced on the ideal $\mathfrak{h} \subset \Lambda^2 T$. As before we denote the classifying endomorphism of the ideal $\mathfrak{g}$ simply by $K := K^{\mathfrak{g}}$.

Of course the definition of the classifying endomorphism $K^{\mathfrak{h}}$ is motivated by Fegan's Lemma 3.3 for the conformal weight operator $B^{\mathfrak{h}}$. Note that for every $\mathfrak{g}$–equivariant map $F :\ T \otimes T \longrightarrow \mathrm{End}\, V$ the map $K^{\mathfrak{h}}(F) :\ T \otimes T \longrightarrow \mathrm{End}\, V$ is again $\mathfrak{g}$–equivariant, because we sum over an orthonormal basis $\{X_\alpha\}$ of the ideal $\mathfrak{h} \subset \mathfrak{g}$ for a $\mathfrak{g}$–invariant scalar product. In the interpretation $\mathfrak{W}(V) = \mathrm{End}_{\mathfrak{g}}(T \otimes V)$ the definition of $K^{\mathfrak{h}}$ reads

$$K^{\mathfrak{h}}(\,F\,)(b \otimes v) \;\;=\;\; -\sum_{\mu\alpha} t_\mu \otimes F_{X_\alpha t_\mu \otimes X_\alpha b}\, v \;\;=\;\; \sum_{\mu\alpha} X_\alpha t_\mu \otimes F_{t_\mu \otimes X_\alpha b}\, v$$

or more succinctly:

$$K^{\mathfrak{h}}(\,F\,) \;\;=\;\; \sum_{\alpha} (\,X_\alpha \otimes \mathrm{id}\,)\, F\, (\,X_\alpha \otimes \mathrm{id}\,)\ . \tag{3.14}$$

In consequence the classifying endomorphisms $K^{\mathfrak{h}}$ and $K^{\tilde{\mathfrak{h}}}$ for two ideals $\mathfrak{h},\, \tilde{\mathfrak{h}} \subset \mathfrak{g}$ commute on the space $\mathfrak{W}(V)$ of Weitzenböck formulas similar to the conformal weights operators. The classifying endomorphisms will be extremely useful in finding the matrix of the twist $\tau :\ \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$ in the basis of $\mathfrak{W}(V)$ given by the orthogonal idempotents pr_ε.

Lemma 3.9 (Eigenvalues of the Classifying Endomorphism)
Consider the decomposition of the tensor product $T \otimes T = \oplus_\alpha W_\alpha$ into irreducible summands. The classifying endomorphisms $K^{\mathfrak{h}}$ are diagonalizable on $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ with eigenspaces $\mathrm{Hom}_{\mathfrak{g}}(\,W_\alpha, \mathrm{End}\, V\,) \subset \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ for relevant α and eigenvalues:

$$\kappa_{W_\alpha} \;\;=\;\; \frac{1}{2}\,\mathrm{Cas}^{\Lambda^2}_{W_\alpha} \;-\; \mathrm{Cas}^{\Lambda^2}_{T}\ .$$

In particular the classifying endomorphisms $K^{\mathfrak{h}}$ acts as $K^{\mathfrak{h}}(F) = \sum_\alpha \kappa_{W_\alpha} F|_{W_\alpha}$ on the space of Weitzenböck formulas $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$.

Proof: From the very definition of $K^{\mathfrak{h}}$ we see that it acts by precomposition with the map $-\sum X_\alpha \otimes X_\alpha$ in the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ of the space of Weitzenböck formulas. The argument used in the proof of Corollary 3.4 shows that $K^{\mathfrak{h}}$ is actually a difference of Casimir operators leading to the stated formula for its eigenspaces and eigenvalues. $\square$

In the case of the holonomies $\mathfrak{so}_n, \mathfrak{g}_2$ and $\mathfrak{spin}_7$ we have $T \otimes T = \mathbb{C} \oplus \mathrm{Sym}^2_0 T \oplus \mathfrak{g} \oplus \mathfrak{g}^\perp$ and using Lemma 3.9 we find the following K–eigenvalues:

	$\kappa_{\mathbb{C}}$	$\kappa_{\mathrm{Sym}^2_0 T}$	$\kappa_{\mathfrak{g}}$	$\kappa_{\mathfrak{g}^\perp}$
$\mathfrak{so}_n$	$-(n-1)$	1	-1	—
$\mathfrak{g}_2$	-4	$\frac{2}{3}$	0	-2
$\mathfrak{spin}_7$	$-\frac{21}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{9}{4}$

(3.15)

Note that all these K–eigenvalues are different and consequently the twist τ is a polynomial in the classifying endomorphism K. Moreover a given invariant homomorphism $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ is an eigenvector of K if and only if F is different from zero one precisely one summand $W_\alpha \subset T \otimes T$, so $\mathbf{1}$ and B are clearly K–eigenvectors.

Lemma 3.10 (Properties of the Classifying Endomorphism)
The classifying endomorphism $K : \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$ is a symmetric endomorphism commuting with the twist map τ on the space $\mathfrak{W}(V)$ of Weitzenböck formulas equipped with the scalar product $\langle F, \tilde{F} \rangle := \frac{1}{\dim V} \mathrm{tr}_{T \otimes V} F\tilde{F}$. The special endomorphisms $\mathbf{1}$ and B for the same ideal $\mathfrak{h} \subset \mathfrak{g}$ are K–eigenvectors:

$$K(\mathbf{1}) \;=\; \mathrm{Cas}^{\Lambda^2}_T \mathbf{1} \qquad\qquad K(B) \;=\; (\mathrm{Cas}^{\Lambda^2}_T - \frac{1}{2}\mathrm{Cas}^{\Lambda^2}_{\mathfrak{h}})\, B$$

Proof: The symmetry of K is a trivial consequence of equation (3.14) in the form

$$\langle K(F), \tilde{F} \rangle \;=\; \frac{1}{\dim V} \sum_\nu \mathrm{tr}_{T \otimes V}\Big((X_\nu \otimes \mathrm{id})\, F\, (X_\nu \otimes \mathrm{id})\, \tilde{F} \Big)$$

and the cyclic invariance of the trace, moreover K commutes with τ by definition. Coming to the explicit determination of $K(\mathbf{1})$ and $K(B)$ we observe that the unit of $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ becomes the equivariant map $\mathbf{1}(a \otimes b) = \langle a, b \rangle \mathrm{id}_V$ in $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ and so:

$$(K\mathbf{1})(a \otimes b) \;=\; -\sum_\nu \langle X_\nu a, X_\nu b \rangle\, \mathrm{id}_V \;=\; \sum_\nu \langle a, X_\nu^2 b \rangle\, \mathrm{id}_V \;=\; \mathrm{Cas}^{\Lambda^2}_T \mathbf{1}(a \otimes b)\ .$$

The conformal weight operator B considered as an element of $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ lives by definition in the eigenspace $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{End}\, V)$ for the eigenvalue $-\frac{1}{2}\mathrm{Cas}^{\Lambda^2}_{\mathfrak{g}} + \mathrm{Cas}^{\Lambda^2}_T$ of K, where $\mathrm{Cas}^{\Lambda^2}_{\mathfrak{g}}$ is the Casimir eigenvalue of the adjoint representation. $\square$

On a manifold with holonomy algebra $\mathfrak{g} \subset \mathfrak{so}_n \cong \Lambda^2 T$ the Riemannian curvature tensor takes values in $\mathfrak{g}$, i.e. it can be considered as an element of $\mathrm{Sym}^2\mathfrak{g}$. This fact has the following important consequence:

Proposition 3.11 (Bochner Identities)
Suppose $F \in \mathfrak{W}(V)$ is an invariant homomorphism $T \otimes T \longrightarrow \mathrm{End}\, V$ factoring through the projection to the orthogonal complement $\mathfrak{g}^\perp \subset \Lambda^2 T \subset T \otimes T$ of $\mathfrak{g} \subset \Lambda^2 T$. The curvature expression $F(\nabla^2) = 0$ vanishes regardless of what the curvature tensor R is.

We will call a Weitzenböck formula $F \in \mathfrak{W}(V_\lambda)$ corresponding to an invariant homomorphism $T \otimes T \longrightarrow \mathrm{End}\, V$ factoring through $\mathfrak{g}^\perp$ a *Bochner identity.* Writing such an invariant homomorphism F in terms of the basis $\{\mathrm{pr}_\varepsilon\}$ as $F = \sum f_\varepsilon\, \mathrm{pr}_\varepsilon$ we get the following explicit form of the Bochner identity:

$$\sum_\varepsilon f_\varepsilon\, T_\varepsilon^* T_\varepsilon \;=\; 0\;.$$

The Bochner identities of $\mathbf{G}_2$– and $\mathbf{Spin}(7)$–holonomy correspond to eigenvectors of the classifying endomorphism K for the eigenvalues -2 and $\frac{9}{4}$ respectively. Since the zero weight space of $\mathfrak{g}^\perp$ is in both cases one-dimensional it follows from Lemma 2.3 that

$$\dim \mathrm{Hom}_{\,\mathfrak{g}}(\mathfrak{g}^\perp, \mathrm{End}\, V_\lambda) \;\leq\; 1\;, \tag{3.16}$$

i. e. there is at most one Bochner identity. Moreover the K–eigenvector $\mathbf{1} \in \mathrm{End}_{\,\mathfrak{g}}(T \otimes V_\lambda)$ spans the K–eigenspace $\mathrm{Hom}_{\,\mathfrak{g}}(\mathbb{C}, \mathrm{End}\, V_\lambda) \cong \mathbb{C}$. Because the zero weight space of $\mathfrak{g}$ itself is the fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, an application of Lemma 2.3 results in the estimates:

$$\dim \mathrm{Hom}_{\,\mathfrak{g}_2}(\mathfrak{g}_2, \mathrm{End}\, V_\lambda) \;\leq\; 2 \qquad\qquad \dim \mathrm{Hom}_{\,\mathfrak{spin}_7}(\mathfrak{spin}_7, \mathrm{End}\, V_\lambda) \;\leq\; 3\;.$$

4 The Recursion Procedure for $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$

The definitions of the conformal weight operator B and the classifying endomorphism K given in the previous section are very similar. Given this similarity it should not come as a surprise that the actions of B and K on the space $\mathfrak{W}(V)$ of Weitzenböck formulas obey a simple relation, which is the corner stone of the treatment of Weitzenböck formulas proposed in this article. In the present section we first prove this relation and then use it to construct recursively an basis of K–eigenvectors of $\mathfrak{W}(V_\lambda)$ for the holonomy groups $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$. A different interpretation of the same relation between B and K is studied in the next section concerning Kähler manifolds.

4.1 The basic recursion procedure

Recall that the twist τ is defined in the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\,\mathfrak{g}}(T \otimes T \otimes V, V)$ of the space of Weitzenböck formulas as linear maps $T \otimes T \otimes V \longrightarrow V$ by precomposition with the endomorphism $\tau : a \otimes b \otimes v \longmapsto b \otimes a \otimes v$. Generalizing this precomposition we observe that $\mathfrak{W}(V)$ is a right module over the algebra $\mathrm{End}_{\,\mathfrak{g}}(T \otimes T \otimes V)$ containing τ. Interestingly both the classifying endomorphism K and the (right) multiplication by the conformal weight operator B are induced by precomposition with elements in $\mathrm{End}_{\,\mathfrak{g}}(T \otimes T \otimes V)$, too, say K is the precomposition with the $\mathfrak{g}$–invariant endomorphism

$$K :\;\; T \otimes T \otimes V \;\longrightarrow\; T \otimes T \otimes V, \qquad a \otimes b \otimes v \;\longmapsto\; -\sum_\nu X_\nu a \otimes X_\nu b \otimes v$$

while (right) multiplication by B is precomposition with the $\mathfrak{g}$–invariant endomorphism

$$B: \; T\otimes T\otimes V \;\longrightarrow\; T\otimes T\otimes V, \qquad a\otimes b\otimes v \;\longmapsto\; -\sum_\nu a\otimes X_\nu b\otimes X_\nu v$$

by Fegan's Lemma 3.3. From this description of the action of the classifying endomorphism K and right multiplication of B on $\mathfrak{W}(V)$ we immediately conclude:

$$K \;+\; B \;+\; \tau B\tau \;\;=\;\; -\frac{1}{2}(\,\mathrm{Cas}^{\Lambda^2} \;-\; 2\mathrm{Cas}^{\Lambda^2}_T \;-\; \mathrm{Cas}^{\Lambda^2}_{V_\lambda}\,)$$

Applying once again Schur's Lemma this relation implies our basic Recursion Formula:

Theorem 4.1 (Recursion Formula)
Let V_λ be an irreducible representation of the holonomy algebra $\mathfrak{g}$. Then the action of K, B and τ on $\mathfrak{W}(V_\lambda) \;=\; \mathrm{Hom}_{\,\mathfrak{g}}(T\otimes T\otimes V_\lambda, V_\lambda)$ by precomposition satisfies:

$$K \;+\; B \;+\; \tau B\tau \;\;=\;\; \mathrm{Cas}^{\Lambda^2}_T \;\;=\;\; -\,2\,\frac{\dim\,\mathfrak{h}}{\dim\,T}$$

We will now explain how this theorem yields a recursion formula for K–eigenvectors. In fact given an eigenvector $F \;\in\; \mathfrak{W}(V)$ for the twist τ and the classifying endomorphism K with eigenvalues t and κ, i.e. $\tau F \;=\; tF$ and $KF \;=\; \kappa F$, the Recursion Formula allows us to produce a new τ–eigenvector $\widehat{F}$ with eigenvalue $-t$. This simple prescription suffices to obtain a complete eigenbasis for $\mathfrak{W}(V)$ of τ– and actually K–eigenvectors in general Riemannian geometry $\mathfrak{g} \;=\; \mathfrak{so}_n$ and, with some modifications, also in the exceptional cases $\mathfrak{g} \;=\; \mathfrak{g}_2$ and $\mathfrak{g} \;=\; \mathfrak{spin}_7$. The quaternionic Kähler case can be dealt with similarly.

Corollary 4.2 (Basic Recursion Procedure)
Let $F \;\in\; \mathfrak{W}(V)$ be an eigenvector for the involution τ and the classifying endomorphism K of an ideal $\mathfrak{h} \;\subset\; \mathfrak{g}$, i. e. $K(F) \;=\; \kappa F$ and $\tau(F) \;=\; \pm F$. The new Weitzenböck formula

$$F_{\mathrm{new}} \;\;:=\;\; (\,B \;-\; \frac{\mathrm{Cas}^{\Lambda^2}_T - \kappa}{2}\,)\circ F$$

is again a τ–eigenvector in $\mathfrak{W}(V)$ with $\tau(F_{\mathrm{new}}) \;=\; \mp F_{\mathrm{new}}$. In particular we find:

$$\mathbf{1}_{\mathrm{new}} \;\;=\;\; B \qquad\quad \textit{and} \qquad\quad B_{\mathrm{new}} \;\;=\;\; B^2 \;-\; \frac{1}{4}\mathrm{Cas}^{\Lambda^2}_{\mathfrak{g}}\,B\;.$$

Proof: We observe that the Recursion Formula 4.1 in the form $\tau B\tau \;=\; \mathrm{Cas}^{\Lambda^2}_T - K - B$ implies under the assumptions $K(F) \;=\; \kappa F$ and $\tau(F) \;=\; \pm F$ that

$$\pm\tau(\,BF\,) \;\;=\;\; (\,\mathrm{Cas}^{\Lambda^2}_T - \kappa\,)\,F \;-\; BF$$

and consequently:

$$\pm\tau(\,BF \;-\; \frac{\mathrm{Cas}^{\Lambda^2}_T - \kappa}{2}\,F\,) \;\;=\;\; -\,(\,BF \;-\; \frac{\mathrm{Cas}^{\Lambda^2}_T - \kappa}{2}\,F\,)\;.$$

The formulas for $\mathbf{1}_{\mathrm{new}}$ and B_{new} are immediate consequences of Lemma 3.10. $\square$

Recall that a K–eigenvector is automatically a τ–eigenvector. In general however the new Weitzenböck formula $F_{\mathrm{new}} \in \mathfrak{W}(V)$ does not need to be an eigenvector for K again and there is no way to iterate the recursion. Nevertheless it is possible to avoid termination of the recursion procedure for most of the irreducible non–symmetric holonomy algebras by using appropriate projections.

We note that any $+1$–eigenvector of τ orthogonal to $\mathbf{1}$ is already K–eigenvector in $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Sym}^2_0 T, \mathrm{End}\, V_\lambda)$. This is due to the fact that $\mathbf{1}$ spans the other summand of the $+1$–eigenspace of τ. In particular the orthogonal projection of B_{new} onto the orthogonal complement of $\mathbf{1}$, i. e. the polynomial $B^2 - \frac{1}{4}\mathrm{Cas}^{\Lambda^2}_{\mathfrak{g}} B + \frac{2}{n}\mathrm{Cas}^{\Lambda^2}_{V_\lambda}$, is a K–eigenvector in $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Sym}^2_0 T, \mathrm{End}\, V_\lambda)$. More generally we have:

Corollary 4.3 (Orthogonal recursion procedure)
Let $p_0(B), \ldots, p_k(B)$ be a sequence of polynomials obtained by applying the Gram Schmidt orthonormalization procedure to the powers $1,\, B,\, B^2,\, \ldots,\, B^k$ of the conformal weight operator B. If all these polynomials are τ–eigenvectors and $p_k(B)$ is moreover a K–eigenvector, then the orthogonal projection $p_{k+1}(B)$ of B^{k+1} onto the orthogonal complement of the span of $1,\, B,\, \ldots,\, B^k$ is a again a τ–eigenvector.

Proof: Since $p_k(B)$ is a K–eigenvector the basic recursion procedure shows the existence of a polynomial in B of degree $k+1$, which is a τ–eigenvector. It follows that with $\mathrm{span}\{1,\, B,\, \ldots,\, B^k\} = \mathrm{span}\{p_0(B),\, \ldots,\, p_k(B)\}$ also $\mathrm{span}\{1,\, B,\, \ldots,\, B^{k+1}\}$ is τ–invariant. Clearly the orthogonal projection $p_{k+1}(B)$ of B^{k+1} onto the orthogonal complement of $\mathrm{span}\{1,\, B,\, \ldots,\, B^k\}$ is a polynomial in B of degree $k+1$ with:

$$\mathrm{span}\{1,\, B,\, B^2,\, \ldots,\, B^k\} \oplus \mathbb{C}\, p_{k+1}(\, B\,) \;=\; \mathrm{span}\{1,\, B,\, B^2,\, \ldots, B^k, B^{k+1}\}\;.$$

Now the involution τ is symmetric with respect to the scalar product on $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ and so the orthogonal complement of a τ–invariant space is again τ–invariant. □

4.2 Computation of B–eigenvalues for $\mathbf{SO}(n), \mathbf{G}_2$ and $\mathbf{Spin}(7)$

In this section we will compute the B–eigenvalues for the holonomies $\mathbf{SO}(n), \mathbf{G}_2$ and $\mathbf{Spin}(7)$ by applying the explicit formula of Corollary 3.4. In particular we will see that with only two exceptions all B–eigenvalues are pairwise different. This information is relevant in the proof of Proposition 3.7.

