KILLING FORMS ON $\mathbf{G}_2$– AND $\mathbf{Spin}_7$–MANIFOLDS

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Abstract. Killing forms on Riemannian manifolds are differential forms whose covariant derivative is totally skew-symmetric. We prove that on a compact manifold with holonomy G_2 or $Spin_7$ any Killing form has to be parallel. The main tool is a universal Weitzenböck formula. We show how such a formula can be obtain for any given holonomy group and any representation defining a vector bundle.

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1. Introduction

Killing forms are a natural generalization of Killing vector fields. They are defined as differential forms u, such that ∇u is totally skew-symmetric. More generally one considers twistor forms, as forms in the kernel of an elliptic differential operator, defined similar to the twistor operator in spin geometry. Twistor 1-forms are dual to conformal vector fields.

The notion of Killing forms was introduced by K. Yano in [15], where he already noted that a p–form u is a Killing form if and only if for any geodesic γ the $(p-1)$–form $\dot{\gamma} \lrcorner u$ is parallel along γ. In particular, Killing forms define quadratic first integrals of the geodesic equation, i.e. functions which are constant along geodesics. This motivated an intense study of Killing forms in the physics literature, e.g. in the article [11] of R. Penrose and M. Walker. More recently Killing and twistor forms have been successfully applied to define symmetries of field equations (c.f. [3], [4]).

On the standard sphere the space of twistor forms coincides with the eigenspace of the Laplace operator for the minimal eigenvalue and Killing forms are the coclosed minimal eigenforms. The sphere also realizes the maximal possible number of twistor or Killing forms. There are only very few further examples of compact manifolds admitting Killing p-forms with $p \geq 2$. These are Sasakian, nearly Kähler and weak-G_2 manifolds, and products of them.

The present article is the last step in the study of Killing forms on manifolds with restricted holonomy. It was already known that on compact Kähler manifolds Killing p-forms with $p \geq 2$ are parallel ([14]). Moreover we showed in [8] and [2] that the same is true on compact quaternion-Kähler manifolds and compact symmetric spaces. Here we will prove the corresponding statement for the remaining holonomies $\mathbf{G}_2$ and $\mathbf{Spin}_7$.

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Theorem 1.1. *Let $(M^7,\, g)$ be a compact manifold with holonomy $\mathbf{G}_2$. Then any Killing form and any $*$–Killing form is parallel. Moreover, any twistor p–form, with $p \neq 3, 4$, is parallel.*

Theorem 1.2. *Let $(M^8,\, g)$ be a compact manifold with holonomy $\mathbf{Spin}_7$. Then any Killing form and any $*$–Killing form is parallel. Moreover, any twistor p-form, with $p \neq 3, 4, 5$, is parallel.*

The main tool for proving the two theorems are suitable Weitzenböck formulas for the irreducible components of the form bundle. More generally we prove a universal Weitzenböck formula, i.e. we show how to obtain for any fixed holonomy group G and any irreducible G-representation π, a Weitzenböck formula for the twistor operators acting on sections of the vector bundle defined by π. Our formula is already known in the case of Riemannian holonomy $\mathbf{SO}_n$ (c.f. [6]). However, it seems to be new and so far unused in the case of the exceptional holonomies $\mathbf{G}_2$ and $\mathbf{Spin}_7$. We describe here an approach to Weitzenböck formulas which is further developed and completed in [13].

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2. Twistor forms on Riemannian manifolds

In this section we recall the definition and basic facts on twistor and Killing forms. More details and further references can be found in [12]. Most important for the later application will be the integrability condition given in Proposition 2.2.

Consider a n–dimensional Euclidean vector space $(V,\, \langle\cdot,\cdot\rangle)$. Then the tensor product $V^*\otimes\Lambda^pV^*$ has the following $O(n)$–invariant decomposition:

$$V^*\otimes\Lambda^pV^* \;\cong\; \Lambda^{p-1}V^*\oplus\Lambda^{p+1}V^*\oplus\Lambda^{p,1}V^*$$

where $\Lambda^{p,1}V^*$ is the intersection of the kernels of wedge and inner product. This decomposition immediately translates to Riemannian manifolds $(M^n,\, g)$, where we have

$$T^*M\otimes\Lambda^pT^*M \;\cong\; \Lambda^{p-1}T^*M\oplus\Lambda^{p+1}T^*M\oplus\Lambda^{p,1}T^*M \qquad (1)$$

with $\Lambda^{p,1}T^*M$ denoting the vector bundle corresponding to the representation $\Lambda^{p,1}$. The covariant derivative $\nabla\psi$ of a p–form ψ is a section of $T^*M\otimes\Lambda^pT^*M$. Its projections onto the summands $\Lambda^{p+1}T^*M$ and $\Lambda^{p-1}T^*M$ are just the differential $d\psi$ and the codifferential $d^*\psi$. Its projection onto the third summand $\Lambda^{p,1}T^*M$ defines a natural first order differential operator T, called the *twistor operator*. The twistor operator $T:\Gamma(\Lambda^pT^*M)\to\Gamma(\Lambda^{p,1}T^*M)\subset\Gamma(T^*M\otimes\Lambda^pT^*M)$ is given for any vector field X by the following formula

$$[\,T\psi\,](X) \;:=\; [\mathrm{pr}_{\Lambda^{p,1}}(\nabla\psi)](X) \;=\; \nabla_X\,\psi \;-\; \tfrac{1}{p+1}\,X\lrcorner\,d\psi \;+\; \tfrac{1}{n-p+1}\,X^*\wedge d^*\psi\,. \qquad (2)$$

From now on we will identify TM with T^*M using the metric.

Definition 2.1. *A p–form ψ is called a twistor p–form if and only if ψ is in the kernel of T, i.e. if and only if ψ satisfies*

$$\nabla_X \psi \;=\; \tfrac{1}{p+1}\, X \lrcorner\, d\psi \;-\; \tfrac{1}{n-p+1}\, X \wedge d^*\psi \;, \tag{3}$$

for all vector fields X. If the p–form ψ is in addition coclosed, it is called a Killing p–form. A closed twistor form is called $*$-Killing form.

Twistor forms are also known as *conformal Killing forms* or skew-symmetric *Killing-Yano tensors*. Twistor 1-forms are dual to conformal vector fields and Killing 1-forms are dual to Killing vector fields. Note that the Hodge star-operator $*$ maps twistor p–forms into twistor $(n-p)$–forms. In particular, it interchanges Killing and $*$-Killing forms.

Twistor forms are well understood on compact Kähler manifolds (c.f. [8]). Here they are closely related to Hamiltonian 2-forms recently studied in [1]. In particular, one has examples on the complex projective space in any even degree.

