Sasakian Manifolds:

Differential Forms, Curvature and Conformal Killing Forms

Inaugural-Dissertation
zur
Erlangung der Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von
Christian Stromenger
aus Köln

Berichterstatter :	Prof. Dr. U. Semmelmann
	Prof. Dr. H. Geiges

Tag der mündlichen Prüfung: 29.06.2010

Abstract. The main focus of this work is to study conformal Killing forms on Sasakian manifolds. Our most important tool is the so-called curvature condition, which is a certain formula that arises after differentiating the conformal Killing equation. The curvature of a Sasakian manifold has many symmetries with respect to the Reeb vector field and its covariant derivative. The combination of these symmetries with the curvature condition yields restrictions for conformal Killing forms.

We discuss the space of differential forms on a Sasakian manifold and decompose every horizontal form into (p, q)-forms, similarly to Kähler manifolds. This allows us to define the Dolbeault operators on the subspace of horizontal forms.

After that we replace the Riemannian curvature tensor by another tensor that is better adjusted to the Sasakian structure. This simplifies the calculations that we have to do in the discussion of the conformal Killing form.

For conformal Killing forms on Sasakian manifolds we show that they are always the sum of a Killing and a $*$-Killing form. Furthermore we investigate Killing forms and we decompose every Killing form into the sum of a special Killing form and an eigenform of the Lie derivative in direction of the Reeb vector field. Then we discuss the combination of the Killing equation and the eigenvalue equation and decompose the given Killing form into its (p, q)-parts. We use the Dolbeault operators to gain more information about this case.

Finally we classify conformal Killing forms in several special cases, including Eta-Einstein and Sasaki-Einstein manifolds as well as horizontal and normal conformal Killing forms.

Kurzzusammenfassung. Der Hauptbestandteil dieser Arbeit liegt in der Untersuchung von konformen Killingformen auf Sasaki-mannigfaltigkeiten. Das wichtigste Werkzeug hierzu ist die sog. Krümmungsbedingung, die sich durch Differentiation der konformen Killinggleichung ergibt. Der Krümmungstensor einer Sasaki-Mannigfaltigkeit besitzt viele Symmetrien bzgl. des Reebvektorfeldes und seiner kovarianten Ableitung. Die Kombination dieser Symmetrien mit der Krümmungsbedingung liefert Einschränkungen für die konformen Killingformen.

Wir untersuchen den Raum der Differentialformen auf einer Sasaki-Mannigfaltigkeit und zerlegen jede horizontale Form in (p, q)-Formen, analog zum Vorgehen auf Kähler-Mannigfaltigkeiten. Dies erlaubt uns, die Dolbeault-Operatoren auf Sasaki-Mannigfaltigkeiten einzuführen.

Danach ersetzen wir den Riemann'schen Krümmungstensor durch einen anderen Tensor, der der Sasaki-Struktur besser angepasst ist. Dies vereinfacht die Rechnungen, die wir bei der Untersuchung der konformen Killinggleichung durchführen.

Wir zeigen, dass jede konforme Killingform auf einer Sasaki-Mannigfaltigkeit die Summe einer Killingform und einer $*$-Killingform ist. Weiter untersuchen wir Killingformen und zerlegen jede Killingform in die Summe einer speziellen Killingform und einer Eigenform der Lieableitung in Richtung des Reebvektorfeldes. Dann diskutieren wir die Kombination aus der Killinggleichung und der Eigenwertgleichung und zerlegen die gegebene Killingform in ihre (p, q)-Anteile. Wir verwenden die Dolbeault-Operatoren, um in diesem Fall weitere Informationen zu erhalten

Schließlich klassifizieren wir konforme Killingformen in einigen Spezialfällen. Diese beinhalten Eta-Einstein- und Sasaki-Einstein-Mannigfaltigkeiten sowie horizontale und normale konforme Killingformen.

Contents

0 Introduction

0.1 Introduction

The central objects in this work are Sasakian manifolds, which were introduced in the 1960's by S. Sasaki as an odd-dimensional analogon of Kähler manifolds. Kähler manifolds are a classical object of differential geometry and well studied in literature. Compared to that Sasakian manifolds have only recently become subject of deeper research in mathematics and physics.

Kähler manifolds are the generalization of the complex space $\mathbb{C}^n$ to complex manifolds with respect to the geometric and symplectic properties of $\mathbb{C}^n$. The defining objects of a Kähler manifold N are a Riemannian metric g, a complex structure J and a symplectic form ω that satisfy the compatibility condition $\omega(X,Y) = g(JX,Y)$ for all vector fields X and Y on N.

In order to convey the idea of Kähler manifolds to odd dimensions it is necessary to translate the concepts of complex structures and symplectic forms to odd dimensions. Contact forms and normal almost contact structures are often considered as the natural odd-dimensional analogons of these structures because they both induce the corresponding even-dimensional objects when raising or dropping the dimension by 1: Every contact form on M induces a symplectic form on the product manifold $M \times \mathbb{R}_+$[1], and the kernel of the contact form is a symplectic subbundle of TM. Likewise, every normal almost contact structure (ϕ, η, ξ) induces a complex structure on $M \times \mathbb{R}$ and the restriction of ϕ to the kernel of η is a complex structure. On a Sasakian manifold, the two structures are combined in the sense that the 1-form η of the normal almost contact structure is a contact form.

Surprisingly, it is possible to find a Riemannian metric on M which induces Riemannian metrics on both the manifold $M \times \mathbb{R}_+$ and the bundle $\ker(\eta)$ that are compatible with the complex and symplectic structures in the Kähler sense. If (ϕ, η, ξ, g) is a contact metric structure, then by restricting the metric g, the endomorphism ϕ and the 2-form $\frac{1}{2}d\eta$ to $\ker(\eta)$ the bundle $\ker(\eta)$ behaves like the tangent bundle of a Kähler manifold. Unfortunately, the manifold $M \times \mathbb{R}$ endowed with the standard product metric is not a Kähler manifold since the induced symplectic form and the induced complex structure are not compatible with respect to the metric. But by replacing the product metric with the cone metric it is possible to obtain a Kähler structure on the cone. Conversely, every Kähler structure on a cone $M \times \mathbb{R}_+$ is induced by a normal contact metric structure on M (see [B02]). Therefore there is a $1-1$ correspondence between the two structures and a manifold is called Sasakian if its cone is a Kähler manifold.

The close relationship between Kähler manifolds and Sasakian manifolds naturally leads to the question which objects, methods and theorems can be transfered from one to the other. In this work we deal with this question with special regard to differential forms on Sasakian manifolds.

If the Sasakian manifold is regular in the sense of contact manifolds, the quotient space $B := M/\xi$ is also a Kähler manifold. In this case holds $TB = \ker(\eta)$ and the horizontal forms β on M that are the pull-back of forms on B are called basic and are characterized by $\xi \lrcorner \beta = 0$ and $\mathcal{L}_\xi \beta = 0$. But these conditions can be fulfilled even if the manifold is not regular. Horizontal and basic forms are of special interest on Sasakian manifolds since they behave pointwise (if horizontal) or even locally (if basic) like forms

[1] In fact it is possible to induce a symplectic structure on $M \times I$, where $I \subset \mathbb{R}$ is any open interval: If η is a contact form and $f : I \longrightarrow \mathbb{R}$ a smooth function with $f(r) \neq 0$ and $f'(r) \neq 0$ for all $r \in I$, then $d(f\pi^*\eta)$ is a symplectic form on $M \times I$. In the Sasakian case we have $I = \mathbb{R}_+$ and $f(r) = r^2$.

on Kähler manifolds. Furthermore, if $\Omega^p(H)$ denotes the set of horizontal p-forms, then holds $\Omega^p(M) = \Omega^p(H) \oplus \eta \wedge \Omega^{p-1}(H)$, i.e. every differential form on M is given by two horizontal forms. It is therefore important to know how a given operator deals with this splitting. Of particular interest is the curvature tensor R as well as the exterior differentials d and $\delta = d^*$ and related operators.

On almost complex manifolds every differential form can be be uniquely decomposed into a sum of (p, q)-forms. This translates directly to horizontal forms on a Sasakian manifold. The exterior differential d of a Sasakian manifold induces the horizontal differential d_H on horizontal forms, which can be decomposed into $d_H = \partial + \bar{\partial}$ according to the (p, q)-decomposition. Like on Kähler manifolds there are the corresponding Laplace operators $\Delta_H, \Delta_\partial$ and $\Delta_{\bar{\partial}}$, which on basic forms satisfy the Kähler relation $\Delta_H = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.

The curvature tensor of a Sasakian manifold naturally is one of the most important objects since every Sasakian manifold is a Riemannian manifold. But on Sasakian manifolds it is even more important: It is possible to give a definition of Sasakian manifolds that involves the curvature tensor. Therefore the further properties of the curvature tensor are of special interest. On a Kähler manifold the complex structure J is skew-adjoint and parallel, which leads to many simple symmetries of the Riemannian curvature tensor with respect to J. The endomorphism ϕ of a Sasakian manifold is skew-adjoint and the covariant derivative of ϕ is explicitly known, which makes it still possible to determine the symmetries of R with respect to ϕ, but they are more complicated than in the Kähler case. It is also important to know how the curvature tensor behaves with respect to the interior and exterior product with the Reeb vector field ξ and with the action of the 2-form $d\eta \mathrel{\hat{=}} 2\phi$. It turns out that the curvature tensor R neither respects the decomposition $\Omega^p(M) = \Omega^p(H) \oplus \eta \wedge \Omega^{p-1}(M)$ nor is in any way compatible with the decomposition of horizontal forms in (p, q)-forms.

The main focus of this thesis lies on the investigation of conformal Killing forms on Sasakian manifolds. These are forms $\psi \in \Omega^p(M)$ that satisfy

$$\nabla_X \psi = \tfrac{1}{p+1} X \lrcorner\, d\psi - \tfrac{1}{n-p+1} X^* \wedge \delta\psi$$

for every tangent vector X on M. If the form ψ is additionally coclosed, then it is called a Killing form, and if it is closed, then it is called a $*$-Killing form. An essential property of Sasakian manifolds that has no direct analogon on Kähler manifolds is that the contact form η is a Killing 1-form. Moreover, a Sasakian manifold admits at least one nowhere vanishing conformal Killing form in every degree: The forms $\eta \wedge (d\eta)^k$ and $(d\eta)^k$ are Killing resp. $*$-Killing forms for every k. If the manifold is a 3-Sasakian manifold, it is possible to obtain more conformal Killing forms by combining the different Sasakian structures. Up to now, for $p \neq 1, n-1$ these are all known examples of conformal Killing p-forms on Sasakian manifolds different from the sphere. It is natural to ask if there are any other examples. A partial answer is given by the articles of M. Okumura [O62] and S. Yamaguchi [Y72b], where they show that under suitable conditions every conformal Killing p-form splits into the sum of a Killing and a $*$-Killing form. This in contrast to the situation on compact Kähler manifolds, where every Killing and every $*$-Killing form has to be parallel [Y75].

A common property of the known examples is that they are all special Killing and $*$-Killing forms, i.e. they induce parallel forms on the cone. Another natural question is therefore if all Killing and $*$-Killing forms on Sasakian manifolds have to be special.

Conformal Killing forms on Riemannian manifolds are systematically studied in [S01]. By covariant differentiation of the conformal Killing equation many results about the action of the Riemannian curvature tensor on the conformal Killing form ψ and about the properties of the forms $d\psi, \delta\psi$ and $\Delta\psi$ are obtained. This is the starting point for our discussion of conformal Killing forms on Sasakian manifolds.

0.2 Main results

Let M be a $n = (2m+1)$-dimensional Sasakian manifold. After introducing the curvature tensor

$$A(X,Y) = R(X,Y) + (X^* \wedge Y^*)\bullet$$

on M we show that it respects the splitting $\Omega^p(M) = \Omega^p(H) \oplus \eta \wedge \Omega^{p-1}(H)$ and maps (p,q)-forms to (p,q)-forms.

In [Y72b] S. Yamaguchi shows that every conformal Killing p-form on a compact $(2m+1)$-dimensional Sasakian manifold with $p \leq m$ and $m > 1$ splits into the sum of a Killing and a $*$-Killing form. We show that the compactness and the condition $p \leq m$ are not necessary:

Theorem. *Let (M,g) be a n-dimensional Riemannian manifold with $n > 3$. If (M,g) admits a Sasakian structure then for all $p = 1,\dots,n-1$ every conformal Killing p-form ψ is the sum of a Killing p-form σ and a $*$-Killing p-form τ. If $p \geq 2$, then $\sigma = \frac{1}{(p+1)(n-p)}\delta d\psi$, and if $p \leq n-2$, then $\tau = \frac{1}{p(n-p+1)}d\delta\psi$. Furthermore we have*

$$q(R)\psi = p\,(n-p)\psi.$$

for $2 \leq p \leq n-2$.

As a corollary we have $\Delta\sigma = (p+1)(n-p)\sigma$ for every Killing p-form σ with $p \geq 2$ and $\Delta\tau = p\,(n-p+1)\tau$ for every $*$-Killing p-form τ with $p \leq n-2$.

We investigate the question if every Killing form on a Sasakian manifold has to be a special Killing form and have the following result:

Theorem. *Let (M,g,ξ) be a $(2m+1)$-dimensional Sasakian manifold. Then every Killing p-form with $p \geq m+1$ is special. If M is compact, then every Killing m-form is special and all Killing p-forms are the sum of a special Killing form and a Killing form that is an eigenform of $\mathcal{L}_\xi^2$ with eigenvalue $-(p-1)^2$.*

This leads us to a further investigation of Killing forms σ that satisfy the additional equation $\mathcal{L}_\xi\sigma = i(p-1)\sigma$.

Finally we investigate conformal Killing forms in several special cases and have the following results.

Theorem. *Let (M,g,ξ) be a n-dimensional Sasakian manifold and $\psi \in \mathcal{CK}^p(M,g)$ with $2 \leq p \leq n-2$.*

- *If M is a strictly Eta-Einstein manifold, then $\psi = \eta \wedge (d\eta)^k$ if p is odd and $\psi = (d\eta)^k$ if p is even.*
- *If M is a compact Sasaki-Einstein manifold, then ψ is the sum of a special Killing and a special $*$-Killing form.*
- *If ψ is a normal conformal Killing form, then ψ is the sum of a special Killing and a special $*$-Killing form, and the open submanifold $\{x \in M \mid \psi_x \neq 0\}$ is a Sasaki-Einstein manifold.*

0.3 Overview

We give a more detailed description of our paper.

Section 1. We start with the definition of Sasakian manifolds, introduce the main objects and collect some basic properties and equivalent characterizations. We define Eta-Einstein manifolds and discuss their connection to the Ricci curvature of the cone and to the Ricci curvature of the transversal metric $g|HM$. In Section 1.2 we transport the standard operators of Kähler manifolds, i.e. the interior product Λ and exterior product L with the Kähler form and the action of the complex structure on forms, to Sasakian manifolds, where we replace the Kähler form with the Sasakian form $\omega = \frac{1}{2}d\eta$. We compute the commutators between L, Λ and d, δ and decompose differential forms into horizontal and vertical forms. This induces a decomposition of all $\mathbb{R}$-linear operators on $\Omega^*(M)$ into four $\mathbb{R}$-linear operators on $\Omega^*(H)$, which we make explicit for the exterior differentials d and δ as well as the Levi-Civita connection and the Laplace operator. After that we discuss the eigenforms and eigenvalues of the Sasakian endomorphism ϕ on $\Omega^*_{\mathbb{C}}(M)$, which leads us to the Dolbeault operators $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*$ and to the Laplace operators Δ_∂ and $\Delta_{\bar{\partial}}$. At the end of Section 1 we discuss a certain class of metric connections on M, calculate their curvature and study their parallel forms.

Section 2. The second section is about the curvature tensor of a Sasakian manifold. After explicitly calculating the symmetries of R with respect to ϕ, in particular $R(\phi X, Y) + R(X, \phi Y)$ and $[\phi, R(X,Y)]$, we use the curvature tensor to show that Sasakian manifolds do not admit any parallel forms and therefore have to be irreducible. We calculate several contractions of R, before we introduce the Sasakian curvature tensor A, which is essentially the curvature tensor of the cone. We reformulate all results for R in terms of A and show that A is much better adjusted to the Sasakian structure: It maps horizontal forms to horizontal forms, vertical forms to vertical forms and (p,q)-forms to (p,q)-forms because of $[A(X,Y), \eta\wedge] = [A(X,Y), \xi \lrcorner] = [A(X,Y), \phi_{\mathrm{D}}] = 0$. Then we use the symmetries of A and the fact that $\nabla A = \nabla R$ holds to determine the symmetries of ∇R. As a consequence we see that R is completely determines by ∇R. We conclude this section with the proof that the Ricci form of Sasakian manifolds, $\rho(X,Y) = \mathrm{ric}(\phi X, Y)$, is closed.

Section 3. In Section 3 we recall the definition and some properties of conformal Killing forms on arbitrary Riemannian manifolds. All results we state play an important rolein our investigation of conformal Killing forms on Sasakian manifolds. We analyse the so-called curvature condition for conformal Killing forms and combine it with the properties of the curvature tensor of a Sasakian manifold we derived in Section 2. We see that the curvature condition imposes strong restrictions on conformal Killing forms, which leads to the splitting of conformal Killing forms in Killing forms and $*$-Killing form. Due to the splitting we reduce our discussion to Killing forms. Using again the curvature condition we show that for every Killing p-form the form $\mathcal{L}_\xi^2\sigma + (p-1)^2\sigma$ is a special Killing form, which allows us to further reduce the problem to Killing forms σ with $\mathcal{L}_\xi\sigma = i(p-1)\sigma$. We use this eigenvalue equation and decompose σ in its (p,q)-parts, where we obtain $\sigma \in \Omega^{(p,0)} \oplus \Omega^{(p-1,1)} \oplus \eta \wedge \Omega^{(p-1,0)}$ and find that σ is determined by the $(p-1)$-form $\xi \lrcorner \sigma$. The last part of this section contains the investigation of horizontal and vertical conformal Killing forms as well as conformal Killing forms on Einstein and Eta-Einstein manifolds and normal conformal Killing forms. In all these special cases we use the curvature properties we obtained before and are able to classify conformal

Killing forms up to special Killing and special $*$-Killing forms.

0.4 Notation and conventions

Let (M, g) be a Riemannian manifold.

The Levi-Civita connection of the metric g is denoted by ∇, and $\{e_i\}$ is always a local orthonormal frame. The Ricci curvature, considered as $(2, 0)$-tensor, will be denoted by ric, while we write Ric for the corresponding endomorphism.

The set of complex valued differential forms is defined as $\Omega^*_{\mathbb{C}}(M) := \Omega^*(M)+i\Omega^*(M)$.

For any tangent vector $v \in T_xM$ we define $v^* \in T^*_xM$ by $v^* = g(v, \cdot)$. Conversely, if $\alpha \in T^*_xM$ is a tangent covector we define $\alpha_\sharp \in T_xM$ by $g(\alpha_\sharp, \cdot) = \alpha$. Obviously these two operations are inverse to each other, i.e. $(v^*)_\sharp = v$ and $(\alpha_\sharp)^* = \alpha$.

We extend every endomorphism field $F \in \Gamma(\mathrm{End}(TM))$ to a fibre-wise derivation

$$F_{\mathrm{D}} : \Omega^p(M) \longrightarrow \Omega^p(M)$$

via

$$(F_{\mathrm{D}}\alpha)(X_1, \dots, X_p) := -\sum \alpha(X_1, \dots, FX_i, \dots, X_p).$$

The local expression of F_{D} is given by

$$F_{\mathrm{D}} = -\sum e_i^* \wedge Fe_i \lrcorner = -\sum (F^{\mathrm{ad}}e_i)^* \wedge e_i \lrcorner\,.$$

Note that we choose this sign convention since differential operators are extended in the same way and the relations between differential operators and endomorphisms stay the same after the extension. For example, if $\xi \in \Gamma(TM)$ is any vector field with covariant derivative $\phi := \nabla\xi \in \Gamma(\mathrm{End}(TM))$, the Lie derivative $\mathcal{L}_\xi$ is given by

$$\mathcal{L}_\xi X = [\xi, X] = \nabla_\xi X - \nabla_X \xi = (\nabla_\xi - \phi)\, X,$$

thus $\mathcal{L}_\xi = \nabla_\xi - \phi$. After extending to operators on $\Omega^p(M)$ we still have $\mathcal{L}_\xi = \nabla_\xi - \phi_{\mathrm{D}}$. Another example is the extension of the Riemannian curvature tensor $R(X, Y)$: We can extend it to a derivation of Λ^pT^*M directly by $R(X, Y)\alpha = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ or by using the method described above. Using our convention, both ways give the same object $R(X, Y) : \Omega^p(M) \longrightarrow \Omega^p(M)$.