The $\mathbf{SO}(n)$*–case.* Recall that in Section 2 we fixed the notation for the fundamental weights $\omega_1,\, \ldots,\, \omega_r$ and the weights $\pm\varepsilon_1,\, \ldots,\, \pm\varepsilon_r$ of the defining representation $\mathbb{R}^n$ of $\mathbf{SO}(n)$ with $r := \lfloor \frac{n}{2} \rfloor$. Moreover the scalar product $\langle\, ,\, \rangle$ on the dual of a maximal torus was chosen so that the weights $\pm\varepsilon_1,\, \ldots,\, \pm\varepsilon_r$ are an orthonormal basis. A highest weight can be written $\lambda = \lambda_1\omega_1 + \ldots + \lambda_r\omega_r = \mu_1\varepsilon_1 + \ldots + \mu_r\varepsilon_r$ with integral coefficients $\lambda_1,\, \ldots,\, \lambda_r \geq 0$ and coefficients $\mu_1,\, \ldots,\, \mu_r$, which are either all integral or all half–integral and decreasing. Independent of the parity of n the conformal weights are

$$b_{+\varepsilon_k} \;=\; \mu_k \,-\, k \,+\, 1, \qquad b_{-\varepsilon_k} \;=\; -\mu_k \,-\, n \,+\, k \,+\, 1, \qquad b_0 \;=\; -\,r$$

according to Corollary 3.4, where the zero weight only appears for n odd. With only a few exceptions the conformal weights are totally ordered and thus pairwise different. In the case n odd the coefficients $\mu_1, \ldots, \mu_r$ are decreasing in the sense $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r \geq 0$ so that we find strict inequalities

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < \ldots < b_{-\varepsilon_r} \leq b_0 < b_{+\varepsilon_r} < \ldots < b_{+\varepsilon_1}$$

unless $\mu_r = 0$ or equivalently $\lambda_r = 0$. In the latter case $b_{-\varepsilon_r} \leq b_0$ happens to be an equality, however Lemma 2.2 tells us that the zero weight is irrelevant for highest weights λ with $\lambda_r = 0$. Without loss of generality we may thus assume all conformal weights to be different for n odd. Similar considerations in the case of even n based on the inequalities $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{r-1} \geq |\mu_r|$ satisfied by the coefficients $\mu_1, \ldots, \mu_r$ of λ lead to

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < \ldots < b_{-\varepsilon_{r-1}} < \{ b_{-\varepsilon_r} , b_{+\varepsilon_r} \} < b_{+\varepsilon_{r-1}} < \ldots < b_{+\varepsilon_2} < b_{+\varepsilon_1}$$

where nothing specific can be said about the relation between $b_{-\varepsilon_r}$ and $b_{+\varepsilon_r}$ due to $b_{+\varepsilon_r} - b_{-\varepsilon_r} = 2\mu_r$. This should not come too surprising as the outer automorphism of $\mathbf{SO}(n)$ with n even acts on $\mathfrak{t}^*$ as a reflection along the hyperplane $\mu_r = 0$.

The $\mathbf{G}_2$*–case.* We write the highest weight as $\lambda = a\omega_1 + b\omega_2$ with integers $a, b \geq 0$ and use the scalar product defined in Section 2 by setting $\langle \varepsilon_1, \varepsilon_1 \rangle = 1 = \langle \varepsilon_2, \varepsilon_2 \rangle$ and $\langle \varepsilon_1, \varepsilon_2 \rangle = \frac{1}{2}$, equivalently $\langle \omega_1, \omega_1 \rangle = 1$, $\langle \omega_2, \omega_2 \rangle = 3$ and $\langle \omega_1, \omega_2 \rangle = \frac{3}{2}$. According to Corollary 3.4 the conformal weight of the zero weight is given by $b_0 = -2$, similarly:

$$b_{\pm\varepsilon_1} = -(\tfrac{5}{3} \mp \tfrac{5}{3}) \pm (\tfrac{2}{3}a + b), \quad b_{\pm\varepsilon_2} = -(\tfrac{5}{3} \mp \tfrac{4}{3}) \pm (\tfrac{1}{3}a + b), \quad b_{\pm\varepsilon_3} = -(\tfrac{5}{3} \mp \tfrac{1}{3}) \pm \tfrac{1}{3}a\,.$$

Again all conformal weights or B–eigenvalues are pairwise different and totally ordered:

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < b_{-\varepsilon_3} < b_0 < b_{+\varepsilon_3} < b_{+\varepsilon_2} < b_{+\varepsilon_1}\,.$$

The $\mathbf{Spin}(7)$*–case.* Using the fundamental weights $\omega_1, \omega_2, \omega_3$ and the scalar product $\langle\, , \,\rangle$ introduced in Section 2 in terms of the weights $\pm\eta_1, \pm\eta_2, \pm\eta_3$ of the represention $\mathbb{R}^7$ we write the highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ with integers $a, b, c \geq 0$ and compute

$$\begin{aligned} b_{\pm\varepsilon_1} &= -(\tfrac{9}{4} \mp \tfrac{9}{4}) \pm (\tfrac{1}{2}a + b + \tfrac{3}{4}c)\,, & b_{\pm\varepsilon_2} &= -(\tfrac{9}{4} \mp \tfrac{7}{4}) \pm (\tfrac{1}{2}a + b + \tfrac{1}{4}c)\,, \\ b_{\pm\varepsilon_3} &= -(\tfrac{9}{4} \mp \tfrac{3}{4}) \pm (\tfrac{1}{2}a \qquad + \tfrac{1}{4}c)\,, & b_{\pm\varepsilon_4} &= -(\tfrac{9}{4} \mp \tfrac{1}{4}) \pm (\tfrac{1}{2}a \qquad - \tfrac{1}{4}c)\,. \end{aligned}$$

In this case we obtain the inequalities

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < b_{-\varepsilon_3} < \{ b_{-\varepsilon_4} , b_{+\varepsilon_4} \} < b_{+\varepsilon_3} < b_{+\varepsilon_2} < b_{+\varepsilon_3}\,.$$

however the difference $b_{+\varepsilon_4} - b_{-\varepsilon_4} = a - \frac{1}{2}c + \frac{1}{2}$ does not allow to draw conclusions about the relation between $b_{-\varepsilon_4}$ and $b_{+\varepsilon_4}$. In particular for a heighest weight λ with $c = 2a + 1$ the two conformal weights $b_{-\varepsilon_4} = b_{+\varepsilon_4}$ agree.

4.3 Basic Weitzenböck formulas for $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$

In this section we make the recursion procedure of Corollary 4.2 explicit for the holonomy groups $\mathbf{SO}(n), \mathbf{G}_2$ and $\mathbf{Spin}(7)$. Let us start with the generic Riemannian holonomy algebra $\mathfrak{g} = \mathfrak{so}_n$ with only a single non–trivial ideal $\mathfrak{h} = \mathfrak{g}$. According to table (3.15) its classifying endomorphism K has eigenvalues $(n-1)$, 1 and -1 with eigenspaces $\mathbb{C}\mathbf{1}$, the orthogonal complement of $\mathbf{1}$ in the τ–eigenspace for the eigenvalue 1 and the τ–eigenspace for the eigenvalue -1 respectively. The orthogonal projection of every τ–eigenvector to the orthogonal complement of $\mathbf{1}$ is thus a K–eigenvector. Consequently we can modify the recursion procedure such that it associates to an eigenvector F for τ of eigenvalue -1 the K–eigenvector

$$F_{\mathrm{new}} \;:=\; (\,B \,+\, \frac{n-2}{2}\,)\,F \;-\; \frac{1}{n}\,\langle\, BF,\,\mathbf{1}\,\rangle\,\mathbf{1}$$

for the eigenvalue 1, while a τ–eigenvector F for the eigenvalue $+1$ orthogonal to $\mathbf{1}$ is mapped to the K–eigenvector:

$$F_{\mathrm{new}} \;:=\; (\,B \,+\, \frac{n}{2}\,)\,F$$

for the eigenvalue -1. For an irreducible representation V_λ of $\mathfrak{so}_n$ we thus get a sequence $p_0(B)$, $p_1(B)$, $\ldots$ of K–eigenvectors in $\mathfrak{W}(V_\lambda)$ defined recursively by $p_0(B) := \mathbf{1}$, $p_1(B) := B$ and $p_{k+1}(B) := (p_k(B))_{\mathrm{new}}$ for $k \geq 1$. Evidently the different $p_k(B)$ are polynomials of degree k in B and so the eigenvectors $p_0(B), \ldots, p_{d-1}(B) \in \mathfrak{W}(V_\lambda)$ with $d := \deg\min B$ are necessarily linearly independent. According to Proposition 3.7 we always get a complete basis of τ–eigenvectors with the exception of the case $\mathfrak{g} = \mathfrak{so}_{2r}$ and a representation V_λ with $\lambda_{r-1} = \lambda_r$. Here we still have to add a K–eigenvector F_{spin} spanning the orthogonal complement of the image of $\mathbb{C}[\,B\,]$ in $\mathfrak{W}(V_\lambda)$.

Note that the polynomials $p_{2k+1}(B)$, $k = 0, 1, \ldots$ are in the -1–eigenspace of τ. Hence the corresponding Weitzenböck formulas give a pure curvature term. Let N the number of irreducible components of $T \otimes V_\lambda$, then there are $\lfloor \frac{N}{2} \rfloor$ linearly independent equations of this type. This result, which is clear from our construction, was proved for the first time in [BH02]. The first eigenvectors in this sequence are $p_0(B) = \mathbf{1}$ and $p_1(B) = B$ as well as:

$$p_2(B) \;=\; B^2 \,+\, \tfrac{n-2}{2}\,B \,+\, \tfrac{2}{n}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda} \tag{4.17}$$

$$p_3(B) \;=\; B^3 \,+\, (n-1)\,B^2 \,+\, (\tfrac{2}{n}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda} + \tfrac{n(n-2)}{4})\,B \,+\, \mathrm{Cas}^{\Lambda^2}_{V_\lambda} \tag{4.18}$$

Essentially the same procedure can be used in the case $\mathfrak{g} = \mathfrak{g}_2$ to compute a complete K–eigenbasis for the space $\mathfrak{W}(V_\lambda)$ for an irreducible $\mathfrak{g}_2$–representation V_λ. Again there is only one non–trivial ideal $\mathfrak{h} = \mathfrak{g}_2$ and hence only a single classifying endomorphism K. However the τ–eigenspace in $\mathfrak{W}(V_\lambda)$ for the eigenvalue -1 decomposes into two K–eigenspaces. The recursion procedure gives the K–eigenvectors

$$p_0(B) \;=\; \mathbf{1}, \quad p_1(B) \;=\; B, \quad p_2(B) \;=\; B^2 + 2B + \tfrac{2}{7}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\;. \tag{4.19}$$

Using the recursion procedure again gives a polynom of degree 3 in B. Projecting it onto the orthogonal complement of B we obtain

$$p_3(B) \;=\; B^3 \,+\, \tfrac{13}{3}\,B^2 \,+\, (\tfrac{1}{2}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda} + 4)\,B \,+\, \tfrac{2}{3}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\;. \tag{4.20}$$

We will see in Theorem 8.6 that $p_3(B)$ is in fact a K–eigenvector for the eigenvalue -2, in other words $p_3(B) \in \mathrm{Hom}_{\mathfrak{g}_2}(\mathfrak{g}_2^\perp, V_\lambda)$ is a Bochner identity. Due to the estimate (3.16) any τ–eigenvector orthogonal to $\mathbf{1}$ and F_3 is a K–eigenvector and so we may obtain a complete eigenbasis $p_0(B), \ldots, p_6(B)$ in the $\mathbf{G}_2$–case by applying the Gram–Schmidt orthogonalization to the powers of B and using Corollary 4.3.

In order to make the generalized Bochner identity corresponding to the polynomial $p_3(B)$ explicit we recall that its coefficients as a linear combination of the basis projections pr_ε are the value of the polynomial p_3 at the corresponding B–eigenvalues b_ε. Substituting the explicit formulas for b_ε and for $\mathrm{Cas}^{\Lambda^2}_{V_\lambda}$ (c.f. Remark B.5) we obtain:

$$\begin{aligned} F_{\text{Bochner}} \;:=\; 27\,p_3(B) \;=\; &+ \quad a \quad (a+3b+3)\;(2a+3b+4)\;\mathrm{pr}_{+\varepsilon_1}\\ &-\;(a+2)\;(a+3b+5)\;(2a+3b+6)\;\mathrm{pr}_{-\varepsilon_1}\\ &-\;(a+2)\;(a+3b+3)\;(2a+3b+4)\;\mathrm{pr}_{+\varepsilon_2}\\ &+ \quad a \quad (a+3b+5)\;(2a+3b+6)\;\mathrm{pr}_{-\varepsilon_2}\\ &- \quad a \quad (a+3b+5)\;(2a+3b+4)\;\mathrm{pr}_{+\varepsilon_3}\\ &+\;(a+2)\;(a+3b+3)\;(2a+3b+6)\;\mathrm{pr}_{-\varepsilon_3}\\ &+\;6(\,a^2+3b^2\;+\;3ab+\;5a\;+9b\;+\;6)\;\mathrm{pr}_0 \end{aligned} \tag{4.21}$$

Eventually let us discuss the example of $\mathfrak{g} = \mathfrak{spin}_7$. Here the modified recursion procedure gives the three K–eigenvectors

$$p_0(B) \;=\; \mathbf{1}, \quad p_1(B) \;=\; B, \quad p_2(B) \;=\; B^2 + \tfrac{5}{2}\,B + \tfrac{1}{4}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\;. \tag{4.22}$$

and a τ–eigenvector for the eigenvalue -1, which is of third order as a polynomial in B. After projecting it onto the orthogonal complement of B we obtain:

$$p_3(B) \;=\; B^3 \;+\; \tfrac{11}{2}\,B^2 \;+\; \tfrac{1}{2\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}}\,(\mathrm{Cas}^{[4]}_{V_\lambda} + \tfrac{55}{2}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda})\,B \;+\; \tfrac{3}{4}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\;, \tag{4.23}$$

where $\mathrm{Cas}^{[4]}_{V_\lambda}$ is the eigenvalue of the higher casimir $\mathrm{Cas}^{[4]}$ on the irreducible representation V_λ. Its explicit value is given in the appendix in Remark B.6.

However in difference to the $\mathfrak{g}_2$–case this is no K–eigenvector. Indeed in Section 8 we will see that the space of polynomials in B of degree at most 3 is not invariant under K. Hence there cannot be a further K–eigenvector expressible as a polynomial of order 3 in B. In general the other K–eigenvectors are polynomials of degree 7 in B. They are too complicated to be written down, but surprisingly the K–eigenvector for the eigenvalue $-\frac{9}{4}$, i. e. the Bochner identity, for a representation of highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ has the following simple explicit expression:

$$\begin{aligned} F_{\text{Bochner}} \;=\; &+ \quad c \quad (2b+c+2)\;(2a+2b+c+4)\;\mathrm{pr}_{+\varepsilon_1}\\ &-\;(c+2)\;(2b+c+4)\;(2a+2b+c+6)\;\mathrm{pr}_{-\varepsilon_1}\\ &-\;(c+2)\;(2b+c+2)\;(2a+2b+c+4)\;\mathrm{pr}_{+\varepsilon_2}\\ &+ \quad c \quad (2b+c+4)\;(2a+2b+c+6)\;\mathrm{pr}_{-\varepsilon_2}\\ &- \quad c \quad (2b+c+4)\;(2a+2b+c+4)\;\mathrm{pr}_{+\varepsilon_3}\\ &+\;(c+2)\;(2b+c+2)\;(2a+2b+c+6)\;\mathrm{pr}_{-\varepsilon_3}\\ &+\;(c+2)\;(2b+c+4)\;(2a+2b+c+4)\;\mathrm{pr}_{+\varepsilon_4}\\ &- \quad c \quad (2b+c+2)\;(2a+2b+c+6)\;\mathrm{pr}_{-\varepsilon_4} \end{aligned} \tag{4.24}$$

This formula is proved in Theorem 8.7. Note that the coefficients of $\mathrm{pr}_{+\varepsilon_4}$ and $\mathrm{pr}_{-\varepsilon_4}$ are different. Hence in the critical case with $c = 2a+1$, i. e. where $b_{+\varepsilon_4} = b_{-\varepsilon_4}$, this K–eigenvector $F_{Bochner}$ spans the space orthogonal to $\mathbb{C}[B]$ in $\mathrm{End}_{\,\mathfrak{g}}(T \otimes V_\lambda)$.

5 The Weitzenböck Machine for Kähler Holonomies

The Weitzenböck machine for the Kähler holonomy groups $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ is remarkably different to the machine for the other non–symmetric irreducible holonomies. Perhaps the most distinctive characteristic of Kähler holonomy is that the complexified holonomy representation $T = \bar{E} \oplus E = T^{1,0} \oplus T^{0,1}$ is not irreducible. Consequently the set of generalized gradients $\{T_\varepsilon\}$ on sections of an irreducible vector bundle $V_\lambda M$ falls apart into two subsets, the sets of first order parallel differential operators factorizing over the complementary projections $\mathrm{pr}^{0,1}$ and $\mathrm{pr}^{1,0}$ from $T \otimes V_\lambda$ to $T^{0,1} \otimes V_\lambda$ and $T^{1,0} \otimes V_\lambda$ respectively. In turn the space of Weitzenböck formulas on V_λ is the direct sum

$$\mathfrak{W}(\,V_\lambda\,) \;\;=\;\; \mathfrak{W}^{1,0}(\,V_\lambda\,) \;\oplus\; \mathfrak{W}^{0,1}(\,V_\lambda\,) \tag{5.25}$$

of the spaces $\mathfrak{W}^{1,0}(V_\lambda) := \mathrm{End}_{\,\mathfrak{g}}(\,T^{0,1} \otimes V_\lambda\,)$ and $\mathfrak{W}^{0,1}(V_\lambda) := \mathrm{End}_{\,\mathfrak{g}}(\,T^{1,0} \otimes V_\lambda\,)$ of holomorphic and antiholomorphic Weitzenböck formulas. Evidently the space $\mathfrak{W}^{1,0}(V_\lambda)$ is spanned by the projections pr_ε to the holomorphic relevant weights $\varepsilon \subset \lambda^{1,0}$ defined as the relevant weights among the weights $-\varepsilon_1, \ldots, -\varepsilon_n$ of $T^{0,1}$, in particular $\mathrm{pr}^{0,1}$ is the sum $\mathrm{pr}^{0,1} = \sum_{\varepsilon \subset \lambda^{1,0}} \mathrm{pr}_\varepsilon$ over all relevant holomorphic weights. Similarly the space $\mathfrak{W}^{0,1}(V_\lambda)$ is spanned by the projections to the relevant weights $\varepsilon \subset \lambda^{0,1}$ among the antiholomorphic weights $+\varepsilon_1, \ldots, +\varepsilon_n$ of $T^{1,0}$ with $\mathrm{pr}^{1,0} = \sum_{\varepsilon \subset \lambda^{0,1}} \mathrm{pr}_\varepsilon$. The isomorphisms $T^{0,1} = T^{1,0*}$ and $T^{1,0} = T^{0,1*}$ are reponsible for this apparently skewed notation.

In the Weitzenböck machine we are eventually interested in the matrices of the twist τ and the classifying endomorphism K in the basis pr_ε of idempotents of $\mathfrak{W}(V_\lambda)$, and a simple observation drastically simplifies this task in the case of Kähler holonomies. The subspaces $T^{1,0}$ and $T^{0,1}$ are isotropic subspaces of T so that $\mathrm{Hom}_{\,\mathfrak{u}_n}(T^{1,0} \otimes T^{1,0} \otimes V_\lambda, V_\lambda)$ and $\mathrm{Hom}_{\,\mathfrak{u}_n}(T^{0,1} \otimes T^{0,1} \otimes V_\lambda, V_\lambda)$ are trivial. Hence the decomposition (5.25) becomes

$$\mathfrak{W}^{1,0}(\,V_\lambda\,) \;=\; \mathrm{Hom}_{\,\mathfrak{u}_n}(T^{1,0} \otimes T^{0,1} \otimes V_\lambda, V_\lambda) \qquad \mathfrak{W}^{0,1}(\,V_\lambda\,) \;=\; \mathrm{Hom}_{\,\mathfrak{u}_n}(T^{0,1} \otimes T^{1,0} \otimes V_\lambda, V_\lambda)$$

in the interpretation $\mathfrak{W}(V_\lambda) = \mathrm{Hom}_{\,\mathfrak{u}_n}(T \otimes T \otimes V_\lambda, V_\lambda)$ of the space of Weitzenböck formulas. In this interpretation the endomorphisms τ and K are defined by precomposition with $a \otimes b \otimes \psi \longmapsto b \otimes a \otimes \psi$ and $a \otimes b \otimes \psi \longmapsto -\sum_\alpha X_\alpha a \otimes X_\alpha b \otimes \psi$ respectively so that K preserves the decomposition (5.25), while τ interchanges $\mathfrak{W}^{1,0}(V_\lambda)$ and $\mathfrak{W}^{0,1}(V_\lambda)$. In difference to the holonomies $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$ discussed above the twist τ is thus not a polynomial in the classifying endomorphism K for $\mathbf{U}(n)$ and $\mathbf{SU}(n)$.

