Quaternionic Killing Spinors

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Abstract

In [KSW97] we proved a lower bound for the spectrum of the Dirac operator on quaternionic Kähler manifolds. In the present article we study the limiting case, i. e. manifolds where the lower bound is attained as an eigenvalue. We give an equivalent formulation in terms of a quaternionic Killing equation and show that the only symmetric quaternionic Kähler manifolds with smallest possible eigenvalue are the quaternionic projective spaces.

Contents

1 Introduction

Let (M^{4n}, g), $n \geq 2$ be a compact quaternionic Kähler manifold of positive scalar curvature κ. By definition its holonomy group is then contained in the subgroup $\mathbf{Sp}(n)\cdot\mathbf{Sp}(1) \subset \mathbf{SO}(4n)$. If the quaternionic dimension n is even or if $M = \mathbb{H}P^n$, M is spin and we proved in [KSW97] the following lower bound for the spectrum of the Dirac operator D on M:

$$\lambda^2 \geq \frac{\kappa}{4}\,\frac{n+3}{n+2}\,.$$

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Note that κ is constant on M, since any quaternionic Kähler manifold is automatically Einstein. This was first shown by D. V. Alekseevskii in [Ale68-1] and [Ale68-2] (see also [Ish74]). The estimate is sharp since the lower bound is the first eigenvalue of D^2 on the quaternionic projective space, as follows from the computation of the spectrum done in [Mil92].

The natural task is then to study the limiting case and find all manifolds where $\frac{\kappa}{4}\,\frac{n+3}{n+2}$ is in the spectrum of D^2. In this article we rule out all Wolf spaces besides the quaternionic projective spaces, thus settling the question for all compact symmetric quaternionic Kähler manifolds. Up to now, no other examples of compact quaternionic Kähler manifolds of positive scalar curvature are known, and a common conjecture, proved by C. LeBrun and S. Salamon [LSa94] in quaternionic dimensions $n = 2$ and $n = 3$, says that there are none.

The principal result shows that the existence of an eigenspinor with the minimal eigenvalue is equivalent to the existence of a solution for a suitable quaternionic Killing equation, i. e. a section of a suitable vector bundle which is parallel with respect to a modified connection. The curvature of this Killing connection is precisely the hyperkähler or Weyl part of the curvature tensor. Explicit calculation then shows that no Wolf space besides the quaternionic projective space allows a parallel section for this connection.

A peculiar feature of the quaternionic Killing connection is that unlike its Riemannian or Kählerian counterpart it is not defined on (a subbundle of) the spinor bundle, but involves a non–spinor bundle naturally. These "hidden parameters" account for the fact that the dimension of the space of eigenspinors with minimal eigenvalue on the quaternionic projective space exceeds the dimension of $\mathbf{S}_0(\mathbb{H}P^n) \oplus \mathbf{S}_1(\mathbb{H}P^n)$. As the geometric significance of the additional bundle is obscure it seems difficult to describe the Killing connection in purely geometric terms without using representation theory of $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$.

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2 Spin Geometry of Quaternionic Kähler Manifolds

Let (M^{4n}, g) be a quaternionic Kähler manifold, i. e. the Levi–Civitá connection on M is already defined on a $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$–reduction P of the $\mathbf{SO}(4n)$–bundle of orthonormal frames. Any representation V of $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ locally gives a vector bundle $\mathbf{V}$ associated to (a local two–fold covering of) P. This bundle exists globally iff the representation factors through $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$.

The representation theory of $\mathbf{Sp}(1)$ and $\mathbf{Sp}(n)$ is governed by the defining representations $H := \mathbb{H} \cong \mathbb{C}^2$ and $E := \mathbb{H}^n \cong \mathbb{C}^{2n}$ respectively. More precisely, any irreducible $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$–representation can be realized as a subspace of $H^{\otimes p} \otimes E^{\otimes q}$ for some p and q; those with $p + q$ even factor through $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$. Hence, any vector bundle on M associated to P can be expressed in terms of the local bundles $\mathbf{H}$ and $\mathbf{E}$. For example, the complexified tangent bundle is defined by the representation $H \otimes E$, i. e.

$$TM^{\mathbb{C}} \;=\; \mathbf{H} \otimes \mathbf{E}\,.$$

In this section we will recall some of the definitions and results given in [KSW97]. Besides elementary properties of the representations $\Lambda^s E$ and $\mathrm{Sym}^r H$ we also need the explicit description of the curvature tensor given in [KSW97] as well as the definition of Dirac and twistor operators.

2.1 Preliminaries on $\mathbf{Sp}(n)$–Representations

Let H and E be the defining complex representations of $\mathbf{Sp}(1)$ and $\mathbf{Sp}(n)$ with their invariant symplectic forms $\sigma_H \in \Lambda^2 H^*$ and $\sigma_E \in \Lambda^2 E^*$ and their compatible positive quaternionic structures J, e. g.

$$\begin{array}{rcl} J^2 &=& -1 \\ \sigma_E(Je_1, Je_2) &=& \overline{\sigma_E(e_1, e_2)} \\ \sigma_E(e, Je) &>& 0 \quad \text{for } e \neq 0\ . \end{array}$$

The symplectic form σ_E defines an isomorphism $\sharp : E \to E^*,\ e \mapsto e^\sharp := \sigma_E(e,\,.\,)$ with inverse $\flat : E^* \to E$.

Of particular importance is the representation $\mathrm{Sym}^2 E$. Its real subspace is canonically isomorphic to $\mathfrak{sp}(n)$. Thus $\mathrm{Sym}^2 E$ acts on every complex representation of $\mathbf{Sp}(n)$, e. g. its action on E is given by $(e_1 e_2)e := \sigma_E(e_1, e)e_2 + \sigma_E(e_2, e)e_1$. Analogous statements are true for H.

Let $\{e_i\}$ and $\{de_i\}$ with $de_i(e_j) = \delta_{ij}$ be a dual pair of bases for E, E^* respectively. In terms of this bases the symplectic form and its canonical bivector — associated by $\Lambda^2 E \cong \Lambda^2 E^*$ — can be written as

$$\sigma_E = \tfrac{1}{2} \sum de_i \wedge e_i^\sharp \in \Lambda^2 E^* \qquad\qquad L_E = \tfrac{1}{2} \sum de_i^\flat \wedge e_i \in \Lambda^2 E\,.$$

Wedging with L_E determines a homomorphism $L : \Lambda^{s-2}E \longrightarrow \Lambda^s E$ whereas contracting with σ_E defines its adjoint $\Lambda := L^* : \Lambda^s E \longrightarrow \Lambda^{s-2}E$. The operators L, Λ and $H := [\Lambda, L]$ with $H|_{\Lambda^s E} = (n-s)\mathrm{id}$ satisfy the commutator algebra of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$. Therefore, $\Lambda^s E = \mathrm{im}(L) \oplus \ker(\Lambda)$ splits as $\mathbf{Sp}(n)$–representation and $\Lambda^s_\circ E := \ker(\Lambda)$, the *primitive space*, turns out to be irreducible. Inductively, the complete decomposition of $\Lambda^s E$ is proved to be:

$$\Lambda^s E \quad = \quad \bigoplus_{k=0}^{[\frac{s}{2}]} \Lambda_\circ^{s-2k} E, \qquad 0 \le s \le n\,.$$

The primitive space is stable under contraction with elements of E^* but it is not preserved by the wedge product. Therefore it is necessary to describe the projection $e \wedge_\circ \omega$ of $e \wedge \omega$ onto $\Lambda^s_\circ E$:

Lemma 2.1 *If $\omega \in \Lambda^s_\circ E$ then $e^\sharp \lrcorner\, \omega \in \Lambda_\circ^{s-1}E$. Furthermore*

$$e \wedge_\circ \omega = e \wedge \omega \;-\; \tfrac{1}{n-s+1}\, L_E \wedge (e^\sharp \lrcorner\, \omega)\,.$$

Summarizing the properties of contraction and modified exterior multiplication we have:

Lemma 2.2 *On $\Lambda^s_\circ E$ modified exterior multiplication and contraction operators satisfy the following anti–commutator relations*

$$\{\eta_1 \lrcorner\,, \eta_2 \lrcorner\,\} = 0 \qquad\qquad \{e_1 \wedge_\circ, e_2 \wedge_\circ\} = 0 \qquad\qquad \{\eta \lrcorner\;, e \wedge_\circ\} = \eta(e) + \tfrac{1}{n-s+1} \eta^\flat \wedge_\circ e^\sharp \lrcorner$$

for arbitrary $\eta, \eta_i \in E^*$ and $e, e_i \in E$. In addition the following variants of number operators are defined:

$$\sum e_i \wedge_\circ de_i \lrcorner \;=\; s\,\mathrm{id} \qquad\qquad \sum de_i \lrcorner\; e_i \wedge_\circ \;=\; \tfrac{(2n-s+2)(n-s)}{n-s+1}\,\mathrm{id}\,.$$

On H, there are similar equations which relate contraction and symmetric product with $h \in H$. However, it is convenient to modify contraction. For $\alpha \in H^*$ we define $\alpha \lrcorner_\circ : \mathrm{Sym}^r H \to \mathrm{Sym}^{r-1} H$ by $\alpha \lrcorner_\circ := \frac{1}{r}\, \alpha \lrcorner$.