For the extension of $F \in \Gamma(\mathrm{End}(TM))$ as a derivation on $\Omega^1(M)$ is characterized by

$$F_{\mathrm{D}}X^* = -(F^{\mathrm{ad}}X)^*$$

and

$$F_{\mathrm{D}}(\alpha \wedge \beta) = F_{\mathrm{D}}\alpha \wedge \beta + \alpha \wedge F_{\mathrm{D}}\beta,$$

and the compatibility with the interior product is given by

$$F_{\mathrm{D}}(\alpha \lrcorner \beta) = -F^{\mathrm{ad}}_{\mathrm{D}}\alpha \lrcorner \beta + \alpha \lrcorner F_{\mathrm{D}}\beta.$$

We mention two special cases:

- If F is self-adjoint, then $F_{\mathrm{D}}X^* = -(FX)^*$. An example is the extension of the Ricci curvature, where we have $\mathrm{Ric}_{\mathrm{D}}(X^*) = -\mathrm{Ric}(X)^*$.
- If F is skew-adjoint, then $F_{\mathrm{D}}X^* = (FX)^*$. In this case we write $F_{\mathrm{D}}X^* = FX^*$ with a slight abuse of notation. Here important examples are given by the Riemannian curvature tensor $R(X,Y)$, i.e. we have $R(X,Y)_{\mathrm{D}}Z^* = R(X,Y)Z^*$, and the covariant derivative $\phi = \nabla\xi$ of any Killing vector field ξ, i.e. we have $\phi_{\mathrm{D}}X^* = \phi X^*$.

Every 2-form $\alpha \in \Omega^2(M)$ acts on TM via the metric by

$$\alpha \bullet X := (X \lrcorner \alpha)_\sharp,$$

or equivalently

$$g(\alpha \bullet X, Y) := \alpha(X,Y).$$

Thus α defines an endomorphism field on M. We extend this endomorphism field as described above and obtain an action

$$\alpha\bullet : \Omega^p(M) \longrightarrow \Omega^p(M).$$

In the special case where $\alpha = X^* \wedge Y^*$ we have a direct formula for $\alpha\bullet$:

$$\alpha\bullet = Y^* \wedge X \lrcorner - X^* \wedge Y \lrcorner .$$

In particular we have the following expression which we use frequently on Sasakian manifolds:

$$(X^* \wedge Y^*) \bullet Z = g(X,Z)Y - g(Y,Z)X.$$

1 Sasakian manifolds

1.1 Definition and properties

In this section we collect some equivalent definitions of Sasakian manifolds and introduce the most important objects like the corresponding 2-form $\omega = \frac{1}{2}d\eta$ and the bundle of horizontal vectors $HM = \ker(\eta)$. The material is well covered in the literature, as references may serve [B02], [BG08] and [FOW09].

Definition 1.1.1. *Let (M, g) be a Riemannian manifold. A vector field $\xi \in \Gamma(TM)$ is called a* ***Sasakian structure*** *on (M, g) if the Riemannian cone $(C(M), g_C) = (M \times \mathbb{R}_+, r^2 g + dr^2)$ is a Kähler manifold with Kähler form $\frac{1}{2}d(r^2\xi^*)$, where r is the radial coordinate. In this case the triple (M, g, ξ) is called a* ***Sasakian manifold****.*

The following theorem contains some well-known equivalent characterizations of Sasakian manifolds.

Theorem 1.1.2. *Let (M, g) be a Riemannian manifold and $\xi \in \Gamma(TM)$ any vector field. Then the following statements are equivalent.*

(i) ξ is a Sasakian structure.

(ii) There exist a 1-form $\eta \in \Omega^1(M)$ and an endomorphism field $\phi \in \Gamma(\mathrm{End}(TM))$ satisfying

- $\eta(\xi) = 1$,
- $\phi^2 X = -X + \eta(X)\xi$,
- $g(\phi X, Y) = \frac{1}{2}d\eta(X, Y)$,
- $[\phi, \phi] = d\eta \otimes \xi$

for all vector fields X and Y, where $[\phi, \phi]$ denotes the Nijenhuis tensor.

(iii) ξ is a unit Killing vector field and satisfies

$$R(X, \xi)Y = (\xi^* \wedge X^*)\bullet = g(\xi, Y)X - g(X, Y)\xi$$

for all vector fields X and Y.

(iv) ξ is a unit Killing vector field such that the endomorphism field $\phi \in \Gamma(\mathrm{End}(TM))$, defined by $\phi X := \nabla_X \xi$, satisfies

$$(\nabla_X \phi)Y = (\xi^* \wedge X^*)\bullet = g(\xi, Y)X - g(X, Y)\xi$$

for all vector fields X and Y.

(v) ξ is a unit Killing vector field and satisfies

$$\nabla_X(d\xi^*) = 2\xi^* \wedge X^*$$

for all vector fields X.

The standard examples of Sasakian manifolds are the odd-dimensional spheres S^{2m+1} with cone $C(M) = \mathbb{C}^{2m+2} \setminus \{0\}$. The euclidean space $\mathbb{R}^{2m+1}$ with the flat metric is not a Sasakian manifold because it does not satisfy (iii) of the previous theorem. However, if we equip $\mathbb{R}^{2m+1}$ with its (scaled) standard contact form $\eta = \frac{1}{2}dz - \frac{1}{2}\sum y_i dx_i$ and define a Riemannian metric by $g = \eta \otimes \eta + \frac{1}{4}\sum(dx_i^2 + dy_i^2)$, then $(\mathbb{R}^{2m+1}, g, \eta^\sharp)$ is a Sasakian manifold (see [B02]). Other examples can be constructed as principal S^1-bundles over Kähler manifolds, the construction is described in [B02].

We summarize the main objects on Sasakian manifolds in the following definition. Note that this definition is consistent with the notation used in the previous theorem.

Definition 1.1.3. *Let (M, g, ξ) be a Sasakian manifold in the sense of Definition 1.1.1. We define a 1-form $\eta \in \Omega^1(M)$, an endomorphism field $\phi \in \Gamma(\mathrm{End}(TM))$ and a 2-form $\omega \in \Omega^2(M)$ on M by*

- $\eta(X) := g(\xi, X)$,
- $\phi X := \nabla_X \xi$,
- $\omega(X, Y) := g(\phi X, Y) = \frac{1}{2}(d\xi^*)(X, Y)$

for all vector fields X and Y. We call ϕ the ***Sasakian endomorphism*** *and ω the* ***Sasakian 2-form*** *associated to the Sasakian structure ξ. The vectors in $\ker(\eta)$ are called* ***horizontal*** *and we write $HM := \ker(\eta)$.*

The next proposition contains a collection of some of the well-known properties of Sasakian manifolds.

Proposition 1.1.4. *Let (M, g, ξ) be n-dimensional Sasakian manifold.*

- *(M, η) is a contact manifold with Reeb vector field ξ.*
- *The Sasakian endomorphism ϕ has constant rank $n-1$ and we have $\ker(\phi) = \langle \xi \rangle$ and $\mathrm{im}(\phi) = \ker(\eta) = HM$.*
- *The tangent bundle TM of M splits orthogonally in*

$$\begin{array}{ccccc} TM & = & HM & \oplus & \langle \xi \rangle, \\ X & = & -\phi^2 X & + & \eta(X)\xi. \end{array}$$

- *ϕ is an almost complex structure on HM.*
- *The Ricci tensor satisfies $\mathrm{Ric}(\xi) = (n-1)\xi$, or equivalently $\mathrm{ric}(X, \xi) = (n-1)\eta(X)$.*
- *Every parallel form on M has to vanish.*
- *M is irreducible.*
- *If M is symmetric, then $R(X, Y) = -X^* \wedge Y^*$.*

Since we have $\phi^2 = -\mathrm{id}$ on HM we may decompose the complexified contact structure $HM_\mathbb{C} = H^+ \oplus H^-$, where $H^\pm$ denotes the eigenspace of ϕ with eigenvalue $\pm i$. The decomposition is given explicitly by $U = U^+ + U^-$ with $U^\pm = \frac{1}{2}(U \mp i\phi U)$.

An important subclass of Sasakian manifolds are of course Sasaki-Einstein manifolds. If (M, g, ξ) is a Sasaki-Einstein manifold with $\mathrm{Ric} = \lambda \mathrm{id}$, then necessarily we have $\lambda = n-1$ since $\mathrm{Ric}(\xi) = (n-1)\xi$. Thus the cone of a Sasaki-Einstein manifold is Kähler and Ricci-flat, i.e. a Calabi-Yau manifold.

Another important subclass of Sasakian manifolds is defined in the following definition.

Definition 1.1.5. *A Sasakian manifold (M, g, ξ) is called **Eta-Einstein** if there exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\mathrm{Ric} = c_1 \mathrm{id} + c_2 \eta \otimes \eta.$$

*A **strictly Eta-Einstein** manifold is an Eta-Einstein manifold which is not Einstein, i.e. $c_2 \neq 0$.*

From $\mathrm{Ric}(\xi) = (n-1)\xi$ we obtain $n - 1 = c_1 + c_2$, thus we may rewrite the Eta-Einstein condition as

$$\mathrm{Ric} - (n-1)\mathrm{id} = c \cdot \phi^2.$$

The relevance of Eta-Einstein manifolds is given by the following fact: If M is a regular Sasakian manifold, then M is Eta-Einstein if and only if the quotient space M/ξ is an Einstein manifold. More generally, a Sasakian manifold is Eta-Einstein if the restricted metric $g^T := g|HM$ is an Einstein metric on HM.

1.2 Differential forms

1.2.1 Important operators

The Reeb vector field ξ induces three natural operations on $\Omega^*(M)$, namely the interior product with ξ, the exterior product with η and the Lie derivative $\mathcal{L}_\xi$ in direction of ξ. Likewise we have the interior and exterior product with the Sasakian form ω and, since ω is a 2-form, we also have the action of ω on $\Omega^p(M)$.

All of the operators introduced here play a crucial role in our further work.

Definition 1.2.1. *Let (M, g, ξ) be a Sasakian manifold. We define*

$$\begin{array}{rcccccc} L: & \Omega^p(M) & \longrightarrow & \Omega^{p+2}(M), & \alpha & \longmapsto & \omega \wedge \alpha, \\ \Lambda: & \Omega^p(M) & \longrightarrow & \Omega^{p-2}(M), & \alpha & \longmapsto & \omega \lrcorner \alpha, \\ \phi_{\mathrm{D}}: & \Omega^p(M) & \longrightarrow & \Omega^p(M), & \alpha & \longmapsto & \omega \bullet \alpha. \end{array}$$

The local expressions of these operators are given by

$$\begin{aligned} L &= \tfrac{1}{2} \sum e_i^* \wedge \phi e_i^* \wedge, \\ \Lambda &= \tfrac{1}{2} \sum \phi e_i \lrcorner e_i \lrcorner, \\ \phi_{\mathrm{D}} &= \sum \phi e_i^* \wedge e_i \lrcorner = \sum \phi e_i \lrcorner (e_i^* \wedge \cdot). \end{aligned}$$

Definition 1.2.2. *Let (M, g, ξ) be a Sasakian manifold. A differential form $\alpha \in \Omega^p(M)$ is called*

- ***horizontal*** *if $\xi \lrcorner \alpha = 0$,*
- ***vertical*** *if $\eta \wedge \alpha = 0$,*

- ***primitive*** *if* $\Lambda\alpha = 0$,
- ***coprimitive*** *if* $L\alpha = 0$.

The set of all horizontal p-forms on M will be denoted by $\Omega^p(H)$.

Like on Kähler manifolds we have the following important commutator relation between L and Λ, which implies the injectivity of L and Λ on certain degrees.

Proposition 1.2.3. *Let* (M, g, ξ) *be a* $(2m+1)$*-dimensional Sasakian manifold. Then we have*

$$[L, \Lambda] = (\deg - m)\mathrm{id} - \eta \wedge \xi \lrcorner .$$

Proof. Let $\alpha \in \Omega^p(M)$. Then we have

$$\begin{aligned}
4L\Lambda\alpha &= \sum e_i^* \wedge \phi e_i^* \wedge \phi e_j \lrcorner e_j \lrcorner \alpha \\
&= -\sum e_i^* \wedge \phi e_j \lrcorner (\phi e_i^* \wedge e_j \lrcorner \alpha) + \sum g(\phi e_i, \phi e_j) e_i^* \wedge e_j \lrcorner \alpha \\
&= \sum e_i^* \wedge \phi e_j \lrcorner e_j \lrcorner (\phi e_i^* \wedge \alpha) + 2\sum g(\phi e_i, \phi e_j) e_i^* \wedge e_j \lrcorner \alpha \\
&= -\sum \phi e_j \lrcorner (e_i^* \wedge e_j \lrcorner (\phi e_i^* \wedge \alpha)) \\
&\quad + 2\sum g(\phi e_i, \phi e_j) e_i^* \wedge e_j \lrcorner \alpha - \sum g(\phi e_i, \phi e_j) e_j \lrcorner (e_i^* \wedge \alpha) \\
&= \sum \phi e_j \lrcorner e_j \lrcorner (e_i^* \wedge \phi e_i^* \wedge \alpha) \\
&\quad + 2\sum g(\phi e_i, \phi e_j) e_i^* \wedge e_j \lrcorner \alpha - 2\sum g(\phi e_i, \phi e_j) e_j \lrcorner (e_i^* \wedge \alpha) \\
&= 4\Lambda L\alpha + 4\sum g(\phi e_i, \phi e_j) e_i^* \wedge e_j \lrcorner \alpha - 2\sum g(\phi e_i, \phi e_i)\alpha \\
&= 4\Lambda L\alpha + 4\sum (g(e_i, e_j) - \eta(e_i)\eta(e_j)) e_i^* \wedge e_j \lrcorner \alpha - 2\sum \left(g(e_i, e_i) - \eta(e_i)^2\right)\alpha \\
&= 4\Lambda L\alpha + 4\sum e_i^* \wedge e_i \lrcorner \alpha - 4\eta \wedge \xi \lrcorner \alpha - 2(2m+1-1)\alpha \\
&= 4\Lambda L\alpha + 4(p-m)\alpha - 4\eta \wedge \xi \lrcorner \alpha. \qquad \square
\end{aligned}$$

Proposition 1.2.4. *Let* (M, g, ξ) *be a* $(2m+1)$*-dimensional Sasakian manifold. Then* $L : \Omega^p(M) \longrightarrow \Omega^{p+2}(M)$ *is injective for* $p \leq m-1$ *and* $\Lambda : \Omega^p(M) \longrightarrow \Omega^{p-2}(M)$ *is injective for* $p \geq m+2$.

Proof. First we show that it suffices to prove the injectivity of L on horizontal forms $\Omega^p(H)$ for $p \leq m-1$. Let $\alpha = \beta + \eta \wedge \gamma \in \Omega^p(M)$ with $\beta \in \Omega^p(H)$ and $\gamma \in \Omega^{p-1}(H)$. Since L commutes with $\eta\wedge$, we have $L\alpha = L\beta + \eta \wedge L\gamma$, and since L commutes with $\xi \lrcorner$, $L\beta$ and $L\gamma$ are again horizontal. Therefore, if $L\alpha = 0$ we get $L\beta = 0$ and $L\gamma = 0$. If L is injective on $\Omega^p(H)$ and on $\Omega^{p-1}(H)$, we can conclude $\beta = 0$ and $\gamma = 0$, thus $\alpha = 0$.

Let $\beta \in \Omega^p(H)$ be in the kernel of L and let $p \leq m-1$. The main tool to prove that β has to vanish will be the commutator relation

$$[L^k, \Lambda] = k(\deg - m + k - 1)L^{k-1} + k\, \eta \wedge \xi \lrcorner L^{k-1},$$

which simplifies to

$$[L^k, \Lambda] = k(\deg - m + k - 1)L^{k-1} \tag{1.2.1}$$

on horizontal forms. This formula implies

$$L^k\Lambda^k\beta = a_k\beta \tag{1.2.2}$$

with

$$a_k = k!\prod_{i=1}^{k}(p-m-i+1)$$

for all $k \geq 1$. To see this we use induction on k. For $k=1$ this is immediately clear by (1.2.1). Assuming (1.2.2) holds for a given k, we get

$$\begin{aligned}
L^{k+1}\Lambda^{k+1}\beta &= L^{k+1}\Lambda\Lambda^k\beta = (\Lambda L^{k+1} + [L^{k+1},\Lambda])\Lambda^k\beta \\
&= \Lambda LL^k\Lambda^k\beta + (k+1)((p-2k)-m+(k+1)-1)L^k\Lambda^k \\
&= a_k\Lambda L\beta + (k+1)(p-m-k)a_k\beta \\
&= a_{k+1}\beta.
\end{aligned}$$

The coefficients a_k are all different from zero: Otherwise there exists an $i \geq 1$ such that $p-m-i+1=0$, which is impossible since $p \leq m-1$. By choosing k with $p-2k<0$ we get $\Lambda^k\beta = 0$, thus $\beta = \frac{1}{a_k}L^k\Lambda^k\beta = 0$.

We use the same argumentation to show that Λ is injective on $\Omega^p(M)$ for $p \geq m+2$. Again it suffices to check the situation on $\Omega^p(H)$ and $\Omega^{p-1}(H)$, thus we need to show that Λ is injective on $\Omega^p(H)$ for all $p \geq m+1$. For $\beta \in \Omega^p(H)$ with $\Lambda\beta = 0$ we find

$$\Lambda^k L^k\beta = a_k\beta$$

with

$$a_k = k!\prod_{i=1}^{k}(m-p-i+1)$$

for all $k \geq 1$. Choosing k big enough yields $L^k\beta = 0$, thus $\beta = 0$ since again all a_k are different from zero because of $p \geq m+1$. $\square$

On Kähler manifolds with complex structure J and Levi-Civita connection ∇ the conjugate differentials d^c and δ^c are locally given by $d^c = \sum Je_i^* \wedge \nabla_{e_i}$ and $\delta^c = -\sum Je_i \lrcorner \nabla_{e_i}$. We transfer this definition to Sasakian manifolds.

Definition 1.2.5. *Let (M,g,ξ) be a Sasakian manifold. We define the operators $d^c : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ and $\delta^c : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ by*

$$\begin{aligned}
d^c &:= \sum \phi e_i^* \wedge \nabla_{e_i}, \\
\delta^c &:= -\sum \phi e_i \lrcorner \nabla_{e_i}.
\end{aligned}$$

These operators do not have a direct geometric meaning on Sasakian manifolds: For example we have $(d^c)^2 = -2L\mathcal{L}_\xi + \eta \wedge d^c \neq 0$, and the formal adjoint of d^c is not δ^c but $\delta^c + (n-1)\xi \lrcorner$. But their projection onto their horizontal parts induces the conjugate differentials of the underlying transversal Kähler structure, cf Section 1.3. They turn out to be very useful when calculating commutator relations like $[\phi_{\mathrm{D}}, d]$ and $[L,\delta]$:

Proposition 1.2.6. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then we have*

$$[L, d] = 0, \qquad [\Lambda, d] = -\delta^c - (\deg -1)\xi \lrcorner\,, \quad [\phi_{\mathrm{D}}, d] = d^c - \deg \eta\wedge,$$
$$[L, \delta] = d^c - (n - \deg -1)\eta\wedge, \quad [\Lambda, \delta] = 0, \qquad [\phi_{\mathrm{D}}, \delta] = \delta^c + \ (n - \deg)\xi \lrcorner\,.$$

Proof. These relations follow from straightforward computations using $\nabla_X\omega = \eta \wedge X^*$. We only demonstrate the calculation for $[L, \delta]$, where for all $\alpha \in \Omega^p(M)$ we have

$$\begin{aligned}
\delta L\alpha &= -\sum e_i \lrcorner \nabla_{e_i}(\omega \wedge \alpha) \\
&= -\sum e_i \lrcorner \left(\nabla_{e_i}\omega \wedge \alpha + \omega \wedge \nabla_{e_i}\alpha\right) \\
&= \sum e_i \lrcorner \left(e_i^* \wedge \eta \wedge \alpha\right) - \sum \phi e_i^* \wedge \nabla_{e_i}\alpha - \omega \wedge \sum e_i \lrcorner \nabla_{e_i}\alpha \\
&= L\delta\alpha - d^c\alpha + (n - p - 1)\eta \wedge \alpha. \qquad \square
\end{aligned}$$

Next we discuss the anticommutator relations between d, δ and d^c, δ^c, which will be used in Section 2.1 in order to study the symmetries of the Riemannian curvature tensor.