Differentiating Equation (2) one obtains the two equations

$$\nabla^*\nabla\psi \;=\; \tfrac{1}{p+1}\, d^*d\,\psi \;+\; \tfrac{1}{n-p+1}\, dd^*\,\psi \;+\; T^*T\,\psi \;, \tag{4}$$

$$q(R)\,\psi \;=\; \tfrac{p}{p+1}\, d^*d\,\psi \;+\; \tfrac{n-p}{n-p+1}\, dd^*\,\psi \;-\; T^*T\,\psi \;, \tag{5}$$

where $q(R)$ is the curvature term appearing in the classical Weitzenböck formula for the Laplacian on p–forms: $\Delta = d^*d + dd^* = \nabla^*\nabla + q(R)$. It is the symmetric endomorphism of the bundle of differential forms defined by

$$q(R) \;=\; \sum e_j \wedge e_i \lrcorner\, R_{e_i,e_j}, \tag{6}$$

where $\{e_i\}$ is any local orthonormal frame and R_{e_i,e_j} denotes the curvature of the form bundle. On 1-forms the endomorphism $q(R)$ is just the Ricci curvature. It is important to note that one may define $q(R)$ also in a more general context. For this we first rewrite equation (6) as

$$q(R) \;=\; \sum_{i<j} (e_j \wedge e_i \lrcorner \;-\; e_i \wedge e_j \lrcorner)\, R_{e_i,e_j} \;=\; \sum_{i<j} (e_i \wedge e_j)\, R(e_i \wedge e_j)$$

where the Riemannian curvature R is considered as element of $\mathrm{Sym}^2(\Lambda^2 TM)$ and 2-forms act via the standard representation of the Lie algebra $\mathfrak{so}(T_mM) \cong \Lambda^2 T_mM$ on the space of p–forms. Note that we can replace $\{e_i \wedge e_j\}$ by any basis of $\mathfrak{so}(T_mM)$ orthonormal with respect to the scalar product induced by g on $\mathfrak{so}(T_mM) \cong \Lambda^2 T_mM$.

Let (M, g) be a Riemannian manifold with holonomy group $G = \mathrm{Hol}$. Then the curvature tensor takes values in the Lie algebra $\mathfrak{g}$ of the holonomy group and we can write $q(R)$ as

$$q(R) \;=\; \sum X_i\, R(X_i) \qquad \in\; \mathrm{Sym}^2(\mathfrak{g})$$

where $\{X_i\}$ is any orthonormal basis of $\mathfrak{g}$ acting via the form representation restricted to the holonomy group. It is clear that in this way $q(R)$ gives rise to a symmetric endomorphism on any associated vector bundle defined via a representation of the holonomy group. Moreover this bundle endomorphism preserves any parallel subbundle and its action only depends on the representation defining the subbundle and not on the particular realization.

Integrating Equation (5) yields a characterization of twistor forms on compact manifolds. This generalizes the characterization of Killing vector fields on compact manifolds, as divergence free vector fields in the kernel of $\Delta - 2\mathrm{Ric}$.

Proposition 2.2. *Let $(M^n,\, g)$ a compact Riemannian manifold. Then a p–form ψ is a twistor p–form, if and only if* $q(R)\,\psi = \frac{p}{p+1}\, d^* d\,\psi + \frac{n-p}{n-p+1}\, dd^*\,\psi$ *. A coclosed p–form ψ a Killing form if and only if* $\nabla^*\nabla\psi = \frac{1}{p} q(R)\psi$ *.*

One has similar characterizations for closed twistor forms and for twistor m–forms on 2m–dimensional manifolds. For the later application in the case of compact Ricci-flat manifolds we still mention an immediate consequence of Equation (5).

Corollary 2.3. *Let M be a compact manifold and let $EM \subset \Lambda^p T^* M$ be a parallel subbundle such that $q(R)$ acts trivially on E. Then any twistor and any harmonic form in EM has to be parallel. In particular, there are only parallel twistor resp. harmonic forms in Λ^k_7 for $1 \le k \le 6$ on a compact $\mathbf{G}_2$-manifolds and there are only parallel twistor resp. harmonic forms in Λ^k_7 for $k = 2, 4, 6$ and in Λ^k_8 for $k = 1, 3, 4, 5, 7$ on a compact $\mathbf{Spin}_7$-manifold.*

Proof: It remains to prove that $q(R)$ acts trivially on the given bundles. This is clear for the 7-dimensional bundles in the $\mathbf{G}_2$ case and for the 8-dimensional bundles in the $\mathbf{Spin}_7$-case. Indeed, these bundles are defined via the holonomy representation, where $q(R)$ acts as Ricci-curvature, which is zero in the case of holonomy $\mathbf{G}_2$ or $\mathbf{Spin}_7$.

In the $\mathbf{Spin}_7$-case there is still an other argument for proving the trivial action of $q(R)$ on the 8- and 7-dimensional part: The spinor bundle of a manifold with $\mathbf{Spin}_7$–holonomy splits into the sum of a trivial line bundle, corresponding to the parallel spinor, and the sum of a bundle of rank 7 and a bundle of rank 8. These two bundles are induced by the 8-dimensional holonomy representation and by the 7-dimensional standard representation. It is well-known that $q(R)$ acts as $\frac{s}{16}\mathrm{id}$ on the summands of the spinor bundle. But for $\mathbf{Spin}_7$–manifolds the scalar curvature s is zero and we conclude that $q(R)$ acts trivially on the bundles in question. □

3. A universal Weitzenböck formula

In this section we derive for any manifold with a fixed holonomy one basic Weitzenböck formula. The coefficients of this formula will depend on the holonomy group and the defining representation. This is only the first step of a more general method of producing all possible Weitzenböck formulas on such manifolds (c.f. [13]).

We consider the following situation: let (M^n, g) be an oriented Riemannian manifold with holonomy group $G := \mathrm{Hol}(M, g) \subset \mathbf{SO}(n)$. Then the $\mathbf{SO}(n)$-frame bundle reduces to a G-principal bundle $P_G \to M$ and all natural bundles over M are associated to P_G via representations of G.

If $\pi : G \to \mathrm{Aut}(E)$ is a complex representation of G we denote with EM the corresponding associated bundle over M. In particular, we will denote the complexified holonomy representation of G, given by the inclusion $G \subset \mathbf{SO}(n)$, with T. The associated vector bundle is then of course the complexified tangent bundle TM.

The Levi-Civita connection of (M, g) induces a connection ∇ on any bundle EM and the covariant derivative of a section of EM is a section of $TM \otimes EM$. Hence, we may define natural first order differential operators by composing the covariant derivative with projections onto the components of $TM \otimes EM$. These operators are also known as *Stein-Weiss operators*.

Let $T \otimes E = \sum E_i$ be the decomposition of $T \otimes E$ into irreducible G-representations, where we consider E_i as a subspace of $T \otimes E$. This induces a corresponding decomposition of the tensor product $TM \otimes EM$. For any component $E_i\, M$ we define a *twistor operator* T_i by

$$T_i\, : \Gamma(EM) \to \Gamma(E_i\, M), \qquad T_i\,(\psi) := \mathrm{pr_i}\,(\nabla\psi)\ ,$$

where $\mathrm{pr_i}$ denotes the projection $T \otimes E \to E_i\ \subset T \otimes E$ and the corresponding bundle map. In the following we will make no difference between representations resp. equivariant maps and the corresponding vector bundles resp. bundle maps.