Lemma 2.3 *On $\mathrm{Sym}^r H$ symmetric multiplication and modified contraction operators satisfy the following commutator relations*

$$\begin{array}{rclcrcl} [h_1 \cdot, h_2 \cdot] & = & 0 & \qquad & [\alpha \lrcorner_\circ, h \cdot] & = & -\frac{1}{r+1} \alpha^\flat \cdot h^\sharp \lrcorner_\circ \\ [\alpha_1 \lrcorner_\circ, \alpha_2 \lrcorner_\circ] & = & 0 & & \alpha(h)\mathrm{id} & = & h \cdot \alpha \lrcorner_\circ - \alpha^\flat \cdot h^\sharp \lrcorner_\circ \end{array}$$

for arbitrary $h, h_i \in H$ and $\alpha, \alpha_i \in H^*$. In addition the following variants of number operators are defined:

$$\sum h_i \cdot dh_i \lrcorner_\circ \;=\; \mathrm{id} \qquad \sum dh_i \lrcorner_\circ h_i \cdot \;=\; \tfrac{r+2}{r+1}\,\mathrm{id}\,.$$

2.2 The Curvature Tensor

For later use we need an explicit description of the curvature tensor which we take from [KSW97]. First we recall the definition of the following $\mathrm{End}(\mathbf{H} \otimes \mathbf{E})$-valued 2-forms on $\mathbf{H} \otimes \mathbf{E}$:

$$\begin{array}{rcl} R^H_{h_1 \otimes e_1, h_2 \otimes e_2} & = & \sigma_E(e_1, e_2)(h_1 h_2 \otimes \mathrm{id}_E) \\ R^E_{h_1 \otimes e_1, h_2 \otimes e_2} & = & \sigma_H(h_1, h_2)(\mathrm{id}_H \otimes e_1 e_2) \\ R^{hyper}_{h_1 \otimes e_1, h_2 \otimes e_2} & = & \sigma_H(h_1, h_2)(\mathrm{id}_H \otimes \mathfrak{R}_{e_1, e_2}), \end{array}$$

where $\mathfrak{R} \in \mathrm{Sym}^4 E^*$ induces the endomorphisms $\mathfrak{R}_{e_1,e_2} : e \mapsto \mathfrak{R}(e_1, e_2, e, .)^\flat$ of E.

Lemma 2.4 *The curvature tensor of quaternionic Kähler manifold M^{4n} is given by*

$$R = -\frac{\kappa}{8n(n+2)}(R^H + R^E) + R^{hyper}\,,$$

where κ is the scalar curvature of M and the symmetric 4–form $\mathfrak{R}$ is necessarily the symmetrisation:

$$\mathfrak{R}(e_1, e_2, e_3, e_4) = \frac{1}{24\sigma_H(h_1,h_2)\sigma_H(h_3,h_4)} \sum_{\tau\in S_4} \langle R_{h_1\otimes e_{\tau 1}, h_2\otimes e_{\tau 2}} h_3 \otimes e_{\tau 3}, h_4 \otimes e_{\tau 4}\rangle\,,$$

which is independent of the choice of the h_i as long as $\sigma_H(h_1,h_2)\sigma_H(h_3,h_4) \neq 0$.

2.3 Spinor Bundle and Clifford Multiplication

The spinor module considered as $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$–representation splits into a sum of $n+1$ irreducible components. Hence, the spinor bundle of a $4n$–dimensional quaternionic Kähler manifold decomposes into a sum of $n+1$ subbundles which can be expressed using the locally defined bundles $\mathbf{E}$ and $\mathbf{H}$.

Proposition 2.1 [BaS83],[HiM95],[Wan89] *The spinor bundle $\mathbf{S}(M)$ of a quaternionic Kähler manifold M decomposes as*

$$\mathbf{S}(M) \;=\; \bigoplus_{r=0}^{n} \mathbf{S}_r(M) \;:=\; \bigoplus_{r=0}^{n} \mathrm{Sym}^r\mathbf{H} \otimes \Lambda_\circ^{n-r}\mathbf{E}\,.$$

The rank of the subbundle $\mathbf{S}_r(M)$ is given by

$$\mathrm{rank}\,(\mathbf{S}_r(M)) = (r+1)\left(\binom{2n}{n-r} - \binom{2n}{n-r-2}\right).$$

Note that the covariant derivative on $\mathbf{S}(M)$ induced by the Levi–Civitá connection on (M,g) respects the decomposition given above. The following proposition presents the Clifford multiplication in terms of the E–H–formalism.

Proposition 2.2 [KSW97] *For any tangent vector $h \otimes e \in \mathbf{H} \otimes \mathbf{E} = TM^{\mathbb{C}}$, the Clifford multiplication $\mu(h \otimes e) : \mathbf{S}(M) \to \mathbf{S}(M)$ is given by:*

$$\mu(h\otimes e) \quad = \quad \sqrt{2}\,(h\,\cdot \otimes e^\sharp \lrcorner \;+\; h^\sharp \lrcorner_\circ \otimes e \wedge_\circ)\,.$$

In particular, the Clifford multiplication maps the subbundle $\mathbf{S}_r(M)$ to the sum $\mathbf{S}_{r-1}(M) \oplus \mathbf{S}_{r+1}(M)$.

Thus, Clifford multiplication splits into two components:

$$\mu^+_- : \quad TM \otimes \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r+1}(M) \qquad \text{and} \qquad \mu^-_+ : \quad TM \otimes \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r-1}(M)\,,$$

where $\mu^+_-(e \otimes h \otimes \psi) = \sqrt{2}\,(h\,\cdot \otimes e^\sharp \lrcorner\,)\,\psi$ and $\mu^-_+(e \otimes h \otimes \psi) = \sqrt{2}\,(h^\sharp \lrcorner_\circ \otimes e \wedge_\circ)\,\psi$. We note that this definition makes sense also for $\mathbf{S}_r(M)$ replaced by $\mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E}$. In this spirit it is possible to define two operations similar to Clifford multiplication:

$$\begin{array}{rccc} \mu^+_+ : & TM \otimes \mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E} & \longrightarrow & \mathrm{Sym}^{r+1}\mathbf{H} \otimes \Lambda^{s+1}_\circ\mathbf{E} \\ & h \otimes e \otimes \psi & \longmapsto & \sqrt{2}\,(h\,\cdot \otimes e \wedge_\circ)\psi \end{array}$$

and

$$\begin{array}{rccc} \mu^-_- : & TM \otimes \mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E} & \longrightarrow & \mathrm{Sym}^{r-1}\mathbf{H} \otimes \Lambda^{s-1}_\circ\mathbf{E} \\ & h \otimes e \otimes \psi & \longmapsto & \sqrt{2}\,(h^\sharp \lrcorner_\circ \otimes e^\sharp \lrcorner\,)\psi\,. \end{array}$$

Using the number operators of Lemmata 2.2 and 2.3 it is easy to prove the following useful formulas:

Lemma 2.5 *The following relations are satisfied on* $\mathrm{Sym}^r\mathbf{H}\otimes\Lambda^s_\circ\mathbf{E}$*:*

$$\begin{aligned}\sum\mu^+_+(X_a)\mu^-_-(X_a) &= 2s\\ \sum\mu^+_-(X_a)\mu^-_+(X_a) &= -2\,\tfrac{(2n-s+2)(n-s)}{n-s+1}\\ \sum\mu^-_+(X_a)\mu^+_-(X_a) &= -2s\,\tfrac{r+2}{r+1}\\ \sum\mu^-_-(X_a)\mu^+_+(X_a) &= 2\,\tfrac{(2n-s+2)(n-s)}{n-s+1}\,\tfrac{r+2}{r+1},\end{aligned}$$

where the sum is over a local orthonormal base $\{X_a\}$ *of* TM*. All other combinations of the partial Clifford multiplications vanish upon summation over* $\{X_a\}$*.*

2.4 Dirac and Twistor Operators

In this section we recall the definition of quaternionic Dirac and twistor operators. For defining these operators we have to decompose $TM\otimes\mathbf{S}_r(M)$ into irreducible components and to project the covariant differential of a spinor onto the different summands. For $r\neq 0,n$ we have the following decomposition

$$\begin{aligned}TM\otimes\mathbf{S}_r(M) &\cong (\mathbf{H}\otimes\mathbf{E})\otimes(\mathrm{Sym}^r\mathbf{H}\otimes\Lambda^{n-r}_\circ\mathbf{E})\\ &\cong \mathbf{S}_{r+1}(M)\oplus\mathbf{S}_{r-1}(M)\oplus(S^+_r\oplus S^-_r\oplus V^+_r\oplus V^-_r)\,.\end{aligned}\tag{2.1}$$

Here we used the notation $S^\pm_r=\mathrm{Sym}^{r\pm1}\mathbf{H}\otimes\Lambda^{n-r\pm1}_\circ\mathbf{E}$ and $V^\pm_r=\mathrm{Sym}^{r\pm1}\mathbf{H}\otimes K^{n-r}\mathbf{E}$, where $K^{n-r}\mathbf{E}$ is the summand corresponding to the sum of the highest weights in the decomposition of $E\otimes\Lambda^{n-r}_\circ E$. In the case $r=0$ and $r=n$ four of the above summands vanish and we obtain:

$$(\mathbf{H}\otimes\mathbf{E})\otimes\Lambda^n_\circ\mathbf{E}\cong\mathbf{S}_1(M)\oplus V^+_0\qquad\text{and}\qquad(\mathbf{H}\otimes\mathbf{E})\otimes\mathrm{Sym}^n\mathbf{H}\cong\mathbf{S}_{n-1}(M)\oplus S^+_n\,.\tag{2.2}$$

The two components of the Clifford multiplication define natural projections onto the first two summands appearing in the decomposition (2.1). The remaining four summands constitute the kernel of the Clifford multiplication. The projections onto S^+_r resp. S^-_r are given by μ^+_+ resp. μ^-_- and we denote the projections onto $V^\pm$ by $pr_{V^\pm}$. Applying these projectors to the section $\nabla\psi\in\Gamma(TM\otimes\mathbf{S}(M))$ we get the two components of the Dirac operator:

$$D^+_-:=\mu^+_-\circ\nabla:\mathbf{S}_r(M)\longrightarrow\mathbf{S}_{r+1}(M)\qquad D^-_+:=\mu^-_+\circ\nabla:\mathbf{S}_r(M)\longrightarrow\mathbf{S}_{r-1}(M)\,,\tag{2.3}$$

where $D=D^+_-+D^-_+$ is the Dirac operator, and four twistor operators:

$$\begin{aligned}D^+_+ &:= \mu^+_+\circ\nabla:\mathbf{S}_r(M)\longrightarrow S^+_r & D^-_- &:= \mu^-_-\circ\nabla:\mathbf{S}_r(M)\longrightarrow S^-_r\\ T^+ &:= pr_{V^+}\circ\nabla:\mathbf{S}_r(M)\longrightarrow V^+ & T^- &:= pr_{V^-}\circ\nabla:\mathbf{S}_r(M)\longrightarrow V^-\,.\end{aligned}\tag{2.4}$$

The square of the Dirac operator respects the splitting of the spinor bundle, i. e. $D^2:\mathbf{S}_r(M)\longrightarrow\mathbf{S}_r(M)$. In particular, we have: $D^+_-D^+_-=0=D^-_+D^-_+$.