Proposition 1.2.7. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then we have*

$$\{d, d^c\} = \eta \wedge d + 2\deg\, L, \qquad \{\delta, d^c\} = \eta \wedge \delta - (n - \deg -1)\mathcal{L}_\xi,$$
$$\{d, \delta^c\} = -\xi \lrcorner\, d - (\deg -1)\mathcal{L}_\xi, \qquad \{\delta, \delta^c\} = -\xi \lrcorner\, \delta - 2(n - \deg)\Lambda.$$

The formal adjoints of d^c and δ^c are given by

$$\begin{aligned}
(d^c)^* &= \delta^c + (n-1)\xi \lrcorner\,, \\
(\delta^c)^* &= d^c + (n-1)\eta \wedge\,.
\end{aligned}$$

Proof. Let $\alpha \in \Omega^p(M)$. Using Proposition 1.2.6 yields

$$\begin{aligned}
dd^c\alpha + d^c d\alpha &= d([\phi_{\mathrm{D}}, d] + p\,\eta\wedge)\alpha + ([\phi_{\mathrm{D}}, d] + (p+1)\,\eta\wedge)\,d\alpha \\
&= p\,d(\eta \wedge \alpha) + (p+1)\eta \wedge d\alpha \\
&= \eta \wedge d\alpha + 2p\,L\alpha,
\end{aligned}$$

$$\begin{aligned}
d\delta^c\alpha + \delta^c d\alpha &= d(-[\Lambda, d] - (p-1)\xi \lrcorner\,)\alpha + (-[\Lambda, d] - p\,\xi \lrcorner\,)d\alpha \\
&= -(p-1)d(\xi \lrcorner\, \alpha) - p\,\xi \lrcorner\, d\alpha \\
&= -\xi \lrcorner\, d\alpha - (p-1)\mathcal{L}_\xi\alpha,
\end{aligned}$$

$$\begin{aligned}
\delta d^c\alpha + d^c\delta\alpha &= \delta([L, \delta] + (n-p-1)\eta\wedge)\alpha + ([L, \delta] + (n - (p-1) - 1)\eta\wedge)\delta\alpha \\
&= (n-p-1)\delta(\eta \wedge \alpha) + (n-p)\eta \wedge \delta\alpha \\
&= \eta \wedge \delta\alpha - (n-p-1)\mathcal{L}_\xi\alpha,
\end{aligned}$$

$$\begin{aligned}
\delta\delta^c\alpha + \delta^c\delta\alpha &= \delta([\phi, \delta] - (n-p)\xi \lrcorner\,)\alpha + ([\phi, \delta] - (n - (p-1))\xi \lrcorner\,)\delta\alpha \\
&= -(n-p)\delta(\xi \lrcorner\, \alpha) - (n-p+1)\xi \lrcorner\, \delta\alpha \\
&= -\xi \lrcorner\, \delta\alpha - 2(n-p)\Lambda\alpha.
\end{aligned}$$

To determine the formal adjoint of d^c, let $\alpha' \in \Omega^{p-1}(M)$ and let α and α' be compactly supported. Then we obtain

$$\int_M g((d^c)^*\alpha, \alpha')\mathrm{vol} = \int_M g(\alpha, d^c\alpha')\mathrm{vol}$$

$$
\begin{aligned}
&= \int_M g(\alpha, [\phi_{\mathrm{D}}, d]\alpha' + (p-1)\eta \wedge \alpha')\mathrm{vol} \\
&= \int_M g([\phi_{\mathrm{D}}, \delta]\alpha + (p-1)\xi \lrcorner \alpha, \alpha')\mathrm{vol} \\
&= \int_M g(\delta^c \alpha + (n-1)\xi \lrcorner \alpha, \alpha')\mathrm{vol}.
\end{aligned}
$$

The calculation for $(\delta^c)^*$ is analogous. $\square$

In order to decompose d, δ, d^c, δ^c and Δ into their horizontal and vertical part, cf Section 1.2.2, we need the (anti-)commutators between d, δ, d^c, δ^c and Δ on one hand and $\eta\wedge, \xi \lrcorner$ on the other hand.

Lemma 1.2.8. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then we have*

$$
\begin{aligned}
&\{\eta\wedge, d\} = 2L, && \{\xi \lrcorner, d\} = \mathcal{L}_\xi, \\
&\{\eta\wedge, \delta\} = -\mathcal{L}_\xi, && \{\xi \lrcorner, \delta\} = 2\Lambda \\
&[\eta\wedge, \Delta] = 2d^c - 2(n - \deg -1)\eta\wedge, && [\xi \lrcorner, \Delta] = -2\delta^c - 2(\deg -1)\xi \lrcorner,
\end{aligned}
$$

$$
\begin{aligned}
&\{\eta\wedge, d^c\} = 0, && \{\xi \lrcorner, d^c\} = \deg \cdot \mathrm{id} - \eta \wedge \xi \lrcorner, \\
&\{\eta\wedge, \delta^c\} = -(n - \deg -1)\mathrm{id} - \eta \wedge \xi \lrcorner, && \{\xi \lrcorner, \delta^c\} = 0.
\end{aligned}
$$

Proof. Let $\alpha \in \Omega^p(M)$. Since d is an antiderivation we clearly have

$$d(\eta \wedge \alpha) = d\eta \wedge \alpha - \eta \wedge d\alpha = 2L\alpha - \eta \wedge d\alpha,$$

which yields

$$2\Lambda = 2L^* = \{\eta\wedge, d\}^* = \{\xi \lrcorner, \delta\}.$$

Cartan's formula states $\{\xi \lrcorner, d\} = \mathcal{L}_\xi$, and we have

$$\{\eta\wedge, \delta\} = \{\xi \lrcorner, d\}^* = \mathcal{L}_\xi^* = -\mathcal{L}_\xi,$$

where we used the fact that ξ is a Killing vector field. Because of $\Delta = \{d, \delta\}$ we obtain

$$
\begin{aligned}
[\eta\wedge, \Delta] &= [\eta\wedge, \{d, \delta\}] \\
&= [\{\eta\wedge, d\}, \delta] + [\{\eta\wedge, \delta\}, d] \\
&= 2[L, \delta] - [\mathcal{L}_\xi, d] \\
&= 2d^c - 2(n - \deg -1)\eta \wedge .
\end{aligned}
$$

The relation $d^c = [\phi_{\mathrm{D}}, d] + \deg \eta\wedge$ yields

$$
\begin{aligned}
\{\eta\wedge, d^c\} &= \{\eta\wedge, [\phi_{\mathrm{D}}, d]\} \\
&= \{[\eta\wedge, \phi], d\} - [\{\eta\wedge, d\}, \phi_{\mathrm{D}}] \\
&= -2[L, \phi_{\mathrm{D}}] \\
&= 0,
\end{aligned}
$$

and from $\delta^c = [\phi_{\mathrm{D}}, \delta] - (n - \deg)\xi \lrcorner$ we obtain

$$
\begin{aligned}
\{\eta\wedge, \delta^c\} &= \{\eta\wedge, [\phi_{\mathrm{D}}, \delta]\} - \{\eta\wedge, (n - \deg)\xi \lrcorner\} \\
&= \{[\eta\wedge, \phi], \delta\} - [\{\eta\wedge, \delta\}, \phi_{\mathrm{D}}] - (n - \deg -1) \cdot \mathrm{id} - \eta \wedge \xi \lrcorner \\
&= [\mathcal{L}_\xi, \phi_{\mathrm{D}}] - (n - \deg -1) \cdot \mathrm{id} - \eta \wedge \xi \lrcorner \\
&= -(n - \deg -1) \cdot \mathrm{id} - \eta \wedge \xi \lrcorner .
\end{aligned}
$$

The formulas for $[\xi \lrcorner, \Delta], \{\xi \lrcorner, d^c\}$ and $\{\xi \lrcorner, \delta^c\}$ follow similarly. $\square$

1.2.2 Horizontal and vertical Forms

The interior product between vector fields and differential forms is an antiderivation, i.e. it satisfies

$$X \lrcorner (\alpha_1 \wedge \alpha_2) = (X \lrcorner \alpha_1) \wedge \alpha_2 + (-1)^{\deg(\alpha_1)} \alpha_1 \wedge (X \lrcorner \alpha_2)$$

for all vector fields X and differential forms α_1 and α_2. In the special case of a Sasakian manifold we choose $X = \xi$ and $\alpha_1 = \eta$ to get

$$\alpha_2 = \xi \lrcorner (\eta \wedge \alpha_2) + \eta \wedge (\xi \lrcorner \alpha_2) \in \Omega^p(H) \oplus \eta \wedge \Omega^{p-1}(H)$$

because of $\xi \lrcorner \eta = g(\xi, \xi) = 1$. This leads us to the following definition.

Definition 1.2.9. *For all $\alpha \in \Omega^p(M)$ we call*

- $\mathrm{h}(\alpha) := \xi \lrcorner (\eta \wedge \alpha) \in \Omega^p(H)$ *the **horizontal part** of α,*
- $\mathrm{v}(\alpha) := \xi \lrcorner \alpha \in \Omega^{p-1}(H)$ *the **vertical part** of α.*

For a vector field X we define $\mathrm{h}(X) := -\phi^2 X$ and $\mathrm{v}(X) := \eta(X)$.

Note that the vertical part of a form is always horizontal and that we have $\mathrm{h}(X^*) = \mathrm{h}(X)^*$. Due to this definition we can decompose every form $\alpha \in \Omega^p(M)$ into

$$\alpha = \mathrm{h}(\alpha) + \eta \wedge \mathrm{v}(\alpha),$$

which allows us to identify $\Omega^p(M)$ with $\Omega^p(H) \oplus \Omega^{p-1}(H)$. For a p-form $\beta + \eta \wedge \gamma \in \Omega^p(M)$ with $\beta \in \Omega^p(H)$ and $\gamma \in \Omega^{p-1}(H)$ we write

$$\begin{pmatrix} \beta \\ \gamma \end{pmatrix} := \beta + \eta \wedge \gamma.$$

Every $\mathbb{R}$-linear mapping $F : \Omega^*(M) \longrightarrow \Omega^*(M)$ can be represented by a (2×2)-matrix whose entries are $\mathbb{R}$-linear mappings $\Omega^*(H) \longrightarrow \Omega^*(H)$, i.e.

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

which is equivalent to

$$F(\beta + \eta \wedge \gamma) = F_1\beta + F_2\gamma + \eta \wedge (F_3\beta + F_4\gamma) \in \Omega^*(H) \oplus \eta \wedge \Omega^*(H).$$

We are especially interested in $F \in \{d, \delta, d^c, \delta^c, \nabla_X, \Delta\}$. In order to decompose these operators, we need the following definition.

Definition 1.2.10. *Let (M, g, ξ) be a Sasakian manifold. We define the **horizontal differential***

$$d_H : \Omega^p(H) \longrightarrow \Omega^{p+1}(H)$$

by

$$d_H := \mathrm{h} \circ d|\Omega^p(H).$$

*The operators δ_H, d^c_H and δ^c_H arise from δ, d^c and δ^c in the same manner. The **Sasakian connection** D is defined by*

$$D_X = \nabla_X - (\eta \wedge \phi X^*)\bullet,$$

*and the **horizontal Laplace operator** $\Delta_H : \Omega^p(H) \longrightarrow \Omega^p(H)$ is given by*

$$\Delta_H := d_H\delta_H + \delta_H d_H.$$

Remark. Note that the Sasakian connection is the horizontal projection of the Levi-Civita connection when acting an horizontal vector fields, extended trivially on the Sasakian structure ξ:

$$DU = \mathrm{h}(\nabla U) \qquad \text{and} \qquad D\xi = 0$$

for all horizontal vector fields U.

Proposition 1.2.11. *Let (M, g, ξ) be a Sasakian manifold. According to the identification*

$$\Omega^p(M) = \Omega^p(H) \oplus \Omega^{p-1}(H)$$

we have

$$d = \begin{pmatrix} d_H & 2L \\ \mathcal{L}_\xi & -d_H \end{pmatrix}, \qquad \delta = \begin{pmatrix} \delta_H & -\mathcal{L}_\xi \\ 2\Lambda & -\delta_H \end{pmatrix},$$

$$d^c = \begin{pmatrix} d^c_H & 0 \\ \deg \cdot \mathrm{id} & -d^c_H \end{pmatrix}, \qquad \delta^c = \begin{pmatrix} \delta^c_H & -(n - \deg)\mathrm{id} \\ 0 & -\delta^c_H \end{pmatrix},$$

$$\nabla_X = \begin{pmatrix} D_X & \phi X^* \wedge \\ -\phi X \lrcorner & D_X \end{pmatrix},$$

$$\Delta = \begin{pmatrix} \Delta_H - (\mathcal{L}_\xi)^2 + 4L\Lambda & -2d^c_H \\ -2\delta^c_H & \Delta_H - (\mathcal{L}_\xi)^2 + 4\Lambda L \end{pmatrix}.$$

Proof. Let $\alpha = \beta + \eta \wedge \gamma \in \Omega^p(M) = \Omega^p(H) + \eta \wedge \Omega^{p-1}(H)$ be any p-form. The anticommutator relations $\{\xi \lrcorner, d\} = \mathcal{L}_\xi$ and $\{\eta\wedge, d\} = 2L$ yield

$$\begin{aligned} d\alpha &= d(\beta + \eta \wedge \gamma) \\ &= d\beta + 2L\gamma - \eta \wedge d\gamma \\ &= \mathrm{h}(d\beta) + \eta \wedge \mathrm{v}(d\beta) + 2L\gamma - \eta \wedge \mathrm{h}(d\gamma) \\ &= d_H\beta + \eta \wedge \xi \lrcorner d\beta + 2L\gamma - \eta \wedge d_H\gamma \\ &= d_H\beta + 2L\gamma + \eta \wedge (\mathcal{L}_\xi\beta - d_H\gamma), \end{aligned}$$

whereas from $\{\eta\wedge, \delta\} = -\mathcal{L}_\xi$ and $\{\xi \lrcorner, \delta\} = 2\Lambda$ we get

$$\begin{aligned} \delta\alpha &= \delta(\beta + \eta \wedge \gamma) \\ &= \delta\beta - \mathcal{L}_\xi\gamma - \eta \wedge \delta\gamma \\ &= \mathrm{h}(\delta\beta) + \eta \wedge \mathrm{v}(\delta\beta) - \mathcal{L}_\xi\gamma - \eta \wedge \mathrm{h}(\delta\gamma) \\ &= \delta_H\beta + \eta \wedge \xi \lrcorner \delta\beta - \mathcal{L}_\xi\gamma - \eta \wedge \delta_H\gamma \\ &= \delta_H\beta - \mathcal{L}_\xi\gamma + \eta \wedge (2\Lambda\beta - \delta_H\gamma). \end{aligned}$$

The other statements of the proposition follow similarly. $\square$

If the Sasakian manifold is regular, an important class of differential forms on M is given by those which are the pull-back of forms of the Kähler quotient space M/ξ. This condition is fulfilled for a form α if and only if it is horizontal and constant along the flowlines of ξ, in other words, $\xi \lrcorner \alpha = 0$ and $\mathcal{L}_\xi \alpha = \xi \lrcorner d\alpha = 0$. This motivates the following definition.

Definition 1.2.12. *Let (M, g, ξ) be a Sasakian manifold. A form $\alpha \in \Omega^*(M)$ is called* ***basic*** *if both α and $d\alpha$ are horizontal. The subspace of $\Omega^p(M)$ consisting of basic forms is denoted by $\Omega^p_B(H)$.*

1.2.3 Eigenvalues of ϕ_{D} and (p, q)-forms

In this section we discuss the eigenvalue equation

$$\phi_{\mathrm{D}}\alpha = \lambda\alpha$$

with $\alpha \in \Omega^k_{\mathbb{C}}(M)$ and $\lambda \in \mathbb{C}$.

Proposition 1.2.13. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and $\alpha \in \Omega^k_{\mathbb{C}}(H)$ an eigenform of ϕ_{D} with eigenvalue $\lambda \in \mathbb{C}$, i.e. we have*

$$\phi_{\mathrm{D}}\alpha = \lambda\alpha$$

with $\alpha \neq 0$. Then α is either horizontal or vertical.

- *If α is a horizontal k-form, then $\lambda = i(q-p)$ with $p, q \in \mathbb{Z}$, $0 \leq p, q \leq m$ and $p + q = k$.*
- *If α is a vertical k-form, then $\lambda = i(q-p)$ with $p, q \in \mathbb{Z}$, $0 \leq p, q \leq m$ and $p + q = k - 1$.*

Proof. We decompose $\alpha = \mathrm{h}(\alpha) + \eta \wedge \mathrm{v}(\alpha)$ and obtain

$$\phi_{\mathrm{D}}\mathrm{h}(\alpha) = \lambda \mathrm{h}(\alpha),$$
$$\phi_{\mathrm{D}}\mathrm{v}(\alpha) = \lambda \mathrm{v}(\alpha).$$

From complex Linear Algebra we know that for all $x \in M$ the eigenvalues of ϕ_{D} on $\Lambda^k(H^*_x M)$ are given by $\lambda = i(q-p)$ with $p, q \in \mathbb{Z}$, $0 \leq p, q \leq m$ and $p + q = k$ because ϕ_x is a complex structure on the real $2m$-dimensional vector space $H_x M$. Assuming that both $\mathrm{h}(\alpha)$ and $\mathrm{v}(\alpha)$ do not vanish we have $\lambda = i(q-p) = i(q'-p')$ with $p + q = k$ and $p' + q' = k - 1$ since $\mathrm{h}(\alpha)$ is a k- and $\mathrm{v}(\alpha)$ is a $(k-1)$-form. We obtain $2p = 2p' + 1$, which is a contradiction. Thus we have $\mathrm{h}(\alpha) = 0$ or $\mathrm{v}(\alpha) = 0$. If α is a horizontal form we obtain $\lambda = i(q-p)$ with $p + q = k$, and if α is vertical we obtain $\lambda = i(q-p)$ with $p + q = k - 1$. ☐

This proposition leads us to the following definition.

Definition 1.2.14. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold. We define the spaces $\Lambda^{(p,q)}(H^*_x M)$ for all $x \in M$ and for $p, q \in \mathbb{Z}$ with $0 \leq p, q \leq m$ by*

$$\Lambda^{(p,q)}(H^*_x M) := \left\{\alpha \in \Lambda^{p+q}(H^*_x M) \mid \phi_{\mathrm{D}}\alpha = i(q-p)\alpha\right\}.$$

The set of sections in $\Lambda^{(p,q)}(H^*M)$ is denoted by $\Omega^{(p,q)}(H) := \Gamma\left(\Lambda^{(p,q)}(H^*M)\right)$, and the elements in $\Omega^{(p,q)}(H)$ are called (p, q)-forms.