Since we are on a Riemannian manifold we have for any twistor operator T_i its formal adjoint $T_i^*\, : \Gamma(E_i\, M) \to \Gamma(EM)$. The aim of this section is to derive a Weitzenböck formula for the second order operators $T_i^* \circ T_i$, i.e. a linear combination $\sum_i\ c_i\ T_i^* \circ T_i$ with real numbers c_i, which is of zero order, i.e. a curvature term. The coefficients c_i will depend on the holonomy group G and the representation E.

Our approach to Weitzenböck formulas, further developed in [13], is motivated by the following remarks. Let ψ be any section of EM, then $\nabla^2\psi$ is a section of the bundle $TM \otimes TM \otimes EM$. Any G-equivariant homomorphism $F \in \mathrm{Hom}_G(T \otimes T \otimes E, E)$ defines by $\psi \mapsto F(\nabla^2\psi)$ a second order differential operator acting on sections of EM. For describing these homomorphisms it is rather helpful to use the natural identification

$$\mathrm{Hom}_G(T \otimes T \otimes E, E)\ \cong\ \mathrm{End}_G(T \otimes E)\ \cong\ \mathrm{Hom}_G(T \otimes T, \mathrm{End}E)\ .$$

A homomorphisms $F : T \otimes T \to \mathrm{End}E$ is mapped onto the endomorphism F of $T \otimes E$ defined by $F(a \otimes s) = \sum e_i \otimes F_{e_i \otimes a}(s)$, for any orthonormal basis $\{e_i\}$ of T and any $a \in T, s \in E$. Conversely an endomorphism F is mapped to the homomorphism F with $F_{a \otimes b}(s) = a \lrcorner F(b \otimes s)$. In particular, the identity of $T \otimes E$ is mapped onto $\mathrm{id}_{a\otimes b} = g(a,\, b)\, \mathrm{id}_E$. Finally, $F \in \mathrm{Hom}\,(T \otimes T \otimes E, E)$ is defined as $F(a \otimes b \otimes s) = F_{a\otimes b}s$.

Beside the identity $\mathrm{id}_{T \otimes E}$ we have the projections $\mathrm{pr_i} : \mathrm{T} \otimes \mathrm{E} \to \mathrm{E_i} \subset \mathrm{T} \otimes \mathrm{E}$ as important examples of invariant endomorphisms. The following proposition describes the corresponding second order differential operators.

Proposition 3.1. *Let $T \otimes E = \sum E_i$ the decomposition into irreducible summands, with corresponding twistor operators T_i . Then any section ψ of EM satisfies*

$$\begin{aligned} (1) \qquad \mathrm{id}(\nabla^2\psi) &= -\nabla^*\nabla\psi \\ (2) \qquad \mathrm{pr_i}\,(\nabla^2\psi) &= -\mathrm{T_i^*}\ \mathrm{T_i}\,(\psi) \end{aligned}$$

Proof: Let $\{e_i\}$ be a parallel local ortho-normal frame of TM, then $\nabla^2 = \sum e_i \otimes e_j \otimes \nabla_{e_i}\nabla_{e_j}$ and $\mathrm{id}(\nabla^2\psi) = \sum g(e_i, e_j)\nabla_{e_i}\nabla_{e_j}\psi = -\nabla^*\nabla\psi$, which proves Equation (1).

Equation (2) is a direct consequence of the following more general statement.

Lemma 3.2. *Let $(M,\, g)$ be a Riemannian manifold and let E, F be hermitian vector bundles over M, equipped with metric connections ∇. If $D : \Gamma(E) \to \Gamma(F)$ is a differential operator defined as $D = p \circ \nabla$, where $p : TM \otimes E \to F$ is some parallel linear map. Then the adjoint operator for D is $D^* = \nabla^* \circ p^*$ and $D^*D = -\,\mathrm{tr} \circ (\mathrm{id} \otimes p^*p)\,\nabla^2$.*

The second equation follows with $F = TM \otimes EM$ and an orthogonal projection $p = \mathrm{pr_i}$ onto a subbundle $EM_i \subset TM \otimes EM$. Indeed, in this case we have $p^*p = p^2 = p$ and it follows $(T_i)T_i\psi = -\mathrm{tr} \circ (\mathrm{id} \otimes \mathrm{pr_i})\nabla^2\psi = -\mathrm{pr_i}\,(\nabla^2\,\psi)$.

We will now prove the lemma. Note, that the formal adjoint ∇^* of the covariant derivative $\nabla : \Gamma(E) \to \Gamma(TM \otimes E)$ is given as the composition of the following differential operators: $\Gamma(TM \otimes E) \xrightarrow{\nabla} \Gamma(TM \otimes TM \otimes E) \xrightarrow{-\mathrm{tr}} \Gamma(E)$, where ∇ also denotes the tensor product connection, i.e $\nabla := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla$. Indeed, we obtain for a vector field $X \in \Gamma(TM)$ and a section $\psi \in \Gamma(E)$ that

$$X \otimes \psi \longmapsto \sum_i \left(e_i \otimes \nabla_{e_i}X \otimes \psi + e_i \otimes X \otimes \nabla_{e_i}\psi\right) \longmapsto \sum_i (-\mathrm{div}(X)\,\psi - \nabla_X\psi)$$

and it is easily seen that this composition is formally adjoint to ∇. Since D is defined as $D = p \circ \nabla$ we have $D^* = (p \circ \nabla)^* = \nabla^* \circ p^*$, thus $D^*\,D = (p \circ \nabla)^*(p \circ \nabla) = \nabla^* \circ p^*p \circ \nabla$. Since p is a parallel map it commutes with ∇^*. Hence we can substitute the formula for ∇^* to obtain $D^*D = \nabla^* \circ p^*p \circ \nabla = -\,\mathrm{tr} \circ \nabla \circ p^*p \circ \nabla = -\,\mathrm{tr} \circ (\mathrm{id} \otimes p^*p) \circ \nabla^2$.

□

Since we obviously have $\mathrm{id} = \sum \mathrm{pr_i}$, the proposition above immediately implies a rather useful formula for the operator $\nabla^*\nabla$, which corresponds to Equation (4) in the case $G = \mathbf{SO}_n$ and $E = \Lambda^pT$.

Corollary 3.3. $\nabla^*\nabla = \sum_i T_i^* \circ T_i$

Let G be the holonomy group of an irreducible, non-symmetric Riemannian manifold. It is then well-known that any isotypic component of $T \otimes E$ is irreducible, i.e. in the decomposition $T \otimes E = \sum E_i$ any summand E_i occurs only once. As a consequence the projection maps $\{\mathrm{pr_i}\}$ form a basis of $\mathrm{End}_G(T \otimes E)$ and any invariant endomorphism F of $T \otimes E$ can be written as $F = \sum f_i\,\mathrm{pr_i}$, with $F|_{E_i} = f_i\,\mathrm{id}$.