According to the definition of the Dirac and twistor operators by decomposition (2.1) it is possible to reconstruct the covariant differential of a spinor with the help of these operators. As this is a prerequisite for deriving Killing equations we state the final formula with the help of right inverses $\iota^\mp_\mp$ for the partial Clifford multiplications $\mu^\pm_\pm$:

Lemma 2.6

$$\nabla\phi=\iota^-_+(D^+_-\phi)+\iota^+_-(D^-_+\phi)+\iota^-_-(D^+_+\phi)+\iota^+_+(D^-_-\phi)+T^+\phi+T^-\phi$$

where the embeddings $\iota^{\mp}_{\mp}$ are defined as the right inverses of $\mu^{\pm}_{\pm}$. With the help of Lemma 2.5 their explicit form is readily established:

$$\begin{array}{rcll} \iota^-_+ : \mathbf{S}_{r+1}(M) & \longrightarrow & TM \otimes \mathbf{S}_r(M) \\ \phi & \longmapsto & -\frac{r+2}{2(n+r+3)(r+1)} \sum X_a \otimes \mu^-_+(X_a)\phi \end{array} \qquad \begin{array}{rcll} \iota^-_- : S^+_r & \longrightarrow & TM \otimes \mathbf{S}_r(M) \\ \phi & \longmapsto & \frac{1}{2(n-r+1)} \sum X_a \otimes \mu^-_-(X_a)\phi \end{array}$$

$$\begin{array}{rcll} \iota^+_- : \mathbf{S}_{r-1}(M) & \longrightarrow & TM \otimes \mathbf{S}_r(M) \\ \phi & \longmapsto & -\frac{r}{2(n-r+1)(r+1)} \sum X_a \otimes \mu^+_-(X_a)\phi \end{array} \qquad \begin{array}{rcll} \iota^+_+ : S^-_r & \longrightarrow & TM \otimes \mathbf{S}_r(M) \\ \phi & \longmapsto & \frac{r(r+2)}{2(n+r+3)(r+1)^2} \sum X_a \otimes \mu^+_+(X_a)\phi \end{array}$$

where $\{X_a\}$ is a local orthonormal base of TM.

3 Weitzenböck Formulas

The central result of [KSW97] is a Weitzenböck formula in matrix form which relates two sets of naturally defined 2nd order differential operators from the spinor bundle to itself. The idea is to cope with the abundance of natural 2nd order operators defined for spin manifolds with special holonomy by replacing the Lichnerowicz Weitzenböck formula of general holonomy by the linear space of all Weitzenböck formulas adapted to the holonomy in question. In the case of quaternionic Kähler manifolds the final matrix equation is an identity of differential operators defined on sections of the bundles $\mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E}$ depending on the quaternionic dimension n:

$$\begin{pmatrix} -\nabla^*\nabla\psi \\ \frac{\kappa}{4}\frac{r(r+2)}{n+2}\psi \\ \frac{\kappa}{4}\frac{s(2n-s+2)}{n(n+2)}\psi \\ \mathcal{C}\psi \\ \mathcal{L}\psi \\ 0 \end{pmatrix} = \mathcal{W}_H(r) \otimes \mathcal{W}_E(s) \cdot \begin{pmatrix} -\frac{1}{2}(D^+_+)^*D^+_+\psi \\ \frac{1}{2}D^+_-D^-_+\psi \\ \frac{1}{2}D^-_+D^+_-\psi \\ -\frac{1}{2}(D^-_-)^*D^-_-\psi \\ -(T^+)^*T^+\psi \\ (T^-)^*T^-\psi \end{pmatrix}, \tag{3.5}$$

where $\mathcal{W}_H(r) \otimes \mathcal{W}_E(s)$ is the Kronecker product of the two matrices

$$\mathcal{W}_H(r) = \begin{pmatrix} 1 & -\frac{r}{r+1} \\ r & \frac{r(r+2)}{r+1} \end{pmatrix} \qquad \text{and} \qquad \mathcal{W}_E(s) = \begin{pmatrix} \frac{1}{s+1} & -\frac{n-s+2}{(n-s+1)(2n-s+3)} & 1 \\ -\frac{s}{s+1} & \frac{(n-s+2)(2n-s+2)}{(n-s+1)(2n-s+3)} & 1 \\ -\frac{(n+1)s}{n(s+1)} & -\frac{(n+1)(n-s)(2n-s+2)}{n(n-s+1)(2n-s+3)} & \frac{n-s}{n} \end{pmatrix}. \tag{3.6}$$

The proof given in [KSW97] for $s = n-r$ goes through without modification in the general case, only Lemma 4.4 of [KSW97] has to be reformulated. We remark that this formula simplifies in case $r = 0$ or $s = 0, n$, because some of the operators involved vanish by definition. Though this Weitzenböck formula is powerful enough to prove the eigenvalue estimate for the Dirac operator, it turns out to be insufficient to derive the quaternionic Killing equations of the next section.

For that purpose we need additional Weitzenböck formulas between 2nd order differential operators between different vector bundles, which are not covered by (3.5). Nevertheless, the basic idea of [KSW97] can be applied to derive these additional formulas.

We consider for $s \geq 2$ the isotypical $\mathrm{Sym}^r\mathbf{H} \otimes K^{s-1}\mathbf{E}$–component of the second covariant differential $\nabla^2\psi \in \Gamma(\mathbf{H} \otimes \mathbf{E} \otimes \mathbf{H} \otimes \mathbf{E} \otimes \mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E})$ of a section ψ of $\mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E}$. As this isotypical component contains four copies unless $r = 0$ the resulting formula will in general relate two sets of four projectors.

As in [KSW97] the problem can be split into two parts dealing with $\mathbf{Sp}(1)$ and $\mathbf{Sp}(n)$–representations only. Representation theory of $\mathbf{Sp}(1)$, however, is very simple and no arguments beyond [KSW97] are needed. For this reason we briefly recall that two copies of $\mathrm{Sym}^r H$ are contained in $H \otimes H \otimes \mathrm{Sym}^r H$, but the projectors onto these two copies are not unique. A first pair of projectors is obtained by decomposing $H \otimes H$ into irreducibles, which then act as endomorphisms on $\mathrm{Sym}^r H$:

$$\begin{array}{rccc} pr_{\mathbb{C}}: & H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & \mathrm{Sym}^r H \\ & h_1 \otimes h_2 \otimes s & \longmapsto & \sigma_H(h_1, h_2)s, \\ & & & \\ pr_{\mathrm{Sym}^2 H}: & H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & \mathrm{Sym}^r H \\ & h_1 \otimes h_2 \otimes s & \longmapsto & (h_1 h_2)(s). \end{array}$$

We get a second pair of projectors through the operation of H on $\mathrm{Sym}^r H$ by the H-part of Clifford multiplication:

$$\begin{array}{rccccc} pr_{-+}: & H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & H \otimes \mathrm{Sym}^{r+1} H & \longrightarrow & \mathrm{Sym}^r H \\ & h_1 \otimes h_2 \otimes s & \longmapsto & h_1 \otimes h_2 \cdot s & \longmapsto & h_1^\sharp \lrcorner_\circ (h_2 \cdot s), \\ & & & & & \\ pr_{+-}: & H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & H \otimes \mathrm{Sym}^{r-1} H & \longrightarrow & \mathrm{Sym}^r H \\ & h_1 \otimes h_2 \otimes s & \longmapsto & h_1 \otimes h_2^\sharp \lrcorner_\circ s & \longmapsto & h_1 \cdot (h_2^\sharp \lrcorner_\circ s). \end{array}$$

These two pairs of projectors are related by the matrix $\mathcal{W}_H(r)$:

$$\begin{pmatrix} pr_{\mathbb{C}} \\ \\ pr_{\mathrm{Sym}^2 H} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{r}{r+1} \\ \\ r & \frac{r(r+2)}{r+1} \end{pmatrix} \begin{pmatrix} pr_{-+} \\ \\ pr_{+-} \end{pmatrix}. \tag{3.7}$$

Turning now to the second part of the problem dealing with $\mathbf{Sp}(n)$–representations, we have to look at the isotypical $K^{s-1}E$–component of $E \otimes E \otimes \Lambda^2_\circ E$ containing two copies. Looking at the decomposition $E \otimes E \simeq \mathrm{Sym}^2 E \oplus \Lambda^2_\circ E \oplus \mathbb{C}$ we can write down two projectors immediately:

$$\begin{array}{rccccc} pr_{K\mathrm{Sym}^2 E}: & E \otimes E \otimes \Lambda^s_\circ E & \longrightarrow & \mathrm{Sym}^2 E \otimes \Lambda^s_\circ E & \longrightarrow & K^{s-1}E \\ & e_1 \otimes e_2 \otimes \phi & & \longmapsto & & \widetilde{pr}_K(e_2 \otimes e_1^\sharp \lrcorner\, \phi + e_1 \otimes e_2^\sharp \lrcorner\, \phi) \\ & & & & & \\ pr_{K\Lambda^2_\circ E}: & E \otimes E \otimes \Lambda^s_\circ E & \longrightarrow & \Lambda^2_\circ E \otimes \Lambda^s_\circ E & \longrightarrow & K^{s-1}E \\ & e_1 \otimes e_2 \otimes \phi & & \longmapsto & & \widetilde{pr}_K(e_2 \otimes e_1^\sharp \lrcorner\, \phi - e_1 \otimes e_2^\sharp \lrcorner\, \phi). \end{array}$$

On the other hand, we can first project $E \otimes \Lambda^s_\circ E$ onto $\Lambda^{s-1}_\circ E$ resp. $K^s E$ and then look what the second E-factor can do:

$$\begin{array}{rccccc} pr_{K-}: & E \otimes E \otimes \Lambda^s_\circ E & \longrightarrow & E \otimes \Lambda^{s-1}_\circ E & \longrightarrow & K^{s-1}E \\ & e_1 \otimes e_2 \otimes \phi & & \longmapsto & & \widetilde{pr}_K(e_1 \otimes e_2^\sharp \lrcorner\, \phi), \\ & & & & & \\ pr_{-K}: & E \otimes E \otimes \Lambda^s_\circ E & \longrightarrow & E \otimes K^s E & \longrightarrow & K^{s-1}E \\ & e_1 \otimes e_2 \otimes \phi & & \longmapsto & & \widetilde{pr}_K\big((\mathrm{id} \otimes e_1^\sharp \lrcorner\,)\widetilde{pr}_K(e_2 \otimes \phi)\big). \end{array}$$

