1. Preliminaries

1.1. **Lefschetz–decomposition of forms.** Now suppose that $(V, \langle\cdot,\cdot\rangle)$ is a $2m$–dimensional Euclidean vector space with compatible almost complex structure J. We fix an orthonormal basis $x_1, \dots, x_m, y_1, \dots, y_m$ of V such that $Jx_j = y_j$ for all $1 \le j \le m$ and set $z_j := 1/2(x_j - iy_j) \in V^{\mathbb{C}}$ as well as $\overline{z}_j := \overline{z_j}$. Note that $z_1, \dots, z_m, \overline{z}_1, \dots, \overline{z}_m$ is a $\mathbb{C}$–basis of $V^{\mathbb{C}}$.

Now if $x^1, \dots, x^m, y^1, \dots, y^m$ is the $\mathbb{R}$–basis dual to $x^1, \dots, x^m, y^1, \dots, y^m$, then $z^j := x^j + iy^j$ is the $\mathbb{C}$–basis dual to z_j while $\overline{z}^j := x^j - iy^j$ is dual to $\overline{z}_j$. The 2–form on V given by $(v, w) \mapsto \langle Jv, w\rangle$ corresponds to an element ω in $\wedge^2 V^*$. Namely,

$$\begin{aligned}\omega &= \sum_{1\le k<\ell\le m} \langle Jx^k, x^\ell\rangle x^k \wedge x^\ell + \langle Jx^k, y^\ell\rangle x^k \wedge y^\ell + \langle Jy^k, y^\ell\rangle y^k \wedge y^\ell \\ &= \sum_{k=1}^m x^k \wedge y^k\end{aligned}$$

or in terms of the basis $z^1, \dots, z^m, \overline{z}^1, \dots, \overline{z}^m$

$$\omega = \frac{i}{4}\sum_{k=1}^m (z^k + \overline{z}^k) \wedge (\overline{z}^k - z^k) = \frac{i}{2}\sum_{k=1}^m z^k \wedge \overline{z}^k.$$

Now define the operator $L\colon \wedge^k V^* \to \wedge^{k+2} V^*$ by $L(\alpha) = \omega \wedge \alpha$ and let $\Lambda\colon \wedge^{k+2} V^* \to \wedge^k V^*$ be its adjoint with respect to the inner product on $\wedge^* V^*$ induced by $\langle\cdot,\cdot\rangle$. We further define $B\colon \wedge^k V^* \to \wedge^k V^*$ by $B|_{\wedge^k V^*} = (m-k)\mathrm{id}$.

Proposition 1.1. The following identities hold:

$$[B, \Lambda] = 2\Lambda,\ [B, L] = -2L \text{ and } [\Lambda, L] = B.$$

Thus, $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \mapsto \Lambda$, $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) \mapsto L$, and $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) \mapsto B$ define a representation of $\mathfrak{sl}(2,\mathbb{R})$ on $\wedge^* V^*$.

Proof. We have

$$[B, \Lambda]|_{\wedge^{k+2}V^*} = (m-k)\Lambda|_{\wedge^{k+2}V^*} - (m-k-2)\Lambda|_{\wedge^{k+2}V^*} = 2\Lambda|_{\wedge^{k+2}V^*}.$$

Similarly, one concludes that $[B, L] = -2L$. Now let α, β be arbitrary k–forms. We have

$$\langle \Lambda(L(\alpha)), \beta\rangle = \langle L\alpha, L\beta\rangle = \sum_{k=1}^m \langle \omega \wedge \alpha, x^k \wedge y^k \wedge \beta\rangle.$$

We further compute

$$\begin{aligned}\langle \omega \wedge \alpha, x^k \wedge y^k \wedge \beta\rangle &= \langle (i_{x_k}\omega) \wedge \alpha + \omega \wedge i_{x_k}\alpha, y^k \wedge \beta\rangle \\ &= \langle y^k \wedge \alpha + \omega \wedge i_{x_k}\alpha, y^k \wedge \beta\rangle \\ &= \langle \alpha - y^k \wedge i_{y_k}\alpha - x^k \wedge i_{x_k}\alpha + \omega \wedge i_{y_k} i_{x_k}\alpha, \beta\rangle.\end{aligned}$$

