1. Preliminaries

1.1. Lefschetz–decomposition of forms. Now suppose that $(V, \langle \cdot, \cdot \rangle)$ is a $2m$–dimensional Euclidean vector space with compatible almost complex structure $J$. We fix an orthonormal basis $x_1, \ldots, x_m, y_1, \ldots, y_m$ of $V$ such that $Jx_j = y_j$ for all $1 \leq j \leq m$ and set $z_j := 1/2(x_j - iy_j) \in V^\mathbb{C}$ as well as $\overline{z}_j := \overline{z}_j$. Note that $z_1, \ldots, z_m, \overline{z}_1, \ldots, \overline{z}_m$ is a $\mathbb{C}$–basis of $V^\mathbb{C}$.

Now if $x^1, \ldots, x^m, y^1, \ldots, y^m$ is the $\mathbb{R}$–basis dual to $x^1, \ldots, x^m, y^1, \ldots, y^m$, then $z^j := x^j + iy^j$ is the $\mathbb{C}$–basis dual to $z_j$ while $\overline{z}^j := x^j - iy^j$ is dual to $\overline{z}_j$. The $2$–form on $V$ given by $(v, w) \mapsto \langle Jv, w \rangle$ corresponds to an element $\omega$ in $\Lambda^2 V^\ast$. Namely,

$$
\omega = \sum_{1 \leq k < l \leq m} \langle Jx_k, x_l \rangle x^k \wedge x^l + \langle Jx_k, y_l \rangle x^k \wedge y^l + \langle Jy_k, y_l \rangle y^k \wedge y^l
$$

or in terms of the basis $z^1, \ldots, z^m, \overline{z}^1, \ldots, \overline{z}^m$

$$
\omega = \frac{i}{4} \sum_{k=1}^m (z^k + \overline{z}^k) \wedge (z^k - \overline{z}^k) = \frac{i}{2} \sum_{k=1}^m z^k \wedge \overline{z}^k.
$$

Now define the operator $L \colon \Lambda^k V^\ast \rightarrow \Lambda^{k+2} V^\ast$ by $L(\alpha) = \omega \wedge \alpha$ and let $A \colon \Lambda^{k+2} V^\ast \rightarrow \Lambda^k V^\ast$ be its adjoint with respect to the inner product on $\Lambda^* V^\ast$ induced by $\langle \cdot, \cdot \rangle$. We further define $B \colon \Lambda^k V^\ast \rightarrow \Lambda^k V^\ast$ by $B|_{\Lambda^k V^\ast} = (m-k)\text{id}$.

**Proposition 1.1.** The following identities hold:

$$
[B, A] = 2A, \quad [B, L] = -2L \quad \text{and} \quad [A, L] = B.
$$

Thus, $(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}) \mapsto A$, $(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}) \mapsto L$, and $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \mapsto B$ define a representation of $\mathfrak{sl}(2, \mathbb{R})$ on $\Lambda^* V^\ast$.

**Proof.** We have

$$
[B, A]|_{\Lambda^{k+2} V^\ast} = (m-k)A|_{\Lambda^{k+2} V^\ast} - (m-k-2)A|_{\Lambda^{k+2} V^\ast} = 2A|_{\Lambda^{k+2} V^\ast}.
$$

Similarly, one concludes that $[B, L] = -2L$. Now let $\alpha, \beta$ be arbitrary $k$–forms. We have

$$
\langle A(L(\alpha)), \beta \rangle = \langle L\alpha, L\beta \rangle = \sum_{k=1}^m (\omega \wedge \alpha, x^k \wedge y^k \wedge \beta).
$$

We further compute

$$
\langle \omega \wedge \alpha, x^k \wedge y^k \wedge \beta \rangle = (i_z x_k \omega) \wedge \alpha + \omega \wedge i_{z_k} \alpha, y^k \wedge \beta
$$

$$
= (y^k \wedge \alpha + \omega \wedge i_{z_k} \alpha, y^k \wedge \beta
$$

$$
= (\alpha - y^k \wedge i_{y_k} \alpha - x^k \wedge i_{x_k} \alpha + \omega \wedge i_{y_k} i_{z_k} \alpha, \beta).
$$