The projector pr_{-K} is not yet in its final form. To simplify its definition an operator identity on $\Lambda^{s-1}_\circ E$ comes in handy:

$$\begin{array}{rcl} e_1^\sharp \lrcorner\, e \wedge_\circ & = & -e \wedge_\circ e_1^\sharp \lrcorner\, + \sigma_E(e_1, e) + \frac{1}{n-s+2} e_1 \wedge_\circ e^\sharp \lrcorner \\ & = & -\frac{(n-s+1)(n-s+3)}{(n-s+2)^2} e \wedge_\circ e_1^\sharp \lrcorner\, + \frac{n-s+1}{n-s+2}\sigma_E(e_1, e) - \frac{1}{n-s+2} e^\sharp \lrcorner\, e_1 \wedge_\circ\,, \end{array}$$

which is obtained by applying the anticommutator rules of Lemma 2.2 twice. We remark that by definition the projector $\widetilde{pr}_K : E \otimes \Lambda^s_\circ E \to K^s E$,

$$\widetilde{pr}_K(e \otimes \phi) := e \otimes \phi - \tfrac{1}{s+1} \sum_i e_i \otimes de_i \lrcorner\, e \wedge_\circ \phi - \tfrac{n-s+2}{(2n-s+3)(n-s+1)} \sum_i de_i^\flat \otimes e_i \wedge_\circ e^\sharp \lrcorner\, \phi,$$

kills elements of the form $\sum_i e_i \otimes de_i \lrcorner\, \phi$ and $\sum_i de_i^\flat \otimes e_i \wedge_\circ \phi$. With this in mind the projector pr_{-K} can be made completely explicit:

$$\begin{aligned}
&pr_{-K}(e_1 \otimes e_2 \otimes \phi)\\
&= \widetilde{pr}_K\Big(e_2 \otimes e_1^\sharp \lrcorner\, \phi - \tfrac{1}{s+1} \sum_i e_i \otimes e_1^\sharp \lrcorner\, de_i \lrcorner\, e_2 \wedge_\circ \phi - \tfrac{n-s+2}{(2n-s+3)(n-s+1)} \sum_i de_i^\flat \otimes e_1^\sharp \lrcorner\, e_i \wedge_\circ e_2^\sharp \lrcorner\, \phi\Big)\\
&= \widetilde{pr}_K(e_2 \otimes e_1^\sharp \lrcorner\, \phi)\\
&\quad - \tfrac{n-s+2}{(2n-s+3)(n-s+1)} \widetilde{pr}_K\Big(\sum_i de_i^\flat \otimes (-\tfrac{(n-s+1)(n-s+3)}{(n-s+2)^2} e_i \wedge_\circ e_1^\sharp \lrcorner + \tfrac{n-s+1}{n-s+2} \sigma_E(e_1, e_i) - \tfrac{1}{n-s+2} e_i^\sharp \lrcorner\, e_1 \wedge_\circ) e_2^\sharp \lrcorner\, \phi\Big)\\
&= \widetilde{pr}_K(e_2 \otimes e_1^\sharp \lrcorner\, \phi) - \tfrac{1}{2n-s+3} \widetilde{pr}_K(e_1 \otimes e_2^\sharp \lrcorner\, \phi).
\end{aligned}$$

With this simpler form of pr_{-K} the relations between the projectors become obvious:

$$\begin{pmatrix} pr_{K\mathrm{Sym}^2 E} \\ \\ pr_{K\Lambda^2_\circ E} \end{pmatrix} = \begin{pmatrix} \frac{2n-s+4}{2n-s+3} & 1 \\ \\ -\frac{2n-s+2}{2n-s+3} & 1 \end{pmatrix} \begin{pmatrix} pr_{K-} \\ \\ pr_{-K} \end{pmatrix}. \tag{3.8}$$

In a final step the differential operators associated to the projectors have to be identified. This is simple for the right–hand side projectors, because by definition the associated operators are products of 1st order differential operators. To write down the result, we have to define two new 1st order differential operators, which appear naturally in this way:

$$\theta^\pm : \Gamma(\mathrm{Sym}^r \mathbf{H} \otimes K^s \mathbf{E}) \to \Gamma(\mathrm{Sym}^{r\pm 1} \mathbf{H} \otimes K^{s-1} \mathbf{E})$$

are the composition of the covariant differential $\nabla : \Gamma(\mathrm{Sym}^r \mathbf{H} \otimes K^s \mathbf{E}) \to \Gamma((\mathbf{H} \otimes \mathbf{E}) \otimes \mathrm{Sym}^r \mathbf{H} \otimes K^s \mathbf{E})$ with linear maps

$$\begin{array}{ccccc}
(\mathbf{H} \otimes \mathbf{E}) \otimes \mathrm{Sym}^r \mathbf{H} \otimes K^s \mathbf{E} & \hookrightarrow & (\mathbf{H} \otimes \mathbf{E}) \otimes \mathrm{Sym}^r \mathbf{H} \otimes \mathbf{E} \otimes \Lambda^s_\circ \mathbf{E} & \to & \mathrm{Sym}^{r\pm 1} \mathbf{H} \otimes K^{s-1} \mathbf{E} \\
& & (h \otimes e) \otimes s \otimes \tilde{e} \otimes \phi & \mapsto & \left\{ \begin{array}{c} h \cdot s \\ h^\sharp \lrcorner_\circ s \end{array} \right\} \otimes \widetilde{pr}_K(\tilde{e} \otimes e^\sharp \lrcorner\, \phi)\,.
\end{array}$$

Then the right–hand projectors define the following operator products:

$$\begin{aligned}
pr_{-+} \otimes pr_{K-}(\nabla^2 \phi) &= \tfrac{1}{\sqrt{2}} T^- D^+_- \phi \\
pr_{+-} \otimes pr_{K-}(\nabla^2 \phi) &= \tfrac{1}{\sqrt{2}} T^+ D^-_- \phi \\
pr_{-+} \otimes pr_{-K}(\nabla^2 \phi) &= \theta^- T^+ \phi \\
pr_{+-} \otimes pr_{-K}(\nabla^2 \phi) &= \theta^+ T^- \phi\,.
\end{aligned}$$

The left–hand projectors provide two new 2nd order differential operators:

$$\begin{aligned}
T_{\mathcal{C}} \phi &:= pr_{\mathrm{Sym}^2 H} \otimes pr_{K\mathrm{Sym}^2 E}(\nabla^2 \phi) \\
T_{\mathcal{L}} \phi &:= pr_{\mathbb{C}} \otimes pr_{K\Lambda^2_\circ E}(\nabla^2 \phi)\,,
\end{aligned}$$

which are of no particular importance for the time being, and two linear operators depending only on curvature: $(pr_{\mathrm{Sym}^2 H} \otimes pr_{K\Lambda^2_\circ E}) \circ \nabla^2$ and $(pr_{\mathbb{C}} \otimes pr_{K\mathrm{Sym}^2 E}) \circ \nabla^2$.

It is easy to see that $(pr_{\mathrm{Sym}^2 H} \otimes pr_{K\Lambda^2_\circ E}) \circ \nabla^2$ is the zero operator, because the curvature tensor of a quaternionic Kähler manifold takes values only in the complement $\mathrm{Sym}^2 H \otimes \mathbb{C}$ of $\mathrm{Sym}^2 H \otimes \Lambda^2_\circ E$ in $\mathrm{Sym}^2 H \otimes \Lambda^2 E$. We now claim that the operator $(pr_{\mathbb{C}} \otimes pr_{K\mathrm{Sym}^2 E}) \circ \nabla^2$ is trivial, too.

Of the three summands of the curvature tensor of a quaternionic Kähler manifold according to Lemma 2.4, only R^E and R^{hyper} take values in $\mathbb{C} \otimes \mathrm{Sym}^2 E$. Tracing down the definitions of $(pr_{\mathbb{C}} \otimes pr_{K\mathrm{Sym}^2 E}) \circ \nabla^2$ the contribution of R^{hyper} can be written in terms of $\mathfrak{R} \in \mathrm{Sym}^4 E^*$ as the following morphism from $\mathrm{Sym}^r H \otimes \Lambda^s_\circ E$ to $\mathrm{Sym}^r H \otimes K^{s-1} E$:

$$(pr_{\mathbb{C}} \otimes pr_{K\mathrm{Sym}^2 E}) \circ \nabla^2 \, (s \otimes \phi) = s \otimes \Big(\sum_{ij} \big(\widetilde{pr}_K \circ (de_i^\flat \otimes de_j \lrcorner + de_j^\flat \otimes de_i \lrcorner\,) \circ \mathfrak{R}_{e_i, e_j} \big) \, \phi \Big) .$$

Expanding $\mathfrak{R}$ in fourth powers $\frac{1}{24}\alpha^4$, $\alpha \in E^*$ the contribution from R^{hyper} is seen to vanish because already without projecting $\alpha^\flat \otimes \alpha \lrcorner \, \alpha^\flat \wedge_\circ \alpha \lrcorner \, \phi = 0$ for all ϕ. The contribution of R^E to $(pr_{\mathbb{C}} \otimes pr_{K\mathrm{Sym}^2 E}) \circ \nabla^2$ can be written down similarly. The resulting homomorphism is immediately seen to be $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$–equivariant and has to vanish, because there is no non–trivial way to map $\mathrm{Sym}^r H \otimes \Lambda^s_\circ E$ to $\mathrm{Sym}^r H \otimes K^{s-1} E$ equivariantly.