It turns out that for certain invariant endomorphisms F the operator $F \circ \nabla^2$ is in fact a zero order term. Hence, in these cases F gives rise to the Weitzenböck formula $F \circ \nabla^2 = \sum f_i \, T_i^* \, T_i$. The following lemma will provides us with an easy criterion for deciding which invariant endomorphisms F lead to Weitzenböck formulas.

Lemma 3.4. *Let F be an equivariant endomorphism of $T \otimes E$ considered as element of* $\mathrm{Hom}_G(T \otimes T, \mathrm{End} E)$. *Then $F \circ \nabla^2$ defines a zero order operator if and only if $F_{a\otimes b} = -F_{b\otimes a}$ for any vectors $a, b \in T$.*

Proof: Let R be the curvature of EM and let $\{e_i\}$ be a parallel local frame, then

$$F\circ\nabla^2 = \sum F(e_i\otimes e_j)\nabla_{e_i}\nabla_{e_j} = \tfrac{1}{2}\sum F(e_i\otimes e_j)(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}) = \tfrac{1}{2}\sum F(e_i\otimes e_j)R_{e_i,e_j}$$

□

We show in [13] that $\mathrm{End}_G(T \otimes E)$ is in many cases, including the exceptional holonomies $\mathbf{G}_2$ and $\mathbf{Spin}_7$, the quotient of a polynomial algebra generated by one special endomorphism, the *conformal weight operator* B.

Definition 3.5. *The conformal weight operator* $B \in \mathrm{End}_G(T \otimes E) \cong \mathrm{Hom}_G(T \otimes T, \mathrm{End} E)$ *is for any $a, b \in T, s \in E$ defined as*

$$B_{a\otimes b}\, s \;:=\; \mathrm{pr}_{\mathfrak{g}}(a \wedge b)\, s\ ,$$

where $\mathfrak{g}$ is the Lie algebra of the holonomy group G and $\mathrm{pr}_{\mathfrak{g}}$ denotes the projection $\Lambda^2 T \to \mathfrak{g} \subset \mathfrak{so}_n \cong \Lambda^2 T$. Here $\mathfrak{g}$ acts via the differential of the representation π on E.

However, for the present article it is only important to note that B defines a Weitzenböck formula, since obviously $B_{a\otimes b} = -B_{b\otimes a}$, for any $a, b \in T$. We will later apply this formula for proving that any Killing form on a compact manifold with exceptional holonomy has to be parallel.

The curvature term defined by B turns out to be the endomorphism $q(R)$ already introduced in Section 2. In fact, Equation (5) can be considered as the Weitzenböck formula corresponding to B in the special case of $G = \mathbf{SO}_n$ and $E = \Lambda^p T$.

Lemma 3.6. $$B \circ \nabla^2 \quad = \quad q(R)$$

Proof: Let $\{X_i\}$ be an ortho-normal basis for the induced scalar product on $\mathfrak{g} \subset \Lambda^2 T$ and let $\{e_i\}$ be a local ortho-normal frame. Then

$$\begin{aligned} B \circ \nabla^2 \;&=\; \sum \mathrm{pr}_{\mathfrak{g}}(e_i \wedge e_j)\, \nabla^2_{e_i,e_j} \;=\; \tfrac{1}{2} \sum \mathrm{pr}_{\mathfrak{g}}(e_i \wedge e_j)(\nabla^2_{e_i,e_j} \,-\, \nabla^2_{e_j,e_i}) \\ &=\; \sum_{i<j} \mathrm{pr}_{\mathfrak{g}}(e_i \wedge e_j)\, R_{e_i,e_j} \;=\; \sum X_i \cdot R(X_i)\cdot \;=\; q(R)\ . \end{aligned}$$

□

In order to obtain the general Weitzenböck formula defined by B we have to write $B = \sum b_i \,\mathrm{pr}_i$ and to determine the coefficients b_i . We first describe the conformal weight operator as an element of $\mathrm{End}_G(T \otimes E)$.

Lemma 3.7. *Let $\{X_i\}$ be an orthonormal basis for the induced scalar product on $\mathfrak{g} \subset \Lambda^2 T$. Then $B = -\sum X_i \otimes X_i$, where X_i is acting on T resp. E via the holonomy representation resp. the representation E.*

Proof: Using the formula $\langle X, a \wedge b\rangle = \langle Xa, b\rangle$, for $a, b \in T$ and $X \in \Lambda^2 T \cong \mathfrak{so}_n$, we may write B as

$$\begin{aligned} B(a \otimes s) &= \sum e_i \otimes \mathrm{pr}_{\mathfrak{g}}(\mathrm{e_i} \wedge \mathrm{a})\,\mathrm{s} \;=\; \sum \mathrm{e_i} \otimes \langle \mathrm{e_i} \wedge \mathrm{a},\, \mathrm{X_j}\rangle\, \mathrm{X_j}\,\mathrm{s} \\ &= \sum \langle X_j e_i,\, a\rangle\, e_i \otimes X_j\, s \;=\; -\sum \langle e_i,\, X_j a\rangle\, e_i \otimes X_j\, s \\ &= -\big(\sum X_j \otimes X_j\big)\, a \otimes s \end{aligned}$$

□

Let G be a compact semi–simple Lie group, with Lie algebra $\mathfrak{g}$ and let $\pi : G \to \mathrm{Aut}(V)$ be a representation of G on the complex vector space V. If $\{X_i\}$ is a basis of $\mathfrak{g}$, orthonormal with respect to an invariant scalar product g, the *Casimir operator* $\mathrm{Cas}^g_\pi \in \mathrm{End}(V)$ is defined as

$$\mathrm{Cas}^g_\pi \;:=\; \sum \pi_*(X_i) \circ \pi_*(X_i) \;=\; \sum X_i^2 \;,$$

where $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$ denotes the differential of the representation π. It is well-known that $\mathrm{Cas}^g_\pi = c^g_\pi\, \mathrm{id}_V$, if the representation π is irreducible. Moreover, the Casimir eigenvalues c^g_π can be expressed in terms of the highest weight of π.

