TWISTOR FORMS ON RIEMANNIAN PRODUCTS

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Abstract. We study twistor forms on products of compact Riemannian manifolds and show that they are defined by Killing forms on the factors. The main result of this note is a necessary step in the classification of compact Riemannian manifolds with non-generic holonomy carrying twistor forms.

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1. Twistor Forms on Riemannian Manifolds

A *twistor p-form* on a Riemannian manifold $(M^n,\, g)$ is a smooth section ψ of $\Lambda^p T^*M$ whose covariant derivative only depends on its differential $d\psi$ and codifferential $\delta\psi$. More precisely, ψ satisfies the equation

$$\nabla_X \psi \;=\; \tfrac{1}{p+1} X \lrcorner\, d\psi - \tfrac{1}{n-p+1} X^\flat \wedge \delta\psi, \tag{1}$$

for all vector fields X, where $X^\flat$ denotes the metric dual of X.

If the p-form ψ is in addition coclosed (*i.e.* $\delta\psi = 0$), then it is called a *Killing p-form*. We denote by $\mathfrak{T}(M)$, $\mathfrak{K}(M)$ and $\mathfrak{P}(M)$ the spaces of twistor, Killing and parallel forms on M respectively. Notice that $\mathfrak{T}(M)$ is preserved by Hodge duality, and that the Hodge dual of a Killing form is a closed twistor form. For a comprehensive introduction to twistor forms, see [8].

A few years ago, a program of classification of twistor forms on compact manifolds was started. By the de Rham decomposition theorem, every simply connected Riemannian manifold is a Riemannian product of *irreducible* manifolds. Moreover, the Berger-Simons holonomy theorem (see [2], p. 300) implies that any simply connected irreducible Riemannian manifold is either symmetric or has holonomy $\mathrm{SO_n}$, $\mathrm{U_m}$, $\mathrm{SU_m}$, $\mathrm{Sp_k}$, $Sp_k \cdot \mathrm{Sp_1}$, G_2 or $Spin_7$. Killing forms on symmetric spaces were studied in [1]. Twistor forms on Kähler manifolds (covering the holonomies $\mathrm{U_m}$, $\mathrm{SU_m}$, and $\mathrm{Sp_k}$) were described in [4], and Killing forms on quaternion-Kähler manifolds (holonomy $Sp_k \cdot \mathrm{Sp_1}$) or Joyce manifolds (holonomies G_2 or $Spin_7$) were studied in [5] and [9] respectively. In Theorem 2.1 below, we prove that the general case (twistor forms on a Riemannian product of compact manifolds) reduces to the study of Killing forms on the factors. By the discussion above, besides the case of generic holonomy ($\mathrm{SO_n}$), all other cases are fully understood.

2. The Main Result

Let $M = M_1 \times M_2$ be the Riemannian product of two compact Riemannian manifolds (M_1, g_1) and (M_2, g_2) of dimensions m and n respectively. We denote by π_i the projection $\pi_i : M \to M_i$. From (1) it is clear that $\pi_i^*(\mathfrak{K}(M_i)) \subset \mathfrak{K}(M)$, so the space

$$\mathfrak{K}_0(M) := \pi_1^*(\mathfrak{K}(M_1)) + \pi_2^*(\mathfrak{K}(M_2)) + \mathfrak{P}(M)$$

is a subspace of $\mathfrak{K}(M)$. For later use, we give the following description of $\pi_i^*(\mathfrak{K}(M_i))$:

$$\pi_1^*(\mathfrak{K}(M_1)) = \{u \in \mathfrak{K}(M) \mid \nabla_X u = 0, \ \forall X \in TM_2\} \tag{2}$$

and

$$\pi_2^*(\mathfrak{K}(M_2)) = \{u \in \mathfrak{K}(M) \mid \nabla_X u = 0, \ \forall X \in TM_1\}. \tag{3}$$

The aim of this note is to prove the following result:

Theorem 2.1. *Every twistor form on M is a sum of forms of the following types: parallel forms, pull-backs of Killing forms on M_1 or M_2, and Hodge duals of them. In other words, $\mathfrak{T}(M) = \mathfrak{K}_0(M) + *\mathfrak{K}_0(M)$.*