We conclude this section in tensoring equations (3.7) and (3.8) to get the following twistor Weitzenböck formula:

$$\begin{pmatrix} 0 \\ T_{\mathcal{C}} \\ T_{\mathcal{L}} \\ 0 \end{pmatrix} = \mathcal{W}_{twist} \begin{pmatrix} \frac{1}{\sqrt{2}} T^- D^+_- \\ \frac{1}{\sqrt{2}} T^+ D^-_- \\ \theta^- T^+ \\ \theta^+ T^- \end{pmatrix} \qquad \mathcal{W}_{twist} = \begin{pmatrix} \frac{2n-s+4}{2n-s+3} & -\frac{r}{r+1} \frac{2n-s+4}{2n-s+3} & 1 & -\frac{r}{r+1} \\ r \frac{2n-s+4}{2n-s+3} & \frac{r(r+2)}{r+1} \frac{2n-s+4}{2n-s+3} & r & \frac{r(r+2)}{r+1} \\ -\frac{2n-s+2}{2n-s+3} & \frac{r}{r+1} \frac{2n-s+2}{2n-s+3} & 1 & -\frac{r}{r+1} \\ -r \frac{2n-s+2}{2n-s+3} & -\frac{r(r+2)}{r+1} \frac{2n-s+2}{2n-s+3} & r & \frac{r(r+2)}{r+1} \end{pmatrix} .$$

Corollary 3.1 *The following operator identity holds on sections of the bundle* $\mathrm{Sym}^r \mathbf{H} \otimes \Lambda^s_\circ \mathbf{E}$ *with* $s \geq 2$*:*

$$\frac{1}{\sqrt{2}} \frac{2n-s+4}{2n-s+3} \Big(T^- D^+_- - \frac{r}{r+1} T^+ D^-_- \Big) + \Big(\theta^- T^+ - \frac{r}{r+1} \theta^+ T^- \Big) \quad = \quad 0.$$

This identity is trivially satisfied for $s = 1$*, because* $T^- D^+_-$*,* $T^+ D^-_-$*,* $\theta^- T^+$ *and* $\theta^+ T^-$ *all vanish separately.*

4 The Quaternionic Killing Equation

The matrix Weitzenböck formula (3.5) generates a linear space of operator identities of 2nd order differential operators from a bundle $\mathbf{S}_r(M)$ to itself by multiplying it from the left the an arbitrary row vector. A particularly important identity in this linear spaces leads to the following key identity of operator norms for any section ψ_r of $\mathbf{S}_r(M)$:

$$\tfrac{(r+2)(n+r+2)}{n+2} \tfrac{\kappa}{4} \, \|\psi_r\|^2 \;=\; -\, \tfrac{r+1}{n-r+1} \, \|D^+_+ \psi_r\|^2 + (r+2) \, \|D^-_+ \psi_r\|^2 + \tfrac{(r+2)(n+r+2)}{n+r+3} \, \|D^+_- \psi_r\|^2 - 2(r+1) \, \|T^+ \psi_r\|^2. \tag{4.9}$$

Since D^2 respects the splitting of $\mathbf{S}(M)$ into the subbundles $\mathbf{S}_r(M)$ an eigenspinor of D can be assumed to be localized in $\mathbf{S}_r(M) \oplus \mathbf{S}_{r+1}(M)$. If $\psi = \psi_r + \psi_{r+1}$ is such an eigenspinor with eigenvalue λ, then $D^+_- \psi_r = \lambda \psi_{r+1}$, $D^-_+ \psi_r = 0$ and identity (4.9) implies for ψ_r:

$$\tfrac{\kappa}{4} \tfrac{n+r+3}{n+2} \|\psi_r\|^2 = \lambda^2 \|\psi_{r+1}\|^2 - \tfrac{r+1}{r+2} \tfrac{n+r+3}{n+r+2} \Big(\tfrac{1}{n-r+1} \|D^+_+ \psi_r\|^2 + 2 \|T^+ \psi_r\|^2 \Big).$$

In the same vein identity (4.9) implies for ψ_{r+1}:

$$\tfrac{\kappa}{4} \tfrac{n+r+3}{n+2} \|\psi_{r+1}\|^2 = \lambda^2 \|\psi_r\|^2 - \tfrac{r+2}{r+3} \Big(\tfrac{1}{n-r} \|D^+_+ \psi_{r+1}\|^2 + 2 \|T^+ \psi_{r+1}\|^2 \Big).$$

These identities prove:

Theorem 4.1 *Let (M^{4n}, g) be a compact quaternionic Kähler spin manifold of positive scalar curvature κ and let $\psi = \psi_r + \psi_{r+1} \in \Gamma(\mathbf{S}_r(M) \oplus \mathbf{S}_{r+1}(M))$ be an eigenspinor for D with eigenvalue λ. Then*

$$\lambda^2 \;\geq\; \frac{\kappa}{4}\,\frac{n+r+3}{n+2}$$

with equality if and only if ψ_r and ψ_{r+1} are both minimal in the sense that $D^+_+\psi_r = 0 = D^+_+\psi_{r+1}$ and $T^+\psi_r = 0 = T^+\psi_{r+1}$.

Thus an eigenspinor ψ of D with smallest possible eigenvalue λ with $\lambda^2 = \frac{\kappa}{4}\frac{n+3}{n+2}$ has to be of the form $\psi = \psi_0 + \psi_1 \in \Gamma(\mathbf{S}_0(M) \oplus \mathbf{S}_1(M))$. As ψ_0 is a section of $\mathbf{S}_0(M)$ its covariant differential $\nabla\psi$ splits in only two pieces $D^+_-\psi$ and $T^+\psi$ according to Lemma 2.1. Minimality implies $T^+\psi_0 = 0$ and Lemma 2.6 reconstructs $\nabla\psi_0$ from $D^+_-\psi_0 = \lambda\psi_1$:

$$\nabla\psi_0 \quad = \quad \lambda\iota^-_+\psi_1$$

The covariant differential of ψ_1 is more complicated as it splits into six pieces: $D^\pm_\pm\psi_1$ and $T^\pm\psi_1$. Minimality implies $D^+_+\psi_1 = 0$ and $T^+\psi_1 = 0$, and as part of the Dirac operator $D^+_-\psi_1 = 0$. Plugging this into the Weitzenböck formula (3.5) the second, third and sixth row read:

$$\begin{pmatrix} \frac{\kappa}{4}\frac{3}{n+2}\psi_1 \\ \frac{\kappa}{4}\frac{(n+3)(n-1)}{n(n+2)}\psi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2n} & -\frac{9}{4(n+4)} & \frac{3}{2} \\ \frac{n-1}{2n} & -\frac{3(n+3)}{4(n+4)} & -\frac{1}{2} \\ -\frac{3(n-1)(n+1)}{2n^2} & -\frac{3(n+3)(n+1)}{4n(n+4)} & \frac{3}{2n} \end{pmatrix} \begin{pmatrix} \frac{1}{2}D^+_-D^-_+\psi_1 \\ -\frac{1}{2}(D^-_-)^*D^-_-\psi_1 \\ (T^-)^*T^-\psi_1 \end{pmatrix}$$

With $D^+_-D^-_+\psi_1 = \frac{\kappa}{4}\frac{n+3}{n+2}\psi_1$ this system of equations may be solved for $(D^-_-)^*D^-_-\psi_1$ and $(T^-)^*T^-\psi_1$ to find:

$$(D^-_-)^*D^-_-\psi_1 = \tfrac{\kappa}{2}\tfrac{(n+4)(n-1)}{n(n+2)}\psi_1 \qquad \text{and} \qquad (T^-)^*T^-\psi_1 = 0\,.$$

But, since we are on a compact manifold this implies $T^-\psi_1 = 0$. Independently, $T^-\psi_1 = 0$ can be shown by applying Corollary 3.1 to ψ_0. With the help of Lemma 2.6 the covariant differential $\nabla\psi_1$ is reconstructed from $D^-_+\psi_1 = \lambda\psi_0$ and $D^-_-\psi_1$:

$$\nabla\psi_1 \;=\; \lambda\,\iota^+_-(\psi_0) \;+\; \iota^+_+(D^-_-\psi_1).$$

This in turn provides a first version of the quaternionic Killing equations:

$$\begin{aligned} \nabla_X\psi_0 \quad &= \quad -\tfrac{\lambda}{n+3}\,\mu^-_+(X)\,\psi_1 \\ \nabla_X\psi_1 \quad &= \quad -\tfrac{\lambda}{4n}\,\mu^+_-(X)\,\psi_0 \;+\; \tfrac{3}{8(n+4)}\,\mu^+_+(X)\,D^-_-\psi_1. \end{aligned}$$

Unfortunately it is not possible to say much about the section $D^-_-\psi_1$. The idea to overcome this obstacle is to include this special section and to consider a quaternionic Killing equation for two spinors and an auxiliary section of the bundle $\Lambda^{n-2}_\circ\mathbf{E}$. This yields indeed a useful Killing equation due to the following proposition:

Proposition 4.1

$$\nabla_X(D^-_-\psi_1) \;=\; -\tfrac{\kappa}{4}\,\tfrac{n+4}{n(n+2)}\,\mu^-_-(X)\,\psi_1.$$

Proof. Since $\psi_- := D^-_-\psi_1$ is a section of $\Lambda^{n-2}_\circ\mathbf{E}$ its covariant differential $\nabla\psi_-$ splits into three pieces: $D^\pm_+\psi_-$ and $T^+\psi_-$. However, from Corollary 3.1 applied to ψ_1 we conclude $T^+\psi_- = T^+D^-_-\psi_1 = 0$. With $D^+_+ = -(D^-_-)^*$ we have in addition:

$$(D^+_+)^*D^+_+\psi_- = D^-_-(D^-_-)^*D^-_-\psi_1 = \tfrac{\kappa}{2}\tfrac{(n+4)(n-1)}{n(n+2)}\psi_-.$$

Now the third row of the Weitzenböck formula (3.5) applied to ψ_- reads

$$\tfrac{\kappa}{4}\tfrac{(n+4)(n-2)}{n(n+2)}\psi_- = \tfrac{n-2}{2(n-1)}(D_+^+)^* D_+^+\psi_- + \tfrac{2(n+4)}{3(n+5)}D_+^- D_-^+\psi_- = \tfrac{\kappa}{4}\tfrac{(n+4)(n-2)}{n(n+2)}\psi_- + \tfrac{2(n+4)}{3(n+5)}(D_-^+)^* D_-^+\psi_-.$$

On the compact manifold M this implies $D_-^+\psi_- = 0$. Thus the covariant differential of ψ_- can be reconstructed from $D_+^+\psi_- = -(D_-^-)^* D_-^-\psi_1 = -\frac{k}{2}\frac{(n+4)(n-1)}{n(n+2)}\psi_1$ in the spirit of Lemma 2.6:

$$\nabla\psi_- = -\tfrac{\kappa}{2}\tfrac{(n+4)(n-1)}{n(n+2)}\iota_-^-(\psi_1).$$