It follows from the lemma above that the conformal weight operator B can be written as a linear combination of Casimir operators, which leads to

Corollary 3.8. *Let $T \otimes E = \oplus E_i$ be the decomposition of the tensor product into irreducible components. Then the conformal weight operator B is given as*

$$B \;=\; \sum b_i\, \mathrm{pr_i} \qquad \textit{with} \qquad \mathrm{b_i} \;=\; \frac{1}{2}\,(\mathrm{c}^{\Lambda^2}_{\mathrm{T}} \;+\; \mathrm{c}^{\Lambda^2}_{\mathrm{E}} \;-\; \mathrm{c}^{\Lambda^2}_{\mathrm{E_i}})\;, \tag{7}$$

for Casimir eigenvalues $c^{\Lambda^2}_\pi$ computed with respect to the induced scalar product on $\mathfrak{g} \subset \Lambda^2 T$. The corresponding universal Weitzenböck formula on sections of EM is

$$q(R) \;=\; -\sum b_i\, T_i^*\, T_i \;. \tag{8}$$

Proof: Expanding the Casimir operator $\mathrm{Cas}^{\Lambda^2}_{T\otimes E} = \sum X_i^2$ acting on $T \otimes E$ we obtain

$$\mathrm{Cas}^{\Lambda^2}_{T\otimes E} \;=\; \sum \big(X_i^2 \otimes \mathrm{id}_E \;+\; 2X_i \otimes X_i \;+\; \mathrm{id}_T \otimes X_i^2\big)$$

Hence, Lemma 3.7 implies that the conformal weight operator can be written as

$$B \;=\; -\frac{1}{2}\,\big(\mathrm{Cas}^{\Lambda^2}_{T\otimes E} \;-\; \mathrm{Cas}^{\Lambda^2}_{T} \otimes \mathrm{id}_E \;-\; \mathrm{id}_T \otimes \mathrm{Cas}^{\Lambda^2}_{E}\big)\;.$$

which yields the formula above after restriction to the irreducible components E_i. □

Remark 3.9. *In the case of Riemannian holonomy $G = \mathbf{SO}_n$ the Weitzenböck formula (8) was considered for the first time in* [6]*. In this article one can also find the conformal weight operator and its expression in terms of Casimir operators. The operator B appears also in* [5]*. Similar results can be found in* [7]*.*

It remains to compute the Casimir eigenvalues. For doing so we first recall how to compute them for an irreducible representation of highest weight λ and with respect to the scalar product $(\cdot,\cdot)$ defined by the Killing form B. Let ρ be the half sum of the positive roots of $\mathfrak{g}$, then

$$c^B_\pi \;=\; \|\rho\|^2 \;-\; \|\lambda+\rho\|^2 \;=\; -\,(\lambda,\, \lambda\,+\,2\rho)\;. \tag{9}$$

For the application of Corollary 3.8 we need the Casimir eigenvalues $c^{\Lambda^2}_\pi$ defined with respect to the induced scalar product on $\mathfrak{g}\subset\Lambda^2T$. The relation between these Casimir eigenvalues is contained in the following normalization lemma.

Lemma 3.10. *Let $\mathfrak{g}$ be the Lie algebra of a compact simple Lie group and let V be an irreducible real $\mathfrak{g}$-representation with invariant scalar product $\langle\cdot,\cdot\rangle$. If π is any other irreducible $\mathfrak{g}$-representation with invariant scalar product g, then*

$$c^{\Lambda^2}_\pi \;=\; -\,2\,\frac{\dim\mathfrak{g}}{\dim V}\,\frac{1}{c^g_V}\;c^g_\pi\;.$$

In particular, the Casimir eigenvalue of the representation V is given as $c^{\Lambda^2}_V = -2\frac{\dim\mathfrak{g}}{\dim V}$.

Proof: Since we assume V to be equipped with a $\mathfrak{g}$–invariant scalar product $\langle\cdot,\cdot\rangle$ we have $\mathfrak{g}\subset\mathfrak{so}(V)\cong\Lambda^2V$. Restricting the induced scalar product onto $\mathfrak{g}\subset\Lambda^2V$ defines the natural scalar product $\langle\cdot,\cdot\rangle_{\Lambda^2}$ on $\mathfrak{g}$. Note that $\langle\alpha,\,\beta\rangle_{\Lambda^2} = -\frac{1}{2}\,\mathrm{tr}_V(\alpha\circ\beta) = \frac{1}{2}\,\langle\alpha,\,\beta\rangle_{\mathrm{End}V}$. Let $\{X_a\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{\Lambda^2}$ and let $\{e_i\}$ be an orthonormal basis of V. Then $\mathrm{Cas}^{\Lambda^2}_V(v) = c^{\Lambda^2}_V v = \sum_a X^2_a(v)$, for any $v\in V$, and we obtain

$$\dim V\, c^{\Lambda^2}_V = \sum_{a,j}\langle X^2_a(e_j),\, e_j\rangle = -\sum_{a,j}\langle X_a(e_j),\, X_a(e_j)\rangle = -\,2\sum|X_a|^2_{\Lambda^2} = -\,2\,\dim\mathfrak{g}$$

which proves the lemma in the case $\pi = V$. Since $\mathfrak{g}$ is a simple Lie algebra it follows that two Casimir operators defined with respect to different scalar products differ only by a factor independent from the irreducible representation π. Hence $\frac{c^{\Lambda^2}_\pi}{c^g_\pi} = \frac{c^{\Lambda^2}_V}{c^g_V}$ and the statement of the lemma follows from the special case $\pi = V$. □

In the remaining part of this section we will consider the holonomy groups $\mathbf{G}_2$ and $\mathbf{Spin}_7$ and make the Weitzenböck formula (8) explicit for certain representations appearing in the decomposition of the form spaces.

3.1. **The group $\mathbf{G}_2$.** The group $\mathbf{G}_2 \subset \mathbf{SO}(7)$ is a compact simple Lie group of dimension 14 and of rank 2. As fundamental weights one usually considers ω_1 corresponding to the 7-dimensional holonomy representation T and ω_2 corresponding to the 14-dimensional adjoint representation $\mathfrak{g}_2$. The half-sum of positive roots is the sum of the fundamental weights, i.e. $\rho = \omega_1 + \omega_2$. Any other irreducible $\mathbf{G}_2$-representation can be parameterized as $\Gamma_{a,b} = a\omega_1 + b\omega_2$, e.g. the trivial representation is $\Gamma_{0,0} = \mathbb{C}$. Further examples are

$$\Gamma_{0,1} = \Lambda^2_{14} = \mathfrak{g}_2, \qquad \Gamma_{2,0} = \Lambda^3_{27}, \qquad \Gamma_{1,1} = V_{64}, \qquad \Gamma_{3,0} = V^-_{77} \;,$$

where the subscripts denote the dimension of the representation, which is unique up to dimension 77. In dimension 77 on has two irreducibel $\mathbf{G}_2$–representations, denoted by V^+_{77} and V^-_{77}. Below we need the following decomposition of the spaces of 2- and 3-forms

$$\Lambda^2 T \;\cong\; \Lambda^5 T \;\cong\; T \oplus \Lambda^2_{14}, \qquad\qquad \Lambda^3 T \;\cong\; \Lambda^4 T \;\cong\; \mathbb{C} \oplus T \oplus \Lambda^3_{27} \;. \tag{10}$$