Since ϕ_{D} is a skew-adjoint endomorphism of $\Lambda^k(H^*M)$ we can decompose $\Lambda^k(H^*_x M)$ into the eigenspaces of ϕ_{D}, which leads us to

$$\Omega^k_{\mathbb{C}}(M) = \left(\bigoplus_{p+q=k} \Omega^{(p,q)}(H)\right) \oplus \eta \wedge \left(\bigoplus_{p+q=k-1} \Omega^{(p,q)}(H)\right).$$

1.3 Dolbeault operators

In this section we will first give local expressions for d_H, δ_H, d^c_H and δ^c_H which allow us to introduce the Dolbeault operators $\partial, \bar\partial, \partial^*$ and $\bar\partial^*$ on a Sasakian manifold. Then we investigate the relation between the horizontal Laplace operator

$$\Delta_H = d_H\delta_H + \delta_H d_H$$

and the Dolbeault-Laplace operators

$$\Delta_\partial := \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

It turns out that the Kähler relation $\Delta_H = 2\Delta_\partial = 2\Delta_{\bar\partial}$ only is guaranteed on basic forms, thus is not true in general. Nevertheless we are able to determine the difference explicitly.

Proposition 1.3.1. *Let (M, g, ξ) be a Sasakian manifold. If $\{f_j\}$ is a local orthonormal basis of HM, then*

(i) $d_H = \sum f_j^* \wedge D_{f_j}, \qquad d^c_H = \sum \phi f_j^* \wedge D_{f_j},$

(ii) $\delta_H = -\sum f_j \lrcorner D_{f_j}, \qquad \delta^c_H = -\sum \phi f_j \lrcorner D_{f_j}.$

Proof. Let $\beta \in \Omega^*(H)$. By definition we have

$$D_U\beta = \xi \lrcorner (\eta \wedge \nabla_U\beta) \qquad \text{and} \qquad d_H = \xi \lrcorner (\eta \wedge d\beta).$$

Since $\{\xi, f_j\}$ is a local orthonormal basis of HM we can express d as

$$d = \sum f_j^* \wedge \nabla_{f_j} + \eta \wedge \nabla_\xi,$$

thus

$$\begin{aligned} d_H\beta &= \xi \lrcorner \left(\eta \wedge \left(\sum f_j^* \wedge \nabla_{f_j}\beta + \eta \wedge \nabla_\xi\beta\right)\right) \\ &= -\sum \eta(f_j)\eta \wedge \nabla_{f_j}\beta + \sum f_j^* \wedge \xi \lrcorner \left(\eta \wedge \nabla_{f_j}\beta\right) \\ &= \sum f_j^* \wedge D_{f_j}\beta. \end{aligned}$$

The proofs of the other expressions are similar. $\square$

Recall that we defined $U^\pm = \frac{1}{2}(U \mp i\phi U)$ for every horizontal tangent vector U.

Definition 1.3.2. *Let (M, g, ξ) be a Sasakian manifold and $\{f_j\}$ a local orthonormal basis of HM. We define*

$$\partial, \bar\partial : \Omega^p_{\mathbb{C}}(H) \longrightarrow \Omega^{p+1}_{\mathbb{C}}(H) \qquad \text{and} \qquad \partial^*, \bar\partial^* : \Omega^p_{\mathbb{C}}(H) \longrightarrow \Omega^{p-1}_{\mathbb{C}}(H)$$

as

$$\begin{aligned} \partial &:= \sum (f_j^-)^* \wedge D_{f_j}, \qquad & \bar\partial &:= \sum (f_j^+)^* \wedge D_{f_j}, \\ \partial^* &:= -\sum f_j^+ \lrcorner D_{f_j}, \qquad & \bar\partial^* &:= -\sum f_j^- \lrcorner D_{f_j}. \end{aligned}$$

Proposition 1.3.3. *Let* (M, g, ξ) *be a Sasakian manifold. Then we have*

$$d_H = \partial + \bar{\partial}, \qquad \delta_H = \partial^* + \bar{\partial}^*,$$
$$d_H^c = -i(\partial - \bar{\partial}), \qquad \delta_H^c = i(\partial^* - \bar{\partial}^*),$$

and the restriction of $\partial, \bar{\partial}, \partial^*$ *and* $\bar{\partial}^*$ *to* (p, q)*-forms leads to*

$$\partial : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p+1,q)}(H), \qquad \partial^* : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p-1,q)}(H),$$
$$\bar{\partial} : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p,q+1)}(H), \qquad \bar{\partial}^* : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p,q-1)}(H).$$

Proof. The first part is obvious in view of Proposition 1.3.1. The second part follows from the following facts:

- For every $U \in HM$ the vector U^+ is an eigenvector of ϕ with eigenvalue i, thus

$$(U^+)^* \wedge : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p,q+1)}(H), \qquad U^+ \lrcorner : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p-1,q)}(H).$$

- For every $U \in HM$ the vector U^- is an eigenvector of ϕ with eigenvalue $-i$, thus

$$(U^-)^* \wedge : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p+1,q)}(H), \qquad U^- \lrcorner : \Omega^{(p,q)}(H) \longrightarrow \Omega^{(p,q-1)}(H).$$

Together with the fact that D commutes with ϕ_{D} on $\Omega^*(H)$, this finishes the proof. □

Unfortunately, the operators d_H and δ_H do not square to zero on $\Omega^*(H)$, as we see in the next proposition. Nevertheless we still have $\partial^2 = \bar{\partial}^2 = 0$.

Proposition 1.3.4. *Let* (M, g, ξ) *be a Sasakian manifold. Then we have*

$$d_H^2 = -2L\mathcal{L}_\xi, \quad \delta_H^2 = 2\Lambda\mathcal{L}_\xi;$$

in particular

$$\partial^2 = 0, \qquad \bar{\partial}^2 = 0, \qquad \{\partial, \bar{\partial}\} = -2L\mathcal{L}_\xi,$$
$$(\partial^*)^2 = 0, \qquad (\bar{\partial}^*)^2 = 0, \qquad \{\partial^*, \bar{\partial}^*\} = 2\Lambda\mathcal{L}_\xi.$$

Proof. Let $\beta \in \Omega^{(p,q)}(H)$. Because of

$$\partial\beta + \bar{\partial}\beta = d_H\beta = d\beta - \eta \wedge \mathcal{L}_\xi\beta$$

and

$$\partial^*\beta + \bar{\partial}^*\beta = \delta_H\beta = \delta\beta - 2\eta \wedge \Lambda\beta$$

we get

$$\begin{aligned}
\partial^2\beta + \{\partial, \bar{\partial}\}\beta + \bar{\partial}^2\beta &= (\partial + \bar{\partial})^2 \\
&= d_H^2\beta \\
&= d^2\beta - d(\eta \wedge \mathcal{L}_\xi\beta) - \eta \wedge \mathcal{L}_\xi d\beta \\
&= -2L\mathcal{L}_\xi\beta
\end{aligned}$$

and

$$(\partial^*)^2\beta + \{\partial^*, \bar{\partial}^*\}\beta + (\bar{\partial}^*)^2\beta = (\partial^* + \bar{\partial}^*)^2$$

$$
\begin{aligned}
&= \delta_H^2\beta \\
&= \delta^2\beta - 2\delta(\eta\wedge\Lambda\beta) - 2\eta\wedge\Lambda\delta\beta \\
&= 2\Lambda\mathcal{L}_\xi\beta,
\end{aligned}
$$

which yields all results via type comparison. $\square$

Proposition 1.3.3 allows us to compute the commutator and anticommutator relations between $\phi_{\mathrm{D}}, L, \Lambda, d_H, \delta_H, d_H^c, \delta_H^c, \partial, \bar\partial, \partial^*$ and $\bar\partial^*$. A complete list can be found in Appendix D. All relations are identical with the ones for the analogous operators on Kähler manifolds except for $\{\partial, \bar\partial\}$ and $\{\partial^*, \bar\partial^*\}$ as we just saw.

Since ω is D-parallel, we get

$$
\begin{aligned}
\Lambda d_H\beta &= \omega \lrcorner \sum f_j^* \wedge D_{f_j}\beta \\
&= \sum f_j^* \wedge \omega \lrcorner D_{f_j}\beta + \sum \phi f_j \lrcorner D_{f_j}\beta \\
&= \sum f_j^* \wedge D_{f_j}(\omega \lrcorner \beta) - \delta_H^c\beta \\
&= d_H\Lambda\beta - \delta_H^c\beta.
\end{aligned}
$$

Restricting to (p,q)-forms immediately yields

$$
[\Lambda, \partial] = i\bar\partial^*, \quad [\Lambda, \bar\partial] = -i\partial^*
$$

because of $d_H = \partial + \bar\partial$ and $\delta_H^c = i(\partial - \bar\partial)$. This gives

$$
\begin{aligned}
\{\bar\partial, \partial^*\} &= \bar\partial\partial^* + \partial^*\bar\partial \\
&= i\bar\partial[\Lambda, \bar\partial] + i[\Lambda, \bar\partial]\bar\partial \\
&= i\bar\partial\Lambda\bar\partial - i\bar\partial^2\Lambda + i\Lambda\bar\partial^2 - i\bar\partial\Lambda\bar\partial \\
&= 0
\end{aligned}
$$

and similarly

$$
\{\partial, \bar\partial^*\} = 0.
$$

Thus on one hand it follows

$$
\begin{aligned}
\Delta_H &= \{d_H, \delta_H\} \\
&= \{\partial + \bar\partial, \partial^* + \bar\partial^*\} \\
&= \{\partial, \partial^*\} + \{\partial, \bar\partial^*\} + \{\bar\partial, \partial^*\} + \{\bar\partial, \bar\partial^*\} \qquad (1.3.1) \\
&= \Delta_\partial + \Delta_{\bar\partial},
\end{aligned}
$$

and on the other hand we have

$$
\begin{aligned}
\Delta_\partial &= \partial\partial^* + \partial^*\partial \\
&= i\partial[\Lambda, \bar\partial] + i[\Lambda, \bar\partial]\partial \\
&= i\partial\Lambda\bar\partial - i\partial\bar\partial\Lambda + i\Lambda\bar\partial\partial - i\bar\partial\Lambda\partial \\
&= i\partial\Lambda\bar\partial + i\bar\partial\partial\Lambda - i\Lambda\partial\bar\partial - i\bar\partial\Lambda\partial + i\Lambda\{\partial, \bar\partial\} - i\{\partial, \bar\partial\}\Lambda \\
&= i[\partial, \Lambda]\bar\partial + i\bar\partial[\partial, \Lambda] - 2i\Lambda L\mathcal{L}_\xi + 2iL\Lambda\mathcal{L}_\xi \\
&= \bar\partial^*\bar\partial + \bar\partial\bar\partial^* + 2i\mathcal{L}_\xi[L, \Lambda] \\
&= \Delta_{\bar\partial} - i(n - 2\deg - 1)\mathcal{L}_\xi,
\end{aligned}
$$

where n denotes the dimension of M. This together with (1.3.1) yields the following result.

Theorem 1.3.5. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then we have*

$$2\Delta_{\partial} = \Delta_H - i(n - 2\deg -1)\mathcal{L}_\xi$$

and

$$2\Delta_{\bar{\partial}} = \Delta_H + i(n - 2\deg -1)\mathcal{L}_\xi$$

on $\Omega^*_{\mathbb{C}}(H)$.

1.4 Adapted connections

Since every Sasakian manifold is a Riemannian manifold with a fixed metric we have the Levi-Civita connection ∇. A key feature of Sasakian manifolds is that the natural 2-form $\omega = \frac{1}{2}d\eta$ is not parallel. The bundle $HM = \ker(\eta)$ consisting of all horizontal vectors behaves like the tangent bundle of a Kähler manifold with Kähler form $\omega|\ker(\eta)$. It is therefore natural to ask for connections $\widetilde{\nabla}$ on M with $\widetilde{\nabla}\omega = 0$. Examples are the Levi-Civita connection ∇^T of the transversal Kähler metric g^T and the unique metric connection $\overline{\nabla}$ that has totally skewsymmetric torsion such that ω is $\overline{\nabla}$-parallel. This connection is described in [FI02]. Another example is given by the Sasakian connection D introduced in Definition 1.2.10, which is essentially the projection of the Levi-Civita connection on its horizontal part. These connections belong to 1-parameter family of connections, whose members we call adapted connections.

Definition 1.4.1. *Let (M, g, ξ) be a Sasakian manifold. An* ***adapted connection*** $\nabla^{(a)}$ *is a connection of the form*

$$\nabla^{(a)}_X = D_X + a\eta(X)\phi_{\mathrm{D}}$$

with $a \in \mathbb{R}$.

Recall that the Sasakian connection D on M is defined as

$$D_X = \nabla_X - (\eta \wedge \phi X^*) \bullet .$$

In this notation we have

$$D = \nabla^{(0)}, \qquad \overline{\nabla} = \nabla^{(1)}, \qquad \nabla^T = \nabla^{(-1)}.$$

In the next proposition we calculate the torsion tensor and the curvature of $\nabla^{(a)}$.

Proposition 1.4.2. *Let (M, g, ξ) be a Sasakian manifold and $\nabla^{(a)}$ an adapted connection. Then we have*

$$\nabla^{(a)}g = 0, \qquad \nabla^{(a)}\xi = 0, \qquad \nabla^{(a)}\phi = 0,$$

and the torsion tensor and the curvature tensor of $\nabla^{(a)}$ are given by

$$T^{(a)}(X, Y) = 2\omega(X, Y)\xi - (a + 1)\phi R(X, Y)\xi,$$
$$R^{(a)}(X, Y) = R(X, Y) + (R(X, Y)\eta \wedge \eta - \phi X^* \wedge \phi Y^*) \bullet +2a\omega(X, Y)\phi_{\mathrm{D}}.$$

Proof. All claims are proven straightforwardly using the definition of $\nabla^{(a)}$ as

$$\nabla^{(a)}_X = \nabla_X - (\eta \wedge \phi X^*) \bullet +a\eta(X)\phi_{\mathrm{D}}.$$

We only demonstrate the parallelity of g, ξ and ω with respect to $\nabla^{(a)}$, where we have

$$\begin{aligned}\left(\nabla^{(a)}_X g\right)(X,Y) &= X(g(Y,Z)) - g(\nabla_X Y - (\eta\wedge\phi X^*)\bullet Y + a\eta(X)\phi Y, Z)\\ &\qquad\qquad - g(Y, \nabla_X Z - (\eta\wedge\phi X^*)\bullet Z + a\eta(X)\phi Z)\\ &= (\eta\wedge\phi X^*)(Y,Z) - a\eta(X)g(\phi Y, Z)\\ &\quad + (\eta\wedge\phi X^*)(Z,Y) - a\eta(X)g(Y,\phi Z)\\ &= 0,\end{aligned}$$

$$\begin{aligned}\left(\nabla^{(a)}_X \omega\right)(X,Y) &= X(\omega(Y,Z)) - \omega(\nabla_X Y - (\eta\wedge\phi X^*)\bullet Y + a\eta(X)\phi Y, Z)\\ &\qquad\qquad - \omega(Y, \nabla_X Z + (\eta\wedge\phi X^*)\bullet Z + a\eta(X)\phi Z)\\ &= (\nabla_X\omega)(Y,Z) + \omega((\eta\wedge\phi X^*)\bullet Y, Z) - a\eta(X)\omega(\phi Y, Z)\\ &\quad + \omega(Y, (\eta\wedge\phi X^*)\bullet Z) - a\eta(X)\omega(Y,\phi Z)\\ &= (\eta\wedge X^*)(Y,Z) - (\eta\wedge\phi X^*)(Y,\phi Z)\\ &\quad + (\eta\wedge\phi X^*)(Z,\phi Y)\\ &= \eta(Y)g(X,Z) - \eta(Z)g(X,Y) - \eta(Z)g(\phi X,\phi Y) + \eta(Z)g(\phi X,\phi Y)\\ &= 0,\end{aligned}$$

$$\begin{aligned}\nabla^{(a)}_X\xi &= \nabla_X\xi - \eta(\xi)\phi X\\ &= 0. \qquad\qquad \square\end{aligned}$$

If the manifold is regular, then the transversal connection $\nabla^T = \nabla^{(-1)}$ is the pullback of the Levi-Civita connection of the quotient Kähler manifold M/ξ. With Proposition 1.4.2 we determine the relation between the curvature tensor of M and the curvature tensor of M/ξ.

Corollary 1.4.3. *If (M, g, ξ) is a n-dimensional regular Sasakian manifold, the curvature tensor of the quotient Kähler manifold $B = M/\xi$ is determined by*

$$\begin{aligned}R_B(T\pi_B U, T\pi_B V)T\pi_B W = T\pi_B(&R(U,V)W\\ &- \omega(U,W)\phi V + \omega(V,W)\phi U - 2\omega(U,V)\phi W)\end{aligned}$$

for all horizontal tangent vectors U, V, W on M, where $\pi_B : M \longrightarrow B$ denotes the projection. The Ricci curvatures are related by

$$\mathrm{Ric}^B(T\pi_B U) = T\pi_B(\mathrm{Ric}(U) + 2U).$$

The Levi-Civita connection of a Sasakian manifold does not admit any non-trivial parallel forms, while every adapted connection has ω^k and $\eta\wedge\omega^k$ as parallel forms. We conclude this section with the investigation of $\nabla^{(a)}$-parallel forms. A form is $\nabla^{(a)}$-parallel if and only if its horizontal and vertical part are $\nabla^{(a)}$-parallel. Therefore we discuss only horizontal $\nabla^{(a)}$-parallel forms.

Proposition 1.4.4. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and let $\nabla^{(a)}$ be an adapted connection. Let $\beta \in \Omega^p(H)$ be a $\nabla^{(a)}$-parallel horizontal form. Then the norm of β is constant on M. If $p \neq m$, then β is closed and $\nabla^{(-1)}$-parallel. If additionally $a \neq -1$ and p is even, then $\phi_{\mathrm{D}}\beta = 0$, and if p is odd, then $\beta = 0$.*

Proof. The key argument of our proof is that the forms $\phi_{\mathrm{D}}\beta$ and $\Lambda\beta$ are again $\nabla^{(a)}$-parallel and horizontal, thus all results for β automatically also hold for $\phi_{\mathrm{D}}\beta$ and $\Lambda\beta$.

From
$$\nabla_X\beta = \phi X \lrcorner (\eta \wedge \beta) - a\eta(X)\phi_{\mathrm{D}}\beta,$$
we obtain
$$\begin{aligned} X\left(\|\beta\|^2\right) &= 2g(\nabla_X\beta, \beta) \\ &= 2g(\phi X \lrcorner (\eta \wedge \beta), \beta) - 2a\eta(X)g(\phi_{\mathrm{D}}\beta, \beta) \\ &= 2g(\beta, -\phi X^* \wedge \xi \lrcorner \beta) \\ &= 0 \end{aligned}$$
and
$$\begin{aligned} d\beta &= \sum e_i^* \wedge \nabla_{e_i}\beta \\ &= \sum e_i^* \wedge \phi e_i \lrcorner (\eta \wedge \beta) - a\sum \eta(e_i)e_i^* \wedge \phi_{\mathrm{D}}\beta \\ &= -(a+1)\eta \wedge \phi_{\mathrm{D}}\beta. \end{aligned}$$
If $a = -1$, then β is closed. In the remainder of the proof we assume $a \neq -1$. Since $d\beta$ and $d\phi_{\mathrm{D}}\beta$ are vertical we get
$$\begin{aligned} 0 &= \tfrac{1}{a+1} d^2\phi_{\mathrm{D}}\beta \\ &= -d(\eta \wedge \phi_{\mathrm{D}}\beta) \\ &= -2L\phi_{\mathrm{D}}\beta + \eta \wedge d\phi_{\mathrm{D}}\beta \\ &= -2L\phi_{\mathrm{D}}\beta, \end{aligned}$$
i.e. $L\phi_{\mathrm{D}}\beta = 0$. Since $\Lambda\beta$ is a $\nabla^{(a)}$-parallel horizontal form we also have $L\Lambda\phi_{\mathrm{D}}\alpha = 0$. Then the commutator relation $[L, \Lambda] = (\deg - m)\mathrm{id} - \eta \wedge \xi \lrcorner$ yields $(p-m)\phi_{\mathrm{D}}\beta = 0$, thus $\phi_{\mathrm{D}}\beta = 0$ for $p \neq m$. This implies $d\beta = 0$ and $\nabla^{(-1)}\beta = 0$. $\square$

We have the following two Corollaries.