Now note that for $\eta = x^{i_1} \wedge \ldots \wedge x^{i_r} \wedge y^{j_1} \wedge \ldots \wedge y^{j_s}$ we have

$$
\sum_{k=1}^m x^k \wedge i_{x_k} \eta + y^k \wedge i_{y_k} \eta = (r+s)\eta,
$$

and therefore

$$
\langle A(L(\alpha)), \beta \rangle = (m-k)\langle \alpha, \beta \rangle + \langle \omega \wedge L(\alpha), \beta \rangle = \langle B\alpha + L(A(\alpha)), \beta \rangle.
$$

Complex linearly extending the maps $A$, $L$ and $B$, we hence also obtain a representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\Lambda^k V^\ast := (\Lambda^* V^\ast) \otimes_{\mathbb{R}} \mathbb{C}$. 
\hfill \Box
Remark 1.4. If we extend the inner product on $\Lambda^k V^*$ linearly. Since $\alpha \in \Lambda^k V^*$ for all $\alpha \in \Lambda^k V^*$, and the space of all real primitive $k$–forms is denoted $P^k$. 

Theorem 1.3 (Lefschetz decomposition).

1. We have decompositions

$$\Lambda^k V^* = \bigoplus_{j \geq 0} L^j (P^{k-2j})$$

2. The primitive space $P^k$ is trivial for $k > m$.

3. The map $L^k : \Lambda^{m-k} V^* \to \Lambda^{m+k} V^*$ is an isomorphism for all $k \geq 0$.

4. We have $P^k = \{ \alpha \in \Lambda^k V^* | L^{m-k+1} \alpha = 0 \}$ for all $k \leq m$.

Proof. We consider $\Lambda^k V^*$ as a (finite–dimensional) $\mathfrak{sl}(2, \mathbb{C})$–representation. Then

$$\Lambda^k V^* = \bigoplus_{\ell \geq 0} \bigoplus_{r=1}^{N_\ell} W_{\ell, r},$$

where each $W_{\ell, r} \subseteq \Lambda^k V^*$ is an irreducible $\mathfrak{sl}(2, \mathbb{C})$–representation of highest weight $\ell$. By construction, $\Lambda^{m-k} V^*$ is the eigenspace of $B$ for the eigenvalue $k$, and thus

$$\Lambda^{m-k} V^* = \bigoplus_{\ell \geq 0} \bigoplus_{r=1}^{N_\ell} (W_{\ell, r})_k$$

Taking $k = m-t$, the first identity follows, as the canonical map $P^t \otimes_{\mathbb{R}} \mathbb{C} \to P^t_{\mathbb{C}}$ is an isomorphism.

Since $L^k |_{(W_{\ell, r})_k} : (W_{\ell, r})_k \to (W_{\ell, r})_{k-t}$ is always an isomorphism, the third claim follows. Likewise, the map $\Lambda^1 |_{(W_{\ell, r})_{-t}}$, is an isomorphism, whence $\Lambda$ is necessarily injective on $(W_{\ell, r})_{-t}$. Since $\Lambda^{m+t} V^*$ is the sum of all $(W_{\ell, r})_{-t}$, it thus follows that $P^{m+t} = 0$, proving the second assertion. Finally, note that $P^k$ is the sum of all $(W_{m-k, r})_{m-k}$, which is precisely the kernel of $L^{m-k+1}$ on $\Lambda^k V^*$, provided that $k \leq m$. 

Remark 1.4. If we extend the inner product on $\Lambda^* V$ complex linearly, then $\langle \alpha, \varphi \Lambda \beta \rangle = \langle i \varphi \alpha, \beta \rangle$ for all $\alpha \in \Lambda^{k+1} V^*$, $\beta \in \Lambda^k V^*$, and $\varphi \in \Lambda^k V^*$, where we also extended $\Lambda : V^* \to V$ complex linearly. Since

$$\langle 2i z_k, v \rangle = \langle x_k + iy_k, v \rangle = z^k(v),$$

and similarly $2(z_k)^* = \pi_k$, we see that $\langle \alpha, \beta \rangle \neq 0$ for $\alpha, \beta$ of type $(p, q)$ and $(r, s)$ only if $(p, q) = (s, r)$, that is, if $\beta$ is of type $(q, p)$.