The proposition follows. $\square$

Changing slightly the notation we obtain

Theorem 4.2 *Let (M^{4n}, g) be a compact quaternionic Kähler spin manifold of quaternionic dimension n and with positive scalar curvature κ. Let $\psi = \psi_0 + \psi_1 \in \Gamma(\mathbf{S}_0(M) \oplus \mathbf{S}_1(M))$ be an eigenspinor for the smallest possible eigenvalue λ, i. e. $\lambda^2 = \frac{\kappa}{4}\frac{n+3}{n+2}$. Then ψ_0, ψ_1 and $\psi_- := \frac{1}{4\lambda}\frac{n+3}{n+4}D_-^-\psi_1 \in \Gamma(\Lambda_\circ^{n-2}\mathbf{E})$ solve the following quaternionic Killing equation for the parameter λ:*

$$\begin{aligned}
\nabla_X\psi_0 &= && -\tfrac{\lambda}{n+3}\,\mu_+^-(X)\,\psi_1\\
\nabla_X\psi_1 &= -\tfrac{\lambda}{4n}\,\mu_-^+(X)\,\psi_0 && +\tfrac{3\lambda}{2(n+3)}\,\mu_+^+(X)\,\psi_-\\
\nabla_X\psi_- &= -\tfrac{\lambda}{4n}\,\mu_-^-(X)\,\psi_1 &&
\end{aligned}$$

Conversely, if the triple ψ_0, ψ_1, ψ_- is a solution of these equations for any $\lambda \neq 0$ then

$$\begin{array}{lll}
D_-^+\psi_0 \;=\; \lambda\,\psi_1 & D_+^-\psi_1 \;=\; \lambda\,\psi_0 & D_+^+\psi_- \;=\; -\lambda\,\tfrac{n-1}{2n}\,\psi_1\\
T^+\psi_0 \;=\; 0 & D_-^-\psi_1 \;=\; 4\lambda\,\tfrac{n+4}{n+3}\,\psi_- & D_-^+\psi_- \;=\; 0\\
 & D_\pm^+\psi_1 \;=\; 0 & T^+\psi_- \;=\; 0\\
 & T^\pm\psi_1 \;=\; 0 &
\end{array}$$

In particular, $\psi_0 + \psi_1$ is an eigenspinor for the smallest possible eigenvalue.

We will call solutions of the quaternionic Killing equations *quaternionic Killing spinors.* These are sections $\psi = (\psi_0, \psi_1, \psi_-)$ of the bundle

$$\mathbf{S}^{Killing}(M) \;:=\; \mathbf{S}_0(M) \oplus \mathbf{S}_1(M) \oplus \Lambda_\circ^{n-2}\mathbf{E} \;\cong\; \Lambda_\circ^n\mathbf{E} \oplus (\mathbf{H}\otimes\Lambda_\circ^{n-1}\mathbf{E}) \oplus \Lambda_\circ^{n-2}\mathbf{E}.$$

As in the Riemannian or the Kähler case it is possible to define a Killing connection $\nabla^{Killing}$ for which quaternionic Killing spinors are parallel. On sections of the bundle $\mathbf{S}^{Killing}(M)$ this connection is given by $\nabla_X^{Killing} := \nabla_X + A_X$ where A_X is the following matrix

$$A_X = \begin{pmatrix} 0 & \frac{\lambda}{n+3}\,\mu_+^-(X) & 0\\[2mm] \frac{\lambda}{4n}\,\mu_-^+(X) & 0 & -\frac{3\lambda}{2(n+3)}\,\mu_+^+(X)\\[2mm] 0 & \frac{\lambda}{4n}\,\mu_-^-(X) & 0 \end{pmatrix}.$$

Hence, quaternionic Killing spinors $\psi = (\psi_0, \psi_1, \psi_-)$ are annihilated by the curvature of the Killing connection, i. e. $R_{X,Y}^{Killing}\psi = 0$ for any vector fields X, Y. The following proposition is crucial for further investigations of the Killing equations:

Proposition 4.2

$$R^{Killing} \;=\; R^{hyper}$$

Proof. As the partial Clifford multiplications are $\mathbf{Sp}(n)\cdot\mathbf{Sp}(1)$-equivariant, they are parallel with respect to the Levi–Civitá connection, and so is the endomorphism–valued 1-form A. Defining for arbitrary endomorphism–valued 1-forms B and C the endomorphism–valued 2–form $(B\wedge C)_{X,Y} := B_X\circ C_Y - B_Y\circ C_X$ the usual formula relating the curvature of ∇ and $\nabla^{Killing} = \nabla + A$ reads

$$R^{Killing} \;=\; R + [A,A] \;=\; R + \frac{\lambda^2}{4n(n+3)}\begin{pmatrix} \mu_+^-\wedge\mu_-^+ & 0 & \frac{6n}{n+3}\mu_+^-\wedge\mu_+^+ \\ 0 & \mu_-^+\wedge\mu_+^- - \frac{3}{2}\mu_+^+\wedge\mu_-^- & 0 \\ \frac{n+3}{4n}\mu_-^-\wedge\mu_-^+ & 0 & -\frac{3}{2}\mu_-^-\wedge\mu_+^+ \end{pmatrix}. \quad (4.10)$$

The entries of this matrix are endomorphism–valued 2-forms, which can be simplified further using Lemmas 2.2 and 2.3. Considering tangent vectors of the form $X = h_1\otimes e_1$ and $Y = h_2\otimes e_2$ we get:

$$\begin{aligned} (\mu_-^-\wedge\mu_-^+)_{X,Y} &= 2\sigma_H(h_1,h_2)(e_1^\sharp\lrcorner\, e_2^\sharp\lrcorner\, + e_2^\sharp\lrcorner\, e_1^\sharp\lrcorner\,) &= 0 \\ (\mu_+^-\wedge\mu_+^+)_{X,Y} &= 2\sigma_H(h_1,h_2)(e_1\wedge_\circ e_2\wedge_\circ + e_2\wedge_\circ e_1\wedge_\circ) &= 0 \end{aligned}$$

and

$$\begin{aligned} (\mu_+^-\wedge\mu_-^+)_{X,Y} &= 2\sigma_H(h_1,h_2)(e_1\wedge_\circ e_2^\sharp\lrcorner\, + e_2\wedge_\circ e_1^\sharp\lrcorner\,) &= 2R^E_{X,Y} \\ (\mu_-^-\wedge\mu_+^+)_{X,Y} &= 2\sigma_H(h_1,h_2)(e_1^\sharp\lrcorner\, e_2\wedge_\circ + e_2^\sharp\lrcorner\, e_1\wedge_\circ) & \\ &= -\tfrac{4}{3}\sigma_H(h_1,h_2)(e_1\wedge_\circ e_2^\sharp\lrcorner\, + e_2\wedge_\circ e_1^\sharp\lrcorner\,) &= -\tfrac{4}{3}R^E_{X,Y} \end{aligned}$$

with $e_1^\sharp\lrcorner\, e_2\wedge_\circ = -e_2\wedge_\circ e_1^\sharp\lrcorner\, + \sigma_E(e_1,e_2) + \frac{1}{3}e_1\wedge_\circ e_2^\sharp\lrcorner$ on $\Lambda_\circ^{n-2}E$. Using the same argument the calculation of the last matrix entry reduces to

$$\begin{aligned} &(\mu_-^+\wedge\mu_+^- - \tfrac{3}{2}\mu_+^+\wedge\mu_-^-)_{X,Y} \\ &= 2\Big(h_1\cdot h_2^\sharp\lrcorner_\circ\otimes(-e_2\wedge_\circ e_1^\sharp\lrcorner\, + \sigma_E(e_1,e_2) - e_1\wedge_\circ e_2^\sharp\lrcorner\,) - h_2\cdot h_1^\sharp\lrcorner_\circ\otimes(-e_1\wedge_\circ e_2^\sharp\lrcorner\, + \sigma_E(e_2,e_1) - e_2\wedge_\circ e_1^\sharp\lrcorner\,)\Big) \\ &= 2\sigma_E(e_1,e_2)(h_1\cdot h_2^\sharp\lrcorner_\circ + h_2\cdot h_1^\sharp\lrcorner_\circ)\otimes\mathrm{id} + 2(h_2\cdot h_1^\sharp\lrcorner_\circ - h_1\cdot h_2^\sharp\lrcorner_\circ)\otimes(e_1\wedge_\circ e_2^\sharp\lrcorner_\circ + e_2\wedge_\circ e_1^\sharp\lrcorner_\circ) \\ &= 2(R^H + R^E)_{X,Y} \end{aligned}$$

using $h_2\cdot h_1^\sharp\lrcorner_\circ - h_1\cdot h_2^\sharp\lrcorner_\circ = \sigma_H(h_1,h_2)$ on H. From Lemma 2.4 we know the explicit form of the curvature tensor R. Combining this with the above computations of the matrix entries in formula (4.10) for $R^{Killing}$ we have

$$R^{Killing} = R + \frac{\kappa}{16n(n+2)}\begin{pmatrix} 2R^E & 0 & 0 \\ 0 & 2(R^H+R^E) & 0 \\ 0 & 0 & 2R^E \end{pmatrix} = R^{hyper}$$

because R^H annihilates $\Lambda_\circ^n\mathbf{E}$ and $\Lambda_\circ^{n-2}\mathbf{E}$. $\square$

Corollary 4.1 *The vector bundle $\mathbf{S}^{Killing}(\mathbb{H}P^n)$ is trivial. Any constant section projects to a unique eigenspinor on $\mathbb{H}P^n$ with minimal eigenvalue.*

Let $\psi = (\psi_0,\psi_1,\psi_-)$ be a quaternionic Killing spinor. We will now show how to construct eigenfunctions of the Laplace operator as linear combinations of the length functions of the three components. Similar constructions were already considered in the Riemannian and the Kähler case. The main tool is the following formula which holds for any section ψ of an hermitean vector bundle with hermitean connection ∇:

$$\Delta\,|\psi|^2 \;=\; 2\,\mathrm{Re}\,(\nabla^*\nabla\,\psi,\,\psi) \;-\; 2|\nabla\,\psi|^2.$$

From this formula and the quaternionic Killing equations it is easy to derive

$$\Delta \begin{pmatrix} |\psi_0|^2 \\ |\psi_1|^2 \\ |\psi_-|^2 \end{pmatrix} = \frac{2\lambda^2}{n+3} \begin{pmatrix} 1 & -1 & 0 \\ -\frac{n+3}{4n} & 1 & -\frac{6(n+4)}{n+3} \\ 0 & -\frac{(n+3)(n-1)}{8n^2} & \frac{n+4}{n} \end{pmatrix} \begin{pmatrix} |\psi_0|^2 \\ |\psi_1|^2 \\ |\psi_-|^2 \end{pmatrix}.$$