Before calculating the B–eigenvalues let us recall that the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_n$ of the holonomy representation T form an orthonormal basis for a scalar product $\langle\, ,\, \rangle$ on the dual $\mathfrak{t}^*$ of a maximal torus $\mathfrak{t} \subset \mathfrak{u}_n$. The decomposition $\mathfrak{u}_n = i\mathbb{R} \oplus \mathfrak{su}_n$ of $\mathfrak{u}_n$ into center $i\mathbb{R}$ and simple ideal $\mathfrak{su}_n$ is accompanied by an analoguous decomposition $\lambda = \lambda^{i\mathbb{R}} + \lambda^{\mathfrak{su}}$ of a weight $\lambda \in \mathfrak{t}^*$ along complementary orthogonal projections defined on the basis by:

$$\varepsilon_k^{i\mathbb{R}} \;:=\; \frac{1}{n}(\,\varepsilon_1 + \ldots + \varepsilon_n\,) \qquad\qquad \varepsilon_k^{\mathfrak{su}} \;:=\; \varepsilon_k - \frac{1}{n}(\,\varepsilon_1 + \ldots + \varepsilon_n\,)\,.$$

For the calculations below it is useful to keep the following formulas in mind for all k, l:

$$\langle\, \varepsilon_k^{i\mathbb{R}}, \varepsilon_l^{i\mathbb{R}}\,\rangle \;=\; \frac{1}{n} \qquad\qquad \langle\, \varepsilon_k^{i\mathbb{R}}, \varepsilon_l^{\mathfrak{su}}\,\rangle \;=\; 0 \qquad\qquad \langle\, \varepsilon_k^{\mathfrak{su}}, \varepsilon_l^{\mathfrak{su}}\,\rangle \;=\; \delta_{kl} - \frac{1}{n}\,.$$

Interestingly the half sum $\rho = \frac{n-1}{2}\varepsilon_1 + \frac{n-3}{2}\varepsilon_2 + \ldots + \frac{1-n}{2}\varepsilon_n$ of positive roots of $\mathfrak{u}_n$ is not equal to the sum of the fundamental weights $\omega_1 := \varepsilon_1$, $\omega_2 := \varepsilon_1 + \varepsilon_2$, $\omega_3 := \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ etc. specified in Section 2. Nevertheless $\rho = \rho^{\mathfrak{su}}$ is orthogonal to a weight $\lambda^{i\mathbb{R}} \in \mathfrak{t}^*$ due to

$$\rho \;=\; \omega_1^{\mathfrak{su}} \;+\; \omega_2^{\mathfrak{su}} \;+\; \ldots \;+\; \omega_n^{\mathfrak{su}}\,,$$

in particular $\langle\, \varepsilon_k^{\mathfrak{su}},\, 2\rho\,\rangle = \langle\, \varepsilon_k,\, 2\rho\,\rangle = n - 2k + 1$. Having established this much of the notation we need to verify that the chosen scalar product $\langle\,,\,\rangle$ is suitable for calculating Casimir eigenvalues. The problem is that there are two linearly independent scalar products on $\mathfrak{u}_n$ and so we have to check the normalization condition (3.8) for the Casimir eigenvalues for both ideals $i\mathbb{R}$ and $\mathfrak{su}_n$ of $\mathfrak{u}_n$ separately. Recalling that the holonomy representation T is the complexification of a real irreducible representation $T_{\mathbb{R}}$ we observe that the Casimir eigenvalues of the normalized Casimirs $\mathrm{Cas}^{\Lambda^2, i\mathbb{R}}$ and $\mathrm{Cas}^{\Lambda^2, \mathfrak{su}}$ of the two ideals $i\mathbb{R}$ and $\mathfrak{su}_n$ are the same on $T, T^{1,0}$ and $T^{0,1}$. Taking the irreducible subspace $T^{1,0}$ of highest weight ε_1 for convenience we verify (3.8)

$$\begin{aligned}
\mathrm{Cas}_T^{\Lambda^2, i\mathbb{R}} &= \mathrm{Cas}_{T^{1,0}}^{\Lambda^2, i\mathbb{R}} = -\,2\,\frac{\dim i\mathbb{R}}{\dim T} = -\frac{1}{n} = -\,\langle\, \varepsilon_1^{i\mathbb{R}},\, \varepsilon_1^{i\mathbb{R}}\,\rangle \\
\mathrm{Cas}_T^{\Lambda^2, \mathfrak{su}} &= \mathrm{Cas}_{T^{1,0}}^{\Lambda^2, \mathfrak{su}} = -\,2\,\frac{\dim \mathfrak{su}}{\dim T} = -\,\frac{n^2-1}{n} = -\,\langle\, \varepsilon_1^{\mathfrak{su}} + 2\rho,\, \varepsilon_1^{\mathfrak{su}}\,\rangle
\end{aligned}$$

for the normalized Casimir:

$$\mathrm{Cas}_{V_\lambda}^{\Lambda^2} \;=\; -\,\langle\, \lambda + 2\rho,\, \lambda\,\rangle \;=\; -\,\langle\, \lambda^{i\mathbb{R}},\, \lambda^{i\mathbb{R}}\,\rangle \;-\; \langle\, \lambda^{\mathfrak{su}} + 2\rho,\, \lambda^{\mathfrak{su}}\,\rangle\,. \tag{5.26}$$

Corollary 3.4 allows us to calculate the eigenvalues of the conformal weight operator B aka conformal weights. With respect to the orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathfrak{t}^*$ the highest weight of an irreducible representation V_λ can be written $\lambda = \mu_1\varepsilon_1 + \ldots + \mu_n\varepsilon_n$ with decreasing integral coefficients $\mu_1 \geq \ldots \geq \mu_n$ so that the conformal weights are

$$b_{-\varepsilon_k} \;=\; -\mu_k + k - n \qquad\qquad b_{+\varepsilon_k} \;=\; \mu_k - k + 1 \tag{5.27}$$

for the conformal weight operator B of the full holonomy algebra $\mathfrak{u}_n$. In particular both the holomorphic and antiholomorphic conformal weights are separately ordered

$$b_{-\varepsilon_n} \;>\; b_{-\varepsilon_{n-1}} \;>\; \ldots \;>\; b_{-\varepsilon_1} \qquad\qquad b_{+\varepsilon_1} \;>\; b_{+\varepsilon_2} \;>\; \ldots \;>\; b_{+\varepsilon_n}\,,$$

but we can not exclude the equality between the conformal weights for a holomorphic and an antiholomorphic weight, moreover there are certainly dominant integral weights λ with $b_{-\varepsilon_k} = b_{+\varepsilon_{n-k}}$ for all $k = 1, \ldots, n$. For this reason it is prudent to study the central conformal weight operator $B^{i\mathbb{R}}$ besides the conformal weight operator B of the holonomy algebra $\mathfrak{u}_n$, whose eigenvalues or central conformal weights

$$b_\varepsilon^{i\mathbb{R}} \;=\; \frac{1}{2}\Big(\,\langle\, \lambda^{i\mathbb{R}} + \varepsilon^{i\mathbb{R}},\, \lambda^{i\mathbb{R}} + \varepsilon^{i\mathbb{R}}\,\rangle \;-\; \langle\, \lambda^{i\mathbb{R}},\, \lambda^{i\mathbb{R}}\,\rangle \;-\; \langle\, \varepsilon_1^{i\mathbb{R}},\, \varepsilon_1^{i\mathbb{R}}\,\rangle\Big) \;=\; \langle\, \lambda^{i\mathbb{R}},\, \varepsilon^{i\mathbb{R}}\,\rangle$$

equal $\pm\frac{1}{n}(\mu_1 + \ldots + \mu_n)$ for holomorphic $(-)$ and antiholomorphic $(+)$ weights respectively. The algebra homomorphism $\Phi:\ \mathbb{C}[B^{i\mathbb{R}}, B^{\mathfrak{su}}] \longrightarrow \mathrm{End}_{\,\mathfrak{u}_n}(T \otimes V_\lambda)$ is thus surjective for all dominant integral weights λ with $\lambda^{i\mathbb{R}} \neq 0$.

In order to find the eigenspaces of the classifying endomorphism K and their dimensions we recall that K preserves the orthogonal subspaces $\mathfrak{W}^{1,0}(V_\lambda)$ and $\mathfrak{W}^{0,1}(V_\lambda)$ of holomorphic and antiholomorphic Weitzenböck formulas. The eigenvalues of K on $\mathfrak{W}^{1,0}(V_\lambda)$ agree with the conformal weights for the relevant antiholomorphic weights for $T^{0,1} = V_{-\varepsilon_n}$ with associated projections in $T^{1,0} \otimes T^{0,1} \subset T \otimes T^{0,1}$. The eigenvalues of K on the complementary subspace $\mathfrak{W}^{0,1}(V_\lambda)$ correspond similarly to the conformal weights for the relevant holomorphic weights for $T^{1,0} = V_{+\varepsilon_1}$. Using the decision criterion of Lemma 2.2 we find the relevant antiholomorphic weights $+\varepsilon_1$ and $+\varepsilon_n$ for $\lambda = -\varepsilon_n$ and the relevant holomorphic weights $-\varepsilon_1$ and $-\varepsilon_n$ for $\lambda = +\varepsilon_1$ with corresponding decompositions

$$T^{1,0} \otimes V_{-\varepsilon_n} \;=\; (\mathfrak{su}_n \otimes_{\mathbb{R}} \mathbb{C}) \,\oplus\, \mathbb{C} \qquad\qquad T^{0,1} \otimes V_{+\varepsilon_1} \;=\; \mathbb{C} \,\oplus\, (\mathfrak{su}_n \otimes_{\mathbb{R}} \mathbb{C})$$

where $\mathfrak{su}_n \otimes_{\mathbb{R}} \mathbb{C} = V_{+\varepsilon_1 - \varepsilon_n}$ is the adjoint representation. Using the explicit values (5.27) for the conformal weights we conclude that K has eigenvalues $-n$ and 0 with multiplicities 1 and $n-1$ respectively on $\mathfrak{W}^{1,0}(V_\lambda)$, in other words K vanishes on the orthogonal complement of the one–dimensional eigenspace of eigenvalue $-n$ in $\mathfrak{W}^{1,0}(V_\lambda)$. The distinguished projections $\mathrm{pr}^{0,1} \in \mathfrak{W}^{0,1}(V_\lambda)$ and $\mathrm{pr}^{1,0} \in \mathfrak{W}^{0,1}(V_\lambda)$ are natural candidates for eigenvectors of K for the eigenvalue $-n$ and in fact we find in the interpretation $\mathfrak{W}^{1,0}(V_\lambda) = \mathrm{Hom}_{\,\mathfrak{u}_n}(T^{1,0} \otimes T^{0,1} \otimes V_\lambda,\, V_\lambda)$

$$\begin{aligned}(\,K\mathrm{pr}^{0,1}\,)(a \otimes b \otimes v) \;&=\; -\sum_\alpha \langle\, X_\alpha a,\, \mathrm{pr}^{0,1}(X_\alpha b)\,\rangle\, v \;&=&\; \sum_\alpha \langle\, X_\alpha\, X_\alpha a,\, \mathrm{pr}^{0,1} b\,\rangle\, v \\ &=\; \mathrm{Cas}^{\Lambda^2}_{T^{1,0}}\, \mathrm{pr}^{0,1}(a \otimes b \otimes v) \;&=&\; -\,n\, \mathrm{pr}^{0,1}(a \otimes b \otimes v)\,,\end{aligned}$$

where the X_α are an orthonormal basis for $\mathfrak{u}_n$ with respect to the scalar product induced from $\mathfrak{u}_n \subset \Lambda^2 T$. Turning to antiholomorphic Weitzenböck formulas in $\mathfrak{W}^{0,1}(V_\lambda)$ a similar argument implies that the classifying endomorphism K vanishes on the orthogonal complement of the eigenvector $\mathrm{pr}^{1,0}$ with eigenvalue $-n$. Put differently this result reads:

Lemma 5.1 (Explicit Form of the Classifying Endomorphism)
On the basis vectors $\{\mathrm{pr}_\varepsilon\}$ of the space $\mathfrak{W}(V_\lambda)$ of Weitzenböck formulas on an irreducible representation V_λ the classifying endomorphism K of the holonomy algebra $\mathfrak{u}_n$ acts by

$$K\,\mathrm{pr}_\varepsilon \;=\; -\,\frac{\dim\, V_{\lambda+\varepsilon}}{\dim\, V_\lambda}\,\mathrm{pr}^{0,1} \qquad \textit{or} \qquad K\,\mathrm{pr}_\varepsilon \;=\; -\,\frac{\dim\, V_{\lambda+\varepsilon}}{\dim\, V_\lambda}\,\mathrm{pr}^{1,0}$$

depending on whether ε is a holomorphic or antiholomorphic weight of T respectively.

Proof: Taking the stated formula as a definition of a linear map $K : \mathfrak{W}(V_\lambda) \longrightarrow \mathfrak{W}(V_\lambda)$ we apply it to a holomorphic Weitzenböck formula $F^{1,0} = \sum_{\varepsilon \subset \lambda^{1,0}} f_\varepsilon \mathrm{pr}_\varepsilon$ and find

$$K(\sum_{\varepsilon \subset \lambda^{1,0}} f_\varepsilon\, \mathrm{pr}_\varepsilon\,) \;=\; -\Big(\sum_{\varepsilon \subset \lambda^{1,0}} f_\varepsilon\, \frac{\dim\, V_{\lambda+\varepsilon}}{\dim V_\lambda}\Big)\,\mathrm{pr}^{0,1} \;=\; -\,\langle\, \mathrm{pr}^{0,1},\, F^{1,0}\,\rangle\, \mathrm{pr}^{0,1}$$

so that $K\mathrm{pr}^{0,1} = -n\, \mathrm{pr}^{0,1}$ and $KF^{1,0} = 0$ for $F^{1,0} \in \mathfrak{W}^{1,0}(V_\lambda)$ orthogonal to $\mathrm{pr}^{0,1}$. Hence the endomorphism K implicitly defined above agrees with the classifying endomorphism on the subspace $\mathfrak{W}^{1,0}(V_\lambda)$. Mutatis mutandis we check agreement of K with the classifying endomorphism on the subspace $\mathfrak{W}^{0,1}(V_\lambda)$. □

In the case of Kähler holonomies the twist τ is not a polynomial on the classifying endomorphism K and the explicit description of K in Lemma 5.1 does not seem to be of any use in understanding τ. It turns out however that the Recursion Formula 4.1 considered as a matrix equation for the twist τ already has a unique solution. More precisely the Recursion Formula 4.1 can be put into the form

$$\tau K \;=\; -(\,\tau B \;+\; B\tau \;+\; n\tau\,) \tag{5.28}$$

because τ is an involution and $-2\frac{\dim \mathfrak{u}_n}{\dim T} = -n$. Moreover τ interchanges the subspaces $\mathfrak{W}^{1,0}(V_\lambda)$ and $\mathfrak{W}^{0,1}(V_\lambda)$ of holomorphic and antiholomorphic Weitzenböck formulas, thus

$$\tau\,\mathrm{pr}_{-\varepsilon} \;=\; \sum_{+\tilde\varepsilon} \tau_{-\varepsilon,+\tilde\varepsilon}\,\mathrm{pr}_{+\tilde\varepsilon} \qquad\qquad \tau\,\mathrm{pr}_{+\varepsilon} \;=\; \sum_{-\tilde\varepsilon} \tau_{+\varepsilon,-\tilde\varepsilon}\,\mathrm{pr}_{-\tilde\varepsilon}$$

with as yet unknown rational matrix coefficients $\tau_{+\varepsilon,-\tilde\varepsilon}$ and $\tau_{-\varepsilon,+\tilde\varepsilon}$. Here and in the following the notation $-\varepsilon$ refers exclusively to holomorphic, $+\varepsilon$ to antiholomorphic weights. Besides interchanging $\mathfrak{W}^{1,0}(V_\lambda)$ and $\mathfrak{W}^{0,1}(V_\lambda)$ the twist interchanges $\mathrm{pr}^{0,1}$ and $\mathrm{pr}^{1,0}$:

$$(\,\tau\,\mathrm{pr}^{0,1}\,)(a\otimes b\otimes v) \;=\; \langle\, b,\,\mathrm{pr}^{0,1}a\,\rangle\, v \;=\; \langle\, a,\,\mathrm{pr}^{1,0}b\,\rangle\, v \;=\; \mathrm{pr}^{1,0}(a\otimes b\otimes v)\,.$$

In consequence Lemma 5.1 tells us the matrix coefficients of the operator τK

$$(\,\tau K\,)\,\mathrm{pr}_{-\varepsilon} \;=\; -\,\frac{\dim V_{\lambda-\varepsilon}}{\dim V_\lambda}\,\tau\,\mathrm{pr}^{0,1} \;=\; -\sum_{+\tilde\varepsilon}\frac{\dim V_{\lambda-\varepsilon}}{\dim V_\lambda}\,\mathrm{pr}_{+\tilde\varepsilon}$$

while B is diagonal in the basis pr_ε with eigenvalues b_ε. All in all equation (5.28) becomes:

$$(\,\tau K\,)_{-\varepsilon,+\tilde\varepsilon} \;=\; -\,\frac{\dim V_{\lambda-\varepsilon}}{\dim V_\lambda} \;=\; -\,(\,b_{-\varepsilon} \;+\; b_{+\tilde\varepsilon} \;+\; n\,)\,\tau_{-\varepsilon,+\tilde\varepsilon}$$

This equation has a unique solution $\tau_{-\varepsilon,+\tilde\varepsilon}$, because $b_{-\varepsilon} + b_{+\tilde\varepsilon} + n$ is never zero. In fact

$$b_{-\varepsilon_k} \;+\; b_{+\varepsilon_l} \;+\; n \;=\; (\,\mu_k \;-\; \mu_l\,) \;+\; (\,l \;-\; k\,) \;+\; 1 \qquad\qquad 1 \;\leq\; k,\, l \;\leq\; n$$

equals 1 for $k = l$ and is positive for $k < l$ as the coefficients $\mu_1 \geq \ldots \geq \mu_n$ are decreasing. Similarly $(\mu_k - \mu_l) + (l - k) + 1$ is negative for $k > l$ unless $k = l + 1$ and $\mu_l = \mu_k$, in which case Lemma 2.2 ensures that $-\varepsilon_l$ is irrelevant by $\lambda_{l+1} = 0$. Mutatis mutandis the same argument applies to the remaining matrix coefficients $\tau_{+\varepsilon,-\tilde\varepsilon}$ of the twist operator.

Theorem 5.2 (Matrix Coefficients of the Twist Operator)
The matrix coefficients $\tau_{-\varepsilon,+\tilde\varepsilon}$ and $\tau_{+\varepsilon,-\tilde\varepsilon}$ of the twist operator in the basis pr_ε of projections in $\mathfrak{W}(V_\lambda)$ defined by $\tau\mathrm{pr}_{-\varepsilon} = \sum_{+\tilde\varepsilon}\tau_{-\varepsilon,+\tilde\varepsilon}\,\mathrm{pr}_{+\tilde\varepsilon}$ and $\tau\mathrm{pr}_{+\varepsilon} = \sum_{-\tilde\varepsilon}\tau_{+\varepsilon,-\tilde\varepsilon}\,\mathrm{pr}_{-\tilde\varepsilon}$ are given by:

$$\tau_{-\varepsilon,+\tilde\varepsilon} \;=\; \frac{1}{b_{-\varepsilon} + b_{+\tilde\varepsilon} + n}\,\frac{\dim V_{\lambda-\varepsilon}}{\dim V_\lambda} \qquad\qquad \tau_{+\varepsilon,-\tilde\varepsilon} \;=\; \frac{1}{b_{+\varepsilon} + b_{-\tilde\varepsilon} + n}\,\frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}$$

Another interesting consequence of the decomposition of the space $\mathfrak{W}(V_\lambda)$ into the subspaces $\mathfrak{W}^{1,0}(V_\lambda)$ and $\mathfrak{W}^{0,1}(V_\lambda)$ of holomorphic and antiholomorphic Weitzenböck formulas should not pass unnoticed. As the twist τ interchanges $\mathfrak{W}^{1,0}(V_\lambda)$ and $\mathfrak{W}^{0,1}(V_\lambda)$ every eigenvector for τ is determined by its summand in the subspace $\mathfrak{W}^{1,0}(V_\lambda)$. In fact an arbitrarily chosen $F^{1,0} \in \mathfrak{W}^{1,0}(V_\lambda)$ gives rise to an eigenvector

$$F \;:=\; F^{1,0} \;\pm\; \tau(\,F^{1,0}\,)$$

for τ of eigenvalue ± 1. Of course we are primarily interested in the eigenvalue -1:

Corollary 5.3 (Pure Curvature Terms in Kähler Holonomy)
Recall that the eigenvectors $F = \sum_\varepsilon f_\varepsilon \mathrm{pr}_\varepsilon$ of the twist operator τ for the eigenvalue -1 in the space $\mathfrak{W}(V_\lambda)$ of Weitzenböck formulas on an irreducible representation V_λ correspond to Weitzenböck formulas reducing to a pure curvature term, compare (3.4) and (3.5):

$$F(\,\nabla^2\,) \;=\; -\sum_\varepsilon T_\varepsilon^* T_\varepsilon \;=\; \frac{1}{2}\sum_{\mu,\nu} F_{t_\mu \otimes t_\nu}\, R_{t_\mu,\, t_\nu}\;.$$

In Kähler holonomies there is a bijection $F^{1,0} \longmapsto F^{1,0} - \tau(F^{1,0})$ between $\mathfrak{W}^{1,0}(V_\lambda)$ and the eigenspace of τ for eigenvalue -1 or pure curvature term Weitzenböck formulas.