This observation shows that $\Lambda$ must map forms of type $(p, q)$ to forms of type $(p-1, q-1)$. In fact, if $\alpha$ is of type $(p, q)$ and $\beta$ is not of type $(q-1, p-1)$, then

$$\langle \Lambda \alpha, \beta \rangle = \langle \alpha, L \beta \rangle = 0,$$
since $L$ is homogeneous of bidegree $(1,1)$. Therefore, $A_0$ must be of type $(p-1,q-1)$, and so we have $P^p_\ell = \bigoplus_{p+q=k} P^{p,q}$, with $P^{p,q} = P^{p+q} \cap \wedge^{p,q} V^*$. It follows that
\[
\wedge^{p,q} V^* = \bigoplus_{j \geq 0} (L^j(P^{p+q-2j}_\ell) \cap \wedge^{p,q} V^*) = \bigoplus_{j \geq 0} L^j(P^{p-j,q-j}).
\]

1.2. Consequences for Kähler manifolds. Let $(M,g,J)$ be a compact Kähler manifold of real dimension $2m$ with fundamental form $\omega \in \Omega^2(M)$. The operator $L: \Omega^*(M) \to \Omega^*(M)$, $\alpha \mapsto \omega \wedge \alpha$, coincides, in each fiber, with the operator $L$ defined in the previous section. Let $\Lambda$ be the adjoint of $L$ with respect to the inner product on $\Omega^*(M)$.

Theorem 1.5 (Hard Lefschetz Theorem). The map $L^k: H^{m-k}(M;\mathbb{K}) \to H^{m+k}(M;\mathbb{K})$ is an isomorphism for every $k \geq 0$. Furthermore, there is a decomposition
\[
H^\ell(M;\mathbb{K}) = \bigoplus_{j \geq 0} L^j(H^{\ell-2j}(M;\mathbb{K})_0),
\]
where $H^\ell(M;\mathbb{K})_0 = \ker L^{m-\ell+1}|_{H^\ell(M;\mathbb{K})}$ for $\ell \leq m$ and $H^\ell(M;\mathbb{K})_0 = 0$ for $\ell > m$ is the space of primitive forms. The space of primitive forms further decomposes as
\[
H^\ell(M;\mathbb{K})_0 = \bigoplus_{p+q = \ell} H^{p,q}(M;\mathbb{K})_0,
\]
where $H^\ell(M;\mathbb{K})_0^{p,q} = H^\ell(M;\mathbb{K})_0 \cap H^{p,q}(M;\mathbb{C})$.

Proof. Denote by $\mathcal{H}^k \subseteq \Omega^k(M;\mathbb{K})$ the space of harmonic $k$–forms with respect to the exterior derivative $d$ (and the metric $g$). By the Hodge–Theorem, the map $\mathcal{H}^k \to H^k(M;\mathbb{K})$, $\alpha \mapsto [\alpha]$, is an isomorphism. Moreover, since $M$ is Kähler, the Laplacian $\Delta$ commutes with $L$, and thus $L$ maps harmonic forms to harmonic forms. Since $L^k: \Omega^{m-k}(M;\mathbb{K}) \to \Omega^{m+k}(M;\mathbb{K})$ is an isomorphism by the Lefschetz decomposition (theorem 1.3), so is $L^k: \mathcal{H}^{m-k} \to \mathcal{H}^{m+k}$. Therefore, we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{H}^{m-k} & \xrightarrow{L^k} & \mathcal{H}^{m+k} \\
\downarrow \cong & & \downarrow \cong \\
H^{m-k}(M;\mathbb{K}) & \xrightarrow{L^k} & H^{m+k}(M;\mathbb{K}),
\end{array}
\]
and since the upper row is an isomorphism, so must be the lower row. This proves the first statement. For the second statement, we use theorem 1.3 again: we know that
\[
\Omega^k(M) = \bigoplus_{j \geq 0} L^j(P^{k-2j}),
\]
where $P^\ell = \{\alpha \in \Omega^\ell(M) \mid L^{m-\ell+1} \alpha = 0\}$ for $\ell \leq m$ and $P^\ell = 0$ for $\ell > m$. Since $L$ commutes with $\Delta$, we hence also have $\mathcal{H}^\ell = \bigoplus_{j} L^j(P^{k-2j} \cap \mathcal{H}^{k-2j})$, and the isomorphism $\mathcal{H}^{k-2j} \to H^{k-2j}(M;\mathbb{K})$ takes $P^{k-2j} \cap \mathcal{H}^{k-2}$ to $H^{k-2}(M;\mathbb{K})_0$. \hfill $\square$