Since we want to apply the Weitzenböck formula for the bundles $\Lambda^2_{14}T$ and $\Lambda^3_{27}T$ we still need the following tensor product decompositions

$$T \otimes \Lambda^2_{14} \;\cong\; T \oplus \Lambda^3_{27} \oplus V_{64}, \qquad T \otimes \Lambda^3_{27} \;\cong\; T \oplus \Lambda^4_{27} \oplus \Lambda^2_{14} \oplus V_{64} \oplus V^-_{77} \tag{11}$$

There is a suitable invariant bilinear form g on $\mathfrak{g}_2$, which induces the scalar products

$$g(\omega_1, \omega_1) \;=\; 1, \qquad g(\omega_2, \omega_2) \;=\; 3, \qquad g(\omega_1, \omega_2) \;=\; \tfrac{3}{2} \;.$$

Using Equation (9) and Lemma 3.10 we obtain the following Casimir eigenvalues

$$c^{\Lambda^2}_{\Gamma_{a,b}} \;=\; -\tfrac{2}{3}\, c^{g}_{\Gamma_{a,b}} \;=\; -\tfrac{2}{3}\,(a^2 \;+\; 3\,b^2 \;+\; 3\,a\,b \;+\; 5\,a \;+\; 9\,b) \;.$$

In particular we have $c^{\Lambda^2}_{\Lambda^3_{27}} = -\,\frac{28}{3},\; c^{\Lambda^2}_{\Lambda^2_{14}} = -\,8,\; c^{\Lambda^2}_{T} = -\,4,\; c^{\Lambda^2}_{V_{64}} = -\,14,\; c^{\Lambda^2}_{V^-_{77}} = -\,16$.

Finally, we use (7) to obtain the Weitzenböck formula on the bundles Λ^2_{14} and Λ^3_{27}. Recall that the twistor operator T_i is the projection of the covariant derivative onto the i.th summand in the tensor product decomposition of $T \otimes E$, i.e we will number the operators T_i according to the numbering of the summands in this decomposition, which has to be fixed in order to make the notation unique.

Here we will consider the tensor product decomposition given in (11), e.g. in the case of the representation Λ^2_{14} the operator T_3 denotes the projection of the covariant derivative onto the summand V_{64}, whereas for the representation Λ^3_{27} the operator T_3 denotes the projection of the covariant derivative onto the summand Λ^2_{14}.

Proposition 3.11. *Let $\{T_i\}$ be the twistor operators defined corresponding to the decompositions in (11). Then the following Weitzenböck formulas hold*

$$\textit{on}\quad \Lambda^2_{14}: \qquad q(R) \;=\; 4\,T_1^*T_1 \;+\; \tfrac{4}{3}\,T_2^*T_2 \;-\; T_3^*T_3$$

$$\textit{on}\quad \Lambda^3_{27}: \qquad q(R) \;=\; \tfrac{14}{3}\,T_1^*T_1 \;+\; 2\,T_2^*T_2 \;+\; \tfrac{8}{3}\,T_3^*T_3 \;-\; \tfrac{1}{3}\,T_4^*T_4 \;-\; \tfrac{4}{3}\,T_5^*T_5$$

3.2. **The group $\mathbf{Spin}_7$.** Let e_1, e_2, e_3 be the weights of the 7-dimensional standard representation of $\mathbf{Spin}_7$. Then the fundamental weights are defined as

$$\omega_1 = e_1, \quad \omega_2 = e_1 + e_2, \quad \omega_3 = \tfrac{1}{2}(e_1 + e_2 + e_3)$$

corresponding to the representations $\Lambda^1\mathbb{R}^7, \Lambda^2\mathbb{R}^7$ and the spin representation. All other irreducible $\mathbf{Spin}_7$-representations are parameterized as $\Gamma_{a,b,c} = a\omega_1 + b\omega_2 + c\omega_3$. The half-sum of positive roots is given as $\rho = \omega_1 + \omega_2 + \omega_3 = \frac{1}{2}(5, 3, 1)$.

The holonomy group $\mathbf{Spin}_7$ is considered as subgroup of $\mathbf{SO}_8$ such that the holonomy representation T is given by the 8-dimensional real spin representation. This leads to the following decompositions of the form spaces $\Lambda^k T$.

$$\Lambda^2 T \cong \Lambda^2_7 \oplus \Lambda^2_{21}, \qquad \Lambda^3 T \cong \Lambda^3_8 \oplus \Lambda^3_{48}, \qquad \Lambda^4 T \cong \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35} \tag{12}$$

Again, the subscripts denote the dimensions of the representations and of course we have $\Lambda^2_7 \cong \Lambda^4_7$ and $\Lambda^3_8 = T$. For the investigation of twistor forms on $\mathbf{Spin}_7$-manifolds we need Weitzenböck formulas on the bundles corresponding to Λ^2_{21}, Λ^3_{48}, Λ^4_{27} and Λ^4_{35}. The decompositions of the tensor products $T \otimes E$ are given as

$$\begin{aligned} T \otimes \Lambda^2_{21} \;&\cong\; T \oplus \Lambda^3_{48} \oplus V^a_{112}, \qquad T \otimes \Lambda^4_{27} \cong \Lambda^3_{48} \oplus V_{168} \\ T \otimes \Lambda^3_{48} \;&\cong\; \Lambda^4_{35} \oplus \Lambda^2_{21} \oplus \Lambda^2_7 \oplus \Lambda^4_{27} \oplus V_{105} \oplus V_{189} \\ T \otimes \Lambda^4_{35} \;&\cong\; T \oplus \Lambda^3_{48} \oplus V^a_{112} \oplus V^b_{112} \end{aligned} \tag{13}$$

There are two 112-dimensional irreducible $\mathbf{Spin}_7$-representation, which we denote with V^a_{112} and V^b_{112}. In terms of fundamental weights the representations appearing in the above decompositions are given as follows

$$\Lambda^2_7 = \Gamma_{1,0,0}, \quad \Lambda^2_{21} = \Gamma_{0,1,0}, \quad \Lambda^3_{48} = \Gamma_{1,0,1}, \quad \Lambda^4_{27} = \Gamma_{2,0,0}, \quad \Lambda^4_{35} = \Gamma_{0,0,2}$$

$$V^a_{112} = \Gamma_{0,1,1}, \quad V^b_{112} = \Gamma_{0,0,3}, \quad V_{168} = \Gamma_{2,0,1}, \quad V_{105} = \Gamma_{1,1,0}, \quad V_{189} = \Gamma_{1,0,2}$$