Corollary 1.4.5. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold. Then all adapted connections $\nabla^{(a)}$ with $a \neq 1$ have the same parallel p-forms for $p \neq m$.*

Corollary 1.4.6. *Let (M, g, ξ) be a regular $(2m+1)$-dimensional Sasakian manifold. Let $B = M/\xi$ be the quotient Kähler manifold and $\pi_B : M \longrightarrow B$ be the projection. Let $p \neq m$ and β be a parallel p-form of an adapted connection. Then there exists a parallel p-form β' on (B, g_B) such that $\beta = \pi_B^*\beta'$.*

2 Curvature of Sasakian manifolds

In this section we will prove several identities concerning the curvature tensor R of a Sasakian manifold (M, g, ξ). It turns out that it is much more convenient to replace the Riemannian curvature tensor R by a new tensor A defined by

$$A(X, Y)Z := R(X, Y)Z + g(X, Z)Y - g(Y, Z)X, = (R(X, Y) + (X^* \wedge Y^*)\bullet)Z$$

which we call the Sasakian curvature tensor. After calculating the behaviour of R under various natural operations concerning the Sasakian structure, we reformulate all identities for R we obtained so far in terms of the Sasakian curvature tensor A.

2.1 Riemannian curvature tensor

Proposition 2.1.1. *Let (M, g, ξ) be a Sasakian manifold. Then we have*

(i) $R(X, Y)\omega = \phi_{\mathrm{D}}(X^* \wedge Y^*)$,

(ii) $[R(X, Y), \phi_{\mathrm{D}}] = \phi_{\mathrm{D}}(X^* \wedge Y^*)\bullet$,

(iii) $R(\phi X, \phi Y) = R(X, Y) + (X^* \wedge Y^* - \phi X^* \wedge \phi Y^*)\bullet$,
$R(\phi X, Y) + R(X, \phi Y) = -\phi_{\mathrm{D}}(X^* \wedge Y^*)\bullet$,

(iv) $R(\phi X, \phi Y, Z, W) = R(X, Y, \phi Z, \phi W)$

for all vector fields X, Y, Z and W on M.

Proof. To calculate $R(X, Y)\omega$ we use

$$\nabla_X \omega = \eta \wedge X^*,$$

which yields

$$\begin{aligned} R(X, Y)\omega &= \nabla_X(\eta \wedge Y^*) - \nabla_Y(\eta \wedge X^*) - \eta \wedge [X, Y]^* \\ &= \phi X^* \wedge Y^* - \phi Y^* \wedge X^* \\ &= \phi_{\mathrm{D}}(X^* \wedge Y^*). \end{aligned}$$

To verify (ii) and (iii) it suffices to check them on vector fields, where we have

$$\begin{aligned} g([R(X, Y), \phi]Z, W) &= -g(\phi Z, R(X, Y)W) + g(R(X, Y)Z, \phi W) \\ &= -\omega(Z, R(X, Y)W) + \omega(W, R(X, Y)Z) \\ &= (R(X, Y)\omega)(Z, W) \\ &= (\phi_{\mathrm{D}}(X^* \wedge Y^*))(Z, W) \\ &= g(\phi_{\mathrm{D}}(X^* \wedge Y^*) \bullet Z, W) \end{aligned}$$

and

$$\begin{aligned} g(R(\phi X, \phi Y)Z, W) &= g(R(Z, W)\phi X, \phi Y) \\ &= g(\phi(R(Z, W)X), \phi Y) + g([R(Z, W), \phi]X, \phi Y) \\ &= g(R(Z, W)X, Y) - \eta(R(Z, W)X)\eta(Y) \\ &\qquad + g((\phi_{\mathrm{D}}(Z^* \wedge W^*)) \bullet X, \phi Y) \\ &= g(R(X, Y)Z, W) + \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(Z)g(X, W) \end{aligned}$$

$$
\begin{aligned}
&+ g(\phi Z, X)g(\phi Y, W) - g(X, W)g(\phi Y, \phi Z)\\
&+ g(X, Z)g(\phi Y, \phi W) - g(X, \phi W)g(\phi Y, Z)\\
&= g(R(X, Y)Z + g(X, Z)Y - g(Y, Z)X, W)\\
&+ g(g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y, W)\\
&= g(R(X, Y)Z + (X^* \wedge Y^* - \phi X^* \wedge \phi Y^*) \bullet Z, W).
\end{aligned}
$$

We replace Y by ϕY to obtain

$$
R(\phi X, \phi^2 Y) = R(X, \phi Y) + (X^* \wedge \phi Y^* - \phi X^* \wedge \phi^2 Y^*)\bullet,
$$

which is equivalent to

$$
\begin{aligned}
R(\phi X, Y) + R(X, \phi Y) &= -\phi(X^* \wedge Y^*) \bullet + \eta(Y)R(\phi X, \xi) + \eta(Y)(\phi X^* \wedge \eta)\bullet\\
&= -\phi(X^* \wedge Y^*) \bullet .
\end{aligned}
$$

The last statement also follows from (iii):

$$
\begin{aligned}
R(\phi X, \phi Y, Z, W) &= g(R(X, Y)Z, W) + g((X^* \wedge Y^* - \phi X^* \wedge \phi Y^*) \bullet Z, W)\\
&= g(R(X, Y)Z, W) + g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\\
&\quad - g(\phi X, Z)g(\phi Y, W) + g(\phi Y, Z)g(\phi X, W)\\
&= g(R(Z, W)X, Y) + g(Z, X)g(W, Y) - g(Z, Y)g(W, X)\\
&\quad - g(\phi Z, X)g(\phi W, Y) + g(\phi Z, Y)g(\phi W, X)\\
&= g(R(Z, W)X, Y) + g((Z^* \wedge W^* - \phi Z^* \wedge \phi W^*) \bullet X, Y)\\
&= R(\phi Z, \phi W, X, Y)\\
&= R(X, Y, \phi Z, \phi W). \qquad \square
\end{aligned}
$$

Next we show $[\mathrm{Ric}, \phi] = 0$, which we need to introduce the Ricci form ρ.

Corollary 2.1.2. *Let (M, g, ξ) be a Sasakian manifold. Then we have*

$$
[\mathrm{Ric}, \phi] = 0.
$$

Proof. Let $X \in TM$. With (ii) and (iii) from Proposition 2.1.1 we calculate

$$
\begin{aligned}
\mathrm{Ric}(\phi X) &= \sum R(\phi X, e_i)e_i\\
&= -\sum R(X, \phi e_i)e_i - \sum \phi_{\mathrm{D}}(X^* \wedge e_i^*) \bullet e_i\\
&= \sum R(X, e_i)\phi e_i - \sum \phi_{\mathrm{D}}(X^* \wedge e_i^*) \bullet e_i\\
&= \sum \phi(R(X, e_i)e_i) + \sum \phi_{\mathrm{D}}(X^* \wedge e_i^*) \bullet e_i - \sum \phi_{\mathrm{D}}(X^* \wedge e_i^*) \bullet e_i\\
&= \phi(\mathrm{Ric}(X)).
\end{aligned}
$$

Definition 2.1.3. *Let (M, g, ξ) be a Sasakian manifold. We define the **Ricci form** ρ by*

$$
\rho(X, Y) := \omega(\mathrm{Ric}(X), Y) = \mathrm{ric}(\phi X, Y).
$$

Remark. Another way to express this 2-form is by $\rho = -\frac{1}{2}\mathrm{Ric}_{\mathrm{D}}(\omega)$ or by its action on vector fields, which is given by $\rho \bullet X = \mathrm{Ric}(\phi X)$. As in the Kähler case we have $d\rho = 0$, which we will prove later using the Sasakian curvature tensor.

Proposition 2.1.4. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then we have*

$$\sum R(X, e_i)\phi e_i = \rho \bullet X - (n-2)\phi X$$

for all vector fields X on M and

$$\begin{aligned}\sum R(e_i, \phi e_i) &= -2\rho \bullet + 2(n-2)\phi_{\mathrm{D}},\\ \sum \nabla^2_{\phi e_i, e_i} &= \rho \bullet - (n-2)\phi_{\mathrm{D}},\\ \sum \nabla^2_{e_i, \phi e_i} &= -\rho \bullet + (n-2)\phi_{\mathrm{D}}\end{aligned}$$

on $\Omega^*(M)$.

Proof. We prove the first claim directly with Proposition 2.1.1:

$$\begin{aligned}\sum R(X, e_i)\phi e_i &= -\sum R(X, \phi e_i)e_i\\ &= \sum R(\phi X, e_i)e_i\\ &\quad + \sum (g(\phi X, e_i)e_i - g(e_i, e_i)\phi X + g(X, e_i)\phi e_i - g(\phi e_i, e_i)X)\\ &= \mathrm{Ric}(\phi X) - (n-2)\phi X.\end{aligned}$$

It suffices to check the second identity on vector fields. Because of the first Bianchi identity

$$\begin{aligned}\sum R(e_i, \phi e_i)X &= -\sum R(\phi e_i, X)e_i - \sum R(X, e_i)\phi e_i\\ &= -2\sum R(X, e_i)\phi e_i\\ &= -2\rho \bullet X + 2(n-2)\phi X\end{aligned}$$

holds for every vector field X on M. The last two identities follow directly since

$$\sum \nabla^2_{\phi e_i, e_i} = -\sum \nabla^2_{e_i, \phi e_i}. \qquad \square$$

Definition 2.1.5. *Let (M, g) be a Riemannian manifold with curvature tensor R. We define the following operators on $\Omega^*(M)$:*

$$\begin{aligned}R^+(X) &:= \sum e_i^* \wedge R(X, e_i),\\ R^-(X) &:= \sum e_i \lrcorner R(X, e_i),\\ q(R) &:= -\sum e_i \lrcorner R^+(e_i) = -\sum e_i^* \wedge R^-(e_i).\end{aligned}$$

Remark. The operators R^+ and R^- appear in many equations concerning conformal Killing forms, and $q(R)$ is the curvature endomorphism from the classical Weitzenböck formula

$$\Delta = \nabla^*\nabla + q(R).$$

The next two propositions are about these three operators on a Sasakian manifold (M, g, ξ). First we calculate $R^\pm(\xi)$ explicitly and then give some symmetry properties of $R^\pm$ and $q(R)$ with respect to ϕ_{D}. Most of the identities given here will be crucial in our discussion of conformal Killing forms on Sasakian manifolds in Chapter 3.

Proposition 2.1.6. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then we have*

$$
\begin{aligned}
R^+(\xi) &= -\deg \cdot \eta \wedge, \\
R^-(\xi) &= -(n - \deg) \cdot \xi \lrcorner \,.
\end{aligned}
$$

Proof. This is an easy consequence of $R(X, \xi) = (\eta \wedge X^*)\bullet$:

$$
\begin{aligned}
R^+(\xi) &= \sum e_i^* \wedge R(\xi, e_i) = \sum e_i^* \wedge (\eta \wedge e_i \lrcorner - e_i^* \wedge \xi \lrcorner) \\
&= -\deg \cdot \eta \wedge
\end{aligned}
$$

and

$$
\begin{aligned}
R^-(\xi) &= \sum e_i \lrcorner R(\xi, e_i) = \sum e_i \lrcorner (\eta \wedge e_i \lrcorner - e_i \wedge \xi \lrcorner) \\
&= -(n - \deg) \cdot \xi \lrcorner \,. \qquad \square
\end{aligned}
$$

Corollary 2.1.7. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then every parallel p-form has to vanish for $p \neq 0, n$.*

Proof. Let $\alpha \in \Omega^p(M)$ with $\nabla \alpha = 0$. Then we have $R(X, Y)\alpha = 0$ for all vector fields X and Y, and it follows

$$
\begin{aligned}
0 &= R^+(\xi)\alpha = -p\, \eta \wedge \alpha, \\
0 &= R^-(\xi)\alpha = -(n - p)\xi \lrcorner \alpha.
\end{aligned}
$$

We obtain $\alpha = \xi \lrcorner (\eta \wedge \alpha) + \eta \wedge \xi \lrcorner \alpha = 0.$ $\square$

The non-existence of parallel forms shows that all Sasakian manifolds are irreducible:

Corollary 2.1.8. *Every Sasakian manifold is irreducible.*

Proof. Assume that M is reducible, i.e. locally holds $(M, g) = (M_1, g_1) \times (M_2, g_2)$. Then the volume forms of M_1 and M_2 are parallel on M, thus M can not be a Sasakian manifold. $\square$

Proposition 2.1.9. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then for all vector fields X on M we have*

(i) $\sum \phi e_i^* \wedge R(X, e_i) = R^+(\phi X) + 2X \lrcorner L - X^* \wedge \phi_{\mathrm{D}} + (\deg - 2)\phi X^* \wedge,$

(ii) $\sum \phi e_i^* \wedge R^+(e_i) = 2 \deg L,$

(iii) $\sum \phi e_i^* \wedge R^-(e_i) = -\rho \bullet + (\deg - 1)\phi_{\mathrm{D}};$

(iv) $\sum \phi e_i \lrcorner R(X, e_i) = R^-(\phi X) - 2X^* \wedge \Lambda - X \lrcorner \phi_{\mathrm{D}} + (n - \deg - 2)\phi X \lrcorner \,,$

(v) $\sum \phi e_i \lrcorner R^-(e_i) = -2(n - \deg)\, \Lambda,$

(vi) $\sum \phi e_i \lrcorner R^+(e_i) = -\rho \bullet + (n - \deg - 1)\phi_{\mathrm{D}}.$

Proof. Let $\alpha \in \Omega^p(M)$. Without restriction we may assume that $\{e_i\}$ is parallel at a point. Identity (i) is proven straightforward:

$$\begin{aligned}
\sum \phi e_i^* \wedge R(X, e_i)\alpha &= -\sum e_i^* \wedge R(X, \phi e_i)\alpha \\
&= \sum e_i^* \wedge (R(\phi X, e_i)\alpha + \phi_{\mathrm{D}}(X^* \wedge e_i^*) \bullet \alpha) \\
&= R^+(\phi X)\alpha + \sum e_i^* \wedge (e_i^* \wedge \phi X \lrcorner \alpha - \phi X^* \wedge e_i \lrcorner \alpha) \\
&\quad + \sum e_i^* \wedge (\phi e_i^* \wedge X \lrcorner \alpha - X^* \wedge \phi e_i \lrcorner \alpha) \\
&= R^+(\phi X)\alpha + \phi X^* \wedge \sum e_i^* \wedge e_i \lrcorner \alpha \\
&\quad + \sum e_i^* \wedge \phi e_i^* \wedge X \lrcorner \alpha + X^* \wedge \sum e_i^* \wedge \phi e_i \lrcorner \alpha \\
&= R^+(\phi X)\alpha + (p-2)\phi X^* \wedge \alpha + 2X \lrcorner L\alpha - X^* \wedge \phi_{\mathrm{D}}\alpha.
\end{aligned}$$

To prove (ii), recall the anticommutator relation

$$\{d, d^c\} = \eta \wedge d + 2 \deg L$$

and calculate

$$\begin{aligned}
dd^c\alpha &= \sum e_i^* \wedge \nabla_{e_i} \left(\phi e_j^* \wedge \nabla_{e_j}\alpha\right) \\
&= \sum e_i^* \wedge ((\nabla_{e_i}\phi)e_j^*) \wedge \nabla_{e_j}\alpha + \sum e_i^* \wedge \phi e_j^* \wedge \nabla_{e_i}\nabla_{e_j}\alpha \\
&= \sum e_i^* \wedge (\eta(e_j)e_i^* - \delta_{ij}\eta) \wedge \nabla_{e_j}\alpha + \sum e_i^* \wedge \phi e_j^* \wedge \nabla_{e_j}\nabla_{e_i}\alpha \\
&\quad + \sum e_i^* \wedge \phi e_j^* \wedge R(e_i, e_j)\alpha \\
&= \eta \wedge \sum e_i^* \wedge \nabla_{e_i}\alpha - \sum \phi e_j^* \wedge \nabla_{e_j}(e_i^* \wedge \nabla_{e_i}\alpha) \\
&\quad + \sum \phi e_j^* \wedge e_i^* \wedge R(e_j, e_i)\alpha \\
&= \eta \wedge d\alpha - d^c d\alpha + \sum \phi e_j^* \wedge R^+(e_j)\alpha \\
&= dd^c\alpha - 2\deg \cdot L + \sum \phi e_j^* \wedge R^+(e_j)\alpha,
\end{aligned}$$

which yields $\sum \phi e_j^* \wedge R^+(e_j)\alpha = 2\deg \cdot L$.

The proof of (iii) is similar: With

$$\{\delta, d^c\} = -(n - \deg -1)\mathcal{L}_\xi + \eta \wedge \delta$$

we obtain

$$\begin{aligned}
\delta d^c\alpha &= -\sum e_i \lrcorner \nabla_{e_i}(\phi e_j^* \wedge \nabla_{e_j}\alpha) \\
&= -\sum e_i \lrcorner \left(((\nabla_{e_i}\phi)e_j^*) \wedge \nabla_{e_j}\alpha\right) - \sum e_i \lrcorner (\phi e_j^* \wedge \nabla_{e_i}\nabla_{e_j}\alpha) \\
&= -\sum e_i \lrcorner \left((\eta(e_j)e_i^* - \delta_{ij}\eta) \wedge \nabla_{e_j}\alpha\right) \\
&\quad - \sum g(e_i, \phi e_j)\nabla_{e_i}\nabla_{e_j}\alpha + \sum \phi e_j^* \wedge e_i \lrcorner \nabla_{e_i}\nabla_{e_j}\alpha \\
&= -\sum e_i \lrcorner (e_i^* \wedge \nabla_\xi\alpha) + \sum e_i \lrcorner (\eta \wedge \nabla_{e_i}\alpha) - \sum \nabla_{\phi e_j}\nabla_{e_j}\alpha \\
&\quad + \sum \phi e_j^* \wedge e_i \lrcorner \nabla_{e_j}\nabla_{e_i}\alpha + \sum \phi e_j^* \wedge e_i \lrcorner R(e_i, e_j)\alpha
\end{aligned}$$

$$
\begin{aligned}
&= -(n-p-1)\nabla_\xi\alpha + \eta\wedge\delta\alpha - S_{\mathrm{D}}\alpha + (n-2)\phi_{\mathrm{D}}\alpha \\
&\qquad + \sum \phi e_j^* \wedge \nabla_{e_j}(e_i \lrcorner \nabla_{e_i}\alpha) - \sum \phi e_j^* \wedge e_i \lrcorner R(e_j, e_i)\alpha \\
&= -(n-p-1)\nabla_\xi\alpha + \eta\wedge\delta\alpha - S_{\mathrm{D}}\alpha + (n-2)\phi_{\mathrm{D}}\alpha - d^c\delta\alpha - \sum \phi e_j^* \wedge R^-(e_j) \\
&= \delta d^c - \rho \bullet + (p-1)\phi_{\mathrm{D}} - \sum \phi e_j^* \wedge R^-(e_j),
\end{aligned}
$$

thus $\sum \phi e_j^* \wedge R^-(e_j) = -\rho \bullet + (p-1)\phi_{\mathrm{D}}$.

The remaining equations (iv), (v) and (vi) are proven exactly like the first three.□

Similarly to above we are able to determine the commutators between $R^+, R^-, q(R)$ on one hand and $\xi \lrcorner, \eta\wedge, \Lambda, L, \phi_{\mathrm{D}}$ on the other hand. However, we wait until we introduce the Sasakian curvature tensor in order to obtain much shorter calculations.

During our investigation of conformal Killing Forms on Sasakian manifolds in Section 3 we need a certain contraction of the covariant derivative denoted by δR:

Definition 2.1.10. *Let (M, g) be a Riemannian manifold. Then we define*

$$
\begin{aligned}
(\delta R)(X) &:= -\sum (\nabla_{e_i} R)(e_i, X), \\
(\delta R)^+ &:= \sum e_i^* \wedge (\delta R)(e_i), \\
(\delta R)^- &:= \sum e_i \wedge (\delta R)(e_i).
\end{aligned}
$$

The following lemma connects δR to the covariant derivative of the Ricci tensor. The proof is an elementary calculation and can be found in [S01].

Lemma 2.1.11. *Let (M, g) be a Riemannian manifold. Then*

$$
g((\delta R)(X)Y, Z) = g((\nabla_Y \mathrm{Ric})Z - (\nabla_Z \mathrm{Ric})Y, X)
$$

for all vector fields X, Y, Z on M.