Remark 1.6. Note that $\dim H^\ell(M;\mathbb{K})_0 = \dim H^\ell(M;\mathbb{K}) - \dim H^{\ell-2}(M;\mathbb{K}) = b_0(M) - b_{\ell-2}(M)$.

2. Hodge–Riemann bilinear relations

2.1. The Hodge–Riemann bilinear form on Hermitian vector spaces. Let $(V,\langle \cdot,\cdot \rangle)$ be a $2m$–dimensional Euclidean vector space with compatible almost complex structure $J$ and fundamental form $\omega$. Endow $V$ with the canonical orientation induced by $J$ and let $vol \in \wedge^{2m} V^*$ be the corresponding volume form, i.e. $vol = (x^1 \wedge y^1) \wedge \ldots \wedge (x^m \wedge y^m)$, where $x^1, \ldots, x^m, y^1, \ldots, y^m$ is the basis dual to an orthonormal basis $x_1, \ldots, x_m, y_1, \ldots, y_m$ with $Jx_k = y_k$ for all $k$. 


Theorem 2.6. Let $(\omega^r)^{k\times k}.$ The intersection form on K"ahler manifolds.

since for $\alpha$ is of type $(p,q)$ and $\beta$ is of type $(r,s)$, with $k = p + q$, then $\alpha \land \beta \land \omega^{m-k}$ is of type $(p+r+m-k, q+s+m-k)$ and a multiple of the volume form, which is of type $(m,m)$. Thus, in order for $Q(\alpha, \beta)$ to be non-zero, we must have $p + r + m - k = m$ or equivalently $r = k - p = q$.
Similarly, one concludes that $s = p$.

Proposition 2.3. Let $\alpha \in \wedge^k \mathbb{R}^* \times \wedge^k \mathbb{R}^* \to \mathbb{C}$ is defined by

$$Q(\alpha, \beta) \cdot \text{vol} = (-1)^{\frac{k(k-1)}{2}} \cdot \alpha \land \beta \land \omega^{m-k}.$$  

Remark 2.2. Note that $Q$ is zero when restricted to $\wedge^p V^* \times \wedge^q V^*$, unless $(p,q) = (s,r)$. Indeed, if $\alpha$ is of type $(p,q)$ and $\beta$ is of type $(r,s)$, with $k = p + q$, then $\alpha \land \beta \land \omega^{m-k}$ is of type $(p+r+m-k, q+s+m-k)$ and a multiple of the volume form, which is of type $(m,m)$. Thus, in order for $Q(\alpha, \beta)$ to be non-zero, we must have $p + r + m - k = m$ or equivalently $r = k - p = q$.
Similarly, one concludes that $s = p$.

Theorem 2.4. Let $\alpha \in \wedge^k \mathbb{R}^* \times \wedge^k \mathbb{R}^* \to \mathbb{C}$ is defined by

$$Q(\alpha, \beta) = (m-k)! \cdot (\alpha, \alpha \cdot \beta) > 0,$$  

where $(\gamma, \beta)_\mathbb{C} = (\gamma, \beta)$ is the Hermitian product induced by $(\cdot, \cdot)$.