Now note that for $\eta = x^{i_1} \wedge \ldots \wedge x^{i_r} \wedge y^{j_1} \wedge \ldots \wedge y^{j_s}$ we have

$$\sum_{k=1}^m x^k \wedge i_{x_k}\eta + y^k \wedge i_{y_k}\eta = (r+s)\eta,$$

and therefore

$$\langle \Lambda(L(\alpha)), \beta\rangle = (m-k)\langle \alpha, \beta\rangle + \langle \omega \wedge \Lambda(\alpha), \beta\rangle = \langle B\alpha + L(\Lambda(\alpha)), \beta\rangle.$$

□

Complex linearly extending the maps Λ, L and B, we hence also obtain a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\wedge^*_{\mathbb{C}} V^* := (\wedge^* V^*) \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 1.2. A form $\alpha \in \wedge^k_{\mathbb{C}} V^*$ is called *primitive*, if $\Lambda\alpha = 0$. The space of all primitive k–forms is denoted $P^k_{\mathbb{C}}$, and the space of all real primitive k–forms is denoted P^k.

Theorem 1.3 (Lefschetz decomposition)**.**

(1) We have decompositions

$$\wedge^k V^* = \bigoplus_{j\geq 0} L^j(P^{k-2j}) \text{ and } \wedge^k_{\mathbb{C}} V^* = \bigoplus_{j\geq 0} L^j(P^{k-2j}_{\mathbb{C}}).$$

(2) The primitive space $P^k_{\mathbb{C}}$ is trivial for $k > m$.
(3) The map $L^k\colon \wedge^{m-k}_{\mathbb{C}} V^* \to \wedge^{m+k}_{\mathbb{C}} V^*$ is an isomorphism for all $k \geq 0$.
(4) We have $P^k_{\mathbb{C}} = \{\alpha \in \wedge^k_{\mathbb{C}} V^* \mid L^{m-k+1}\alpha = 0\}$ for all $k \leq m$.

Proof. We consider $\wedge^*_{\mathbb{C}} V^*$ as a (finite–dimensional) $\mathfrak{sl}(2,\mathbb{C})$–representation. Then

$$\wedge^*_{\mathbb{C}} V^* = \bigoplus_{\ell\geq 0} \bigoplus_{r=1}^{N_\ell} W_{\ell,r},$$

where each $W_{\ell,r} \subseteq \wedge^*_{\mathbb{C}} V^*$ is an irreducible $\mathfrak{sl}(2,\mathbb{C})$–subrepresentation of highest weight ℓ. By construction, $\wedge^{m-k}_{\mathbb{C}} V^*$ is the eigenspace of B for the eigenvalue k, and thus

$$\begin{aligned}
\wedge^{m-k}_{\mathbb{C}} V^* &= \bigoplus_{\ell\geq 0} \bigoplus_{r=1}^{N_\ell} (W_{\ell,r})_k \\
&= \bigoplus_{j\geq 0} \bigoplus_{r=1}^{N_{k+2j}} (W_{k+2j,r})_k \\
&= \bigoplus_{j\geq 0} \bigoplus_{r=1}^{N_{k+2j}} L^j((W_{k+2j,r})_{k+2j}) \\
&= \bigoplus_{j\geq 0} L^j\left(P^{m-(k+2j)}_{\mathbb{C}}\right).
\end{aligned}$$

Taking $k = m-t$, the first identity follows, as the canonical map $P^t \otimes_{\mathbb{R}} \mathbb{C} \to P^t_{\mathbb{C}}$ is an isomorphism.

Since $L^k|_{(W_{\ell,r})_k}\colon (W_{\ell,r})_k \to (W_{\ell,r})_{-k}$ is always an isomorphism, the third claim follows. Likewise, the map $\Lambda^t|_{(W_{\ell,r})_{-t}}$ is an isomorphism, whence Λ is necessarily injective on $(W_{\ell,r})_{-t}$. Since $\wedge^{m+t}_{\mathbb{C}} V^*$ is the sum of all $(W_{\ell,r})_{-t}$, it thus follows that $P^{m+t} = 0$, proving the second assertion. Finally, note that P^k is the sum of all $(W_{m-k,r})_{m-k}$, which is precisely the kernel of L^{m-k+1} on $\wedge^k_{\mathbb{C}} V^*$, provided that $k \leq m$. □