2.2 Sasakian curvature tensor

This section contains most results of Section 2.1 reformulated in terms of the Sasakian curvature tensor which is introduced in the following definition.

Definition 2.2.1. *Let (M, g, ξ) be a Sasakian manifold. We define the **Sasakian curvature tensor** A by*

$$
A(X, Y) = R(X, Y) + (X^* \wedge Y^*)\bullet
$$

for all vector fields X, Y on M. Additionally we introduce the following operators:

$$
\begin{aligned}
A^+(X) &:= \sum e_i^* \wedge A(X, e_i), \\
A^-(X) &:= \sum e_i \lrcorner A(X, e_i), \\
q(A) &:= -\sum e_i \lrcorner A^+(e_i) = -\sum e_i^* \wedge A^-(e_i), \\
\mathrm{Ric}^A(X) &:= \sum A(X, e_i)e_i, \\
\rho^A(X, Y) &:= \omega(\mathrm{Ric}^A(X), Y) = \mathrm{ric}^A(\phi X, Y).
\end{aligned}
$$

Remark. The Sasakian curvature tensor A is essentially the curvature tensor R_C of the Riemannian cone $C(M)$: For all $\alpha \in \Omega^p(M)$ we have

$$R_C(X,Y)\pi_C^*\alpha = \pi_C^* A(X,Y)\alpha$$

for all tangent vectors X and Y on M, cf Lemma B.0.12 in Appendix B. Here $\pi_C : C(M) \longrightarrow M$ denotes the projection.

In Proposition 2.2.2 we note that the Sasakian curvature tensor A has the same covariant derivative as the Riemannian curvature tensor, i.e. $\nabla A = \nabla R$, and satisfies all curvature identities. Lemma 2.2.3 contains the difference between the operators defined above and the corresponding ones of the Riemannian curvature.

Proposition 2.2.2. *Let (M, g, ξ) be a Sasakian manifold. Then the Sasakian curvature tensor A satisfies*

- $A(X,Y) + A(Y,X) = 0,$
- $A(X,Y)Z + A(Y,Z)X + A(Z,X)Y = 0,$
- $A(X,Y,Z,W) = -A(X,Y,W,Z),$
- $A(X,Y,Z,W) = A(Z,W,X,Y),$
- $\nabla A = \nabla R,$
- $(\nabla_X A)(Y,Z) + (\nabla_Y A)(Z,X) + (\nabla_Z A)(X,Y) = 0$

for all vector fields X, Y, Z and W on M.

Proof. This is an immediate consequence of Definition 2.2.1 because - up to a sign - the additional term

$$(X^* \wedge Y^*)\bullet$$

can be considered as the curvature tensor of the standard sphere. □

Lemma 2.2.3. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. Then we have*

- $A^+(X) = R^+(X) + \deg \cdot X^* \wedge,$
- $A^-(X) = R^-(X) + (n - \deg) \cdot X \lrcorner\,,$
- $q(A) = q(R) - \deg(n - \deg)\mathrm{id},$
- $\mathrm{Ric}^A = \mathrm{Ric} - (n-1)\mathrm{id},$
- $\rho^A = \rho - (n-1)\omega.$

In particular we have $A^+(\xi) = A^-(\xi) = 0$.

Proof. This is an easy application of Definition 2.2.1. □

Analogue to Proposition 2.1.1 we have the following symmetry properties of A.

Theorem 2.2.4. *Let (M, g, ξ) be a Sasakian manifold. Then we have*

- $A(\xi, X)Y = A(X,Y)\xi = 0,$

- $A(X,Y)\omega = 0$,
- $[A(X,Y), \phi_{\mathrm{D}}] = 0$,
- $A(\phi X, \phi Y) = A(X,Y)$,
- $A(\phi X, Y) + A(X, \phi Y) = 0$,
- $A(\phi X, \phi Y, Z, W) = A(X, Y, \phi Z, \phi W)$

for all vector fields X, Y, Z and W on M.

Likewise, the Propositions 2.1.4 and 2.1.9 yield the next theorem.

Theorem 2.2.5. *Let (M, g, ξ) be a Sasakian manifold. Then for all vector fields X on M we have*

- $\sum A(X, e_i)\phi e_i = \rho^A \bullet X$,
- $\sum A(e_i, \phi e_i) = -2\rho^A \bullet$,
- $\sum \phi e_i^* \wedge A(X, e_i) = A^+(\phi X)$,
- $\sum \phi e_i \lrcorner A(X, e_i) = A^-(\phi X)$,
- $\sum \phi e_i^* \wedge A^+(e_i) = \sum \phi e_i \lrcorner A^-(e_i) = 0$,
- $\sum \phi e_i \lrcorner A^+(e_i) = \sum \phi e_i^* \wedge A^-(e_i) = -\rho^A \bullet$.

Proof. With 2.1.9 this is evident. $\square$

With the results obtained so far we can easily determine the commutator and anti-commutator relations between $A^+, A^-, q(A)$ on one hand and $\xi \lrcorner, \eta \wedge, \Lambda, L, \phi_{\mathrm{D}}$ on the other hand.

Proposition 2.2.6. *Let (M, g, ξ) be a Sasakian manifold. Then we have*

- $[\Lambda, A^+(X)] = A^-(\phi X)$,
- $[L, A^-(X)] = -A^+(\phi X)$,
- $[L, \mathrm{Ric}_D^A] = -2\rho^A \wedge$,
- $[\Lambda, \mathrm{Ric}_D^A] = 2\rho^A \lrcorner$
- $[\phi_{\mathrm{D}}, A^{\pm}(X)] = A^{\pm}(\phi X)$.

All other commutators between $q(A), \mathrm{Ric}_{\mathrm{D}}^A$ and $\rho^A \bullet$ on one hand and $\eta \wedge, \xi \lrcorner, L, \Lambda$ and ϕ_{D} on the other hand vanish, as well as the anticommutators of $A^{\pm}(X)$ with $\xi \lrcorner$ and $\eta \wedge$.

Proof. This is a straightforward calculation. For $\alpha \in \Omega^*(M)$ we have

$$\begin{aligned} A^+(X)\Lambda\alpha &= \sum e_i^* \wedge A(X, e_i)(\omega \lrcorner \alpha) \\ &= \sum e_i^* \wedge \omega \lrcorner A(X, e_i)\alpha \\ &= \omega \lrcorner \sum e_i^* \wedge A(X, e_i)\alpha - \sum \phi e_i \lrcorner A(X, e_i)\alpha \end{aligned}$$

$$
\begin{aligned}
&= \Lambda A^+(X)\alpha + \sum e_i \lrcorner A(X, \phi e_i)\alpha \\
&= \Lambda A^+(X)\alpha - \sum e_i \lrcorner A(\phi X, e_i)\alpha \\
&= \Lambda A^+(X)\alpha - A^-(\phi X)\alpha.
\end{aligned}
$$

and the self-adjointness of Ric^A leads to

$$
\begin{aligned}
\mathrm{Ric}^A_{\mathrm{D}} \Lambda\alpha &= \mathrm{Ric}^A_{\mathrm{D}}(\omega \lrcorner \alpha) \\
&= -(\mathrm{Ric}^A_{\mathrm{D}}\omega) \lrcorner \alpha + \omega \lrcorner \mathrm{Ric}^A_{\mathrm{D}}\alpha \\
&= -2\rho^A \lrcorner \alpha + \Lambda \mathrm{Ric}^A_{\mathrm{D}}\alpha.
\end{aligned}
$$

The proofs of the other equations are similar. □

The relations of Proposition 2.2.6 immediately yield the corresponding relations concerning the Riemannian curvature tensor R. A complete list can be found in Appendix D.

In [K71], T. Kashiwada combines the symmetries of the Riemannian curvature with respect to the three different Sasakian structures of a 3-Sasakian manifold in order to show the following theorem. We reformulate his proof in terms of the Sasakian curvature tensor A.

Theorem 2.2.7. *Every* 3*-Sasakian manifold is an Einstein manifold.*

Proof. Let $(M, g, \xi_1, \xi_2, \xi_3)$ be a 3-Sasakian manifold, then in particular we have

$$
\phi_1 \circ \phi_2 = \phi_3 + \eta_2 \otimes \xi_1.
$$

The definition of the Sasakian curvature tensor A depends only on the metric g, therefore the Theorems 2.2.4 and 2.2.5 yield

$$
\begin{aligned}
-2\rho_3^A &= \sum A(e_i, \phi_3 e_i) \\
&= \sum A(e_i, \phi_1\phi_2 e_i) - \sum \eta_2(e_i) A(e_i, \xi_1) \\
&= -\sum A(\phi_2 e_i, \phi_1 e_i) \\
&= \sum A(\phi_1\phi_2 e_i, e_i) \\
&= \sum A(\phi_3 e_i, e_i) + \sum \eta_2(e_i) A(\xi_1, e_i) \\
&= 2\rho_3^A.
\end{aligned}
$$

We obtain $\mathrm{Ric}^A \circ \phi_3 = 0$, and since $TM = \mathrm{im}(\phi_3) \oplus \langle \xi_3 \rangle$, we have $\mathrm{Ric}^A = 0$, i.e. $\mathrm{Ric} = (n-1)\mathrm{id}$. □

Remark. Another way to see that 3-Sasakian manifolds are Einstein is to use the fact that every hyperkähler manifold is Ricci-flat.

2.3 Covariant derivative of the curvature tensor

With Theorem 2.2.4 we research the symmetries of the covariant derivative $\nabla R = \nabla A$ of the curvature tensor of a Sasakian manifold with respect to the Reeb vector field ξ and to the Sasakian endomorphism ϕ. We show $\nabla_\xi R = 0$ and that the covariant derivative of R determines R. In particular we have $\nabla A = 0$ if and only if $A = 0$.

Proposition 2.3.1. *Let (M, g, ξ) be a Sasakian manifold. Then we have*

- $(\nabla_Z A)(X, \xi) = A(\phi X, Z)$,
- $(\nabla_Z A)(\phi X, Y) + (\nabla_Z A)(X, \phi Y) = -A(\eta(Y)X - \eta(X)Y, Z)$

for all vector fields X, Y and Z.

Proof. By Theorem 2.2.4 the Sasakian curvature tensor satisfies $A(X, \xi) = 0$ and $A(\phi X, Y) + A(X, \phi Y) = 0$. Taking the covariant derivative of these two identities in direction of Z we obtain

$$\begin{aligned} 0 &= \nabla_Z(A(X, \xi)) \\ &= (\nabla_Z A)(X, \xi) + A(\nabla_Z X, \xi) + A(X, \nabla_Z \xi) \\ &= (\nabla_Z A)(X, \xi) + A(X, \phi Z) \end{aligned}$$

and

$$\begin{aligned} 0 &= \nabla_Z(A(\phi X, Y) + A(X, \phi Y)) \\ &= (\nabla_Z A)(\phi X, Y) + (\nabla_Z A)(X, \phi Y) + A((\nabla_Z \phi)X, Y) + A(X, (\nabla_Z \phi)Y) \\ &\quad + A(\phi \nabla_Z X, Y) + A(\nabla_Z X, \phi Y) + A(\phi X, \nabla_Z Y) + A(X, \phi \nabla_Z Y) \\ &= (\nabla_Z A)(\phi X, Y) + (\nabla_Z A)(X, \phi Y) + A(\eta(X)Z - g(X, Z)\xi, Y) \\ &\quad + A(X, \eta(Y)Z - g(Y, Z)\xi) \\ &= (\nabla_Z A)(\phi X, Y) + (\nabla_Z A)(X, \phi Y) + A(\eta(Y)X - \eta(X)Y, Z). \end{aligned}$$ □

Corollary 2.3.2. *Let (M, g, ξ) be a Sasakian manifold. Then we have*

$$\nabla_\xi R = \nabla_\xi A = 0.$$

If the curvature tensor is parallel, i.e. $\nabla R = \nabla A = 0$, then we have $A = 0$.

Proof. With the second Bianchi identity we calculate

$$\begin{aligned} (\nabla_\xi R)(X, Y) &= (\nabla_\xi A)(X, Y) \\ &= -(\nabla_X A)(Y, \xi) - (\nabla_Y A)(\xi, X) \\ &= -\nabla_X(A(Y, \xi) + A(\nabla_X Y, \xi) + A(Y, \nabla_X \xi) \\ &\quad - \nabla_Y(A(\xi, X) + A(\nabla_Y \xi, X) + A(\xi, \nabla_Y X) \\ &= A(Y, \phi X) + A(\phi Y, X) \\ &= 0. \end{aligned}$$

If $\nabla R = \nabla A = 0$ we obtain $0 = (\nabla_{\phi X} A)(Y, \xi) = A(\phi X, \phi Y) = A(X, Y)$. □

2.4 Ricci form

The Ricci form of a Kähler manifold is always a closed 2-form, which is a consequence of the symmetries of the Riemannian curvature tensor of a Kähler manifold with respect to its complex structure combined with the Bianchi identities. Since the Sasakian curvature tensor A has the same symmetries with respect to the Sasakian endomorphism ψ and still satisfies the Bianchi identities, we can copy the proof of the Kähler situation and are able to show that the 2-form ρ^A is always closed. This implies that the Ricci form ρ is closed since the difference of ρ and ρ^A is a constant multiple of the closed Sasakian 2-form ω.

Proposition 2.4.1. *Let (M, g, ξ) be a Sasakian manifold. Then the Ricci form ρ is closed and we have*

$$\rho^A(X,Y) = \tfrac{1}{2}\mathrm{tr}\left(A(X,Y)\circ\phi\right).$$

Proof. We already saw $\mathrm{tr}\left(A(X,Y)\circ\phi\right) = \sum g(A(X,Y)\phi e_i, e_i) = 2\rho^A(X,Y)$. We differentiate this covariantly and obtain

$$\begin{aligned} 2\left(\nabla_Z\rho^A\right)(X,Y) &= \mathrm{tr}\left((\nabla_Z A)(X,Y)\circ\phi\right) + \mathrm{tr}\left(A(X,Y)\circ\nabla_Z\phi\right) \\ &= \mathrm{tr}\left((\nabla_Z A)(X,Y)\circ\phi\right), \end{aligned}$$

where we used

$$\begin{aligned} \mathrm{tr}\left(A(X,Y)\circ\nabla_Z\phi\right) &= \sum g(A(X,Y)(\nabla_Z\phi)e_i, e_i) \\ &= \sum g(A(X,Y)(\eta(e_i)Z - g(Z,e_i)\xi), e_i) \\ &= g(A(X,Y)Z,\xi) - g(A(X,Y)\xi, Z) \\ &= 0. \end{aligned}$$

From the second Bianchi identity we thus get

$$\begin{aligned} 2d\rho(X,Y,Z) &= 2d\rho^A(X,Y,Z) + 2(n-1)d\omega(X,Y,Z) \\ &= 2\left(\left(\nabla_X\rho^A\right)(Y,Z) + \left(\nabla_Y\rho^A\right)(Z,X) + \left(\nabla_Z\rho^A\right)(X,Y)\right) \\ &= \mathrm{tr}\left(\left(\nabla_X A\right)(Y,Z)\circ\phi + \left(\nabla_Y A\right)(Z,X)\circ\phi + \left(\nabla_Z A\right)(X,Y)\circ\phi\right) \\ &= 0. \qquad \square \end{aligned}$$

We conclude this section with some further simple properties of ρ^A.

Proposition 2.4.2. *Let (M, g, ξ) be a Sasakian manifold. Then ρ^A is a basic $(1,1)$-form and we have $\Lambda\rho^A = \frac{1}{2}\mathrm{scal}^A$.*

Proof. The Ricci form ρ^A is clearly horizontal, thus basic, and we compute

$$\begin{aligned} \phi_{\mathrm{D}}\rho^A &= \tfrac{1}{2}\sum \phi e_i^* \wedge e_i \lrcorner \left(e_j^* \wedge \mathrm{Ric}^A(\phi e_j)^*\right) \\ &= \tfrac{1}{2}\sum \phi e_i^* \wedge \mathrm{Ric}^A(\phi e_i)^* - \tfrac{1}{2}\sum g(e_i, \mathrm{Ric}^A(\phi e_j))\phi e_i^* \wedge e_j^* \\ &= \sum \phi e_i^* \wedge \mathrm{Ric}^A(\phi e_i)^* \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \Lambda\rho^A &= \tfrac{1}{4}\sum \phi e_i \lrcorner\, e_i \lrcorner \left(e_j^* \wedge \mathrm{Ric}^A(\phi e_j)^*\right) \\ &= \tfrac{1}{4}\sum g(\phi e_i, \mathrm{Ric}^A(\phi e_i)) - \tfrac{1}{4}\sum g(e_i, \mathrm{Ric}^A(\phi e_j))g(\phi e_i, e_j) \\ &= \tfrac{1}{2}\sum g(\phi e_i, \mathrm{Ric}^A(\phi e_i)) \\ &= \tfrac{1}{2}\mathrm{scal}^A. \qquad \square \end{aligned}$$

3 Conformal Killing forms on Sasakian manifolds

In this section we study the consequences of the existence of a Sasakian structure on a Riemannian manifold (M, g) for its conformal Killing forms. Up to now, all known examples of conformal Killing p-forms on Sasakian manifolds different form the sphere for $p \neq 1, \dim(M) - 1$ are directly linked to Sasakian structures, confer Section 3.2. This leads to the question whether these forms are the only possible conformal Killing forms on Sasakian manifolds. In the 1970's, S. Yamaguchi studied this question and found some results which we will recall in Section 3.2. We give index-free proofs of his statements and extend most of his results to weaker conditions.

After recalling some properties of conformal Killing forms that are important to us we show that every conformal Killing p-form on a Sasakian manifold is the sum of a Killing and a $*$-Killing form. This reduces the discussion of conformal Killing forms to Killing forms, which we then study intensively.

3.1 Preliminaries: Conformal Killing forms on Riemannian manifolds

We give the definition of conformal vector fields and conformal Killing forms and related objects on arbitrary Riemannian manifolds. After that we quote some curvature properties of conformal Killing forms which will be crucial in our discussion of conformal Killing forms on Sasakian manifolds. Most of the results in this section are proven in [S01].

Definition 3.1.1. *Let (M, g) be a Riemannian manifold. A vector field $v \in \Gamma(TM)$ is called a* ***conformal vector field*** *if there exists a function $f \in \mathcal{C}^\infty(M)$ with*

$$\mathcal{L}_v g = 2fg,$$

and v is called a ***Killing vector field*** *if $f = 0$ holds.*

Definition 3.1.2. *Let (M, g) be a Riemannian manifold. A p-form is called a* ***conformal Killing form*** *if it satisfies*

$$\nabla_X \psi = \tfrac{1}{p+1} X \lrcorner\, d\psi - \tfrac{1}{n-p+1} X^* \wedge \delta\psi$$

for all vector fields X on M. Coclosed conformal Killing forms are called ***Killing forms****, and closed conformal Killing forms are called $*$-****Killing forms****. We denote by $\mathcal{CK}^p(M, g), \mathcal{K}^p(M, g)$ and $*\mathcal{K}^p(M, g)$ the space of conformal Killing, Killing and $*$-Killing p-forms, respectively. A Killing form $\sigma \in \mathcal{K}^p(M, g)$ is called* ***special*** *if it satisfies the additional equation*

$$\nabla_X(d\sigma) = cX^* \wedge \sigma$$

for some constant $c \in \mathbb{R}$. Similarly, a $*$-Killing form $\tau \in *\mathcal{K}^p(M, g)$ is called ***special*** *if it satisfies the additional equation*

$$\nabla_X(\delta\tau) = cX \lrcorner\, \tau$$

for some constant $c \in \mathbb{R}$.