Next we have to calculate all the necessary Casimir eigenvalues. We choose on $\mathfrak{spin}_7$ an invariant scalar product g_0 inducing the Euclidean scalar product on the Lie algebra of the maximal torus, which is identified with $\mathbb{R}^3$. Let us start with the representation $V = \Gamma_{1,0,0} = \Lambda^2_7$ with highest weight e_1. Formula (9) yields the Casimir eigenvalue $c^{g_0}_V = -6$ and from Lemma 3.10, for $T = V$, we also get $c^{\Lambda^2}_V = -6$. Hence, we conclude $c^{\Lambda^2}_\pi = c^{g_0}_\pi$ for any irreducible $\mathbf{Spin}_7$-representation π. Using the standard scalar product on $\mathbb{R}^3$ it is now easy to calculate the following Casimir eigenvalues

$$c^{\Lambda^2}_{\Lambda^2_{21}} = -10, \quad c^{\Lambda^2}_T = -\tfrac{21}{4}, \quad c^{\Lambda^2}_{\Lambda^4_{27}} = -14, \quad c^{\Lambda^2}_{\Lambda^4_{35}} = -12, \quad c^{\Lambda^2}_{\Lambda^3_{48}} = -\tfrac{49}{4}$$

$$c^{\Lambda^2}_{V^a_{112}} = -\tfrac{69}{4}, \quad c^{\Lambda^2}_{V_{105}} = -18, \quad c^{\Lambda^2}_{V_{189}} = -20, \quad c^{\Lambda^2}_{V_{168}} = -\tfrac{85}{4}, \quad c^{\Lambda^2}_{V^b_{112}} = -\tfrac{81}{4},$$

Finally, we use (7) to compute the coefficients of the Weitzenböck formula on the bundles Λ^2_{21}, Λ^3_{48}, Λ^4_{27} and Λ^4_{35}. As in the $\mathbf{G}_2$-case we number the twistor operators corresponding to the decomposition (13), e.g. for the representation Λ^2_{21} the twistor operator T_1 denotes the projection of the covariant derivative onto the summand T and for the representation Λ^3_{48} it denotes the projection onto the summand Λ^4_{35}.

Proposition 3.12. *Let $\{T_i\}$ be the twistor operators defined corresponding to the decompositions in (13). Then the following Weitzenböck formulas hold*

$$
\begin{array}{llcl}
\textit{on} & \Lambda^2_{21}: & q(R) = & 10\,T_1^*T_1 + 3\,T_2^*T_2 - T_3^*T_3 \\
\textit{on} & \Lambda^3_{48}: & q(R) = & \frac{11}{4}\,T_1^*T_1 + \frac{15}{4}\,T_2^*T_2 + \frac{23}{4}\,T_3^*T_3 + \frac{7}{4}\,T_4^*T_4 - \frac{1}{4}\,T_5^*T_5 - \frac{5}{4}T_6^*T_6 \\
\textit{on} & \Lambda^4_{27}: & q(R) = & \frac{7}{2}\,T_1^*T_1 - 2\,T_2^*T_2 \\
\textit{on} & \Lambda^4_{35}: & q(R) = & 6\,T_1^*T_1 + \frac{5}{2}\,T_2^*T_2 - \frac{3}{2}\,T_4^*T_4
\end{array}
$$

4. Proof of the Theorems

In this section we will prove Theorems 1.1 and 1.2 using the Weitzenböck formulas of Proposition 3.11 and 3.12. We will first show that on manifolds with holonomy $\mathbf{G}_2$ and $\mathbf{Spin}_7$ any Killing form can be decomposed into a sum of Killing forms belonging to the parallel subbundles of the form bundle. Hence we may assume that the Killing form is a section of one of the irreducible components. Moreover we only have to consider summands where the endomorphism $q(R)$ acts non-trivially. Otherwise the twistor form has to be parallel, as follows from Corollary 2.3.

It is easy to show that on Einstein manifolds the codifferential of any twistor 2-form is either zero or dual to a Killing vector field. However, on a compact Ricci-flat manifold any Killing vector field has to be parallel and it follows after integration that the twistor 2-form has to be coclosed. This argument then implies the more general statement on twistor forms, contained in Theorems 1.1 and 1.2.

4.1. **The holonomy decomposition.** Let (M^n, g) be a manifold with holonomy G, which is assumed to be a proper subgroup of $\mathbf{SO}_n$. In this situation the bundle of p-forms decomposes into a sum of parallel subbundles, $\Lambda^pTM = \oplus V_i$ and correspondingly, any p-form u has a holonomy decomposition $u = \sum u_i$. If u is a twistor form, or even a Killing form, it remains in general not true for the components u_i. Nevertheless we have such a property in the case of the exceptional holonomies.

Lemma 4.1. *Let (M, g) be a compact manifold with holonomy $\mathbf{G}_2$ or $\mathbf{Spin}_7$ and let u be any form with holonomy decomposition $u = \sum u_i$. Then u is a Killing form or a $*$-Killing form if and only if the same is true for all components u_i.*

Proof: We will use the characterization of Killing forms given in Proposition 2.2. Since the decomposition $\Lambda^pTM = \oplus V_i$ is parallel, it is preserved by $\nabla^*\nabla$ and $q(R)$. Thus for a Killing form u all the components satisfy the equation $\nabla^*\nabla u_i = \frac{1}{p}q(R)u_i$ and it remains to verify whether the components are coclosed.

Since manifolds with holonomy $\mathbf{G}_2$ or $\mathbf{Spin}_7$ are Ricci-flat we do not have to consider twistor 1-forms, which are automatically parallel due to Corollary 2.3. We start with the $\mathbf{G}_2$-case, where it is enough to consider Killing forms. The proof for $*$-Killing follows then from the duality under the Hodge star operator. Let $u = u_7 + u_{14}$ be the decomposition of a Killing 2-form according to (10), then Corollary 2.3 shows that u_7 is parallel and thus $u_{14} = u - u_7$ is coclosed. In the case of a Killing 3-form we have the decomposition $u = u_1 + u_7 + u_{27}$ and, as for 2-forms, it follows that u_1 and u_7 have to be parallel, implying that u_{27} is coclosed. The argument is the same for Killing forms in degree 4 and 5, since the same representations are involved.

We now turn to the case of holonomy $\mathbf{Spin}_7$. In the case of twistor 4-forms it follows from the remark after Proposition 2.2 that any component is again a twistor form. The further analysis will be given below. The argument for forms in all other degrees is the same as in the $\mathbf{G}_2$-case.

4.2. **Twistor forms on G_2-manifolds.** For G_2-manifolds we only have to consider two cases: twistor forms in Λ^2_{14} and in Λ^3_{27}. In the first case we have three twistor operators (corresponding to the decomposition (11)) and T_3 vanishes on twistor forms, since the third summand in the decomposition (11) of $T \otimes \Lambda^2_{14}$ belongs neither to the Λ^1 nor to Λ^3. In the second case we have five twistor operators and here T_4 and T_5 vanish on twistor forms, since the last two summands in the decomposition (11) of $T \otimes \Lambda^3_{27}$ do not appear in the form spaces. From Proposition 3.11 we have one Weitzenböck formula for each case. However, to prove Theorem 1.1 we still need to compare the twistor operators with the differential and codifferential.