On a Kähler manifold M the holonomy algebra bundle $\mathfrak{u}M$ can be identified with the bundle of skew–symmetric endomorphisms of the tangent bundle commuting with the parallel complex structure I, it has a parallel subbundle $i\mathbb{R}M \subset \mathfrak{u}M$ associated to the center $i\mathbb{R} \subset \mathfrak{u}_n$ of the holonomy algebra and spanned by I in every point. It is well–known that the curvature tensor R of a Kähler manifold is a $(1,1)$–form in the sense $R_{IX,IY}Z \;=\; R_{X,Y}Z$ for all $X, Y, Z \in TM$, hence the Ricci endomorphism is not only a symmetric endomorphism of the tangent bundle, but commutes with I, too:

$$\mathrm{Ric}(IZ) \;=\; \sum_\mu R_{IZ,t_\mu} t_\mu \;=\; -\sum_\mu R_{Z,It_\mu} t_\mu \;=\; \sum_\mu R_{Z,t_\mu} I t_\mu \;=\; I\,(\mathrm{Ric}\, Z)\,.$$

In turn the skew–symmetric endomorphism $I\mathrm{Ric}$ commuting with I can be thought of as a section of the holonomy algebra bundle $\mathfrak{u}M$ with a natural action on every vector bundle VM associated to the holonomy reduction. A different way to understand this action is to note that $I\mathrm{Ric} \in \Gamma(\mathfrak{u}M)$ is up to sign the result of applying the curvature operator $R:\, \Lambda^2 TM \longrightarrow \mathfrak{u}M$ to the complex structure, because we find the action

$$R(\,I\,)\,Z \;=\; \frac{1}{2}\sum_\mu R_{t_\mu, It_\mu} Z \;=\; -\frac{1}{2}\sum_\mu \Big(R_{Z,t_\mu} I t_\mu \;-\; R_{Z,It_\mu} t_\mu \Big) \;=\; -I\,\mathrm{Ric}\, Z$$

on TM. Using $\langle\, I,\, I\,\rangle = -\frac{1}{2} I^2 = n$ we conclude that the curvature term $q^{i\mathbb{R}}(R)$ reads:

$$q^{i\mathbb{R}}(\,R\,) \;=\; \frac{1}{n}\, I\, R(\,I\,) \;=\; -\,\frac{1}{n}\, I\,(I\,\mathrm{Ric}) \tag{5.29}$$

6 A Matrix Presentation of the Twist Operator τ

In the last section we have studied the Weitzenböck machine for Kähler holonomies and succeeded in describing the twist τ and the classifying endomorphism K explicitly in the natural basis pr_ε of the space of Weitzenböck formulas. The principal idea of this calculation was to read the Recursion Formula 4.1, which we used in Section 4.2 for a recursive construction of an eigenbasis for the classifying endomorphism, directly as a matrix equation for the unknown matrix of τ. It turns out that the same idea allows us to derive a closed matrix expression for the twist operator τ and the classifying endomorphism K in the three other holonomies $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$ considered in this article.

In the case of $\mathbf{SO}(n)$–holonomy the classifying endomorphism K and the twist differ only on the span of the unit $\mathbf{1} \in \mathfrak{W}(V_\lambda)$, which becomes the identity $\mathbf{1} = \mathrm{id}_{T\otimes V_\lambda}$ in the interpretation $\mathfrak{W}(V_\lambda) = \mathrm{End}_{\mathfrak{so}_n}(T \otimes V_\lambda)$ and the connection Laplacian $\mathbf{1}(\nabla^2) = -\nabla^*\nabla$ in the interpretation as a second order parallel differential operator. The norm of $\mathbf{1}$ is simply the dimension $\langle\, \mathbf{1},\, \mathbf{1}\,\rangle = n$ of T and so we conclude

$$K \;=\; \tau \;-\; \mathbf{1} \otimes \mathbf{1} \tag{6.30}$$

where $\mathbf{1} \otimes \mathbf{1}$ denotes the endomorphism:

$$\mathbf{1} \otimes \mathbf{1} : \qquad \mathfrak{W}(V_\lambda) \longmapsto \mathfrak{W}(V_\lambda), \qquad F \longmapsto \langle\, \mathbf{1},\, F\,\rangle\, \mathbf{1}\,.$$

In fact in the interpretation $\mathfrak{W}(V_\lambda) = \mathrm{Hom}_{\mathfrak{so}_n}(T \otimes T \otimes V_\lambda,\, V_\lambda)$ both K and τ act by precomposition with an endomorphism of $T \otimes T$ extended to $T \otimes T \otimes V_\lambda$ in such a way that the eigenvalues of K and τ on the orthogonal complement $\mathrm{Sym}^2_0 T \oplus \Lambda^2$ of $\mathbb{C}$ agree according to Table 3.15. Hence we need only check equation (6.30) on $\mathbf{1}$, where we can use the trivial statement $\tau(\mathbf{1}) = \mathbf{1}$ and $K(\mathbf{1}) = -(n-1)\mathbf{1}$ from Lemma 3.10. In light of equation (6.30) the Recursion Formula 4.1 can be rewritten in the form

$$\tau K \;=\; \mathrm{id} \;-\; \mathbf{1} \otimes \mathbf{1} \;=\; -\,(\; \tau B \;+\; B \tau \;+\; (n-1)\, \tau \;) \tag{6.31}$$

in the case of $\mathbf{SO}(n)$–holonomy, note that τ is an involution with $\tau \circ \mathbf{1} \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$. In order to turn equation (6.31) into a matrix equation for the unknown matrix coefficients

$$\tau\, \mathrm{pr}_\varepsilon \;=:\; \sum_{\tilde{\varepsilon}} \tau_{\varepsilon,\tilde{\varepsilon}}\, \mathrm{pr}_{\tilde{\varepsilon}} \qquad\qquad \tau_{\varepsilon,\tilde{\varepsilon}} \;\in\; \mathbb{Q}$$

of the twist τ we observe that by its very definition $(\mathbf{1} \otimes \mathbf{1})\mathrm{pr}_\varepsilon \;=\; \langle\, \mathbf{1},\, \mathrm{pr}_\varepsilon\,\rangle \mathbf{1} \;=\; \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}\mathbf{1}$. With this additional piece of information equation (6.31) becomes

$$\delta_{\varepsilon,\tilde{\varepsilon}} \;-\; \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda} \;=\; -\,(\; b_\varepsilon \;+\; b_{\tilde{\varepsilon}} \;+\; n \;-\; 1\;)\, \tau_{\varepsilon,\tilde{\varepsilon}}$$

as again the operator B is diagonal in the basis pr_ε. Unlike the Kähler case we can not exclude the possibility that $b_\varepsilon + b_{\tilde{\varepsilon}} + n - 1$ is zero. If we assume ε to be relevant and thus $\dim V_{\lambda+\varepsilon} \neq 0$ however, then this can only happen on the diagonal $\varepsilon = \tilde{\varepsilon}$ under the assumption $\dim V_{\lambda+\varepsilon} \;=\; \dim V_\lambda$. For all relevant, off–diagonal weights $\varepsilon \neq \tilde{\varepsilon}$ the matrix coefficients of τ are thus given by

$$\tau_{\varepsilon,\tilde{\varepsilon}} \;=\; -\,\frac{1}{b_\varepsilon \;+\; b_{\tilde{\varepsilon}} \;+\; n \;-\; 1}\left(\; \delta_{\varepsilon,\tilde{\varepsilon}} \;-\; \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}\;\right)\,. \tag{6.32}$$

and this formula is still true on the diagonal $\varepsilon = \tilde{\varepsilon}$ provided $\dim V_{\lambda+\varepsilon} \neq \dim V_\lambda$. In order to eliminate these rather unpleasant restrictions we note that the equation $\tau(\mathbf{1}) \;=\; \mathbf{1}$ allows us to recover the diagonal coefficients $\tau_{\varepsilon,\varepsilon}$ of τ with $\dim V_{\lambda+\varepsilon} = \dim V_\lambda$. Moreover the matrix coefficients $\tau_{\varepsilon,\tilde{\varepsilon}}$ are rational functions in the highest weight λ, because the conformal weights b_ε and $b_{\tilde{\varepsilon}}$ are linear in λ and the dimensions of V_λ and $V_{\lambda+\varepsilon}$ are polynomials in λ due to Weyl's Dimension Formula. Taken together these arguments imply that the troublesome matrix coefficients $\tau_{\varepsilon,\tilde{\varepsilon}}$ left undefined by equation (6.32) agree with the values of the analytic continuation considered as a rational function of λ to values to the discriminant set, where zeroes of the numerator and the denominator cancel out.

Theorem 6.1 (The Twist Operator in $\mathbf{SO}(n)$–Holonomy)
The matrix coefficients $\tau_{\varepsilon,\tilde{\varepsilon}}$ of the twist $\tau : \mathfrak{W}(V_\lambda) \longrightarrow \mathfrak{W}(V_\lambda)$ in the basis of projections $\mathrm{pr}_\varepsilon \in \mathfrak{W}(V_\lambda)$ are defined by $\tau\,\mathrm{pr}_\varepsilon = \sum_{\tilde{\varepsilon}} \tau_{\varepsilon,\tilde{\varepsilon}}\,\mathrm{pr}_{\tilde{\varepsilon}}$. In $\mathbf{SO}(n)$–holonomy they are given by

$$\tau_{\varepsilon,\tilde{\varepsilon}} \quad = \quad -\,\frac{1}{b_\varepsilon \,+\, b_{\tilde{\varepsilon}} \,+\, n \,-\, 1}\left(\,\delta_{\varepsilon,\tilde{\varepsilon}} \,-\, \frac{\dim\, V_{\lambda+\varepsilon}}{\dim\, V_\lambda}\,\right)$$

provided we interprete $\tau_{\varepsilon,\tilde{\varepsilon}}$ as a rational function in the highest weight λ and cancel zeroes of the numerator and denominator for values of ε and $\tilde{\varepsilon}$, for which $b_\varepsilon + b_{\tilde{\varepsilon}} + n - 1 = 0$.

Turning from the case of $\mathbf{SO}(n)$–holonomy to $\mathbf{G}_2$ we note that the relation between the twist operator τ and the classifying endomorphism K becomes more complicated than equation (6.30) due to the presence of the Bochner identity (4.21). For definiteness we let $\beta := F_{\mathrm{Bochner}}$ be the K–eigenvector for the eigenvalue -2 specified in (4.21), whose coefficients $\beta = \sum_\varepsilon \beta_\varepsilon\,\mathrm{pr}_\varepsilon$ are given explicitly as polynomials in the highest weight $\lambda = a\omega_1 + b\omega_2$, $a,\, b \geq 0$. The argument in the $\mathbf{SO}(n)$–case extends to the relation

$$K \quad = \quad \frac{1}{3}\tau \,+\, \frac{1}{3} \,-\, \frac{2}{3}\,\mathbf{1}\otimes\mathbf{1} \,-\, \frac{2}{\langle\,\beta,\,\beta\,\rangle}\beta\otimes\beta$$

between K and τ in the $\mathbf{G}_2$–case so that the Recursion Formula 4.1 becomes

$$\tau\,K \quad = \quad \frac{1}{3}\,\mathrm{id} \,+\, \frac{1}{3}\tau \,-\, \frac{2}{3}\,\mathbf{1}\otimes\mathbf{1} \,+\, \frac{2}{\langle\,\beta,\,\beta\,\rangle}\beta\otimes\beta \quad = \quad -\,(\,\tau\,B \,+\, B\,\tau \,+\, 4\,\tau\,)\ ,$$

because β is an eigenvector for τ of eigenvalue -1. Moreover the calculation

$$(\,\beta\otimes\beta\,)\,\mathrm{pr}_\varepsilon \quad = \quad \sum_{\tilde{\varepsilon}}\left(\,\beta_\varepsilon\,\frac{\dim\, V_{\lambda+\varepsilon}}{\dim\, V_\lambda}\,\beta_{\tilde{\varepsilon}}\,\right)\mathrm{pr}_{\tilde{\varepsilon}}$$

tells us the matrix coefficients of $\beta\otimes\beta$ and so we end up with the formula

$$\tau_{\varepsilon,\tilde{\varepsilon}} \quad = \quad -\,\frac{1}{3\,b_\varepsilon \,+\, 3\,b_{\tilde{\varepsilon}} \,+\, 13}\left(\,\delta_{\varepsilon\tilde{\varepsilon}} \,-\, 2\left(\,1 \,-\, 3\,\frac{\beta_\varepsilon\,\beta_{\tilde{\varepsilon}}}{\langle\,\beta,\,\beta\,\rangle}\,\right)\frac{\dim\, V_{\lambda+\varepsilon}}{\dim\, V_\lambda}\,\right) \tag{6.33}$$

for the matrix coefficients $\tau_{\varepsilon,\tilde{\varepsilon}}$ of τ defined by $\tau\,\mathrm{pr}_\varepsilon = \sum_{\tilde{\varepsilon}} \tau_{\varepsilon,\tilde{\varepsilon}}\mathrm{pr}_{\tilde{\varepsilon}}$. Again this formula has to be read as a rational function in the highest weight λ to get the right values for $\tau_{\varepsilon,\tilde{\varepsilon}}$ for index tuples ε, $\tilde{\varepsilon}$ satisfying $3b_\varepsilon + 3b_{\tilde{\varepsilon}} + 13 = 0$ by cancelling the zero of the denominator with the apparent zero of the numerator.

Theorem 6.2 (The Twist Operator in $\mathbf{G}_2$–Holonomy)
The matrix coefficients $\tau_{\varepsilon,\tilde{\varepsilon}}$ of the twist $\tau : \mathfrak{W}(V_\lambda) \longrightarrow \mathfrak{W}(V_\lambda)$ in the basis of projections $\mathrm{pr}_\varepsilon \in \mathfrak{W}(V_\lambda)$ are defined by $\tau\,\mathrm{pr}_\varepsilon = \sum_{\tilde{\varepsilon}} \tau_{\varepsilon,\tilde{\varepsilon}}\,\mathrm{pr}_{\tilde{\varepsilon}}$. In $\mathbf{G}_2$–holonomy they are given by

$$\tau_{\varepsilon,\tilde{\varepsilon}} \quad = \quad -\,\frac{1}{3\,b_\varepsilon \,+\, 3\,b_{\tilde{\varepsilon}} \,+\, 13}\left(\,\delta_{\varepsilon\tilde{\varepsilon}} \,-\, 2\left(\,1 \,-\, 3\,\frac{\beta_\varepsilon\,\beta_{\tilde{\varepsilon}}}{\langle\,\beta,\,\beta\,\rangle}\,\right)\frac{\dim\, V_{\lambda+\varepsilon}}{\dim\, V_\lambda}\,\right)$$

where the coefficients β_ε of the Bochner identity $\beta = \sum_\varepsilon \beta_\varepsilon\mathrm{pr}_\varepsilon$ are given by (4.21).

Eventually we want to sketch briefly the corresponding argument in the case of exceptional holonomy $\mathbf{Spin}(7)$. Chosing the Bochner identity $\beta := F_{\mathrm{Bochner}}$ of equation (4.24) with coefficients $\beta = \sum_\varepsilon \beta_\varepsilon \,\mathrm{pr}_\varepsilon$ we can cast the relation between K and τ into the form

$$K \;=\; \frac{1}{2}\tau \;+\; \frac{1}{4} \;-\; \frac{3}{4}\mathbf{1}\otimes\mathbf{1} \;-\; \frac{2}{\langle\,\beta,\,\beta\,\rangle}\beta\otimes\beta$$

so that the Recursion Formula 4.1 is equivalent to the identity:

$$\tau K \;=\; \frac{1}{2}\,\mathrm{id} \;+\; \frac{1}{4}\tau \;-\; \frac{3}{4}\mathbf{1}\otimes\mathbf{1} \;+\; \frac{2}{\langle\,\beta,\,\beta\,\rangle}\beta\otimes\beta \;=\; -\,(\;\tau B \;+\; B\tau \;+\; \frac{21}{4}\tau\;)\;,$$

All in all we end up with the following formula for the matrix coefficients

$$\tau_{\varepsilon,\tilde\varepsilon} \;=\; -\,\frac{1}{2\,b_\varepsilon \;+\; 2\,b_{\tilde\varepsilon} \;+\; 11}\left(\,\delta_{\varepsilon\tilde\varepsilon} \;-\; \left(\,\frac{3}{2} \;-\; 4\,\frac{\beta_\varepsilon\,\beta_{\tilde\varepsilon}}{\langle\,\beta,\,\beta\,\rangle}\,\right)\frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}\,\right) \tag{6.34}$$

of the twist operator τ defined by $\tau\,\mathrm{pr}_\varepsilon = \sum_{\tilde\varepsilon} \tau_{\varepsilon,\tilde\varepsilon}\,\mathrm{pr}_{\tilde\varepsilon}$. Recall that the twist τ classifies all Weitzenböck formulas $F \in \mathfrak{W}(V_\lambda)$, which reduce to a pure curvature term. In consequence the matrix expressions for the the twist τ in the basis $\{\mathrm{pr}_\varepsilon\}$ of $\mathfrak{W}(V_\lambda)$ allows us to check this condition effectively for every given Weitzenböck formula on all Kähler manifolds M and on all manifolds M with holonomy $\mathbf{SO}(n)$, $\mathbf{G}_2$ or $\mathbf{Spin}(7)$.

7 Examples

In this section we will present a few examples of how to obtain for a given representation V_λ the possible Weitzenböck formulas on sections of the associated bundle $V_\lambda M$. The general procedure is as follows: we first determine the relevant weights ε using the diagrams of Section 2. This gives the decomposition of $T\otimes V_\lambda$ into irreducible summands and defines the twistor operators T_ε. Next we compute the B–eigenvalues b_ε, e.g. using the general formula of Corollary 3.4 and obtain the universal Weitzenböck formulas of Proposition 3.6. Other Weitzenböck formulas correspond to the B–polynomials constructed in the preceding section. If $F = F(B)$ is such a polynomial then the coefficient of $T^*_\varepsilon T_\varepsilon$ is given as $-F(b_\varepsilon)$.