Conformal Killing forms can be seen as generalization of conformal vector fields since conformal vector fields are dual to conformal Killing 1-forms:

Proposition 3.1.3. *Let (M, g) be a Riemannian manifold. Then a vector field v is a conformal vector field if and only if the dual 1-form v^* is a conformal Killing form. The correspondence is given by $f = -\frac{1}{n}\delta v^*$. In particular v is a Killing field if and only if v^* is a Killing 1-form.*

There is another connection between conformal vector fields and conformal Killing forms:

Proposition 3.1.4. *Let (M, g) be a Riemannian manifold and v a conformal vector field with $\mathcal{L}_v g = 2fg$. If $\psi \in \mathcal{CK}^p(M, g)$ then also $\mathcal{L}_v\psi - (p+1)f\psi \in \mathcal{CK}^p(M, g)$.*

The space of conformal Killing forms is conformally invariant: If $\psi \in \mathcal{CK}^p(M, g)$ and $f \in C^\infty(M)$, then $e^{(p+1)f}\psi \in \mathcal{CK}^p(M, e^{2f}g)$. Another important property of $\mathcal{CK}^*$ is its closure under the Hodge $*$-operator:

Proposition 3.1.5. *Let (M, g) be a n-dimensional Riemannian manifold. Then we have*

$$*(\mathcal{CK}^p(M, g)) = \mathcal{CK}^{n-p}(M, g)$$

and in particular, since $*$ interchanges coclosed and closed forms,

$$*(\mathcal{K}^p(M, g)) = *\mathcal{K}^{n-p}(M, g),$$

$$*(*\mathcal{K}^p(M, g)) = \mathcal{K}^{n-p}(M, g).$$

Furthermore, if σ is a special Killing form, then $*\sigma$ is special $*$-Killing and vice versa.

The next two propositions contain criteria whether a given differential form is a conformal Killing form or not.

Proposition 3.1.6. *Let (M, g) be a n-dimensional Riemannian manifold. If $\psi \in \mathcal{CK}^p(M, g)$, then*

$$q(R)\psi = \tfrac{p}{p+1}\delta d\psi + \tfrac{n-p}{n-p+1}d\delta\psi.$$

Conversely, if M is compact and $\psi \in \Omega^p(M)$ is an arbitrary p-form satisfying this equation, then ψ is a conformal Killing form.

Proposition 3.1.7. *Let (M, g) be a Riemannian manifold. Then a differential form $\sigma \in \Omega^*(M)$ is a Killing form if and only if*

$$X \lrcorner \nabla_X\sigma = 0$$

for all vector fields X on M. Likewise $\tau \in \Omega^*(M)$ is a $*$-Killing form if and only if

$$X^* \wedge \nabla_X\tau = 0.$$

Proof. If $\sigma \in \Omega^p(M)$ is Killing then $\nabla_X\sigma = \frac{1}{p+1}X \lrcorner d\sigma$, which obviously implies $X \lrcorner \nabla_X\sigma = 0$. Conversely, assume that σ satisfies $X \lrcorner \nabla_X\sigma = 0$. By polarization we get

$$X \lrcorner \nabla_Y\sigma = -Y \lrcorner \nabla_X\sigma$$

for all vector fields X and Y, which implies

$$X \lrcorner d\sigma = X \lrcorner \sum e_i^* \wedge \nabla_{e_i}\sigma$$

$$
\begin{aligned}
&= \sum g(e_i, X)\nabla_{e_i}\sigma - \sum e_i^* \wedge X \lrcorner \nabla_{e_i}\sigma \\
&= \nabla_X\sigma + \sum e_i^* \wedge e_i \lrcorner \nabla_X\sigma \\
&= (p+1)\nabla_X\sigma.
\end{aligned}
$$

The statement about τ is proven analogously. □

We can formulate an immediate corollary:

Corollary 3.1.8. *Let (M, g) be a Riemannian manifold.*

- *If $\sigma \in \mathcal{K}^p(M, g)$ is special, then $d\sigma \in *\mathcal{K}^{p+1}(M, g)$.*
- *If $\tau \in *\mathcal{K}^p(M, g)$ is special, then $\delta\tau \in \mathcal{K}^{p-1}(M, g)$.*

We will see that on Sasakian manifolds the converse statement holds for Killing p-forms with $p \neq 1$ and for $*$-Killing p-forms with $p \neq n-1$.

The importance of special Killing forms is given by the fact that they are in $1-1$ correspondence with parallel forms on the Riemannian cone $(C(M), g_C) = (M \times \mathbb{R}_+, r^2 g + dr^2)$:

Proposition 3.1.9. *Let (M, g) be a Riemannian manifold. Then a form*

$$\widehat{\sigma} \in \Omega^{p+1}(C(M))$$

is parallel on the cone if and only if there exists a special Killing form $\sigma \in K^p(M, g)$ on M such that

$$\widehat{\sigma} = d(r^{p+1}\sigma).$$

Proof. We only show that $\widehat{\sigma}$ is parallel if σ is special. The Levi-Civita connection ∇^C of the cone $C(M)$ acts on differential forms by

$$
\begin{aligned}
\nabla^C_X\alpha &= \nabla_X\alpha + \tfrac{1}{r}X \lrcorner (dr \wedge \alpha), \\
\nabla^C_{\partial_r}\alpha &= -\tfrac{p}{r}\alpha, \\
\nabla^C_X dr &= rX^*, \\
\nabla^C_{\partial_r} dr &= 0,
\end{aligned}
$$

cf Appendix B. We use this and calculate

$$
\begin{aligned}
\nabla^C_{\partial_r}\widehat{\sigma} &= \nabla^C_{\partial_r}\big((p+1)r^p dr \wedge \sigma + r^{p+1}d\sigma\big) \\
&= p\,(p+1)r^{p-1}dr \wedge \sigma + (p+1)r^p dr \wedge \nabla^C_{\partial_r}\sigma \\
&\qquad + (p+1)r^p d\sigma + r^{p+1}\nabla^C_{\partial_r}d\sigma \\
&= p\,(p+1)r^{p-1}dr \wedge \sigma - \tfrac{p}{r}(p+1)r^p dr \wedge \sigma \\
&\qquad + (p+1)r^p d\sigma - \tfrac{p+1}{r}r^{p+1}d\sigma \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\nabla_X\widehat{\sigma} &= \nabla^C_X\big((p+1)r^p dr \wedge \sigma + r^{p+1}d\sigma\big) \\
&= (p+1)r^p\nabla^C_X(dr) \wedge \sigma + (p+1)r^p dr \wedge \nabla^C_X\sigma
\end{aligned}
$$

$$
\begin{aligned}
&+ r^{p+1}\nabla^C_X d\sigma \\
&= (p+1)r^{p+1}X^* \wedge \sigma + (p+1)r^p dr \wedge \nabla_X\sigma \\
&\quad + (p+1)r^{p-1} dr \wedge (X \lrcorner (dr \wedge \sigma)) \\
&\qquad + r^{p+1}\nabla_X d\sigma + r^p X \lrcorner (dr \wedge d\sigma) \\
&= (p+1)r^{p+1}X^* \wedge \sigma + (p+1)r^p dr \wedge \nabla_X\sigma \\
&\quad + r^{p+1}\nabla_X d\sigma + r^p X \lrcorner (dr \wedge d\sigma) \\
&= r^{p+1}\left(\nabla_X(d\sigma) + (p+1)X^* \wedge \sigma\right) \\
&\quad + (p+1)r^p dr \wedge \left(\nabla_X\sigma - \tfrac{1}{p+1}X \lrcorner d\sigma\right) \\
&= 0. \qquad \square
\end{aligned}
$$

Note that the last proposition describes exactly the relation between the Reeb vector field ξ of a Sasakian manifold and the Kähler form $\frac{1}{2}d(r^2\xi^*)$ of the Kähler cone.

With 3.1.9 it is possible to determine all special Killing forms on compact simply connected Riemannian manifolds. The full result can be found in [S01], here we only quote the part we need in our further work.

Proposition 3.1.10. *Let (M, g, ξ) be a compact simply connected Sasakian manifold that is not Einstein. Then all special Killing forms are given by $\eta \wedge \omega^k$.*

By covariantly differentiating the defining equation of conformal Killing forms we obtain an expression for the action of the curvature tensor on conformal Killing forms as well as formulas for the covariant derivatives of $d\psi, \delta\psi$ and $\Delta\psi$. Therefore we recall the expressions for the following curvature operators introduced in Section 2:

- $R^+(X) = \sum e_i^* \wedge R(X, e_i)$,
- $R^-(X) = \sum e_i \lrcorner R(X, e_i)$,
- $q(R) = -\sum e_i \lrcorner R^+(e_i) = -\sum e_i^* \wedge R^-(e_i)$,
- $(\delta R)(X) = -\sum(\nabla_{e_i} R)(e_i, X)$.

The following formula, which is called the curvature condition, is our most important tool in the research of conformal Killing forms on Sasakian manifolds.

Proposition 3.1.11. *Let (M, g) be a n-dimensional Riemannian manifold and $\psi \in \mathcal{CK}^p(M, g)$ with $p \neq 0, n$. Then we have*

$$
\begin{aligned}
R(X,Y)\psi = &\tfrac{1}{p\,(n-p)}(Y^* \wedge X \lrcorner q(R)\psi - X^* \wedge Y \lrcorner q(R)\psi) \\
&+ \tfrac{1}{p}(Y \lrcorner R^+(X)\psi - X \lrcorner R^+(Y)\psi) \\
&+ \tfrac{1}{n-p}(Y^* \wedge R^-(X)\psi - X^* \wedge R^-(Y)\psi)
\end{aligned}
$$

for all vector fields X and Y on M.

On the vector bundle $\Lambda^p(M) \times \Lambda^{p+1}(M) \times \Lambda^{p-1}(M) \times \Lambda^p(M)$ there exists a connection such that $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ is parallel if and only if $\psi_1 \in \mathcal{CK}^p(M, g)$ and $\psi_2 = d\psi_1, \psi_3 = \delta\psi_1, \psi_4 = \Delta\psi_1$ (conf [S01]). As a consequence we have the next two propositions, where the first one contains a dimension bound for $\mathcal{CK}^p$, while the second one states relations for the covariant derivatives of the forms $d\psi, \delta\psi$ and $\Delta\psi$.

Proposition 3.1.12. *Let (M,g) be a n-dimensional Riemannian manifold and $p \neq 0, n$. Then we have*

$$\dim\left(\mathcal{CK}^p(M,g)\right) \leq \left(\begin{array}{c} n+2 \\ p+1 \end{array}\right).$$

In particular $\mathcal{CK}^p(M,g)$ is a finite dimensional vector space.

Proposition 3.1.13. *Let (M,g) be a n-dimensional Riemannian manifold. Then for all $\psi \in \mathcal{CK}^p(M,g)$ with $p \neq 0, n$ we have*

$$\begin{aligned}
\nabla_X(d\psi) &= \tfrac{p+1}{p} R^+(X)\psi + \tfrac{p+1}{p\,(n-p+1)} X^* \wedge d\delta\psi, \\
\nabla_X(\delta\psi) &= -\tfrac{n-p+1}{n-p} R^-(X)\psi - \tfrac{n-p+1}{(p+1)(n-p)} X \lrcorner\, \delta d\psi, \\
\nabla_X(\Delta\psi) &= \tfrac{1}{p+1} X \lrcorner\, d(\Delta\psi) - \tfrac{1}{n-p+1} X^* \wedge \delta(\Delta\psi) + q(\nabla_X R)\psi - (\delta R)(X)\psi.
\end{aligned}$$

Combining the Propositions 3.1.7 and 3.1.13 immediately yields a criterion whether $d\psi, \delta\psi$ and $\Delta\psi$ are again conformal Killing forms.

Corollary 3.1.14. *Let (M,g) be a Riemannian manifold and $\psi \in \mathcal{CK}^p(M,g)$ with $p \neq 0, n$.*

- *$\delta\psi \in \mathcal{K}^{p-1}(M,g)$ if and only if $X \lrcorner\, R^-(X)\psi = 0$ for all vector fields X on M or equivalently $R^-(X)\psi = -\frac{1}{p} X \lrcorner\, q(R)\psi$ for all vector fields X on M.*
- *$d\psi \in *\mathcal{K}^{p+1}(M,g)$ if and only if $X^* \wedge R^+(X)\psi = 0$ for all vector fields X on M or equivalently $R^+(X)\psi = -\frac{1}{n-p} X^* \wedge q(R)\psi$ for all vector fields X on M.*
- *$\Delta\psi \in \mathcal{CK}^p(M,g)$ if and only if $q(\nabla_X R)\psi = (\delta R)(X)\psi$ for all vector fields X on M.*

Proof. With Proposition 3.1.7 it remains to show

$$\begin{aligned}
X \lrcorner\, R^-(X)\psi = 0 \text{ for all } X \quad &\Longleftrightarrow \quad R^-(X)\alpha = -\tfrac{1}{p} X \lrcorner\, q(R)\psi \text{ for all } X, \\
X^* \wedge R^+(X)\psi = 0 \text{ for all } X \quad &\Longleftrightarrow \quad R^+(X)\alpha = -\tfrac{1}{n-p} X^* \wedge q(R)\psi \text{ for all } X.
\end{aligned}$$

We will only show the first equivalence. One direction is obvious, and to prove the other we rewrite $X \lrcorner\, R^-(X)\psi = 0$ as

$$X \lrcorner\, R^-(Y)\psi = -Y \lrcorner\, R^-(X)\psi$$

and obtain

$$\begin{aligned}
X \lrcorner\, q(R)\psi &= -X \lrcorner \sum e_i^* \wedge R^-(e_i)\psi \\
&= -\sum g(e_i, X) R^-(e_i)\psi + \sum e_i^* \wedge X \lrcorner\, R^-(e_i)\psi \\
&= -R^-(X)\psi - \sum e_i^* \wedge e_i \lrcorner\, R^-(X)\psi \\
&= -p\, R^-(X)\psi. \qquad \square
\end{aligned}$$

If ψ is a Killing or a $*$-Killing form, we obtain simplified expressions for $\nabla_X(d\psi)$ and $\nabla_X(\delta\psi)$ in Proposition 3.1.13.

Corollary 3.1.15. *Let (M,g) be a n-dimensional Riemannian manifold.*

- *If $\sigma \in \mathcal{K}^p(M,g)$, then $\nabla_X(d\sigma) = \frac{p+1}{p} R^+(X)\sigma$.*
- *If $\tau \in *\mathcal{K}^p(M,g)$, then $\nabla_X(\delta\tau) = -\frac{n-p+1}{n-p} R^-(X)\tau$.*

The combination of Proposition 3.1.11 with Corollary 3.1.14 yields a simplified curvature condition for Killing and $*$-Killing forms.

Corollary 3.1.16. *Let (M,g) be a n-dimensional Riemannian manifold.*

- *If $\sigma \in \mathcal{K}^p(M,g)$, then*

$$R(X,Y)\sigma = \tfrac{1}{p}\left(Y \lrcorner R^+(X)\sigma - X \lrcorner R^+(Y)\sigma\right).$$

- *If $\tau \in *\mathcal{K}^p(M,g)$, then*

$$R(X,Y)\tau = \tfrac{1}{n-p}\left(Y^* \wedge R^-(X)\tau - X^* \wedge R^-(Y)\tau\right).$$

In order to obtain more information about the curvature properties of Killing forms we differentiate the refined curvature condition of Corollary 3.1.16 and obtain a curvature condition for $d\sigma$.

Proposition 3.1.17. *Let (M,g) be a Riemannian manifold and $\sigma \in \mathcal{K}^p(M,g)$ with $p \neq 1$. Then we have*

$$R(X,Y)d\sigma = \tfrac{1}{p+1}\left(Y \lrcorner R^+(X)d\sigma - X \lrcorner R^+(Y)d\sigma\right) - \sum e_i^* \wedge (\nabla_{e_i}R)(X,Y)\sigma.$$

Proof. This is a lengthy but straightforward computation where we use the definition of a Killing form. Recall that we have

$$R(X,Y)\sigma = \tfrac{1}{p}\left(Y \lrcorner R^+(X)\sigma - X \lrcorner R^+(Y)\sigma\right).$$

We apply the exterior differential $d = \sum e_i^* \wedge \nabla_{e_i}$ to this and use the second Bianchi identity to obtain

$$\begin{aligned}
0 &= \sum e_i^* \wedge \nabla_{e_i}\Big(R(X,Y)\sigma - \tfrac{1}{p}\left(Y \lrcorner R^+(X)\sigma - X \lrcorner R^+(Y)\sigma\right)\Big)\\
&= \sum e_i^* \wedge (\nabla_{e_i}R)(X,Y)\sigma + \sum e_i^* \wedge R(X,Y)\nabla_{e_i}\sigma\\
&\quad - \tfrac{1}{p}\sum e_i^* \wedge Y \lrcorner (e_j^* \wedge (\nabla_{e_i}R)(X,e_j)\sigma) + \tfrac{1}{p}\sum e_i^* \wedge X \lrcorner (e_j^* \wedge (\nabla_{e_i}R)(Y,e_j)\sigma)\\
&\quad - \tfrac{1}{p}\sum e_i^* \wedge Y \lrcorner (e_j^* \wedge R(X,e_j)\nabla_{e_i}\sigma) + \tfrac{1}{p}\sum e_i^* \wedge X \lrcorner (e_j^* \wedge R(Y,e_j)\nabla_{e_i}\sigma)\\
&= \sum e_i^* \wedge (\nabla_{e_i}R)(X,Y)\sigma + \tfrac{1}{p+1}\sum e_i^* \wedge R(X,Y)e_i \lrcorner d\sigma\\
&\quad + \tfrac{1}{p+1}\sum e_i^* \wedge e_i \lrcorner R(X,Y)d\sigma\\
&\quad - \tfrac{1}{p}\sum e_j^* \wedge (\nabla_Y R)(X,e_j)\sigma + \tfrac{1}{p}\sum e_j^* \wedge (\nabla_X R)(Y,e_j)\sigma\\
&\quad + \tfrac{1}{p}Y \lrcorner \sum e_i^* \wedge e_j^* \wedge (\nabla_{e_i}R)(X,e_j)\sigma - \tfrac{1}{p}X \lrcorner \sum e_i^* \wedge e_j^* \wedge (\nabla_{e_i}R)(Y,e_j)\sigma\\
&\quad - \tfrac{1}{p(p+1)}\sum e_j^* \wedge R(X,e_j)Y \lrcorner d\sigma + \tfrac{1}{p(p+1)}\sum e_j^* \wedge R(Y,e_j)X \lrcorner d\sigma
\end{aligned}$$

$$
\begin{aligned}
&- \tfrac{1}{p(p+1)} \sum e_j^* \wedge Y \lrcorner R(X, e_j) d\sigma + \tfrac{1}{p(p+1)} \sum e_j^* \wedge X \lrcorner R(Y, e_j) d\sigma \\
&- \tfrac{1}{p(p+1)} Y \lrcorner \sum e_j^* \wedge e_i^* \wedge R(X, e_j) e_i \lrcorner d\sigma \\
&+ \tfrac{1}{p(p+1)} X \lrcorner \sum e_j^* \wedge e_i^* \wedge R(Y, e_j) e_i \lrcorner d\sigma \\
&- \tfrac{1}{p(p+1)} Y \lrcorner \sum e_j^* \wedge e_i^* \wedge e_i \lrcorner R(X, e_j) d\sigma \\
&+ \tfrac{1}{p(p+1)} X \lrcorner \sum e_j^* \wedge e_i^* \wedge e_i \lrcorner R(Y, e_j) d\sigma
\end{aligned}
$$

$$
\begin{aligned}
= &\sum e_i^* \wedge (\nabla_{e_i} R)(X, Y)\sigma - \tfrac{1}{p+1} R(X, Y) d\sigma + R(X, Y) d\sigma \\
&- \tfrac{1}{p} \sum e_j^* \wedge (\nabla_{e_j} R)(X, Y)\sigma \\
&- \tfrac{1}{p(p+1)} \sum e_j^* \wedge R(X, Y) e_j \lrcorner d\sigma \\
&- \tfrac{1}{p(p+1)} R(X, Y) d\sigma + \tfrac{1}{p(p+1)} R(Y, X) d\sigma \\
&+ \tfrac{1}{p(p+1)} Y \lrcorner \sum e_j^* \wedge R(X, e_j) d\sigma - \tfrac{1}{p(p+1)} X \lrcorner \sum e_j^* \wedge R(Y, e_j) d\sigma \\
&+ \tfrac{1}{p(p+1)} Y \lrcorner \sum e_j^* \wedge R(X, e_j) d\sigma - \tfrac{1}{p(p+1)} X \lrcorner \sum e_j^* \wedge R(Y, e_j) d\sigma \\
&- \tfrac{1}{p} Y \lrcorner \sum e_j^* \wedge R(X, e_j) d\sigma + \tfrac{1}{p} X \lrcorner \sum e_j^* \wedge R(Y, e_j) d\sigma
\end{aligned}
$$

$$
= \tfrac{p-1}{p} \Big(R(X, Y) d\sigma + \sum e_i^* \wedge (\nabla_{e_i} R)(X, Y)\sigma - \tfrac{1}{p+1} \left(Y \lrcorner R^+(X) d\sigma - X \lrcorner R^+(Y) d\sigma \right) \Big).
$$

□

If we use Proposition 3.1.13 on 1-forms we obtain the Kostant formula for conformal vector fields:

Corollary 3.1.18. *Let (M, g) be a Riemannian manifold and v a conformal vector field. Then*

$$
\nabla^2_{X,Y} v = R(X, v)Y + X(f)Y + Y(f)X - g(X, Y)\mathrm{grad}\, f
$$

with $f := -\frac{1}{n}\mathrm{div}\,(v)$.