Consider the case Λ^2_{14}, here the differential splits as $d = d_7 + d_{27}$, e.g. $d_7 = \sum (e_i \wedge \nabla_{e_i})_7$, with subscripts denoting the projection onto the corresponding summand. There is no part d_1, since the trivial representation does not occur in the decomposition of $T \otimes \Lambda^2_{14}$. The projection pr_1 defining T_1 can be written in two ways:

$$\mathrm{pr}_1 : \mathrm{T} \otimes \Lambda^2_{14} \xrightarrow{\pi_1} \mathrm{T} \xrightarrow{\mathrm{j}_1} \mathrm{T} \otimes \Lambda^2_{14}, \qquad \mathrm{pr}_1 : \mathrm{T} \otimes \Lambda^2_{14} \xrightarrow{\pi_2} \Lambda^3_7 \xrightarrow{\mathrm{j}_2} \mathrm{T} \otimes \Lambda^2_{14},$$

with $\pi_1(X \otimes \alpha) = X \lrcorner\, \alpha$ and $\pi_2(X \otimes \alpha) = (X \wedge \alpha)_7$, and the right inverses j_1, j_2. Hence, $du = 0$ or $d^* u = 0$ both imply $T_1 u = 0$. Similarly, $du = 0$ implies $T_2 u = 0$.

Let u be a $*$-Killing form in Λ^2_{14}, then $du = 0$ implies $T_1 u = 0$ and $T_2 u = 0$. Hence all twistor operators vanish on u and the form has to be parallel. Let u be a Killing form in Λ^2_{14}, then only the component $T_2 u$ could be different from 0. But the Weitzenböck formula of Proposition 3.11 and the equation $2\nabla^*\nabla u = q(R)u$ imply

$$0 \;=\; 2\nabla^*\nabla u \,-\, q(R)u \;=\; (2 - \tfrac{4}{3}) T_2^* T_2 u \;.$$

Hence $T_2^* T_2 u = 0$, and after integration also $T_2 u = 0$, i.e. the form u has to be parallel.

Consider now the case Λ^3_{27}, here the differential splits as $d = d_7 + d_{27}$ and the codifferential as $d^* = d^*_7 + d^*_{14}$, again there is no component d_1. With the same arguments as for Λ^2_{14} we see that $du = 0$ implies $T_1 u = 0$ and $T_2 u = 0$ and $d^* u = 0$ implies $T_1 u = 0$ and $T_3 u = 0$. Indeed, the first to summands of the decomposition (11) of $T \otimes \Lambda^3_{27}$ are also summands of the 4-forms. Similarly, the first and the third summand are components of the 2-forms.

Let u be Killing form in Λ^3_{27}, then only $T_2 u$ could be different from zero. But the Weitzenböck formula and the equation $3\nabla^*\nabla u = q(R)u$ implies $(3-2)T_2^*T_2 u = 0$ and u again has to be parallel. Finally, in the case of a $*$-Killing form u in Λ^3_{27}, we have to use the Weitzenböck formula and the equation $4\nabla^*\nabla u = q(R)u$ to show the vanishing of $T_3 u$. □

Remark 4.2. *Using Lemma 3.2 and explicit expressions for the projections onto the irreducible components of the form bundle it is possible to determine the precise relation between the operators $T_i^*T_i$ and similar operators in terms of the components of d and d^*, e.g. on the bundle Λ^2_{14} one finds: $d_7^*d_7 = T_1^*T_1$, $dd^* = 4T_1^*T_1$ and $3d_{27}^*d_{27} = 7T_2^*T_2$.*

4.3. **Twistor forms on $\mathbf{Spin}_7$-manifolds.** Since $q(R)$ acts trivially on the representations $\mathbb{C}$, Λ^2_7 and T, any twistor form in one of these components is automatically parallel. Hence, we only have to consider twistor forms in the subbundles Λ^2_{21}, Λ^3_{48}, Λ^4_{27} and Λ^4_{35}. The argument is now similar to the $\mathbf{G}_2$-case. For any Killing or $*$-Killing form u in one of these bundles we show that all twistor operators vanish on u, such that the form has to be parallel.

Let $u \in \Lambda^2_{21}$ be a twistor 2-form, then $T_3 u = 0$, since the third summand in the decomposition (13) of $T \otimes \Lambda^2_{21}$ belongs neither to Λ^1 nor to Λ^3. The representation $T \cong \Lambda^1$ appears also as summand in the 3-form. Hence, as in the $\mathbf{G}_2$-case, we see that $du = 0$ implies $T_1 u = T_2 u = 0$ and $d^*u = 0$ implies $T_1 u = 0$. Thus $*$-Killing forms are automatically parallel. Let u be a Killing 2-form, then $2\nabla^*\nabla u = q(R)u$ and the Weitzenböck formula of Proposition 3.12 imply $(2-3)T_2^*T_2 u$ and $T_2 u = 0$. Thus also on Killing forms all twistor operators vanish.

Let $u \in \Lambda^3_{48}$ be a twistor 3-form, i.e. $T_5 u = T_6 u = 0$. The representation Λ^2_7, i.e. the third summand in the decomposition (13) of $T \otimes \Lambda^3_{48}$, appears as summand in the 2- and 4-forms. Hence, $du = 0$ implies $T_1 u = T_3 u = T_4 u = 0$ and $d^*u = 0$ implies $T_2 u = T_3 u = 0$. Let u be a $*$-Killing form, then $5\nabla^*\nabla u = q(R)u$ and we find $(5 - \frac{15}{4})T_2^*T_2 u = 0$, thus $T_2 u = 0$. Let u be a Killing form, then $3\nabla^*\nabla u = q(R)u$ and $(3 - \frac{11}{4})T_1^*T_1 + (3 - \frac{7}{4})T_4^*T_4 u = 0$ implies $T_1 u = T_4 u = 0$.

Let $u \in \Lambda^4_{27}$ be a twistor 4-form, i.e. $T_2 u = 0$. Then $4\nabla^*\nabla u = q(R)u$ implies $(4 - \frac{7}{2})T_1^*T_1 u = 0$ and thus $T_1 u = 0$. Hence, any twistor 4-form in Λ^4_{27} has to be parallel, which completes the proof Lemma 4.1. We had already seen that any component of a twistor 4-form is again a twistor form. Now we see that three of the four components have to be parallel. Hence, for a Killing form all components are again coclosed, i.e. again Killing forms.

Finally, let $u \in \Lambda^4_{35}$ be a twistor 4-form, i.e. $T_3 u = T_4 u = 0$. Since both remaining summands T and V^b_{112} are subbundles of $\Lambda^3 \cong \Lambda^5$ we see that T_1 and T_2 vanish for closed or coclosed twistor forms, thus they have to be parallel. □

Remark 4.3. *For the representation Λ^4_{35} we find in [13] an additional Weitzenböck formula (with vanishing curvature term). This equation then shows that indeed any twistor 4-form on a $\mathbf{Spin}_7$-manifold has to be parallel.*

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