As a first example we consider the bundle of p–forms on a Riemannian manifold (M^n, g) for simplicity we assume $n = 2r+1$ and $p \leq r-1$, i.e. $\mathfrak{g} = \mathfrak{so}_{2r+1}$ and $\lambda = \omega_p$, i.e. the associated bundle is the bundle of p–forms. The relevant weights according to the tables of Section 4.2 are ε_1, $-\varepsilon_p$ and ε_{p+1} with the decomposition

$$T\otimes V_\lambda \;=\; V_{\lambda+\varepsilon_1}\,\oplus\,V_{\lambda-\varepsilon_p}\,\oplus\,V_{\lambda+\varepsilon_{p+1}} \;\cong\; V_{\lambda+\varepsilon_1}\,\oplus\,\Lambda^{p-1}\,\oplus\,\Lambda^{p+1}$$

and twistor operators T_{ε_1}, $T_{-\varepsilon_p}$ and T_{ε_p}. To compare our twistor operators with differential and codifferential d, d^* we have to embed Λ^{p-1} resp. Λ^{p+1} into the tensor product $T\otimes\Lambda^p$. This leads to the following formula

$$T^*_{-\varepsilon_p}T_{-\varepsilon_p} \;=\; \tfrac{1}{n-p+1}\,d\,d^*, \qquad T^*_{+\varepsilon_p}T_{+\varepsilon_p} \;=\; \tfrac{1}{p+1}\,d^*\,d$$

Next we take the relevant B–eigenvalues from Section 4.2 they are:

$$b_{+\varepsilon_1} = 1, \qquad b_{-\varepsilon_p} = -n+p, \qquad b_{e_{p+1}} = -p\;.$$

Since we have only three summands in the decomposition of $T \otimes V_\lambda$ we obtain only one Weitzenböck formula with a pure curvature term, which is the formula given in Proposition 3.6:

$$
\begin{aligned}
q(R) &= -\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} + (n-p)\, T^*_{-\varepsilon_p} T_{-\varepsilon_p} + p\; T^*_{+\varepsilon_p} T_{+\varepsilon_p} \\
&= -\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} + \tfrac{n-p}{n-p+1}\; d\, d^* + \tfrac{p}{p+1}\; d^*\, d
\end{aligned}
$$

If we add the Weitzenböck formula (3.4) for $\nabla^*\nabla$ to this expression for $q(R)$ we obtain the classical Weitzenböck formula for the Laplacian on p–forms:

$$
\Delta \;=\; \nabla^*\nabla + q(R) \;=\; (n-p+1)\, T^*_{-\varepsilon_p} T_{-\varepsilon_p} + (p+1)\, T^*_{+\varepsilon_p} T_{+\varepsilon_p} \;=\; d\, d^* + d^*\, d
$$

Let (M^{2r}, g) be a Riemannian spin manifold with spinor bundle $S = S_+ \oplus S_-$. We consider the two bundles defined by the Cartan summand in $S_\pm \otimes T$ with highest weights $\lambda_+ = \omega_1 + \omega_{r-1}$ and $\lambda_- = \omega_1 + \omega_r$. Using the tables of Section 2 we find the relevant weights $+\varepsilon_1$, $-\varepsilon_1$ and $+\varepsilon_2$ for both $\lambda_\pm$ and in addition $-\varepsilon_r$ or $+\varepsilon_r$ for λ_+ or λ_- respectively. The corresponding tensor product decomposition is

$$
T \otimes V_{\lambda_\pm} \;=\; V_{\lambda_\pm + \varepsilon_1} \oplus V_{\lambda_\pm - \varepsilon_1} \oplus V_{\lambda_\pm + \varepsilon_2} \oplus V_{\lambda_\pm \mp \varepsilon_r}\ .
$$

Note that $\lambda_\pm - \varepsilon_1$ is the defining representation for the bundles $S_\pm$ and that $\lambda_\pm \mp \varepsilon_r = \lambda_\mp$. Projecting the covariant derivative of a section of $T \otimes V_{\lambda_\pm}$ onto one of these summands defines four twistor operators. The fourth operator $T_{\mp\varepsilon_r} : \Gamma(V_{\lambda_\pm}) \longrightarrow \Gamma(V_{\lambda_\mp})$ is usually called the *Rarita–Schwinger operator*. A solution of the Rarita–Schwinger equation is by definition a section of $\psi \in \Gamma(V_{\lambda_\pm})$ with both $T_{\mp\varepsilon_r}\psi = 0$ and $T_{-\varepsilon_1}\psi = 0$.

The B–eigenvalues for $\mathfrak{so}_n$–representations were computed in Section 4.2 and in particular:

$$
b_{+\varepsilon_1} = \tfrac{3}{2}, \quad b_{-\varepsilon_1} = -2r + \tfrac{1}{2}, \quad b_{+\varepsilon_2} = -\tfrac{1}{2}, \quad b_{\pm\varepsilon_r} = -r + \tfrac{1}{2}
$$

Since the decomposition of $T \otimes V_{\lambda_\pm}$ has four summands we will obtain two Weitzenböck formulas with a pure curvature term. The first one is again the universal Weitzenböck formula of Proposition 3.6 corresponding to B, whereas the second corresponds to $p_3(B)$, the degree 3 polynomial of the recursion procedure defined in (4.18). Its coefficients are the values $p_3(b_\varepsilon)$ for the relevant weights ε. The Casimir operator of an irreducible $\mathfrak{so}_n$–representation V_λ with highest weight λ is computed as $\mathrm{Cas}_{V_\lambda} = -\langle\, \lambda + 2\rho,\ \lambda\,\rangle$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the standard scalar product on $\mathbb{R}^r$. In particular we have $\mathrm{Cas}_{V_{\lambda_\pm}} = -\frac{1}{4}\, r\,(2r+7)$. Eventually we obtain the following two Weitzenböck formulas on sections of $V_{\lambda_\pm}$:

$$
q(R) \;=\; -\tfrac{3}{2}\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} + (2r - \tfrac{1}{2})\, T^*_{-\varepsilon_1} T_{-\varepsilon_1} + \tfrac{1}{2}\, T^*_{+\varepsilon_2} T_{+\varepsilon_2} + (r - \tfrac{1}{2})\, T^*_{\pm\varepsilon_r} T_{\pm\varepsilon_r}
$$

$$
\begin{aligned}
p_3(B)(\nabla^2) \;=\; & -(\tfrac{3}{2} + r)(r-1)\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} + (2r-1)(r^2-1)\, T^*_{-\varepsilon_1} T_{-\varepsilon_1} \\
& + (r - \tfrac{1}{2})(r+1)\, T^*_{+\varepsilon_2} T_{+\varepsilon_2} + T_{\pm\varepsilon_r}
\end{aligned}
$$

Note that similar Weitzenböck formulas were obtained in [BH02]. More precisely their curvature terms Z_1 and Z_2 are related to B and $p_3(B)$ by the following equations:

$$
Z_1 \;=\; \tfrac{(2r+3)(r-1)}{r(2r+1)}\, B - \tfrac{3}{r(2r+1)}\, p_3(B) \qquad Z_2 \;=\; -\tfrac{(2r-1)(r+1)}{r(2r+1)}\, B + \tfrac{1}{r(2r+1)}\, p_3(B)\ ,
$$

whereas the operators are related by: $T_{+\varepsilon_1} = G_Z,\ T_{-\varepsilon_1} = G_\Sigma,\ T_{+\varepsilon_2} = G_Y,\ T_{\mp\varepsilon_r} = G_T$.

In the last part of this section we want to describe for $\mathbf{G}_2$– and $\mathbf{Spin}(7)$–holonomy all pure curvature Weitzenböck formulas on parallel subbundles of the form bundle. In particular we will present the form Laplacian $\Delta = d^*d + dd^* = \nabla^*\nabla + q(R)$ as a linear combination of the operators $T_\varepsilon^* T_\varepsilon$ and discuss the existence of harmonic forms.

We start with the case of $\mathbf{G}_2$–holonomy. Let $\Gamma_{a,\,b}$ be the irreducible $\mathbf{G}_2$–representation with highest weight $a\omega_1 + b\omega_2$, $a, b \geq 0$, e.g. $\Gamma_{0,\,0} = \mathbb{C}$ is the trivial representation, $\Gamma_{1,\,0} = T$ and $\Gamma_{0,\,1} = \Lambda^2_{14} \cong \mathfrak{g}_2$. Recall that up to dimension 77 irreducible $\mathbf{G}_2$–representations are uniquely determined by their dimension. However there are two different irreducible representations in dimension 77, one of them is $[77]^- := \Gamma_{3,\,0}$, the other is $\Gamma_{0,2}$, the space of $\mathbf{G}_2$-curvature tensors. Moreover

$$\dim \Gamma_{2,\,0} \;=\; 27, \qquad \dim \Gamma_{1,\,1} \;=\; 64\ .$$

The spaces of 2– and 3–forms have the following decompositions:

$$\Lambda^2 T \;\cong\; \Lambda^5 T \;\cong\; T \oplus \Lambda^2_{14}, \qquad\qquad \Lambda^3 T \;\cong\; \Lambda^4 T \;\cong\; \mathbb{C} \oplus T \oplus \Lambda^3_{27}\ , \tag{7.35}$$

where the subscripts denote the dimension of the representation. Next we give the relevant weights for the representations $\Gamma_{1,0}$, $\Gamma_{0,1}$ and $\Gamma_{2,0}$. We start with $\lambda = \omega_2$, i.e. the representation $V_\lambda = \Gamma_{0,\,1} = \Lambda^2_{14}$, here the relevant weights are $\varepsilon = -\varepsilon_2,\ \varepsilon_3,\ \varepsilon_1$ with $\lambda + \varepsilon = \omega_1,\ 2\omega_1,\ \omega_1 + \omega_2$ and the corresponding decomposition reads

$$T \otimes \Gamma_{0,\,1} \;=\; \Gamma_{1,\,0} \oplus \Gamma_{2,\,0} \oplus \Gamma_{1,\,1} \;=\; T^* \oplus \Lambda^3_{27} T^* \oplus [64]\ .$$

In this case the universal Weitzenböck formula of Proposition 3.6) is the only pure curvature Weitzenböck formula. With the explicit B–eigenvalues given in Section 4.2 we find on sections of $\Lambda^2_{14} T^* M$:

$$\begin{aligned} q(R) \;&=\; 4\, T^*_{-\varepsilon_2} T_{-\varepsilon_2} \;+\; \tfrac{4}{3}\, T^*_{+\varepsilon_3} T_{+\varepsilon_3} \;-\; T^*_{+\varepsilon_1} T_{+\varepsilon_1} \\ \Delta \;&=\; 5\, T^*_{-\varepsilon_2} T_{-\varepsilon_2} \;+\; \tfrac{7}{3}\, T^*_{+\varepsilon_3} T_{+\varepsilon_3} \end{aligned}$$

Hence a form ψ in $\Lambda^2_{14} \subset \Lambda^2 T$ is harmonic if and only if $T_{-\varepsilon_2}\psi = 0 = T_{+\varepsilon_3}\psi$ (the manifold is assumed to be compact), i.e. if and and only if $\nabla\psi = T_{+\varepsilon_1}\psi$ or equivalently if and only if $\nabla\psi$ is a section of $\Gamma_{11} = [64]$. This statement corresponds to the fact that on a compact manifold a form is harmonic if and only if it is closed and coclosed.

For $\lambda = \omega_1$, i.e. the representation $V_\lambda = \Gamma_{1,0} = T$ the relevant weights are determined as $\varepsilon = -\varepsilon_1,\ 0,\ +\varepsilon_2,\ +\varepsilon_1$ with $\lambda + \varepsilon = 0,\ \omega_1,\ \omega_2,\ 2\omega_1$ leading to the decomposition:

$$T \otimes \Gamma_{1,0} \;=\; \Gamma_{0,\,0} \oplus \Gamma_{1,\,0} \oplus \Gamma_{0,\,1} \oplus \Gamma_{2,\,0} \;=\; \mathbb{C} \oplus T^* \oplus \Lambda^2_{14} T^* \oplus \Lambda^3_{27} T^*\ .$$

Here we have two pure curvature Weitzenböck formulas. In fact both curvature terms are zero, since $q(R) = \mathrm{Ric} = 0$ on $\Gamma_{1,0} = T$. In addition to the universal Weitzenböck formula of Proposition 3.6) we have the equation corresponding to the polynomial F_3 given in

(4.21) with $a = 1, b = 0$. After substituting the B–eigenvalues we obtain the following Weitzenböck formulas on 1–forms:

$$\begin{array}{rcrcrcrcr} 0 & = & 4\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} & + & 2\,T^*_0 T_0 & & & - & \frac{2}{3}\,T^*_{+\varepsilon_1}T_{+\varepsilon_1} \\ 0 & = & -\;\frac{16}{3}\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} & + & \frac{8}{3}\,T^*_0 T_0 & - & \frac{8}{3}\,T^*_{+\varepsilon_2}T_{+\varepsilon_2} & + & \frac{8}{9}\,T^*_{+\varepsilon_1}T_{+\varepsilon_1} \\ \Delta & = & 5\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} & + & 3\,T^*_0 T_0 & + & T^*_{+\varepsilon_2}T_{+\varepsilon_2} & + & \frac{1}{3}\,T^*_{+\varepsilon_1}T_{+\varepsilon_1} \end{array}$$

It follows that $\Delta \geq \frac{1}{3}\,\nabla^*\nabla$, i. e. there are no non–parallel harmonic 1–forms, which is of course Bochners theorem in the case of $\mathbf{G}_2$–manifolds.

Next we consider the case $\lambda = 2\omega_1$, i. e. the representation $V_\lambda = \Gamma_{2,0} = \Lambda^3_{27}$. Here the relevant weights are $\varepsilon = -\varepsilon_1,\ 0,\ -\varepsilon_3,\ \varepsilon_2,\ \varepsilon_1$ with $\lambda + \varepsilon = \omega_1,\ 2\omega_1,\ \omega_2,\ \omega_1 + \omega_2,\ 3\omega_1$ and with the decomposition

$$\begin{array}{rcl} T \otimes \Gamma_{2,0} & = & \Gamma_{1,0} \oplus\ \Gamma_{2,0}\ \oplus \Gamma_{0,1} \oplus \Gamma_{1,1} \oplus\ \Gamma_{3,0} \\ & = & T^* \ \oplus \Lambda^3_{27}T^* \oplus \Lambda^2_{14} \oplus [\,64\,] \oplus [\,77\,]^- \end{array}$$

Hence we have two pure curvature Weitzenböck formulas on sections of Λ^3_{27}. The first one is the formula for $q(R)$ corresponding to B, while the second corresponds to $-\frac{27}{240}\,p_3(B)$:

$$\begin{array}{rcrcrcrcrcr} q(R) & = & \frac{14}{3}\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} & + & 2\,T^*_0 T_0 & + & \frac{8}{3}\,T^*_{-\varepsilon_3}T_{-\varepsilon_3} & - & \frac{1}{3}\,T^*_{+\varepsilon_2}T_{+\varepsilon_2} & - & \frac{4}{3}\,T^*_{+\varepsilon_1}T_{+\varepsilon_1} \\ 0 & = & -\;\frac{7}{6}\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} & + & \frac{1}{2}\,T^*_0 T_0 & + & \frac{5}{6}\,T^*_{-\varepsilon_3}T_{-\varepsilon_3} & - & \frac{2}{3}\,T^*_{+\varepsilon_2}T_{+\varepsilon_2} & + & \frac{1}{3}\,T^*_{+\varepsilon_1}T_{+\varepsilon_1} \\ \Delta & = & \frac{9}{2}\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} & + & \frac{7}{2}\,T^*_0 T_0 & + & \frac{9}{2}\,T^*_{-\varepsilon_3}T_{-\varepsilon_3} & & & & \end{array}$$

It follows that a form ψ in $\Lambda^3_{27} \subset \Lambda^3 T$ is harmonic if and only if $\nabla\psi$ is a section of $\Gamma_{1,1} \oplus \Gamma_{3,0}$. Note that the expression for Δ was obtained by adding the Bochner identity, i. e. the second Weitzenböck formula, to the equation for $\nabla^*\nabla + q(R)$.

Finally we turn to the case of **Spin**(7)–holonomy. Irreducible **Spin**(7)–representations are parametrized as $\Gamma_{a,\,b,\,c} = a\omega_1 + b\omega_2 + c\omega_2$. Again $\Gamma_{0,\,0,\,0} = \mathbb{C}$ is the trivial representation and $\Gamma_{0,\,0,\,1} = T$ denotes the 8–dimensional holonomy representation. We want to describe the twistor operators for the parallel subbundles of the form bundle. For this we need the following representations, which are also uniquely determined by their dimension:

$$\begin{array}{llll} \dim\Gamma_{1,\,0,\,0} \ = \ 7 & \dim\Gamma_{0,\,1,\,0} \ = \ 21 & \dim\Gamma_{1,\,0,\,1} \ = \ 48 & \dim\Gamma_{1,\,1,\,0} \ = \ 105 \\ \dim\Gamma_{2,\,0,\,0} \ = \ 27 & \dim\Gamma_{0,\,0,\,2} \ = \ 35 & \dim\Gamma_{2,\,0,\,1} \ = \ 168 & \dim\Gamma_{1,\,0,\,2} \ = \ 189\ . \end{array}$$

In dimension 112 there are two different irreducible representations denoted by $[112]^a := \Gamma_{0,\,1,\,1}$ and $[112]^b := \Gamma_{0,\,0,\,3}$. Using Poincaré duality we can decompose the differential forms

$$\begin{array}{rclcl} \Lambda^2 T^* & \cong & \Lambda^2_7 T^* \oplus \Lambda^2_{21}T^* & \cong & \Lambda^6 T^* \\ \Lambda^3 T^* & \cong & T^* \ \oplus \Lambda^3_{48}T^* & \cong & \Lambda^5 T^* \\ \Lambda^4 T^* & \cong & \mathbb{C} \ \oplus \Lambda^4_7 T^* \oplus \Lambda^4_{27}T^* \oplus \Lambda^4_{35}T^* & & \end{array} \qquad (7.36)$$

into irreducible subspaces, where again the subscripts refer to the dimension.

We start with the representation $V_\lambda = \Gamma_{1,0,0} = \Lambda^2_7$ of highest weight $\lambda = \omega_1$. The relevant weights are $+\varepsilon_1$ and $-\varepsilon_4$ with B–eigenvalues $b_{+\varepsilon_1} = \frac{1}{2}$ and $b_{-\varepsilon_4} = -3$ leading to the decomposition:

$$T \otimes \Gamma_{1,0,0} \;=\; \Gamma_{1,0,1} \oplus \Gamma_{0,0,1} \;=\; \Lambda^3_{48} \oplus T \; .$$

Because the bundle defined by $\Gamma_{1,0,0}$ can be considered as the subbundle of the spinor bundle orthogonal to the parallel spinor, the curvature endomorphism $q(R)$ is a multiple of the scalar curvature and hence vanishes. Thus we obtain on sections of Λ^2_7 the only Weitzenböck formula

$$0 \;=\; -\tfrac{1}{2}\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} \;+\; 3\, T^*_{-\varepsilon_4} T_{-\varepsilon_4}$$

It follows that $\Delta \geq \frac{1}{2}\nabla^*\nabla$, i. e. there are no non-parallel harmonic forms in Λ^2_7.

For the second component of the space of 2–forms $\Lambda^2_{21} = \Gamma_{0,1,0}$ the relevant weights are $+\varepsilon_1$, $-\varepsilon_2$, e_3 with B–eigenvalues $1, -5, -\frac{3}{2}$ in the decomposition:

$$T \otimes \Gamma_{0,1,0} \;=\; \Gamma_{0,1,1} \oplus \Gamma_{0,0,1} \oplus \Gamma_{1,0,1} \;=\; [112]^a \oplus T \oplus \Lambda^3_{48} \; .$$

On sections of Λ^2_{21} we have the Weitzenböck formula

$$q(R) \;=\; -\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} \;+\; 5\, T^*_{-\varepsilon_2} T_{-\varepsilon_2} \;+\; \tfrac{3}{2}\, T^*_{+\varepsilon_3} T_{+\varepsilon_3} \; .$$

Hence a form ψ in Λ^2_{21} is harmonic if and only if $\nabla\psi$ is a section of $\Gamma_{0,1,1}$.

The last parallel subbundle of the form bundle with only one pure curvature Weitzenböck formula is $V_\lambda = \Gamma_{2,0,0} = \Lambda^4_{27}$. The relevant weights are $+\varepsilon_1$, $-\varepsilon_4$ with B–eigenvalues $1, -\frac{7}{2}$ and the decomposition

$$T \otimes \Gamma_{2,0,0} \;=\; \Gamma_{2,0,1} \oplus \Gamma_{1,0,1} \;=\; [168] \oplus \Lambda^3_{48} \; ,$$

with $[168] := \Gamma_{2,0,1}$. On sections of Λ^4_{27} we have the Weitzenböck formula:

$$q(R) \;=\; -\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} \;+\; \tfrac{7}{2}\, T^*_{-\varepsilon_2} T_{-\varepsilon_2} \; .$$

Hence a form ψ in Λ^4_{27} is harmonic if and only if $\nabla\psi$ is a section of $\Gamma_{2,0,1}$. For the remaining subbundles we have at least two pure curvature Weitzenböck formulas one of which is a Bochner identity, i. e. with a zero curvature term.