Proof. By definition, 

$$Q(\alpha, \beta) \cdot \text{vol} = (-1)^{\frac{k(k-1)}{2}} \cdot \alpha \land \beta \land \omega^{m-k}$$  

Since $\alpha$ is of type $(p,q)$, we have $J^* \beta = i^{p-q} \cdot \beta$, and thus 

$$L^{m-k} \beta = (-1)^{k(k+1)} \cdot \frac{(m-k)!}{(m-k)!} \cdot i^{p-q} L^{m-k} J^* \beta$$  

Therefore, 

$$Q(\alpha, \beta) = (-1)^k \cdot i^{p-q} \cdot (m-k)! \cdot (\alpha, \alpha \cdot \beta) > 0,$$

and since $i^{2(p-q)} = (-1)^{p+q} = (-1)^k$, the claim follows.

Example 2.5. We have $(\bigwedge^{1,1} V^* \cap \bigwedge^2 V^*) = \mathbb{R} \omega \oplus P^{1,1}$ by the Lefschetz–decomposition. This decomposition is orthogonal, for if $\alpha \in P^{1,1}$, then $\alpha \land \omega \land \omega^{m-2} = L^{m-1} \beta = 0$. Moreover, $Q$ is negative–definite on $\mathbb{R} \omega$ (since $Q(\omega, \omega) \cdot \text{vol} = -\omega^m = -m! \cdot \text{vol}$) and positive–definite on $P^{1,1}$, since for $\alpha \in P^{1,1}$ we have $\beta = \alpha$, so that $Q(\alpha, \alpha) = (m-2)! \cdot (\alpha, \alpha) > 0$.

2.2. The intersection form on K"ahler manifolds.

Theorem 2.6. Let $(M, g, J)$ be a real $2m$–dimensional, compact K"ahler manifold with fundamental form $\omega$. For all closed $k$–forms $\alpha$ with $[\alpha] \in H^k(M; \mathbb{K})_0$ we have 

$$i^{p-q} \cdot (-1)^{\frac{k(k-1)}{2}} \cdot \int_M \alpha \land \beta \land \omega^{m-k} > 0.$$  

Proof. Since the integral only depends on the cohomology class, we may assume $\alpha$ be harmonic and primitive. Then 

$$i^{p-q} \cdot (-1)^{\frac{k(k-1)}{2}} \cdot \int_M \alpha \land \beta \land \omega^{m-k} = \int_M \langle \alpha, \beta \rangle \cdot \text{vol} > 0.$$  

Remark 2.7. Let $(M, g, J)$ be a $2m$–dimensional, compact K"ahler manifold.
(1) Let $\alpha$, $\beta$ be closed $k$–forms of types $(p,q)$ and $(r,s)$, respectively, such that $[\alpha]$, $[\beta]$ are primitive. We have already seen, that $$\tilde{Q}(\alpha,\beta) := (-1)^{\frac{k(k-1)}{2}} \int_M \alpha \wedge \beta \wedge \omega^{m-k} = 0,$$
unless $\beta$ is of type $(q,p)$. Since $\tilde{Q}(L^\ell \alpha, L^\ell \beta) = \tilde{Q}(\alpha,\beta)$, the Hard Lefschetz theorem implies that $\tilde{Q}$ is non–degenerate on $H^\ell(M;K)$ for all $\ell \geq 0$.

(2) If $p+q \equiv 0 \mod 2$, then $p$ and $q$ have the same parity. Thus, with $k = p + q$, we have $$i^{p-q} \cdot (-1)^{\frac{k(k-1)}{2}} = i^{p-q} \cdot (p+q)(p+q-1) = i^{p-q-p-q+(p+q)^2} = (-1)^q,$$
and hence $\tilde{Q}$ is positive–definite on $H^{p,q}(M;\mathbb{K})_0$ with $p, q \equiv 0 \mod 2$, and negative–definite, if $p, q \equiv 1 \mod 2$. 