Remark 1.4. If we extend the inner product on $\wedge^* V$ complex linearly, then $\langle \alpha, \varphi\wedge\beta\rangle = \langle i_{\varphi^\sharp}\alpha, \beta\rangle$ for all $\alpha \in \wedge^{k+1}_{\mathbb{C}} V^*$, $\beta \in \wedge^k_{\mathbb{C}} V^*$, and $\varphi \in \wedge^1_{\mathbb{C}} V^*$, where we also extended $\sharp\colon V^* \to V$ complex linearly. Since

$$\langle 2\overline{z}_k, v\rangle = \langle x_k + iy_k, v\rangle = z^k(v),$$

and similarly $2(z_k)^\sharp = \overline{z}_k$, we see that $\langle \alpha, \beta\rangle \neq 0$ for forms α, β of type (p,q) and (r,s) only if $(p,q) = (s,r)$, that is, if β is of type (q,p).

This observation shows that Λ must map forms of type (p,q) to forms of type $(p-1,q-1)$. In fact, if α is of type (p,q) and β is not of type $(q-1,p-1)$, then

$$\langle \Lambda\alpha, \beta\rangle = \langle \alpha, L\beta\rangle = 0,$$

since L is homogeneous of bidegree $(1,1)$. Therefore, $\Lambda\alpha$ must be of type $(p-1,q-1)$, and so we have $P^k_{\mathbb{C}} = \bigoplus_{p+q=k} P^{p,q}$, with $P^{p,q} = P^{p+q} \cap \wedge^{p,q}V^*$. It follows that

$$\wedge^{p,q}V^* = \bigoplus_{j\geq 0}\big(L^j(P^{p+q-2j}_{\mathbb{C}}) \cap \wedge^{p,q}V^*\big) = \bigoplus_{j\geq 0} L^j(P^{p-j,q-j}).$$

1.2. **Consequences for Kähler manifolds.** Let (M,g,J) be a compact Kähler manifold of real dimension $2m$ with fundamental form $\omega \in \Omega^2(M)$. The operator $L\colon \Omega^*(M) \to \Omega^*(M)$, $\alpha \mapsto \omega \wedge \alpha$, coincides, in each fiber, with the operator L defined in the previous section. Let Λ be the adjoint of L with respect to the inner product on $\Omega^*(M)$.

Theorem 1.5 (Hard Lefschetz Theorem)**.** The map $L^k\colon H^{m-k}(M;\mathbb{K}) \to H^{m+k}(M;\mathbb{K})$ is an isomorphism for every $k \geq 0$. Furthermore, there is a decomposition

$$H^k(M;\mathbb{K}) = \bigoplus_{j\geq 0} L^j(H^{k-2j}(M;\mathbb{K})_0),$$

where $H^\ell(M;\mathbb{K})_0 = \ker L^{m-\ell+1}|_{H^\ell(M;\mathbb{K})}$ for $\ell \leq m$ and $H^\ell(M;\mathbb{K})_0 = 0$ for $\ell > m$ is the space of primitive forms. The space of primitive forms further decomposes as

$$H^\ell(M;\mathbb{K})_0 = \bigoplus_{p+q=\ell} H^{p,q}(M;\mathbb{K})_0,$$

where $H^\ell(M;\mathbb{K})^{p,q}_0 = H^\ell(M;\mathbb{K})_0 \cap H^{p,q}(M;\mathbb{C})$.

Proof. Denote by $\mathcal{H}^k \subseteq \Omega^k(M;\mathbb{K})$ the space of harmonic k–forms with respect to the exterior derivative d (and the metric g). By the Hodge–Theorem, the map $\mathcal{H}^k \to H^k(M;\mathbb{K})$, $\alpha \mapsto [\alpha]$, is an isomorphism. Moreover, since M is Kähler, the Laplacian Δ commutes with L, and thus L maps harmonic forms to harmonic forms. Since $L^k\colon \Omega^{m-k}(M;\mathbb{K}) \to \Omega^{m+k}(M;\mathbb{K})$ is an isomorphism by the Lefschetz decomposition (theorem 1.3), so is $L^k\colon \mathcal{H}^{m-k} \to \mathcal{H}^{m+k}$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{m-k} & \xrightarrow{L^k} & \mathcal{H}^{m+k} \\ \cong\downarrow & & \downarrow\cong \\ H^{m-k}(M;\mathbb{K}) & \xrightarrow{L^k} & H^{m+k}(M;\mathbb{K}), \end{array}$$