We start with the representation $T = \Gamma_{0,0,1}$ describing 1– and 6–forms on M. The relevant weights are $+\varepsilon_1, -\varepsilon_1, +\varepsilon_2, +\varepsilon_4$ conformal weights or B–eigenvalues $\frac{3}{4}, -\frac{21}{4}, -\frac{1}{4}$ and $-\frac{9}{4}$ respectively. The corresponding decomposition of the representation $T \otimes T$ reads:

$$T \otimes \Gamma_{0,0,1} \;=\; \Gamma_{0,0,2} \oplus \Gamma_{0,0,0} \oplus \Gamma_{0,1,0} \oplus \Gamma_{1,0,0} \;=\; \Lambda^4_{35}T^* \oplus \mathbb{C} \oplus \Lambda^2_{21}T^* \oplus \Lambda^2_7 T^* \; .$$

In this case we have the universal Weitzenböck formula and the Bochner identity (4.24) for $(a,b,c) = (0,0,1)$. Since $q(R) = \mathrm{Ric}$ on the tangent bundle we obtain two zero curvature Weitzenböck formulas on sections of T:

$$\begin{aligned} 0 \;&=\; -\; \tfrac{3}{4}\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} \;+\; \tfrac{21}{4}\, T^*_{-\varepsilon_1} T_{-\varepsilon_1} \;+\; \tfrac{1}{4}\, T^*_{+\varepsilon_2} T_{+\varepsilon_2} \;+\; \tfrac{9}{4}\, T^*_{+\varepsilon_4} T_{+\varepsilon_4} \\ 0 \;&=\; +\, 15\, T^*_{+\varepsilon_1} T_{+\varepsilon_1} \;-\; 105\, T^*_{-\varepsilon_1} T_{-\varepsilon_1} \;-\; 45\, T^*_{+\varepsilon_2} T_{+\varepsilon_2} \;+\; 75\, T^*_{+\varepsilon_4} T_{+\varepsilon_4} \end{aligned}$$

Evidently the first equation tells us $\Delta \geq \frac{1}{4}\nabla^*\nabla$ so that every harmonic 1–form is necessarily parallel. Of course this is Bochner's theorem reproved in the case of $\mathbf{Spin}(7)$–manifolds. Another direct consequence is the well–known fact that any Killing vector field on a compact $\mathbf{Spin}(7)$–manifold has to be parallel. Indeed Killing vector fields are vector fields $X \in \Gamma(TM)$, for which $\nabla X^\sharp \in \Gamma(T^*M \otimes T^*M)$ is skew and thus a 2–form. On $\mathbf{Spin}(7)$–manifolds this implies $T_{+\varepsilon_1}X = 0 = T_{-\varepsilon_1}X$ and so all twistor operators vanish on X.

Next we consider the representation $\Lambda^4_{35}T^* = \Gamma_{0,\,0,\,2}$ with relevant weights ε_1, $-\varepsilon_1$, ε_2, ε_4, conformal weights or B–eigenvalues $\frac{3}{2}$, -6, 0, $-\frac{5}{2}$ and decomposition

$$T \otimes \Gamma_{0,\,0,\,2} \;=\; \Gamma_{0,\,0,\,3} \oplus \Gamma_{0,\,0,\,1} \oplus \Gamma_{0,\,1,\,1} \oplus \Gamma_{1,\,0,\,1} \;=\; [112]^b \oplus T^* \oplus [112]^a \oplus \Lambda^3_{48}T^* \,.$$

Thus there are two pure curvature Weitzenböck formulas on sections of Λ^4_{35}. For the second we take $\frac{1}{96}F_{Bochner}$ and obtain

$$\begin{array}{rcl}
q(R) &=& -\ \frac{3}{2}\,T^*_{+\varepsilon_1}T_{+\varepsilon_1} \;+\; 6\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} \;+\; \frac{5}{2}\,T^*_{+\varepsilon_4}T_{+\varepsilon_4} \\
0 &=& \frac{1}{2}\,T^*_{+\varepsilon_1}T_{+\varepsilon_1} \;-\; 2\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} \;-\; T^*_{+\varepsilon_2}T_{+\varepsilon_2} \;+\; \frac{3}{2}\,T^*_{+\varepsilon_4}T_{+\varepsilon_4} \\
\Delta &=& 5\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} \;+\; 5\,T^*_{+\varepsilon_4}T_{+\varepsilon_4}
\end{array}$$

Note that in order to obtain the optimal expression for the operator Δ it was not sufficient to take its definition $\Delta = \nabla^*\nabla + q(R)$, we still had to add a multiple of the Bochner identity. Last but not least we consider the representation $\Lambda^3_{48}T^* = \Gamma_{1,\,0,\,1}$. According to the table at the end of Section 2 the relevant weights are $+\varepsilon_1, -\varepsilon_1, +\varepsilon_2, -\varepsilon_3, \varepsilon_4, -\varepsilon_4$ with conformal weights or B–eigenvalues $\frac{5}{4}$, $-\frac{23}{4}$, $\frac{1}{4}$, $-\frac{15}{4}$, $-\frac{7}{4}$ and $-\frac{11}{4}$ respectively leading to

$$\begin{array}{rcl}
T \otimes \Gamma_{1,\,0,\,1} &=& \Gamma_{1,\,0,\,2} \oplus \Gamma_{1,\,0,\,0} \oplus \Gamma_{1,\,1,\,0} \oplus \Gamma_{0,\,1,\,0} \oplus \Gamma_{2,\,0,\,0} \oplus \Gamma_{0,\,0,\,2} \\
&=& [\,189\,] \oplus \Lambda^2_7T^* \oplus [\,105\,] \oplus \Lambda^2_{21}T^* \oplus \Lambda^4_{27}T^* \oplus \Lambda^4_{35}T^*
\end{array}$$

On sections of the associated bundle $\Lambda^3_{48}T^*M \subset \Lambda^3T^*M$ one has three curvature Weitzenböck formulas, the formula corresponding to B and the Bochner identity $\frac{1}{84}F_{Bochner}$:

$$\begin{array}{rcl}
q(R) &=& -\frac{5}{4}\,T^*_{\varepsilon_1}T_{\varepsilon_1} + \frac{23}{4}\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} - \frac{1}{4}\,T^*_{\varepsilon_2}T_{\varepsilon_2} + \frac{15}{4}\,T^*_{-\varepsilon_3}T_{-\varepsilon_3} + \frac{7}{4}\,T^*_{\varepsilon_4}T_{\varepsilon_4} + \frac{11}{4}\,T^*_{-\varepsilon_4}T_{-\varepsilon_4} \\
0 &=& \frac{1}{4}\,T^*_{\varepsilon_1}T_{\varepsilon_1} - \frac{45}{28}\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} - \frac{3}{4}\,T^*_{\varepsilon_2}T_{\varepsilon_2} + \frac{27}{28}\,T^*_{-\varepsilon_3}T_{-\varepsilon_3} + \frac{5}{4}\,T^*_{\varepsilon_4}T_{\varepsilon_4} - \frac{9}{28}\,T^*_{-\varepsilon_4}T_{-\varepsilon_4} \\
\Delta &=& \frac{36}{7}\,T^*_{-\varepsilon_1}T_{-\varepsilon_1} + \frac{40}{7}\,T^*_{-\varepsilon_3}T_{-\varepsilon_3} + 4\,T^*_{\varepsilon_4}T_{\varepsilon_4} + \frac{24}{7}\,T^*_{-\varepsilon_4}T_{-\varepsilon_4}
\end{array}$$

Consequently a 3–form $\psi \in \Gamma(\Lambda^3_{48}T^*M)$ of type is harmonic if and only if its covariant derivative $\nabla\psi$ takes values in $([189] \oplus [105])M \subset T^*M \otimes \Lambda^3_{48}T^*M$ everywhere.

8 Bochner Identities in $\mathbf{G}_2$– and $\mathbf{Spin}(7)$–Holonomy

Aim of this section is to provide a proof of the Bochner identities for the holonomies $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ and thus to complete the description of the space of Weitzenböck formulas in these cases. Interestingly it seems necessary to introduce a fairly more abstract point of view of Weitzenböck formulas in order to get to this point.

8.1 Universal Weitzenböck Classes and the Kostant Theorem

The essential additional twist we will employ in this section is that we will basis the study of Weitzenböck formulas on the study of the action of central elements of the universal enveloping algebra. As a byproduct we get a explicit formula for a central element of order 4 in the universal enveloping algebra of $\mathfrak{spin}_7$ and a central element of order 4 in the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}_2$.

Recall that the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is the associative algebra with 1 generated freely by the vector space $\mathfrak{g}$ subject only to the commutator relation $X\,Y - Y\,X = [X, Y]$. Thus $\mathcal{U}\,\mathfrak{g}$ is spanned by monomials of the form $X_1 \ldots X_r$ in elements $X_1, \ldots, X_r$ of $\mathfrak{g}$ and the filtration $\mathcal{U}^{\leq\bullet}\mathfrak{g}$ by the degree r of these monomials makes $\mathcal{U}\,\mathfrak{g}$ a filtered algebra. Even more important for our purposes is the Hopf algebra structure of $\mathcal{U}\,\mathfrak{g}$ with the cocommutative comultiplication

$$\Delta:\ \mathcal{U}\,\mathfrak{g} \longrightarrow \mathcal{U}\,\mathfrak{g}\otimes\mathcal{U}\,\mathfrak{g}, \qquad \mathcal{Q} \longmapsto \sum \Delta_L\mathcal{Q}\otimes\Delta_R\mathcal{Q} \tag{8.37}$$

defined as the unique algebra homomorphism sending $X \in \mathfrak{g}$ to $\Delta X := X\otimes 1 + 1\otimes X$ in $\mathcal{U}\,\mathfrak{g}\otimes\mathcal{U}\,\mathfrak{g}$. Defining Δ in this way clearly implies for all $d,\ r \geq 0$:

$$\Delta(\,\mathcal{U}^{\leq d+r}\mathfrak{g}\,) \ \subset\ \mathcal{U}^{<d}\mathfrak{g}\otimes\mathcal{U}\,\mathfrak{g} \;+\; \mathcal{U}\,\mathfrak{g}\otimes\mathcal{U}^{\leq r}\mathfrak{g}\,. \tag{8.38}$$

An integral part of the structure of the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}$ is the algebra homomorphism $\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{End}\,V$ associated to a representation V of $\mathfrak{g}$. For finite–dimensional representations V the images of these algebra homomorphisms are easily characterized.

Lemma 8.1 (Bicommutant Theorem)
Consider a finite dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and the induced representation $\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{End}\,V$ of $\mathcal{U}\,\mathfrak{g}$. The image of this algebra homomorphism is precisely the commutant of the algebra $\mathrm{End}_{\mathfrak{g}}V$ of $\mathfrak{g}$–invariant endomorphisms

$$\mathrm{im}\,(\,\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{End}\,V\,) \;=\; \{\,A\in\mathrm{End}\,V \mid [\,A,\,F\,] = 0 \textit{ for all } F\in\mathrm{End}_{\mathfrak{g}}V\,\}\,.$$

In particular the map $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{Zent}\,\mathrm{End}_{\mathfrak{g}}V$ is surjective for V finite dimensional.

The Bicommutant Theorem is actually a special motivating example of von Neumann's Bicommutant Theorem, observe that every $*$–subalgebra of $\mathrm{End}\,V$ is necessarily von Neumann for a finite dimensional vector space V. The image of $\mathcal{U}\,\mathfrak{g}$ in $\mathrm{End}\,V$ is the subalgebra generated by the $*$–closed subspace $\mathfrak{g}$ of $\mathrm{End}\,V$ and thus von Neumann with commutant $\mathrm{End}_{\mathfrak{g}}V$. We will give a more geometric proof of this theorem based on the Peter–Weyl Theorem in Appendix A.

Coming back to Weitzenböck formulas we conclude that for irreducible representations V_λ the algebra homomorphism $\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{End}\,V_\lambda$ is surjective and hence the same is true for the algebra homomorphism

$$\Phi:\qquad \mathrm{Hom}_{\mathfrak{g}}(\,T\otimes T,\,\mathcal{U}\,\mathfrak{g}\,) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}\,V_\lambda) \;=\; \mathfrak{W}(V_\lambda)$$

where $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}\,V_\lambda)$ is one of the interpretation of the space $\mathfrak{W}(V_\lambda)$ of Weitzenböck formulas on $V_\lambda M$. Motivated by this surjection we will call $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathcal{U}\,\mathfrak{g})$ the space of universal Weitzenböck formulas. With the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}$ being a module over its center the space $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathcal{U}\,\mathfrak{g})$ of universal Weitzenböck formulas is naturally a module for $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$, too, and the filtration $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathcal{U}^{\leq\bullet}\mathfrak{g})$ turns it into a filtered module for the filtration $\mathrm{Zent}^{\leq\bullet}\mathcal{U}\,\mathfrak{g} := \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}\cap\mathcal{U}^{\leq\bullet}\mathfrak{g}$ of the center.

Definition 8.2 (Universal Weitzenböck Classes)
The space of universal Weitzenböck formulas $\mathfrak{W}^{\leq\bullet} := \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathcal{U}^{\leq\bullet}\mathfrak{g})$ *is a filtered module over the center* $\mathrm{Zent}^{\leq\bullet}\mathcal{U}\,\mathfrak{g}$ *of the universal enveloping algebra* $\mathcal{U}\,\mathfrak{g}$*. It splits into the direct sum of filtered* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$*–submodules called universal Weitzenböck classes:*

$$\mathfrak{W}^{\leq\bullet} \;=\; \bigoplus_\alpha \mathfrak{W}^{\leq\bullet}_{W_\alpha} \;:=\; \bigoplus_\alpha \mathrm{Hom}_{\mathfrak{g}}(\, W_\alpha, \mathcal{U}^{\leq\bullet}\mathfrak{g}\,)\,.$$

It is clear from the definition that with $F \in \mathfrak{W}^{\leq k}$ also $F|_{W_\alpha} \in \mathfrak{W}^{\leq k}_{W_\alpha}$. Moreover the powers B^k of the conformal weight operator B are in the image of $\mathfrak{W}^{\leq k}$ under the surjection Φ. Indeed B^k is the image of the nvariant map $p_k : T \otimes T \longrightarrow \mathcal{U}^{\leq k}\mathfrak{g}$ defined by

$$p_k(\, a \otimes b\,) \;=\; \sum_{\mu_1,\ldots,\mu_{k-1}} \mathrm{pr}_{\mathfrak{g}}(\, a \wedge t_{\mu_1}\,)\,\mathrm{pr}(\, t_{\mu_1} \wedge t_{\mu_2}\,)\,\ldots\,\mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-1}} \wedge b\,)\,,$$

where $\mathrm{pr} : T \otimes T \longrightarrow \mathfrak{g} \subset \Lambda^2 T$ is the same orthogonal projection used before in the definition of B. Under the vector space identification $\mathcal{U}\,\mathfrak{g} \cong \mathrm{Sym}\,\mathfrak{g}$ we may consider $p_k(a \otimes b)$ as the polynomial $p_k(a \otimes b)[X] = \langle\, X^k a,\, b\,\rangle$ on $\mathfrak{g}$. Important for our considerations below is that the space of universal Weitzenböck formulas is a free module over $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$:

Theorem 8.3 (Kostant's Theorem)
For every finite dimensional representation V *the space* $\mathrm{Hom}_{\mathfrak{g}}(\, V, \mathcal{U}\,\mathfrak{g}\,)$ *is a free* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$*–module, whose rank over* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$ *agrees with the multiplicity of the zero weight in* V*:*

$$\mathrm{Hom}^{\leq\bullet}_{\mathfrak{g}}(\, V, \mathcal{U}\,\mathfrak{g}\,) \;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g} \,\otimes\, \mathrm{Hom}^{\bullet}_{\mathfrak{t}}(\, V, \mathbb{C}\,)$$

In particular the module $\mathrm{Hom}^{\leq\bullet}_{\mathfrak{g}}(\, \mathfrak{g}, \mathcal{U}\,\mathfrak{g}\,) \cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g} \otimes \mathrm{Prim}^{\bullet+1}\,\mathfrak{g}$ *is generated freely as a filtered* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$*–module by the primitive elements of* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$ *with degrees shifted by* -1.

As an example we consider holonomy $\mathfrak{g}_2$ and the spaces which are mapped under Φ onto the K–eigenspaces. We refer to the appendix for the other holonomies. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}_2}(\quad \mathbb{C} \quad, \mathcal{U}\,\mathfrak{g}_2\,) &\;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2 \\
\mathrm{Hom}_{\mathfrak{g}_2}(\, \mathrm{Sym}^2_0 T\,, \mathcal{U}\,\mathfrak{g}_2\,) &\;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2\,\langle\, F_2,\; F_4,\; F_6\,\rangle \\
\mathrm{Hom}_{\mathfrak{g}_2}(\quad \mathfrak{g}_2 \quad, \mathcal{U}\,\mathfrak{g}_2\,) &\;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2\,\langle\, F_1,\; F_5\,\rangle \\
\mathrm{Hom}_{\mathfrak{g}_2}(\quad \mathfrak{g}_2^{\perp} \quad, \mathcal{U}\,\mathfrak{g}_2\,) &\;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2\,\langle G_3 \rangle
\end{aligned}$$

where $F_1,\, F_2, G_3, F_4, F_5$ and F_6 are free generators of degree $1, 2, \ldots, 6$. The numbers of generators, i. e. the dimension of the corresponding zero weight space, can be read off Table (2.2). The degree of the generators, also called generalized exponents, can be obtained by decomposing $\mathrm{Sym}^k\mathfrak{g}_2$ into irreducible components (e. g. using the program LiE) and by determining the multiplicity of W_α in this decomposition for sufficiently many k. As mentioned in the theorem: the degrees of the generators $F_1,\, F_5$ are the degrees of the generators $C_2,\, C_6$ of $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2$ shifted by one.