Diagonalizing the above matrix leads to the definition of the following functions

$$\begin{aligned} f_0 &:= \tfrac{n+3}{4n}\,|\psi_0|^2 + |\psi_1|^2 + \tfrac{6n}{n+3}\,|\psi_-|^2 \\ f_1 &:= -\tfrac{n+3}{4n}\,|\psi_0|^2 + \tfrac{1}{n}\,|\psi_1|^2 + \tfrac{2(n+4)}{n+3}\,|\psi_-|^2 \\ f_2 &:= \tfrac{n+3}{4n}\,|\psi_0|^2 - \tfrac{n+3}{n}\,|\psi_1|^2 + \tfrac{6(n+4)}{n-1}\,|\psi_-|^2. \end{aligned}$$

Proposition 4.3 *The functions f_0, f_1, f_2 are eigenfunctions of the Laplace operator and the eigenvalues are the first three eigenvalues of the Laplace operator on the quaternionic projective space. More precisely, f_0 is a constant and*

$$\Delta f_1 = \frac{\kappa}{2n}\frac{n+1}{n+2}\, f_1 \quad \textit{and} \quad \Delta f_2 = \frac{\kappa}{2n}\frac{2n+3}{n+2}\, f_2.$$

That f_0 is constant suggests that $\nabla^{Killing}$ is an hermitean connection with respect to a modified scalar product on the bundle $\mathbf{S}^{Killing}(M)$ defined for two sections $\psi = (\psi_0,\, \psi_1,\, \psi_-)$ and $\phi = (\phi_0,\, \phi_1,\, \phi_-)$ by

$$\langle \psi,\, \phi\rangle = \frac{n+3}{4n}\,(\psi_0,\, \phi_0) + (\psi_1,\, \phi_1) + \frac{6n}{n+3}\,(\psi_-,\, \phi_-).$$

Proposition 4.4 *The connection $\nabla^{Killing}$ is hermitean with respect to $\langle,\rangle$.*

Proof. We have $\nabla^{Killing}_X = \nabla_X + A_X$. Since ∇ is an hermitean connection on the different components of $\mathbf{S}^{Killing}(M)$ we only have to prove that A_X is a skew–hermitean endomorphism on $\mathbf{S}^{Killing}(M)$ with respect to $\langle,\rangle$ for all real tangent vectors X or equivalently $\mathrm{Re}\,\langle A_X\psi, \psi\rangle = 0$ for all ψ. This is straightforward use of the general formulas (cf. [KSW97]):

$$(\mu^-_+(X)\,\psi_1,\, \psi_0) = -(\psi_1,\, \mu^+_-(\overline{X})\,\psi_0) \qquad \text{and} \qquad (\mu^+_+(X)\,\psi_-,\, \psi_1) = (\psi_-,\, \mu^-_-(\overline{X})\,\psi_1). \qquad \square$$

The first non–zero eigenvalue of the Laplace operator on a compact quaternionic Kähler manifold of scalar curvature κ is greater or equal to $\frac{\kappa}{2n}\frac{n+1}{n+2}$ and it is a well known fact (cf. [AlMa95] or [LeB95]) that equality is attained if and only if the manifold is isometric to the quaternionic projective space. Hence, the above construction yields the following proposition:

Proposition 4.5 *Let $(M,\, g)$ be a compact quaternionic Kähler spin manifold which admits a quaternionic Killing spinor such that the associated function f_1 is not identically zero. Then M is isometric to the quaternionic projective space.*

5 The Killing Curvature on Wolf Spaces

In this section we determine the eigenvalues of the hyperkähler part of the curvature tensor considered as a symmetric bilinear form on $\mathfrak{sp}(n)$ for the Wolf spaces, i. e. the symmetric compact quaternionic Kähler manifolds. In particular we show that this symmetric bilinear form is always regular except for the quaternionic projective space and conclude:

Lemma 5.1 *Let M be a Wolf space other than $\mathbb{H}P^n$, then the values of R^{hyper} span $\mathfrak{sp}(n)$ at every point $p \in M$ in the following sense*

$$\mathrm{span}\{R^{hyper}_{X,Y} \text{ with } X, Y \in T_pM\} = \mathfrak{sp}(n)$$

Thus there are no parallel sections of $\mathbf{S}^{Killing}(M)$, because if $n > 2$ there are no $\mathfrak{sp}(n)$-invariant elements in the representation $\Lambda^n_\circ E \oplus (H \otimes \Lambda^{n-1}_\circ E) \oplus \Lambda^{n-2}_\circ E$. In case $n = 2$ there is up to scalar a unique $\mathfrak{sp}(n)$-invariant element and in consequence the curvature R^{hyper} vanishes on sections of the trivial line bundle $\Lambda^0_\circ \mathbf{E} \subset \mathbf{S}^{Killing}(M)$. Nevertheless no nontrivial section of this line bundle is parallel with respect to the Killing connection. This shows that there are no quaternionic Killing spinors on Wolf spaces besides $\mathbb{H}P^n$.

5.1 The Curvature Endomorphism on Symmetric Spaces

Let $G/_K$ be an symmetric space without euclidean factors, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding decomposition of the semisimple Lie algebra $\mathfrak{g}$ of G into eigenspaces of the Cartan–involution. The Killing–form B of $\mathfrak{g}$ is non–degenerate on $\mathfrak{g}$ and assumed either negative or positive definite on $\mathfrak{p}$. The Riemannian metric on $G/_K$ is defined accordingly by $g = \mp B$, the upper sign corresponding to the negative definite or compact case, the lower to the positive definite or non–compact case. Despite the choice of metric the inverse isomorphisms $\sharp : \mathfrak{p} \to \mathfrak{p}^*$ and $\flat : \mathfrak{p}^* \to \mathfrak{p}$ are always taken with respect to the Killing–form.

The decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ allow to define two partial Killing–forms $B_\mathfrak{k}$ and $B_\mathfrak{p}$ for $X, Y \in \mathfrak{g}$:

$$B_\mathfrak{k}(X,Y) = \mathrm{tr}_\mathfrak{k}(ad_X \circ ad_Y) \qquad\qquad B_\mathfrak{p}(X,Y) = \mathrm{tr}_\mathfrak{p}(ad_X \circ ad_Y)$$

with $B = B_\mathfrak{k} + B_\mathfrak{p}$ by definition. It is well known and easy to prove that $B_\mathfrak{k}$ and $B_\mathfrak{p}$ are symmetric, vanish on $\mathfrak{k} \times \mathfrak{p}$ and agree on $\mathfrak{p} \times \mathfrak{p}$, in particular $B_\mathfrak{k} = B_\mathfrak{p} = \frac{1}{2}B$ on $\mathfrak{p} \times \mathfrak{p}$. However, no such simple relation is true on $\mathfrak{k} \times \mathfrak{k}$. In fact, if

$$\mathfrak{k} = \bigoplus_{i=1}^r \mathfrak{k}_i$$

is the orthogonal decomposition of $\mathfrak{k}$ into (center and) simple ideals, by Schur's Lemma there exists constants $l_i \in \mathbb{R}$ such that

$$B_\mathfrak{k}|_{\mathfrak{k}_i \times \mathfrak{k}_i} = B_{\mathfrak{k}_i} = l_i \, B|_{\mathfrak{k}_i \times \mathfrak{k}_i}$$

Let L be the endomorphism of $\mathfrak{k}$ defined by $L|_{\mathfrak{k}_i} = l_i \,\mathrm{id}$, then the relation between the partial Killing–forms on $\mathfrak{k} \times \mathfrak{k}$ can be expressed as follows:

$$B_\mathfrak{k}(K_1, K_2) = B(LK_1, K_2) \qquad\qquad B_\mathfrak{p}(K_1, K_2) = B((\mathrm{id} - L)K_1, K_2)$$

Next we define the cobracket $\Delta : \mathfrak{k} \to \Lambda^2\mathfrak{p}$ by

$$B(\Delta K, X \wedge Y) := B(K, [X,Y]) = B([K,X], Y) = B(X^\sharp \lrcorner\, \Delta K, Y)$$

for all $X, Y \in \mathfrak{p}$, in particular Δ is a Lie algebra homomorphism of $\mathfrak{k}$ into $\mathfrak{so}\,\mathfrak{p} \cong \Lambda^2\mathfrak{p}$. The cobracket may be used to define an endomorphism of $\mathfrak{k}$ by $\mathfrak{k} \xrightarrow{\Delta} \Lambda^2\mathfrak{p} \xrightarrow{[,]} \mathfrak{k}$.

Lemma 5.2 *The endomorphism above equals* $\dfrac{L - \mathrm{id}}{2}$.

To prove this lemma one needs to expand the cobracket in terms of a dual pair of bases $\{E_i\}$ and $\{dE_i\}$ of $\mathfrak{p}$ and $\mathfrak{p}^*$, namely $\Delta K = \sum_{i<j} B([K, E_i], E_j) dE_i^\flat \wedge dE_j^\flat$:

$$\begin{aligned} B(\,[\Delta K_1]\,, K_2\,) &= B(\Delta K_1, \Delta K_2) = \frac{1}{2}\sum_{ij} B([K_1, E_i], E_j) B([K_2, dE_i^\flat], dE_j^\flat) \\ &= -\frac{1}{2} B([K_2, [K_1, E_i]], dE_i^\flat) = B(\frac{L - \mathrm{id}}{2} K_1, K_2) \end{aligned}$$

The extensions of the metric or the Killing–form $g = \mp B$ to $\Lambda^2\mathfrak{p}$ are positive definite and in fact agree. Thus the curvature endomorphism $\rho : \Lambda^2\mathfrak{p} \to \Lambda^2\mathfrak{p}$ is uniquely defined by

$$B(\rho(X \wedge Y), Z \wedge W) = B(R_{X,Y}Z, W) = -B([X,Y],[Z,W]) = -B(\Delta[X,Y], Z \wedge W)$$

i. e. $\rho = -\Delta \circ [,]$. Together with $[,] \circ \Delta = \frac{L - \mathrm{id}}{2}$ it follows that ρ is diagonalizable and may be identified with $\frac{\mathrm{id} - L}{2}$ on the image $\Delta\mathfrak{k}$ of $\mathfrak{k}$. Note that due to $B(R_{X,Y}Z, W) = \pm g(R_{Y,X}Z, W)$ the definition of ρ is the standard one only in the compact case. With this in mind the scalar curvature κ of $G/_K$ becomes:

$$\pm\kappa \quad = \quad 2\,\mathrm{tr}_{\Lambda^2\mathfrak{p}}\rho \quad = \quad \mathrm{tr}_{\mathfrak{k}}(\mathrm{id} - L)$$