and since the upper row is an isomorphism, so must be the lower row. This proves the first statement. For the second statement, we use theorem 1.3 again: we know that

$$\Omega^k(M) = \bigoplus_{j\geq 0} L^j(P^{k-2j}),$$

where $P^\ell = \{\alpha \in \Omega^\ell(M) \mid L^{m-\ell+1}\alpha = 0\}$ for $\ell \leq m$ and $P^\ell = 0$ for $\ell > m$. Since L commutes with Δ, we hence also have $\mathcal{H}^k = \bigoplus_j L^j(P^{k-2j} \cap \mathcal{H}^{k-2j})$, and the isomorphism $\mathcal{H}^{k-2j} \to H^{k-2j}(M;\mathbb{K})$ takes $P^{k-2j} \cap \mathcal{H}^{k-2}$ to $H^{k-2j}(M;\mathbb{K})_0$. $\square$

Remark 1.6. Note that $\dim H^\ell(M;\mathbb{K})_0 = \dim H^\ell(M;\mathbb{K}) - \dim H^{\ell-2}(M;\mathbb{K}) = b_\ell(M) - b_{\ell-2}(M)$.

2. Hodge–Riemann bilinear relations

2.1. **The Hodge–Riemann bilinear form on Hermitian vector spaces.** Let $(V, \langle\cdot,\cdot\rangle)$ be a $2m$–dimensional Euclidean vector space with compatible almost complex structure J and fundamental form ω. Endow V with the canonical orientation induced by J and let $\mathrm{vol} \in \wedge^{2m}V^*$ be the corresponding volume form, i. e. $\mathrm{vol} = (x^1 \wedge y^1) \wedge \ldots \wedge (x^m \wedge y^m)$, where $x^1,\ldots,x^m, y^1,\ldots,y^m$ is the basis dual to an orthonormal basis $x_1,\ldots,x_m$, $y_1,\ldots,y_m$ with $Jx_k = y_k$ for all k.

Definition 2.1. The *Hodge–Riemann bilinear form* $Q\colon \wedge^k_{\mathbb{C}} V^* \times \wedge^k_{\mathbb{C}} V^* \to \mathbb{C}$ is defined by

$$Q(\alpha,\beta)\cdot \mathrm{vol} = (-1)^{\frac{k(k-1)}{2}}\cdot \alpha\wedge\beta\wedge\omega^{m-k}.$$

Remark 2.2. Note that Q is zero when restricted to $\wedge^{p,q}V^*\times\wedge^{r,s}V^*$, unless $(p,q)=(s,r)$. Indeed, if α is of type (p,q) and β is of type (r,s), with $k=p+q$, then $\alpha\wedge\beta\wedge\omega^{m-k}$ is of type $(p+r+m-k,q+s+m-k)$ and a multiple of the volume form, which is of type (m,m). Thus, in order for $Q(\alpha,\beta)$ to be non–zero, we must have $p+r+m-k=m$ or equivalently $r=k-p=q$. Similarly, one concludes that $s=p$.

Proposition 2.3. Let $\alpha\in\wedge^k_{\mathbb{C}}V^*$ be primitive. Then

$$*\alpha = (-1)^{\frac{k(k+1)}{2}}\cdot\frac{1}{(m-k)!}\cdot L^{m-k}J^*(\alpha).$$

Theorem 2.4. Let $\alpha\in\wedge^k_{\mathbb{C}}V^*$ be a primitive form of type (p,q). Then we have

$$i^{p-q}\cdot Q(\alpha,\overline{\alpha}) = (m-k)!\langle\alpha,\alpha\rangle_{\mathbb{C}}>0,$$

where $\langle\gamma,\beta\rangle_{\mathbb{C}}=\langle\gamma,\overline{\beta}\rangle$ is the Hermitian product induced by $\langle\cdot,\cdot\rangle$.