It follows from Kostant's theorem that a basis in the eigenspace $\mathfrak{W}_{W_\alpha}(V_\lambda)$ of the classifying endomorphism K may be obtained as the image under the surjective representation map $\mathfrak{W}_{W_\alpha} \longrightarrow \mathfrak{W}_{W_\alpha}(V_\lambda)$ of certain free generators for the universal Weitzenböck classes

$\mathfrak{W}_{W_\alpha}$. Indeed the module multiplication with $\mathcal{Q} \in \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$ in $\mathfrak{W}$ turns in $\mathfrak{W}(V_\lambda)$ into multiplication with the value of the central character for λ on $\mathcal{Q}$, because:

$$\mathcal{Q}|_{V_\lambda} \;=:\; \chi_\lambda(\,\mathcal{Q}\,)\,\mathrm{id}_{V_\lambda}\,. \qquad\qquad \chi_\lambda(\,\mathcal{Q}\,) \;=\; \tfrac{1}{\dim V_\lambda}\,\mathrm{tr}_{V_\lambda}\,\mathcal{Q}$$

In general the value of the central character on $\mathcal{Q} \in \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$ is a polynomial in the highest weight λ invariant under the Weyl group of $\mathfrak{g}$. At least in principle we know the central characters of the higher Casimirs $\mathrm{Cas}^{[k]} \in \mathrm{Zent}^{\leq k}\mathcal{U}\,\mathfrak{g}$ defined in equation (3.12) as traces of the powers of the conformal weight operator, since equation (3.13) implies:

$$\chi_\lambda(\,\mathrm{Cas}^{[k]}\,) \;=\; \sum_\varepsilon b_\varepsilon^k\,\frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}\,. \tag{8.39}$$

In order to proceed we use the diagonal Δ of the Hopf algebra $\mathcal{U}\,\mathfrak{g}$ together with the representation of $\mathcal{U}\,\mathfrak{g}$ on the euclidian vector space T to define an algebra homomorphism

$$\Delta\,:\,\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g} \;\xrightarrow{\Delta}\; (\,\mathcal{U}\,\mathfrak{g}\otimes\mathcal{U}\,\mathfrak{g}\,)^{\mathfrak{g}} \;\longrightarrow\; \mathrm{Hom}_{\,\mathfrak{g}}(\,T\otimes T,\,\mathcal{U}\,\mathfrak{g}\,) \;=\; \mathfrak{W}$$

by $(\Delta\mathcal{Q})_{a\otimes b} = \sum\langle\,a,\,(\Delta_L\mathcal{Q})b\,\rangle\Delta_R\mathcal{Q}$ for all $\mathcal{Q} \in \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$. A particularly nice property of Δ is that the image of $\Delta\mathcal{Q} \in \mathfrak{W}$ under the representation map $\Phi : \mathfrak{W} \longrightarrow \mathfrak{W}(V_\lambda)$ can be written in the following way:

$$\Delta\mathcal{Q} \;=\; \sum_{\varepsilon\subset\lambda}\chi_{\lambda+\varepsilon}(\,\mathcal{Q}\,)\,\mathrm{pr}_\varepsilon \;\in\; \mathfrak{W}(V_\lambda) \tag{8.40}$$

In fact working our way through the identification $\mathrm{Hom}_{\,\mathfrak{g}}(T\otimes T, \mathrm{End}\,V_\lambda) = \mathrm{End}_{\,\mathfrak{g}}(T\otimes V_\lambda)$ we find the usual tensor product action of $\mathcal{Q} \in \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$ on $T\otimes V_\lambda$:

$$\begin{aligned}\Phi(\,\Delta\mathcal{Q}\,)(\,b\otimes v\,) \;&=\; \sum_\mu t_\mu\otimes(\,\Delta\mathcal{Q}\,)_{t_\mu\otimes b}\,v \;=\; \sum_\mu t_\mu\otimes\langle\,t_\mu,\,(\,\Delta_L\mathcal{Q}\,)\,b\,\rangle\,\Delta_R\mathcal{Q}\,v\\ &=\; \sum(\,\Delta_L\mathcal{Q}\,)\,b\otimes(\,\Delta_R\mathcal{Q}\,)\,v \;=\; \Delta\mathcal{Q}(\,b\otimes v\,) \;=\; \mathcal{Q}(\,b\otimes v\,)\end{aligned}$$

The last Δ is the restriction of the comultiplication to $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}\subset\mathcal{U}\,\mathfrak{g}$, which is precisely the action of $\mathcal{Q}\in\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$ on $T\otimes V$. Hence we have the following commutative diagram

$$\begin{array}{ccccc}\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g} & \xrightarrow{\Delta} & \mathrm{Hom}_{\,\mathfrak{g}}(\,T\otimes T,\,\mathcal{U}\,\mathfrak{g}\,) & \cong & \bigoplus_\alpha\,\mathrm{Hom}_{\,\mathfrak{g}}(\,W_\alpha,\,\mathcal{U}\,\mathfrak{g}\,)\\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} & & \big\downarrow\\ \mathrm{End}_{\,\mathfrak{g}}(T\otimes V_\lambda) & \xrightarrow{\cong} & \mathrm{Hom}_{\,\mathfrak{g}}(\,T\otimes T,\,\mathrm{End}\,V_\lambda\,) & \cong & \bigoplus_\alpha\,\mathrm{Hom}_{\,\mathfrak{g}}(\,W_\alpha,\,\mathrm{End}\,V_\lambda\,)\end{array}$$

where the right vertical arrow is the restriction of the representation map Φ onto $\mathfrak{W}_{W_\alpha}$. Recall that the left square consists of algebra and the right square of $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$–module homomorphisms and that moreover all vertical arrows are surjective maps.

Example 8.4 (Conformal Weight Operator)
A special case of this construction is the relation between the conformal weight operator B and the Casimir. The image of the Casimir $\mathrm{Cas}^{\Lambda^2} \in \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$ becomes

$$\Delta(\mathrm{Cas}^{\Lambda^2}) \;=\; -2\,B \;+\; (\,\mathrm{Cas}^{\Lambda^2}_T \;+\; \mathrm{Cas}^{\Lambda^2}_{V_\lambda}\,)$$

because $\Delta(X^2) = X^2 \otimes 1 + 2X \otimes X + 1 \otimes X^2$ *for every* $X \in \mathfrak{g}$ *and Fegan's Lemma 3.3. Moreover since* Δ *is an algebra homomorphism we conclude*

$$p(B) \;=\; \Delta\, p(\,-\tfrac{1}{2}\,\mathrm{Cas}^{\Lambda^2} \;+\; \tfrac{1}{2}(\mathrm{Cas}^{\Lambda^2}_T + \mathrm{Cas}^{\Lambda^2}_{V_\lambda})\,)$$

for every polynomial $p(B)$ *in the conformal weight operator* B*. In particular the space of polynomials in* B *is in the image under* Δ *of the subalgebra generated by* Cas^{Λ^2}*.*

The crucial additional information we get from introducing the universal Weitzenböck classes is the filtration degree of the generators of the $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$–modules $\mathfrak{W}_{W_\alpha}$. In order to prove the Bochner identities for holonomy $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ we still need the following

Lemma 8.5 (Filtration Property of Δ)
Consider the Weitzenböck class $\mathfrak{W}_{W_\alpha}$ *associated to an irreducible subspace* $W_\alpha \subset T \otimes T$*. If there is no non–trivial,* $\mathfrak{g}$*–equivariant map from* W_α *to* $\mathcal{U}^{<d}\mathfrak{g}$ *for some* $d \geq 1$*, i. e. if* $\mathfrak{W}^{<d}_{W_\alpha} = \{0\}$*, then the composition of* Δ *with the restriction* res_{W_α} *to* $\mathfrak{W}_\alpha$ *is filtered*

$$\mathrm{res}_{W_\alpha} \circ \Delta : \quad \mathrm{Zent}^{\,\leq d+\bullet}\mathcal{U}\,\mathfrak{g} \;\longrightarrow\; \mathfrak{W}^{\leq\bullet}_{W_\alpha}, \qquad \mathcal{Q} \;\longmapsto\; \Delta\mathcal{Q}|_{W_\alpha}$$

of degree $-d$*. In particular the restriction* $\Delta\mathcal{Q}|_{W_\alpha} = 0$ *vanishes for all* $\mathcal{Q} \in \mathrm{Zent}^{\,<2d}\mathcal{U}\,\mathfrak{g}$*.*

Proof: By the filtration property (8.38) of the comultiplication we can write the diagonal $\Delta\mathcal{Q}$ of an element $\mathcal{Q} \in \mathrm{Zent}^{\,\leq d+r}\mathcal{U}\,\mathfrak{g}$ in a not necessarily unique way as a sum of two terms $\Delta\mathcal{Q} = \Delta\mathcal{Q}^{<d} + \Delta\mathcal{Q}^{\leq r}$ satisfying $\Delta\mathcal{Q}^{<d} \in (\mathcal{U}^{<d}\mathfrak{g} \otimes \mathcal{U}\,\mathfrak{g})^{\mathfrak{g}}$ and $\Delta\mathcal{Q}^{\leq r} \in (\mathcal{U}\,\mathfrak{g} \otimes \mathcal{U}^{\leq r}\mathfrak{g})^{\mathfrak{g}}$ respectively. Both these summands give rise to $\mathfrak{g}$–equivariant, linear maps $T \otimes T \longrightarrow \mathcal{U}\,\mathfrak{g}$ through the pairing of $T \otimes T$ with the left $\mathcal{U}\,\mathfrak{g}$–factor, explicitly

$$(\,\Delta\mathcal{Q}^{<d}\,)_{a\otimes b} \;:=\; \sum \langle\, a,\, \Delta\mathcal{Q}^{<d}_L\, b\,\rangle\; \Delta\mathcal{Q}^{<d}_R$$

with essentially the same formula for $\Delta\mathcal{Q}^{\leq r}$. By construction the map $T \otimes T \longrightarrow \mathcal{U}\,\mathfrak{g}$ associated to $\Delta\mathcal{Q}^{\leq r}$ maps into $\mathcal{U}^{\leq r}\mathfrak{g}$, while the map associated to $\Delta\mathcal{Q}^{<d}$ vanishes upon restriction to $W_\alpha \subset T \otimes T$, because by assumption there is no non–trivial, $\mathfrak{g}$–invariant, pairing of W_α with the left image of $\Delta\mathcal{Q}^{<d}$ defined by:

$$\mathrm{span}\{\; \Delta\mathcal{Q}^{<d}_L \;|\; \Delta\mathcal{Q}^{<d} \;=\; \sum \Delta\mathcal{Q}^{<d}_L \otimes \Delta\mathcal{Q}^{<d}_R \;\} \;\subset\; \mathcal{U}^{<d}\mathfrak{g}$$

For the second statement we note $\Delta\mathcal{Q}|_{W_\alpha} \in \mathfrak{W}^{<d}_\alpha = \{0\}$ for all $\mathcal{Q} \in \mathrm{Zent}^{\,\leq 2d-1}\mathcal{U}\,\mathfrak{g}$. $\square$

8.2 Proof of the Bochner Identities in Holonomy $\mathfrak{g}_2$ and $\mathfrak{spin}_7$

Let us now discuss the details of the proof of the additional Bochner identity in $\mathbf{G}_2$–holonomy. Applying the Gram–Schmidt orthogonalization process 4.3 to the powers 1, B, B^2 and B^3 of the conformal weight operator we obtained in Equations (4.19) and (4.20) a sequence $p_0(B), p_1(B), p_2(B), p_3(B)$ of τ–eigenvectors. In order to proceed with the recursion procedure it remains to be shown that $p_3(B)$ is a K–eigenvector.

We know that $p_3(B)$ is a -1 eigenvector of τ, orthogonal to B and expressible as polynomial in B of degree 3. Thus $p_3(B)$ is an element in the image of $\mathfrak{W}^{\leq 3}$ in $\mathfrak{W}(V_\lambda)$ and can be written as a sum $p_3(B) = p_3(B)_{\mathfrak{g}_2} + p_3(B)_{\mathfrak{g}_2^\perp}$ of two vectors $p_3(B)_{\mathfrak{g}_2}$ and

$p_3(B)_{\mathfrak{g}_2^\perp}$ in the image of $\mathfrak{W}^{\leq 3}_{\mathfrak{g}_2}$ and $\mathfrak{W}^{\leq 3}_{\mathfrak{g}_2^\perp}$ in $\mathfrak{W}(V_\lambda)$ respectively. However the image of $\mathfrak{W}^{\leq 3}_{\mathfrak{g}_2}$ in $\mathfrak{W}(V_\lambda)$ is spanned by B, because the filtered Zent $\mathcal{U}\,\mathfrak{g}_2$–module $\mathfrak{W}_{\mathfrak{g}_2}$ is generated by two elements in degrees 1 and 5 and the representation $\mathfrak{W}_{\mathfrak{g}_2} \longrightarrow \mathfrak{W}_{\mathfrak{g}_2}(V_\lambda)$ turns module multiplication into multiplication by the central character χ_λ. Consequently the vector $p_3(B)$ is orthogonal to the image of $\mathfrak{W}^{\leq 3}_{\mathfrak{g}_2}$ in $\mathfrak{W}(V_\lambda)$ and lies in the eigenspace $\mathfrak{W}_{\mathfrak{g}_2^\perp}(V_\lambda)$ of the classifying endomorphism K:

Theorem 8.6 (Bochner Identity in G$_2$–Holonomy)
The following cubic polynomial in the conformal weight operator defines an eigenvector for the classifying endomorphisms K of eigenvalue -2:

$$p_3(B) \;:=\; B^3 \;+\; \tfrac{13}{3}\,B^2 \;+\; (\,\tfrac{1}{2}\mathrm{Cas}^{\Lambda^2}_{V_\lambda} \;+\; 4\,)\,B \;+\; \tfrac{2}{3}\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}$$

Inserting the eigenvalues or conformal weights b_ε of B we arrive at equation 4.21.

In the last part of this section we will prove the Bochner identity for holonomy $\mathfrak{spin}_7$. As in the $\mathfrak{g}_2$–case we apply the Gram–Schmidt orthogonalization process 4.3 to the powers 1, B, B^2 and B^3 and obtain in Equations (4.22) and (4.23) a sequence $p_0(B)$, $p_1(B)$, $p_2(B)$, $p_3(B)$ of τ–eigenvectors. Again $p_3(B)$ is a (-1)–eigenvector of τ, orthogonal to B and expressible as polynomial in B of degree 3 so that the summands in the decomposition $p_3(B) = p_3(B)_{\mathfrak{spin}_7} + p_3(B)_{\mathfrak{spin}_7^\perp}$ are in the image of $\mathfrak{W}^{\leq 3}_{\mathfrak{spin}_7}$ and $\mathfrak{W}^{\leq 3}_{\mathfrak{spin}_7^\perp}$ in $\mathfrak{W}(V_\lambda)$ respectively. Of course we want to extract the Bochner identity $p_3(B)_{\mathfrak{spin}_7^\perp}$ from $p_3(B)$. At this point the argument in the **Spin**(7)–case becomes more complicated, because the Zent $\mathcal{U}\,\mathfrak{spin}_7$–module $\mathfrak{W}_{\mathfrak{spin}_7}$ has generators in degree 1, 3 and 5 so that the image of $\mathfrak{W}^{\leq 3}_{\mathfrak{spin}_7}$ in $\mathfrak{W}_{\mathfrak{spin}_7}(V_\lambda)$ has dimension two. Even with $p_3(B)$ orthogonal to B we may thus not conclude that the component $p_3(B)_{\mathfrak{spin}_7} = 0$ vanishes. The idea to cope with this complication is to construct an element $Q_\lambda \in \mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7$ depending polynomially on the highest weight λ such that $\Delta Q_\lambda \in \mathfrak{W}_{\mathfrak{spin}_7}(V_\lambda)$ is orthogonal to B. The Bochner identity is then the projection of $p_3(B)$ onto the orthogonal complement of ΔQ_λ.

In the 4-dimensional space Zent $^{\leq 4}\mathcal{U}\,\mathfrak{spin}_7$ we look for an element $\mathcal{Q}_\lambda$ as a linear combination of the base vectors $\mathbf{1}$, Cas, Cas^2 and Cas$^{[4]}$ with unknown coefficients. We know ΔCas and ΔCas^2 from Example 8.4 and ΔCas$^{[4]}$ from Equations (8.39) and (8.40) so that

$$\langle\,\Delta Q_\lambda,\,\mathbf{1}\,\rangle \;=\; 0 \qquad\qquad \langle\,\Delta Q_\lambda,\,B\,\rangle \;=\; 0 \qquad\qquad \langle\,\Delta Q_\lambda,\,B^2\,\rangle \;=\; 0$$

turn into three linear independent equations for the four unknown coefficients. Using a computer algebra system to do the necessary calculations we find the convenient solution

$$\begin{aligned}\mathcal{Q}_\lambda \;=\;& 2\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\,\mathrm{Cas}^{[4]} \;-\; 160\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\,(\mathrm{Cas}^{\Lambda^2})^2\\ &+(\,320\,(\mathrm{Cas}^{\Lambda^2}_{V_\lambda})^2 \;-\; 1184\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda} \;-\; 4\,\mathrm{Cas}^{[4]}_{V_\lambda}\,)\,\mathrm{Cas}^{\Lambda^2}\\ &+(\,-\,160\,(\mathrm{Cas}^{\Lambda^2}_{V_\lambda})^3 \;+\; 2\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda}\,\mathrm{Cas}^{[4]}_{V_\lambda} \;+\; 1712\,(\mathrm{Cas}^{\Lambda^2}_{V_\lambda})^2 \;-\; 9408\,\mathrm{Cas}^{\Lambda^2}_{V_\lambda} \;-\; 21\,\mathrm{Cas}^{[4]}_{V_\lambda}\,)\end{aligned}$$

in Zent $^{\leq 4}\mathcal{U}\,\mathfrak{spin}_7$, where we denote the eigenvalue of the central element $\mathrm{Cas}^{[4]} \in \mathcal{U}\,\mathfrak{spin}_7$ on the irreducible representation V_λ by $\mathrm{Cas}^{[4]}_{V_\lambda}$ in analogy to the eigenvalues of Cas^{Λ^2}.

By construction $\Delta\mathcal{Q}_\lambda$ is orthogonal to 1, B and B^2 and we conclude that ΔQ_λ is indeed an eigenvector for the classifying endomorphism K. In fact the component $\Delta\mathcal{Q}_\lambda|_{\mathfrak{spin}_7^\perp} = 0$

vanishes according to Lemma 8.5, because $\mathcal{Q}_\lambda$ has degree 4 and there is no non–trivial equivariant map $\mathfrak{spin}_7^\perp \longrightarrow \mathcal{U}^{<3}\mathfrak{spin}_7$. Similarly the components $\Delta\mathcal{Q}_\lambda|_{\mathrm{Sym}^2_\circ T} = 0$ and $\Delta\mathcal{Q}_\lambda|_{\mathbb{C}} = 0$ are trivial, since the image of $\mathfrak{W}^{\leq 2}_{\mathrm{Sym}^2_\circ T}$ in $\mathfrak{W}(V_\lambda)$ has dimension 1 spanned by $p_2(B)$ while $\mathfrak{W}_{\mathbb{C}}(V_\lambda)$ is spanned by $p_0(B)$. With $\Delta\mathcal{Q}_\lambda = \Delta\mathcal{Q}_\lambda|_{\mathfrak{spin}_7}$ being an eigenvector of K orthogonal to $p_1(B) = B$ the problematic component $p_3(B)_{\mathfrak{spin}_7}$ of $p_3(B)$ must be a multiple of $\Delta\mathcal{Q}_\lambda$. In consequence the complementary component $p_3(B)_{\mathfrak{spin}_7^\perp}$ of $p_3(B)$ is the projection of $p_3(B)$ onto the orthogonal complement of ΔQ_λ and may serve as the $\mathfrak{spin}_7$–Bochner identity. Using again a computer algebra system for the necessary calculations we find that this projection of $p_3(B)$ to the orthogonal complement of $\Delta\mathcal{Q}_\lambda$ agrees with the endomorphism $F_{\mathrm{Bochner}} \in \mathfrak{W}(V_\lambda)$ specified in equation (4.24):

Theorem 8.7 (Bochner Identity in Spin(7)–Holonomy)
The endomorphism $F_{\mathrm{Bochner}} \in \mathfrak{W}(V_\lambda)$ specified in equation (4.24) with components

$$\begin{array}{rcrrrrl} F_{\mathrm{Bochner}} & = & + & c & (2b+c+2) & (2a+2b+c+4) & \mathrm{pr}_{+\varepsilon_1} \\ & & - & (c+2) & (2b+c+4) & (2a+2b+c+6) & \mathrm{pr}_{-\varepsilon_1} \\ & & - & (c+2) & (2b+c+2) & (2a+2b+c+4) & \mathrm{pr}_{+\varepsilon_2} \\ & & + & c & (2b+c+4) & (2a+2b+c+6) & \mathrm{pr}_{-\varepsilon_2} \\ & & - & c & (2b+c+4) & (2a+2b+c+4) & \mathrm{pr}_{+\varepsilon_3} \\ & & + & (c+2) & (2b+c+2) & (2a+2b+c+6) & \mathrm{pr}_{-\varepsilon_3} \\ & & + & (c+2) & (2b+c+4) & (2a+2b+c+4) & \mathrm{pr}_{+\varepsilon_4} \\ & & - & c & (2b+c+2) & (2a+2b+c+6) & \mathrm{pr}_{-\varepsilon_4} \end{array}$$

is an eigenvector of the classifying endomorphism K for the eigenvalue $-\frac{9}{4}$.