Proof. Since $\mathfrak{K}_0(M) \subset \mathfrak{K}(M) \subset \mathfrak{T}(M)$ and $*\mathfrak{T}(M) = \mathfrak{T}(M)$, we clearly have $\mathfrak{K}_0(M) + *\mathfrak{K}_0(M) \subset \mathfrak{T}(M)$. It remains to prove the reverse inclusion. Let us define the differential operators

$$d_1 = \sum_{i=1}^{m} e_i^\flat \wedge \nabla_{e_i}, \qquad d_2 = \sum_{j=1}^{n} f_j^\flat \wedge \nabla_{f_j},$$

where $\{e_i\}$ and $\{f_j\}$ denote local orthonormal basis of the tangent distributions to M_1 and M_2. Using the Fubini theorem, we easily see that the adjoint operators to d_1 and d_2 are

$$\delta_1 = -\sum_{i=1}^{m} e_i \lrcorner \nabla_{e_i}, \qquad \delta_2 = -\sum_{j=1}^{n} f_j \lrcorner \nabla_{f_j}.$$

The following relations are straightforward:

$$d^M = d_1 + d_2, \qquad \delta^M = \delta_1 + \delta_2, \qquad (d_1)^2 = (d_2)^2 = (\delta_1)^2 = (\delta_2)^2 = 0,$$

$$0 = d_1 d_2 + d_2 d_1 = \delta_1 \delta_2 + \delta_2 \delta_1, \qquad 0 = d_1 \delta_2 + \delta_2 d_1 = \delta_1 d_2 + d_2 \delta_1.$$

The vector bundle $\Lambda^p M$ decomposes naturally as

$$\Lambda^p M \cong \oplus_{i=0}^{p} \Lambda^{i,p-i} M,$$

where $\Lambda^{i,p-i}M \cong \Lambda^i M_1 \otimes \Lambda^{p-i} M_2$. Obviously, d_1 and δ_1 map $\Lambda^{i,p-i}M$ to $\Lambda^{i+1,p-i}M$ and $\Lambda^{i-1,p-i}M$ respectively, and d_2 and δ_2 map $\Lambda^{i,p-i}M$ to $\Lambda^{i,p-i+1}M$ and $\Lambda^{i,p-i-1}M$ respectively.

With respect to the above decomposition, every p-form can be written $u = u_0 + \ldots + u_p$, where $u_i \in \Lambda^i M_1 \otimes \Lambda^{p-i} M_2$. From now on, u will denote a twistor p-form $u \in \mathfrak{T}(M)$, with $1 \leq p \leq n+m-1$. The twistor equation reads

$$\nabla_X u = \frac{1}{p+1} X \lrcorner (d_1 u + d_2 u) - \frac{1}{m+n-p+1} X \wedge (\delta_1 u + \delta_2 u), \quad \forall X \in TM. \tag{4}$$

By projection onto the different irreducible components of $\Lambda^p M$, (4) can be translated into the following two systems of equations:

$$\nabla_X u_k = \frac{1}{p+1} X \lrcorner (d_1 u_k + d_2 u_{k+1}) - \frac{1}{m+n-p+1} X \wedge (\delta_1 u_k + \delta_2 u_{k-1}), \quad \forall X \in TM_1, \tag{5}$$

and

$$\nabla_X u_k = \frac{1}{p+1} X \lrcorner (d_1 u_{k-1} + d_2 u_k) - \frac{1}{m+n-p+1} X \wedge (\delta_1 u_{k+1} + \delta_2 u_k), \quad \forall X \in TM_2. \tag{6}$$

Recall that if u is any k-form and $\{e_1, \ldots, e_m\}$ is an orthonormal basis on a manifold M, then

$$\sum_{i=1}^{m} e_i^\flat \wedge e_i \lrcorner \omega = k\omega. \tag{7}$$

Taking the wedge product with $X^\flat$ in (5) and summing over an orthonormal basis of TM_1 yields

$$d_1 u_k = \sum_{i=1}^{m} e_i \wedge \nabla_{e_i} u_k = \frac{1}{p+1} \sum_{i=1}^{m} e_i \wedge e_i \lrcorner (d_1 u_k + d_2 u_{k+1}) \overset{(7)}{=} \frac{k+1}{p+1} (d_1 u_k + d_2 u_{k+1})$$

so

$$(p-k) d_1 u_k = (k+1) d_2 u_{k+1}. \tag{8}$$

Similarly, taking the interior product with X and summing over an orthonormal basis of TM_1 yields $\delta_1 u_k = \frac{m-k+1}{m+n-p+1}(\delta_1 u_k + \delta_2 u_{k-1})$, thus

$$(n+k-p)\delta_1 u_k = (m-k+1)\delta_2 u_{k-1}. \tag{9}$$

We distinguish three cases:

Case I. Suppose that p is strictly smaller than m and n. For $k < p$, (8) and (9) imply

$$\delta_1 d_1 u_k = \frac{k+1}{p-k} \delta_1 d_2 u_{k+1} = -\frac{k+1}{p-k} d_2 \delta_1 u_{k+1} = -\frac{(k+1)(m-k)}{(p-k)(n+k-p+1)} d_2 \delta_2 u_k. \tag{10}$$

Integrating over M yields $0 = d_1 u_k = \delta_2 u_k$, $\forall k < p$. Similarly one gets $0 = d_2 u_k = \delta_1 u_k$, $\forall k > 0$. Moreover, we have $0 = \delta_2 u_p = \delta_1 u_0$ (tautologically), so in particular $\delta_1 u_k = \delta_2 u_k = 0$, $\forall k$. From (5) and (6), together with (2) and (3), we see that $u_1, \ldots, u_{p-1} \in \mathfrak{P}(M)$, $u_0 \in \pi_2^*(\mathfrak{K}(M_2))$ and $u_p \in \pi_1^*(\mathfrak{K}(M_1))$, so $u \in \mathfrak{K}_0(M)$.

Case II. Suppose that p is strictly larger than m and n. Since the Hodge dual $*u$ of u is a twistor $(m+n-p)$-form and $m+n-p$ is strictly smaller than m and n, the first case implies that $*u \in \mathfrak{K}_0(M)$, so $u \in *\mathfrak{K}_0(M)$.

Case III. If p is a number between m and n, we may suppose without loss of generality that $m \leq p \leq n$. Obviously $u_{m+1} = \ldots = u_p = 0$. Using (10) and integrating over M, we obtain that $0 = d_1 u_k = \delta_2 u_k$ for $0 \le k \le m-1$ and similarly, $0 = d_2 u_k = \delta_1 u_k$ for $1 \le k \le m$. As before, (5) and (6), together with (2) and (3), show that $u_1, \ldots, u_{m-1} \in \mathfrak{P}(M)$, $u_0 \in \pi_2^*(\mathfrak{K}(M_2))$, and $*u_m \in \pi_2^*(\mathfrak{K}(M_2))$. This proves the theorem.

□

As an application of this result, we have the following:

Proposition 2.2. *Let (M^n, g) be a compact simply connected Riemannian manifold. If M carries a conformal vector field which is not Killing, then* $\mathrm{Hol}(M) = \mathrm{SO_n}$.

Proof. Assume first that $(M, g) = (M_1, g_1) \times (M_2, g_2)$ is a Riemannian product with $\dim(M_1)$, $\dim(M_2) \ge 1$. Then, taking into account that the isomorphism between 1-forms and vector fields defined by the Riemannian metric maps twistor forms to conformal vector fields and Killing forms to Killing vector fields, Theorem 2.1 implies that every conformal vector field on M is a Killing vector field. Thus M is irreducible.

Assume next that $\mathrm{Hol}(M) \neq \mathrm{SO_n}$. From the Berger-Simons holonomy theorem ([2], p. 300), M is either an irreducible symmetric space (in particular Einstein), or its holonomy group is $\mathrm{U_m}$, $\mathrm{SU_m}$, $\mathrm{Sp_k}$, $Sp_k \cdot \mathrm{Sp_1}$, G_2 or $Spin_7$. In the first three cases the manifold is Kähler and in the last three cases it is Einstein. Now, two classical results state that a conformal vector field on a compact manifold M is already a Killing vector field if M is Kähler (see [3], p. 148) or if M is Einstein and not isometric to the round sphere (see [6], [7]).

The only possibility left is therefore $\mathrm{Hol}(M) = \mathrm{SO_n}$.

□

Example. Take any compact simply connected Riemannian manifold (M^n, g) carrying a Killing vector field ξ and let f be a function on M such that $\xi(f)$ is not identically zero. Since $\mathcal{L}_\xi(e^{2f}g) = 2\xi(f)e^{2f}g$, ξ is a conformal vector field on $(M, e^{2f}g)$ which is not Killing. From Proposition 2.2, $(M, e^{2f}g)$ has holonomy $\mathrm{SO_n}$.

Corollary 2.3. *Let (M^n, g) be a compact simply connected homogeneous Riemannian manifold. Then for every non-constant function f on M, $(M, e^{2f}g)$ has holonomy* $\mathrm{SO_n}$.

Proof. Since f is non-constant, there exists $x \in M$ such that $df_x \neq 0$. Killing vector fields on M span the tangent spaces at each point, so in particular there exist a Killing vector field ξ such that $\xi(f)$ is not identically zero. The corollary then follows from the example above. □

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