Combined with the well known formula for the Ricci curvature

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_i B(R_{E_i,X}Y, dE_i^\flat) &= -\sum_i B(dE_i^\flat, [Y,[X,E_i]]) \\ &= -B_{\mathfrak{p}}(X,Y) &= \pm\frac{1}{2}g(X,Y) \end{aligned}$$

implying $\kappa = \pm\frac{\dim \mathfrak{p}}{2}$ this formula relates the dimensions of $\mathfrak{p}$ and of the $\mathfrak{k}_i$ with the eigenvalues of L:

Lemma 5.3

$$\pm\kappa = \frac{\dim \mathfrak{p}}{2} = \mathrm{tr}_{\mathfrak{k}}(\mathrm{id} - L)$$

5.2 The Eigenvalues of the Curvature Endomorphisms

Lemma 5.4 *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{k}^0$ be the Lie algebra decomposition of a Wolf space $G/_K$ of quaternionic dimension $n = \frac{1}{4}\dim \mathfrak{p}$. Then the Killing–forms B of $\mathfrak{g}$ restricted to $\mathfrak{sp}(1)$ and $B_{\mathfrak{sp}(1)}$ of $\mathfrak{sp}(1)$ are related by*

$$B_{\mathfrak{sp}(1)} = \frac{2}{n+2}B$$

i. e. the constant $l_{\mathfrak{sp}(1)}$ is always $\frac{2}{n+2}$.

Proof. From Wolf's construction there exists a base I, J, K of $\mathfrak{sp}(1)$ satisfying $[I,J] = 2K$ etc. such that ad_I is a complex structure on $\mathfrak{p}$. As $\mathfrak{k}^0$ centralizes $\mathfrak{sp}(1)$ the trace of $ad_I \circ ad_I$ on $\mathfrak{g}$ is easily computed:

$$B(I,I) = \text{trace on } \mathfrak{sp}(1) + \text{trace on } \mathfrak{p} = -8 - 4n.$$

On the other hand $B_{\mathfrak{sp}(1)}(I,I) = -8 = \frac{2}{n+2}B(I,I)$. □

This result together with Lemma 5.3 will be used in the sequel to calculate the curvature endomorphisms ρ of all Wolf–spaces:

$\mathbf{Sp}(n+1)/_{\mathbf{Sp}(1) \times \mathbf{Sp}(n)}$: On the quaternionic projective spaces $\mathfrak{k}$ decomposes into $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$. As $\mathfrak{sp}(n)$ is the compact form of $\mathfrak{sp}\,\mathbb{C}^{2n}$ the Killing–form is $(2n+2)\mathrm{tr}_{\mathbb{C}^{2n}}$. Now $\mathfrak{sp}(n)$ is embedded in $\mathfrak{sp}(n+1)$ via the embedding of the defining representations $\mathbb{C}^{2n} \hookrightarrow \mathbb{C}^{2n+2}$. In consequence, the Killing–forms are related by the constant $l_{\mathfrak{sp}(n)} = \frac{2n+2}{2n+4} = \frac{n+1}{n+2}$. Thus the curvature endomorphism of the quaternionic projective space reads:

$$\rho = \frac{n}{2(n+2)}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{1}{2(n+2)}\mathrm{proj}_{\mathfrak{sp}(n)}.$$

$\mathbf{SU}(n+2)/_{\mathbf{S}(\mathbf{U}(2) \times \mathbf{U}(n))}$: On the complex Grassmannian of 2-planes $\mathfrak{k}$ decomposes into $\mathfrak{sp}(1) \oplus \mathfrak{su}(n) \oplus \mathbb{R}$. The constant $l_{\mathbb{R}}$ is certainly 0. As $\mathfrak{su}(n)$ is the compact form of $\mathfrak{sl}(n)$ the

Killing–form is $2n\ \mathrm{tr}_{\mathbb{C}^n}$ and one concludes that $l_{\mathfrak{su}(n)} = \frac{2n}{2n+4} = \frac{n}{n+2}$. The curvature endomorphism of the complex Grassmannian is thus:

$$\rho = \frac{n}{2(n+2)}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{1}{n+2}\mathrm{proj}_{\mathfrak{su}(n)} + \frac{1}{2}\mathrm{proj}_{\mathbb{R}}.$$

$\mathbf{SO}(n+4)/\mathbf{S}(\mathbf{O}(4)\times\mathbf{O}(n))$: On the real Grassmannian of 4-planes $\mathfrak{k}$ decomposes into $\mathfrak{sp}(1)\oplus\widetilde{\mathfrak{sp}}(1)\oplus\mathfrak{so}(n)$. The Killing–form of $\mathfrak{so}(n)$ is $(n-2)\mathrm{tr}_{\mathbb{C}^n}$. Arguing in the same way as before one finds $l_{\mathfrak{so}(n)} = \frac{n-2}{n+2}$. The last constant is then calculated with the help of Lemma 5.3 and agrees with $l_{\mathfrak{sp}(1)} = l_{\widetilde{\mathfrak{sp}}(1)} = \frac{2}{n+2}$, because either of the two subalgebras could be used to define the quaternionic structure. The curvature endomorphism is:

$$\rho = \frac{n}{2(n+2)}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{n}{2(n+2)}\mathrm{proj}_{\widetilde{\mathfrak{sp}}(1)} + \frac{2}{n+2}\mathrm{proj}_{\mathfrak{so}(n)}.$$

$\mathbf{G}_2/\mathbf{SO}(4)$: In this case $\dim\mathfrak{p} = 8$ or $n=2$ and $\mathfrak{k} = \mathfrak{sp}(1)\oplus\widetilde{\mathfrak{sp}}(1)$. However $\mathfrak{sp}(1)$ and $\widetilde{\mathfrak{sp}}(1)$ are definitely distinct, $\widetilde{\mathfrak{sp}}(1)$ does not define a proper quaternionic structure. In fact their constants differ, because Lemma 5.3 implies $l_{\widetilde{\mathfrak{sp}}(1)} = \frac{1}{6}$.

$$\rho = \frac{1}{4}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{5}{12}\mathrm{proj}_{\widetilde{\mathfrak{sp}}(1)}.$$

$\mathbf{F}_4/\mathbf{Sp}(1)\mathbf{Sp}(3)$: This space is not spin, however Lemma 5.1 may be of independent interest. With $\dim\mathfrak{p} = 28$ the quaternionic dimension is $n=7$ and $\mathfrak{k} = \mathfrak{sp}(1)\oplus\mathfrak{sp}(3)$. Using Lemma 5.3 one finds $l_{\mathfrak{sp}(3)} = \frac{4}{9}$ and

$$\rho = \frac{7}{18}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{5}{18}\mathrm{proj}_{\mathfrak{sp}(3)}.$$

$\mathbf{E}_6/\mathbf{Sp}(1)\mathbf{SU}(6)$: With $\dim\mathfrak{p} = 40$ the quaternionic dimension is $n = 10$, $\mathfrak{k} = \mathfrak{sp}(1)\oplus\mathfrak{su}(6)$, $l_{\mathfrak{su}(6)} = \frac{1}{2}$ and

$$\rho = \frac{5}{12}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{1}{4}\mathrm{proj}_{\mathfrak{su}(6)}.$$

$\mathbf{E}_7/\mathbf{Sp}(1)\mathbf{Spin}(12)$: With $\dim\mathfrak{p} = 64$ the quaternionic dimension is $n = 16$, $\mathfrak{k} = \mathfrak{sp}(1)\oplus\mathfrak{so}(12)$, $l_{\mathfrak{so}(12)} = \frac{5}{9}$ and

$$\rho = \frac{4}{9}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{2}{9}\mathrm{proj}_{\mathfrak{so}(12)}.$$

$\mathbf{E}_8/\mathbf{Sp}(1)\mathbf{E}_7$: With $\dim\mathfrak{p} = 112$ the quaternionic dimension is $n = 28$ and $\mathfrak{k} = \mathfrak{sp}(1)\oplus\mathfrak{e}_7$, where $\dim\mathfrak{e}_7 = 133$. Applying Lemma 5.3 one finds $l_{\mathfrak{e}_7} = \frac{3}{5}$ and thus

$$\rho = \frac{7}{15}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{1}{5}\mathrm{proj}_{\mathfrak{e}_7}.$$

To present the argument leading to Theorem 5.1 we recall that for any Wolf space we have a K-invariant subalgebra $\mathfrak{sp}(n)\subset\mathfrak{so}\,\mathfrak{p}$ defined as the centralizer of the subalgebra $\mathfrak{sp}(1)\subset\mathfrak{so}\,\mathfrak{p}$ defining the quaternionic structure and by definition $\mathfrak{k}^0\subset\mathfrak{sp}(n)$. In particular, the curvature endomorphism $\rho^{\mathbb{H}P^n}$ of the quaternionic projective space is well defined on any Wolf space. Bridging the gap between Lie algebra– and E–H–formalism, it is even possible to identify the "curvature" endomorphisms corresponding to R^H and R^E of Lemma 2.4 with:

$$\rho^H = -2n\,\mathrm{proj}_{\mathfrak{sp}(1)} \qquad \text{and} \qquad \rho^E = -2\,\mathrm{proj}_{\mathfrak{sp}(n)}$$

The scalar curvature of a compact Wolf space is $\kappa = 2n$ and Lemma 2.4 implies for the curvature tensor:

$$\rho = -\frac{\kappa}{8n(n+2)}(\rho^H+\rho^E) + \rho^{hyper} = \frac{n}{2(n+2)}\mathrm{proj}_{\mathfrak{sp}(1)} + \frac{1}{2(n+2)}\mathrm{proj}_{\mathfrak{sp}(n)} + \rho^{hyper}.$$

We conclude that the kernel of the hyperkähler part ρ^{hyper} of the curvature endomorphism ρ restricted to $\mathfrak{sp}(n) \subset \mathfrak{so}\,\mathfrak{p}$ is the eigenspace of ρ with eigenvalue $\frac{1}{2(n+2)}$ in $\mathfrak{sp}(n)$. Looking down the list of curvature endomorphisms above one verifies that this eigenvalue does only occur for the curvature endomorphism of the quaternionic projective space itself. $\square$

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