Proof. By definition,

$$\begin{aligned}Q(\alpha,\overline{\alpha})\mathrm{vol} &= (-1)^{\frac{k(k-1)}{2}}\cdot\alpha\wedge\overline{\alpha}\wedge\omega^{m-k}\\ &= (-1)^{\frac{k(k-1)}{2}}\cdot\alpha\wedge L^{m-k}\overline{\alpha}\end{aligned}$$

Since α is of type (p,q), we have $J^*\overline{\alpha}=i^{q-p}\cdot\overline{\alpha}$, and thus

$$\begin{aligned}L^{m-k}\overline{\alpha} &= (-1)^{k(k+1)}\cdot\frac{(m-k)!}{(m-k)!}\cdot i^{p-q}L^{m-k}J^*\overline{\alpha}\\ &= (-1)^{\frac{k(k+1)}{2}}\cdot(m-k)!\cdot i^{p-q}*\overline{\alpha}.\end{aligned}$$

Therefore,

$$Q(\alpha,\overline{\alpha}) = (-1)^k\cdot i^{p-q}\cdot(m-k)!\cdot\langle\alpha,\alpha\rangle_{\mathbb{C}}$$

and since $i^{2(p-q)}=(-1)^{p+q}=(-1)^k$, the claim follows. □

Example 2.5. We have $(\wedge^{1,1}V^*)\cap\wedge^2V^*=\mathbb{R}\omega\oplus P^{1,1}$ by the Lefschetz–decomposition. This decomposition is orthogonal, for if $\alpha\in P^{1,1}$, then $\alpha\wedge\omega\wedge\omega^{m-2}=L^{m-1}\alpha=0$. Moreover, Q is negative–definite on $\mathbb{R}\omega$ (since $Q(\omega,\omega)\mathrm{vol}=-\omega^m=-m!\mathrm{vol}$) and positive–definite on on $P^{1,1}$, since for $\alpha\in P^{1,1}$ we have $\overline{\alpha}=\alpha$, so that $Q(\alpha,\alpha)=(m-2)!\langle\alpha,\alpha\rangle>0$.

2.2. **The intersection form on Kähler manifolds.**

Theorem 2.6. Let (M,g,J) be a real $2m$–dimensional, compact Kähler manifold with fundamental form ω. For all closed k–forms α with $[\alpha]\in H^{p,q}(M;\mathbb{K})_0$ we have

$$i^{p-q}\cdot(-1)^{\frac{k(k-1)}{2}}\cdot\int_M\alpha\wedge\overline{\alpha}\wedge\omega^{m-k}>0$$

Proof. Since the integral only depends on the cohomology class, we may assume α to be harmonic and primitive. Then

$$i^{p-q}\cdot(-1)^{\frac{k(k-1)}{2}}\cdot\int_M\alpha\wedge\overline{\alpha}\wedge\omega^{m-k}=\int_M\langle\alpha,\alpha\rangle_{\mathbb{C}}\mathrm{vol}>0.$$

□

Remark 2.7. Let (M,g,J) be a $2m$–dimensional, compact Kähler manifold.

(1) Let α, β be closed k–forms of types (p,q) and (r,s), respectively, such that $[\alpha]$, $[\beta]$ are primitive. We have already seen, that

$$\tilde{Q}(\alpha,\beta) := (-1)^{\frac{k(k-1)}{2}} \cdot \int_M \alpha \wedge \beta \wedge \omega^{m-k} = 0,$$

unless β is of type (q,p). Since $\tilde{Q}(L^\ell \alpha, L^\ell \beta) = \tilde{Q}(\alpha,\beta)$, the Hard Lefschetz theorem implies that $\tilde{Q}$ is non–degenerate on $H^\ell(M;\mathbb{K})$ for all $\ell \geq 0$.

(2) If $p+q \equiv 0 \bmod 2$, then p and q have the same parity. Thus, with $k = p+q$, we have

$$i^{p-q} \cdot (-1)^{\frac{k(k-1)}{2}} = i^{p-q+(p+q)(p+q-1)} = i^{p-q-p-q+(p+q)^2} = (-1)^q,$$

and hence $\tilde{Q}$ is positive–definite on $H^{p,q}(M;\mathbb{K})_0$ with $p,q \equiv 0 \bmod 2$, and negative–definite, if $p,q \equiv 1 \bmod 2$.