A Geometric Proof of the Bicommutant Theorem

Consider a finite–dimensional representation V of a semisimple Lie group G with Lie algebra $\mathfrak{g}$ and the algebra homomorphism $\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{End}\,V$ associated to the infinitesimal representation of $\mathfrak{g}$ on V. Evidently the image of $\mathcal{U}\,\mathfrak{g}$ under this homomorphism is the subalgebra $\mathcal{A}_{\mathfrak{g}}V$ of $\mathrm{End}\,V$ generated by the image of $\mathfrak{g}$. An alternative characterization of this subalgebra can be given using the notion of the centralizer subalgebra or more succinctly the commutant $\mathrm{Comm}\,\mathcal{A}$ of a subset $\mathcal{A} \subset \mathrm{End}\,V$:

$$\mathrm{Comm}\,\mathcal{A} \;:=\; \{\, F \in \mathrm{End}\,V \mid [\,F,\,A\,] \text{ for all } A \in \mathcal{A} \,\}$$

Von Neumann's famous Bicommutant Theorem states that the commutant of the commutant of a subset $\mathcal{A} \subset \mathrm{End}\,V$ of the star algebra of bounded operators on a complex Hilbert space V is precisely the von Neumann subalgebra generated by $\mathcal{A} \cup \mathcal{A}^*$. Every subalgebra of $\mathrm{End}\,V$ for a finite–dimensional Hilbert space V is von Neumann and with the image of $\mathfrak{g} = \mathfrak{g}^*$ being closed under taking adjoints we can characterize the subalgebra $\mathcal{A}_{\mathfrak{g}}V$ generated by $\mathfrak{g}$ as the commutant of the subalgebra $\mathrm{End}_{\,\mathfrak{g}}V$ of $\mathfrak{g}$–invariant endomorphisms on V:

Lemma A.1 (Bicommutant Theorem)
Let V be a finite–dimensional representation of a complex semisimple Lie algebra $\mathfrak{g}$. The image of the representation homomorphism $\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{End}\,V$ is precisely the commutant $\mathrm{Comm}\,\mathrm{End}_{\,\mathfrak{g}}V$ of the subalgebra $\mathrm{End}_{\,\mathfrak{g}}V \subset \mathrm{End}\,V$ of $\mathfrak{g}$–invariant endomorphisms on V.

Proof: With $\mathfrak{g}$ being a complex semisimple Lie algebra it has a compact real form $\mathfrak{g}_\mathbb{R}$ defined by a real structure $X \longmapsto \overline{X}$ on $\mathfrak{g}$. Being a compact real Lie algebra $\mathfrak{g}_\mathbb{R}$ is the Lie algebra of a compact simply connected Lie group $G_\mathbb{R}$ and hence there exists a $\mathfrak{g}_\mathbb{R}$–invariant hermitian form $(\,,\,)$ on the complex vector space V satisfying $X^* = -\overline{X}$ for all $X \in \mathfrak{g}$. Thus the image of $\mathfrak{g}$ in $\mathrm{End}\, V$ is closed under taking adjoints and by von Neumann's Theorem we conclude that the algebra $\mathcal{A}_\mathfrak{g}V$ generated by $\mathfrak{g}$ is the bicommutant $\mathrm{Comm}^2\mathfrak{g} = \mathrm{Comm}\,\mathrm{End}_\mathfrak{g}V$ of $\mathfrak{g}$.

Actually alluding to von Neumann's theorem may seem somewhat strange for this purely representation theoretic lemma, and for this reason we want to sketch a more elementary proof making use of Schur's Lemma and the Theorem of Peter–Weyl instead. In this way the reader will presumably get a better understanding of the crux of the statement, which interestingly enough was the main observation leading von Neumann to his Bicommutant Theorem in the first place. As V is a representation of a semisimple Lie algebra $\mathfrak{g}$ defined over $\mathbb{C}$ we can decompose V completely into irreducible subrepresentations $V_\lambda \subset V$. With respect to the general form of this decomposition

$$V \quad = \quad \bigoplus_\lambda \mathrm{Hom}_\mathfrak{g}(\, V_\lambda,\, V\,) \otimes V_\lambda$$

the commutant of the image of $\mathfrak{g}$ in $\mathrm{End}\, V$ is precisely the subalgebra

$$\mathrm{End}_\mathfrak{g}V \quad = \quad \bigoplus_\lambda \mathrm{End}\ \mathrm{Hom}_\mathfrak{g}(\, V_\lambda,\, V\,) \otimes \mathrm{id}_{V_\lambda}$$

by Schur's Lemma, which is a direct sum of matrix algebras one for each non–trivial isotypic component. The crucial observation underlying both the theorem above and von Neumann's theorem is the fact that the commutant of such a subalgebra is given by:

$$\mathrm{Comm}\,\mathrm{End}_\mathfrak{g}V \quad = \quad \bigoplus_\lambda \mathrm{id}_{\mathrm{Hom}_\mathfrak{g}(V_\lambda, V)} \otimes \mathrm{End}\, V_\lambda$$

Skipping the details of this argument using only elementary linear algebra we conclude that without loss of generality all isotypic components of V can be assumed irreducible. In other words we may assume that $V = \bigoplus V_\lambda$ is the direct sum of a finite set of irreducible, pairwise non–isomorphic representations V_λ and we want to show that the resulting representation homomorphism is surjective:

$$\mathcal{U}\,\mathfrak{g} \;\longrightarrow\; \bigoplus \mathrm{End}\, V_\lambda \;\;\subset\;\; \mathrm{End}\,\Big(\bigoplus V_\lambda\Big)\ .$$

In order to proceed we reformulate this statement thinking of $\mathcal{U}\,\mathfrak{g}$ as the algebra of left–invariant scalar differential operators on the complex–valued functions $C^\infty(\, G_\mathbb{R},\, \mathbb{C}\,)$ on $G_\mathbb{R}$, in particular there exists a linear map $\mathcal{U}\,\mathfrak{g} \longrightarrow C^\infty(\, G_\mathbb{R},\, \mathbb{C}\,)^*$, $\mathcal{Q} \longmapsto \mathrm{ev}_e \circ \mathcal{Q}$ by composing the action of the differential operator $\mathcal{Q} \in \mathcal{U}\,\mathfrak{g}$ with the evaluation at the identity. On the other hand the Theorem of Peter–Weyl identifies the direct sum $\bigoplus \mathrm{End}\, V_\lambda$ with a subspace of $C^\infty(\, G_\mathbb{R},\, \mathbb{C}\,)$ as all V_λ are irreducible and non–isomorphic. Working out the details of this identification we see that the resulting linear map reads:

$$\mathcal{U}\,\mathfrak{g} \;\longrightarrow\; \Big(\bigoplus \mathrm{End}\, V_\lambda\Big)^*, \qquad \mathcal{Q} \;\longmapsto\; (\,\eta \otimes e \;\longmapsto\; \eta(\,\mathcal{Q}e\,)\,)$$

Now all End $V_\lambda \cong (\,\mathrm{End}\ V_\lambda\,)^*$ are real representations with invariant symmetric bilinear form given by the trace and $\eta(\mathcal{Q}e) = \mathrm{tr}(\,\mathcal{Q}\circ(\eta\otimes e)\,)$ implies that the linear map constructed above is just another interpretation of the representation homomorphism. Moreover it is well–known that all matrix coefficients End $V_\lambda \subset C^\infty(\,G_\mathbb{R},\,\mathbb{C}\,)$ of an irreducible representation are actually analytic functions with respect to the natural analytic structure on $G_\mathbb{R}$. Consequently all functions in the finite direct sum $\bigoplus$ End V_λ are analytic as well and can thus be separated by their partial derivatives at the identity. In other words the second interpretation of the representation homomorphism is surjective simply because the matrix coefficients of an irreducible representation are analytic. □

Corollary A.2
The representation homomorphism $\mathcal{U}\,\mathfrak{g} \longrightarrow \mathrm{End}\ V$ for a finite–dimensional representation V of a complex semisimple Lie algebra $\mathfrak{g}$ restricts to a surjective homomorphism:

$$\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g} \;\longrightarrow\; \mathrm{Zent}\ \mathrm{End}_{\,\mathfrak{g}}V$$

Without giving a formal proof of this corollary we remark that the very definition of the commutant reads $\mathrm{Zent}\,\mathrm{End}_{\,\mathfrak{g}}V = \mathrm{End}_{\,\mathfrak{g}}V \cap \mathcal{A}_\mathfrak{g}V$ while $\mathcal{A}_\mathfrak{g}V$ is the image of the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}$ under the representation homomorphism by Lemma A.1. Consequently every element in $\mathrm{Zent}\,\mathrm{End}_{\,\mathfrak{g}}V$ is the image of some element of $\mathcal{U}\,\mathfrak{g}$. With the representation homomorphism being $\mathfrak{g}$–equivariant and $\mathfrak{g}$ semisimple we can actually choose a $\mathfrak{g}$–invariant preimage $\mathcal{Q} \in \mathcal{U}\,\mathfrak{g}$ for a $\mathfrak{g}$–invariant endomorphism or equivalently a preimage in the center $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}$. Thinking a little bit more about the proof of Lemma A.1 we can formulate a slightly different useful result about the structure of $\mathrm{Zent}\,\mathrm{End}_{\,\mathfrak{g}}V$:

Corollary A.3
The center $\mathrm{Zent}\ \mathrm{End}_{\,\mathfrak{g}}V$ *of the algebra* $\mathrm{End}_{\,\mathfrak{g}}V$ *of $\mathfrak{g}$–invariant endomorphisms on a finite–dimensional representation* $V = \bigoplus \mathrm{Hom}_{\,\mathfrak{g}}(\,V_\lambda,\,V\,)\otimes V_\lambda$ *of a complex semisimple Lie algebra $\mathfrak{g}$ is spanned by the $\mathfrak{g}$–invariant projections onto the isotypical components:*

$$\mathrm{Zent}\,\mathrm{End}_{\,\mathfrak{g}}V \;=\; \bigoplus_\lambda\ \mathbb{C}\ \mathrm{id}_{\mathrm{Hom}_{\,\mathfrak{g}}(V_\lambda,V)}\otimes\mathrm{id}_{V_\lambda}$$

For applications in the Weitzenböck machine the representation V will usually be a tensor product $T\otimes V_\lambda$ of an irreducible representation V_λ of highest weight λ defined over $\mathbb{C}$ with the complexified holonomy representation T of the holonomy group $G_\mathbb{R}$ with Lie algebra $\mathfrak{g}_\mathbb{R}$. In this case the algebra $\mathrm{End}_{\,g}(\,T\otimes V_\lambda\,) = \mathrm{Zent}\ \mathrm{End}_{\,\mathfrak{g}}(\,T\otimes V_\lambda\,)$ is commutative.

B Module Generators and Higher Casimirs

Remark B.1 (Module Generators for $\mathrm{Zent}\ \mathcal{U}\,\mathfrak{so}_{2r+1}$)
The center of the universal enveloping algebra of $\mathfrak{so}_{2r+1}$, $r \geq 1$, is a free polynomial algebra $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r+1} = \mathbb{C}[P^{[2]}, P^{[4]}, \ldots, P^{[2r]}]$ *in r generators of degree* $2,\,4,\,\ldots,\,2r$. *Moreover:*

$$\begin{aligned}
\mathrm{Hom}_{\,\mathfrak{so}_{2r+1}}(\quad\mathbb{C}\quad,\ \mathcal{U}\,\mathfrak{so}_{2r+1}\,) &\;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r+1}\\
\mathrm{Hom}_{\,\mathfrak{so}_{2r+1}}(\ \mathrm{Sym}^2_0T\ ,\ \mathcal{U}\,\mathfrak{so}_{2r+1}\,) &\;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r+1}\,\langle\,F_2,\ F_4,\ \ldots,\ F_{2r}\,\rangle\\
\mathrm{Hom}_{\,\mathfrak{so}_{2r+1}}(\ \mathfrak{so}_{2r+1}\ ,\ \mathcal{U}\,\mathfrak{so}_{2r+1}\,) &\;\cong\; \mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r+1}\,\langle\,F_1,\ F_3,\ \ldots,\ F_{2r-1}\,\rangle
\end{aligned}$$

Remark B.2 (Module Generators for $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r}$**)**
The center of the universal enveloping algebra $\mathcal{U}\,\mathfrak{so}_{2r}$ *of* $\mathfrak{so}_{2r}$, $r \geq 2$, *is a free polynomial algebra* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r} = \mathbb{C}[P^{[2]}, P^{[4]}, \ldots, P^{[2r-2]}, E^{[r]}]$ *in* $r-1$ *generators of degree* $2, 4, \ldots, 2r-2$ *respectively and one additional generator in degree* r*. Moreover:*

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{so}_{2r}}(\ \mathbb{C}\ , \mathcal{U}\,\mathfrak{so}_{2r}) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r} \\
\mathrm{Hom}_{\mathfrak{so}_{2r}}(\mathrm{Sym}^2_0 T, \mathcal{U}\,\mathfrak{so}_{2r}) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r}\,\langle F_2, F_4, \ldots, F_{2r-2}\rangle \\
\mathrm{Hom}_{\mathfrak{so}_{2r}}(\ \mathfrak{so}_{2r}\ , \mathcal{U}\,\mathfrak{so}_{2r}) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{so}_{2r}\,\langle F_1, F_3, \ldots, F_{2r-3}, G_{r-1}\rangle
\end{aligned}$$

Remark B.3 (Module Generators for $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2$**)**
The center of the universal enveloping algebra $\mathcal{U}\,\mathfrak{g}_2$ *of* $\mathfrak{g}_2$ *is a free polynomial algebra* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2 = \mathbb{C}[\mathrm{Cas}^{[2]}, \mathrm{Cas}^{[6]}]$ *in two generators of degree* 2 *and* 6*. Moreover:*

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}_2}(\ \mathbb{C}\ , \mathcal{U}\,\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2 \\
\mathrm{Hom}_{\mathfrak{g}_2}(\mathrm{Sym}^2_0 T, \mathcal{U}\,\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2\,\langle F_2, F_4, F_6\rangle \\
\mathrm{Hom}_{\mathfrak{g}_2}(\ \mathfrak{g}_2\ , \mathcal{U}\,\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2\,\langle F_1, F_5\rangle \\
\mathrm{Hom}_{\mathfrak{g}_2}(\ \mathfrak{g}_2^\perp\ , \mathcal{U}\,\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2\,\langle G_3\rangle
\end{aligned}$$

Remark B.4 (Module Generators for $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7$**)**
The center of the universal enveloping algebra $\mathcal{U}\,\mathfrak{spin}_7$ *of* $\mathfrak{spin}_7$ *is a free polynomial algebra* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7 = \mathbb{C}[\mathrm{Cas}^{[2]}, \mathrm{Cas}^{[4]}, \mathrm{Cas}^{[6]}]$ *in three generators of degree* 2, 4 *and* 6*. Moreover:*

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{spin}_7}(\ \mathbb{C}\ , \mathcal{U}\,\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7 \\
\mathrm{Hom}_{\mathfrak{spin}_7}(\mathrm{Sym}^2_0 T, \mathcal{U}\,\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7\,\langle F_2, F_4, F_6\rangle \\
\mathrm{Hom}_{\mathfrak{spin}_7}(\ \mathfrak{spin}_7\ , \mathcal{U}\,\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7\,\langle F_1, F_3, F_5\rangle \\
\mathrm{Hom}_{\mathfrak{spin}_7}(\ \mathfrak{spin}_7^\perp\ , \mathcal{U}\,\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7\,\langle G_3\rangle
\end{aligned}$$

Remark B.5 (Higher Casimirs for $\mathbf{G}_2$**)**
The eigenvalues of the generators $\mathrm{Cas}^{[2]}$, $\mathrm{Cas}^{[6]}$ *of* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{g}_2$ *of degrees 2 and 6 respectively on the irreducible representation* V_λ *of highest weight* $\lambda = a\omega_1 + b\omega_2$ *are given by:*

$$\begin{aligned}
\frac{3}{4}\,\mathrm{Cas}^{[2]}_{V_\lambda} &= a^2 + 3\,ab + 3\,b^2 + 5\,a + 9\,b \\
\frac{243}{11}\,\mathrm{Cas}^{[6]}_{V_\lambda} &= 4\,a^6 + 36\,a^5b + 117\,a^4b^2 + 162\,a^3b^3 + 81\,a^2b^4 \\
&\quad + 60\,a^5 + 414\,a^4b + 954\,a^3b^2 + 810\,a^2b^3 + 162\,ab^4 \\
&\quad - 408\,a^4 - 2808\,a^3b - 8829\,a^2b^2 - 12636\,ab^3 - 6804\,b^4 \\
&\quad - 6580\,a^3 - 33174\,a^2b - 61362\,ab^2 - 40824\,b^3 \\
&\quad - 6396\,a^2 - 32508\,ab - 27756b^2 \\
&\quad + 56520\,a + 100440\,b
\end{aligned}$$

Remark B.6 (Higher Casimirs for Spin(7)**)**
The eigenvalues of the generators $\mathrm{Cas}^{[2]}$, $\mathrm{Cas}^{[4]}$ *and* $\mathrm{Cas}^{[6]}$ *of* $\mathrm{Zent}\,\mathcal{U}\,\mathfrak{spin}_7$ *of degrees* 2, 4

and 6 *on the irreducible representation* V_λ *of highest weight* $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ *are:*

$$2\,\mathrm{Cas}^{[2]}_{V_\lambda} = 4\,a^2 + 8\,b^2 + 3\,c^2 + 8\,ab + 4\,ac + 8\,bc + 20\,a + 32\,b + 18\,c$$

$$\begin{aligned}
32\,\mathrm{Cas}^{[4]}_{V_\lambda} = {} & 16\,a^4 + 128\,b^4 + 21\,c^4 + 192\,a^2b^2 + 72\,a^2c^2 + 240\,b^2c^2 \\
& + 32\,a^3c + 64\,a^3b + 256\,b^3c + 256\,b^3a + 56\,c^3a + 112\,c^3b \\
& + 192\,a^2bc + 384\,b^2ac + 240\,c^2ab \\
& + 160\,a^3 + 1024\,b^3 + 252\,c^3 + 768\,a^2b + 432\,a^2c + 1536\,b^2a \\
& + 1632\,b^2c + 1056\,c^2b + 552\,c^2a + 1632\,abc \\
& + 800\,a^2 + 1152\,c^2 + 3040\,b^2 + 3040\,ab + 1760\,ac + 3424\,bc \\
& + 2000\,a + 3968\,b + 2376\,c
\end{aligned}$$

$$\begin{aligned}
512\,\mathrm{Cas}^{[6]}_{V_\lambda} = {} & 64\,a^6 + 2048\,b^6 + 183\,c^6 + 384\,a^5b + 192\,a^5c + 6144\,b^5c \\
& + 6144\,b^5a + 732\,c^5a + 1464\,c^5b + 1920\,a^4b^2 + 720\,a^4c^2 + 7680\,b^4a^2 \\
& + 9600\,b^4c^2 + 1260\,c^4a^2 + 4920\,c^4b^2 + 1920\,a^4bc + 15360\,b^4ac \\
& + 4920\,c^4ab + 5120\,a^3b^3 + 1120\,a^3c^3 + 8960\,b^3c^3 + 7680\,a^3b^2c \\
& + 4800\,a^3c^2b + 15360\,b^3a^2c + 19200\,b^3c^2a + 6720\,c^3a^2b + 13440\,c^3b^2a \\
& + 14400\,a^2b^2c^2 \\
& + 960\,a^5 + 24576\,b^5 + 3294\,c^5 + 7680\,a^4b + 4320\,a^4c + 61440\,b^4a \\
& + 65280\,b^4c + 11100\,c^4a + 22080\,c^4b + 30720\,a^3b^2 + 11040\,a^3c^2 \\
& + 61440\,b^3a^2 + 84480\,b^3c^2 + 15120\,c^3a^2 + 60000\,c^3b^2 + 32640\,a^3bc \\
& + 130560\,b^3ac + 60000\,c^3ab + 97920\,a^2b^2c + 63360\,a^2c^2b \\
& + 126720\,b^2c^2a \\
& + 9600\,a^4 + 167424\,b^4 + 32592\,c^4 + 67200\,a^3b + 38400\,a^3c \\
& + 334848\,b^3a + 365568\,b^3c + 88032\,c^3a + 175584\,c^3b + 234624\,a^2b^2 \\
& + 92832\,a^2c^2 + 364128\,b^2c^2 + 257664\,a^2bc + 548352\,b^2ac \\
& + 364128\,c^2ab \\
& + 56000\,a^3 + 684032\,b^3 + 193464\,c^3 + 413952\,a^2b + 251808\,a^2c \\
& + 993024\,b^2a + 1158912\,b^2c + 397968\,c^2a + 790656\,c^2b + 1125888\,abc \\
& + 160000\,a^2 + 1321856\,b^2 + 562848\,c^2 + 1189760\,ab + 759040\,ac \\
& + 1607552\,bc + 200000\,a + 863744\,b + 606240\,c
\end{aligned}$$

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