Proof. By Proposition 3.1.3 we know that v^* is a conformal Killing 1-form, i.e.

$$
\nabla_X v^* = \tfrac{1}{2} X \lrcorner dv^* + f X^*.
$$

We differentiate this covariantly and obtain

$$
\begin{aligned}
\nabla^2_{X,Y} v^* &= \nabla_X \nabla_Y v^* - \nabla_{\nabla_X Y} v^* \\
&= \tfrac{1}{2} Y \lrcorner \nabla_X (dv^*) + X(f) Y^* \\
&= \tfrac{1}{2} Y \lrcorner \left(2R^+(X) v^* + \tfrac{2}{n} X^* \wedge d\delta v^* \right) + X(f) Y^* \\
&= Y \lrcorner R^+(X) v^* - g(X, Y) df + Y(f) X^* + X(f) Y^*
\end{aligned}
$$

which yields the result with $Y \lrcorner R^+(X) v^* = R(X, v) Y^*$. □

With the Kostant formula we derive an expression for the Lie derivative of the curvature tensor in direction of a conformal vector field which will be vital in the discussion of conformal vector fields on Sasakian manifolds:

Corollary 3.1.19. *Let (M, g) be a Riemannian manifold and v a conformal vector field with $\mathcal{L}_v g = 2f\, g$. Then we have*

$$(\mathcal{L}_v R)(X, Y) = (\nabla_X(df) \wedge Y^* - \nabla_Y(df) \wedge X^*) \bullet .$$

Proof. If we set $B(X, Y) := \nabla^2_{X,Y} v - R(X, v)Y$ the Kostant formula reads

$$B(X, Y) = X(f)Y + Y(f)X - g(X, Y)\mathrm{grad}\, f.$$

We use Lemma A.0.11 from Appendix A and obtain

$$\begin{aligned}
(\mathcal{L}_v R)(X, Y)Z &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)\\
&= \nabla_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)\\
&\quad - \nabla_Y(B(X, Z)) + B(\nabla_Y X, Z) + B(X, \nabla_Y Z)\\
&= X(Y(f))Z - (\nabla_X Y)(f)Z + X(Z(f))Y - (\nabla_X Z)(f)Y\\
&\quad - Y(X(f))Z + (\nabla_Y X)(f)Z - Y(Z(f))X + (\nabla_Y Z)(f)X\\
&\quad - g(Y, Z)\nabla_X(\mathrm{grad}\, f) + g(X, Z)\nabla_Y(\mathrm{grad}\, f)\\
&= (\nabla_X(df))(Z)Y - g(Y, Z)\nabla_X(\mathrm{grad}\, f) - (\nabla_Y(df))(Z)X\\
&\quad + g(X, Z)\nabla_Y(\mathrm{grad}\, f)\\
&= (\nabla_X(df) \wedge Y^* - \nabla_Y(df) \wedge X^*) \bullet Z. \qquad \square
\end{aligned}$$

We conclude this section with formulas for the exterior derivatives of $\Delta\psi$ which will be crucial in our discussion of conformal Killing forms on Sasaki-Einstein manifolds:

Proposition 3.1.20. *Let (M, g) be a Riemannian manifold. Then for all $\psi \in \mathcal{CK}^p(M, g)$ with $p \neq 0, n$ we have*

$$\begin{aligned}
d(\Delta\psi) &= -\tfrac{1}{p}\mathrm{Ric}_{\mathrm{D}}(d\psi) + \tfrac{p-1}{p} q(R)(d\psi) + \tfrac{p+1}{p}(\delta R)^+\psi,\\
\delta(\Delta\psi) &= \tfrac{1}{n-p}(\mathrm{scal}\,\delta\psi + \mathrm{Ric}_{\mathrm{D}}(\delta\psi)) + \tfrac{n-p-1}{n-p} q(R)(\delta\psi) - \tfrac{n-p+1}{n-p}(\delta R)^-\psi.
\end{aligned}$$

3.2 Examples and known results on Sasakian manifolds

After giving a list of the known examples of conformal Killing forms on Sasakian manifolds we summarize the results of M. Okumura [O62] and S. Yamaguchi [Y72a], [Y72b].

Proposition 3.2.1. *Let (M, g, ξ) be a Sasakian manifold. Then $\eta \wedge \omega^k$ is a special Killing form and ω^k is a special $*$-Killing form for all k.*

Proof. We prove this directly. Alternatively we could argue in the same way as we do in the proof of Proposition 3.2.2, where we consider parallel forms on the Riemannian cone and use Proposition 3.1.9.

We have $\nabla_X \eta = \phi X^* = X \lrcorner\, \omega$ and $\nabla_X \omega = \eta \wedge X^*$ and obtain

$$\begin{aligned}
\nabla_X(\omega^k) &= k(\nabla_X \omega) \wedge \omega^{k-1}\\
&= -kX^* \wedge \eta \wedge \omega^{k-1},
\end{aligned}$$

$$\nabla_X(\eta \wedge \omega^k) = (X \lrcorner\, \omega) \wedge \omega^k - \eta \wedge \nabla_X(\omega^k)$$

$$= \tfrac{1}{k+1} X \lrcorner \omega^{k+1},$$

thus Proposition 3.1.7 shows that $\eta \wedge \omega^k$ is Killing and ω^k is $*$-Killing. To show that they are special we compute

$$\begin{aligned} d(\eta \wedge \omega^k) &= d\eta \wedge \omega^k \\ &= 2\omega^{k+1}, \end{aligned}$$

$$\begin{aligned} \delta(\omega^k) &= -\sum e_i \lrcorner \nabla_{e_i}(\omega^k) \\ &= k \sum e_i \lrcorner (e_i^* \wedge \eta \wedge \omega^{k-1}) \\ &= 2k(m-k+1)\eta \wedge \omega^{k-1} \end{aligned}$$

and obtain

$$\begin{aligned} \nabla_X(d(\eta \wedge \omega^k)) &= 2\nabla_X(\omega^{k+1}) \\ &= -2(k+1)X^* \wedge \eta \wedge \omega^k, \end{aligned}$$

$$\begin{aligned} \nabla_X(\delta(\omega^k)) &= 2k(m-k+1)\nabla_X(\eta \wedge \omega^{k-1}) \\ &= 2(m-k+1)X \lrcorner \omega^k. \qquad \square \end{aligned}$$

If the manifold is 3-Sasaki we can apply Proposition 3.2.1 thrice and obtain three different types of conformal Killing forms. But in this case it is also possible to combine the different Sasakian structures to create a new kind of conformal Killing forms.

Proposition 3.2.2. *Let $(M, g, \xi_1, \xi_2, \xi_3)$ be a 3-Sasakian manifold. Then for all $a, b, c \in \mathbb{N}$ the form*

$$\begin{aligned} \sigma_{abc} := \eta_1 \wedge (a\omega_1^{a-1}) \wedge \omega_2^b \wedge \omega_3^c \\ + \eta_2 \wedge \omega_1^a \wedge (b\omega_2^{b-1}) \wedge \omega_3^c \\ + \eta_3 \wedge \omega_1^a \wedge \omega_2^b \wedge (c\omega_3^{c-1}) \end{aligned}$$

is a special Killing form.

Proof. According to Propositions 3.1.9 and 3.2.1 the forms

$$\widehat{\omega}_i = d(r^2\eta_i), \qquad i = 1, 2, 3$$

are parallel on the cone, thus trivially also the form

$$\widehat{\omega}_1^a \wedge \widehat{\omega}_2^b \wedge \widehat{\omega}_3^c$$

is parallel on the cone. The relation

$$d(r^{2(a+b+c)-1}\pi^*\sigma_{abc}) = \frac{2(a+b+c)-1}{2^{a+b+c}} \widehat{\omega}_1^a \wedge \widehat{\omega}_2^b \wedge \widehat{\omega}_3^c,$$

which follows from a direct but lengthy computation, yields the claim again by Proposition 3.1.9. $\square$

In his study of conformal Killing forms on Sasakian manifolds in the 1970s, S. Yamaguchi found the following results:

Theorem 3.2.3. *Let (M, g, ξ) be a Sasakian manifold.*

- *Every horizontal conformal Killing form of odd degree is necessarily Killing.*
- *Every horizontal conformal Killing form ψ of even degree $2k$ is uniquely decomposable into*
$$\psi = \sigma + \tau,$$
where σ is a horizontal Killing form and τ is a horizontal $*$-Killing form. In this case holds $\tau = \mathrm{const} \cdot \omega^k$.

Theorem 3.2.4. *Let (M, g, ξ) be a compact $(2m+1)$-dimensional Sasakian manifold with $m > 1$. Then every conformal Killing p-form ψ with $p \leq m$ is uniquely decomposable into*
$$\psi = \sigma + \tau,$$
where σ is a Killing form and τ is a $*$-Killing form. In this case holds $\tau = \frac{1}{p(2m-p+2)} d\delta\psi$. Furthermore $\Lambda^s \delta\psi$ is a special Killing form for all $s = 0, 1, \ldots, \left\lfloor \frac{p-1}{2} \right\rfloor$.

Conformal vector fields on Sasakian manifolds were studied by M. Okumura in [O62]. Translated in the terminology we use here, his result is as follows:

Theorem 3.2.5. *Let (M, g, ξ) be a n-dimensional Sasakian manifold with $n > 3$. Then*
$$\begin{array}{ccccc} \mathcal{CK}^1(M,g) & = & \mathcal{K}^1(M,g) & \oplus & *\mathcal{K}^1(M,g), \\ v^* & = & \left(v^* - \frac{1}{n} d\delta v^*\right) & + & \frac{1}{n} d\delta v^*. \end{array}$$
If additionally (M, g) is connected and complete and admits a non-Killing conformal vector field, then (M, g) is isometric to the standard sphere.

3.3 General case - Reduction to Killing and $*$-Killing forms

In this section we start to investigate the general case of conformal Killing p-forms on arbitrary Sasakian manifolds. Our most important tool is the curvature condition of Proposition 3.1.11. Unfortunately, this condition holds true for all 1- and $(n-1)$-forms and thus no longer allows us to gain any information about conformal Killing forms, which is why we have to treat these cases differently.

For $p = 1$ we give an index-free proof of Okumura's Theorem 3.2.5 where we use Corollary 3.1.19, which states a formula for the Lie derivative of the Riemannian curvature tensor in direction of a conformal vector field. The proof gives a useful property of Killing vector fields needed later in Section 3.4.1. Using the Hodge $*$-operator, we can use Okumura's theorem to cover the case $p = n - 1$.

To study the case $p \neq 1, n-1$ we first reformulate the curvature condition in terms of the Sasakian curvature tensor A introduced in Section 2.2. The properties of this tensor with respect to the Reeb vector field ξ and to the Sasakian endomorphism ϕ combined with the curvature condition imply strong restrictions on the conformal Killing forms. In order to obtain an economic way to prove our results we first discuss the space of forms that satisfy the curvature condition in general, and then turn back to conformal Killing forms.

3.3.1 Conformal vector fields

Proof (of Theorem 3.2.5). Let $v^* \in \mathcal{CK}^1(M,g)$. The main idea of this proof is to combine the formula for the Lie derivative of the curvature tensor R in direction of v with the curvature properties of Sasakian manifolds. Recall that we have

$$(\mathcal{L}_v R)(X,Y) = (\nabla_X(df) \wedge Y^* - \nabla_Y(df) \wedge X^*)\bullet$$

by Corollary 3.1.19. Rewritten in terms of the Sasakian curvature tensor A this reads

$$(\mathcal{L}_v A)(X,Y) = (\nabla_X(df) \wedge Y^* - \nabla_Y(df) \wedge X^* + 2fX^* \wedge Y^*) \bullet .$$

We apply it to ξ and obtain

$$\begin{aligned}(\mathcal{L}_v A)(X,Y)\xi &= (\nabla_X(df) \wedge Y^* - \nabla_Y(df) \wedge X^* + 2fX^* \wedge Y^*) \bullet \xi \\ &= \eta(\nabla_X(\mathrm{grad} f))Y - \eta(Y)\nabla_X(\mathrm{grad} f) - \eta(\nabla_Y(\mathrm{grad} f))X \\ &\quad + \eta(X)\nabla_Y(\mathrm{grad} f) + 2f\eta(X)Y - 2f\eta(Y)X,\end{aligned}$$

which yields

$$\begin{aligned}A(X,Y)\mathcal{L}_v\xi = \eta(Y)\left(\nabla_X(\mathrm{grad} f) + 2fX\right) - \eta(X)\left(\nabla_Y(\mathrm{grad} f) + 2fY\right) \\ + \eta(\nabla_Y(\mathrm{grad} f))X - \eta(\nabla_X(\mathrm{grad} f))Y.\end{aligned} \tag{3.3.1}$$

Using (3.3.1) to calculate $\mathrm{Ric}^A(\mathcal{L}_v\xi)$ in two different ways, the direct approach gives

$$\begin{aligned}\mathrm{ric}^A(\mathcal{L}_v\xi, X) &= \sum g(A(\mathcal{L}_v\xi, e_i)e_i, X) \\ &= \sum g(A(e_i, X)\mathcal{L}_v\xi, e_i) \\ &= \sum \eta(X)g(\nabla_{e_i}(\mathrm{grad} f) + 2fe_i, e_i) - \sum \eta(e_i)g(\nabla_X(\mathrm{grad} f) + 2fX, e_i) \\ &\quad + \sum \eta(\nabla_X(\mathrm{grad} f))g(e_i,e_i) - \sum \eta(\nabla_{e_i}(\mathrm{grad} f))g(X,e_i) \\ &= (n-2)\eta(\nabla_X(\mathrm{grad} f)) - \eta(X)(\Delta f - 2(n-1)f).\end{aligned}$$

This is equivalent to

$$\mathrm{Ric}^A(\mathcal{L}_v\xi) = (n-2)\nabla_\xi(\mathrm{grad} f) - (\Delta f - 2(n-1)f)\xi \tag{3.3.2}$$

since

$$\eta(\nabla_X(\mathrm{grad} f)) = g(\nabla_X(\mathrm{grad} f), \xi) = g(X, \nabla_\xi(\mathrm{grad} f)).$$

We go back to (3.3.1) and contract with $X = e_i, Y = \phi e_i$ to obtain

$$\begin{aligned}-2\phi\mathrm{Ric}^A(\mathcal{L}_v\xi) &= \sum A(e_i, \phi e_i)\mathcal{L}_v\xi \\ &= \sum \eta(\nabla_{\phi e_i}(\mathrm{grad} f))e_i - \sum \eta(\nabla_{e_i}(\mathrm{grad} f))\phi e_i \\ &= -2\phi \sum \eta(\nabla_{e_i}(\mathrm{grad} f))e_i \\ &= -2\phi \sum g(\nabla_\xi(\mathrm{grad} f), e_i)e_i \\ &= -2\phi\nabla_\xi(\mathrm{grad} f),\end{aligned}$$

i.e.

$$\phi\mathrm{Ric}^A(\mathcal{L}_v\xi) = \phi\nabla_\xi(\mathrm{grad} f). \tag{3.3.3}$$

Combining (3.3.2) and (3.3.3) we get

$$\phi\nabla_\xi(\mathrm{grad} f) = 0$$

for $n \neq 3$, which implies

$$\nabla_\xi(\mathrm{grad} f) = \eta(\nabla_\xi(\mathrm{grad} f))\xi$$

and

$$\eta(\nabla_{\phi X}(\mathrm{grad} f)) = g(\nabla_{\phi X}(\mathrm{grad} f), \xi) = g(\nabla_\xi(\mathrm{grad} f), \phi X) = 0.$$

We reformulate this to

$$\eta(\nabla_X(\mathrm{grad} f)) = \eta(X)\eta(\nabla_\xi(\mathrm{grad} f)).$$

In (3.3.1) we set $Y = \xi$ and obtain, since $A(X, \xi)\mathcal{L}_v\xi = 0$,

$$\begin{aligned}\nabla_X(\mathrm{grad} f) = &-2fX + \eta(X)\left(\nabla_\xi(\mathrm{grad} f) + 2f\xi\right)\\ &- \eta(\nabla_\xi(\mathrm{grad} f))X + \eta(\nabla_X(\mathrm{grad} f))\xi\end{aligned}$$

i.e.

$$\nabla_X(\mathrm{grad} f) = f_1 X + f_2\eta(X)\xi, \tag{3.3.4}$$

where we set $f_1 := -2f - \eta(\nabla_\xi(\mathrm{grad} f))$ and $f_2 := 2f + 2\eta(\nabla_\xi(\mathrm{grad} f))$. We differentiate (3.3.4) covariantly in order to calculate $\mathrm{Ric}^A(\mathrm{grad} f)$. A direct calculation leads to

$$\begin{aligned}A(X, Y)\mathrm{grad} f = &\, g(X, \mathrm{grad}(f + f_1))Y - g(Y, \mathrm{grad}(f + f_1))X\\ &+ X(f_2)\eta(Y)\xi - Y(f_2)\eta(X)\xi + 2f_2 g(\phi X, Y)\xi\\ &+ f_2\eta(Y)\phi X - f_2\eta(X)\phi Y,\end{aligned} \tag{3.3.5}$$

which yields

$$\begin{aligned}-2\phi\mathrm{Ric}^A(\mathrm{grad} f) &= \sum A(e_i, \phi e_i)\mathrm{grad} f\\ &= 2\sum g(e_i, \mathrm{grad}(f + f_1))\phi e_i + 2f_2 \sum g(\phi e_i, \phi e_i)\xi\\ &= 2\phi(\mathrm{grad}(f + f_1)) + 2(n-1)f_2\xi.\end{aligned}$$

Applying η to this we get $f_2 = 0$ and $f_1 = -2f - \eta(\nabla_\xi(\mathrm{grad} f)) = -f$. We go back to (3.3.4) and see $\nabla_X(\mathrm{grad} f) = -fX$, or, if we translate it to 1-forms,

$$\nabla_X(df) = -fX^*. \tag{3.3.6}$$

By Proposition 3.1.7 $d\delta v = -ndf$ has to be a $*$-Killing form and $v - \frac{1}{n}d\delta v$ is a Killing form since $\delta(v - \frac{1}{n}d\delta v) = -nf + \Delta f = 0$, where we used (3.3.6).

If (M, g) is complete and connected then by Obata's sphere theorem we either have $f = 0$ or $(M, g) \cong (S^n, g_{\mathrm{st}})$ since $f \in \mathcal{C}^\infty(M)$ satisfies (3.3.6). □

From the proof of Theorem 3.2.5 we obtain useful curvature properties of conformal vector fields on Sasakian manifolds:

Corollary 3.3.1. *Let (M, g) be a n-dimensional Riemannian manifold with $n > 3$ and let $v \in \Gamma(TM)$ be a conformal vector field. If (M, g) admits a Sasakian structure ξ, then* $\mathrm{grad}(\delta v^*) \in V(A)$ *and $\mathcal{L}_\xi v \in V(A)$, where $V(A)$ denotes the curvature nullity of the Sasakian curvature tensor A.*

Proof. This is an immediate consequence of (3.3.1) and (3.3.5) since we have (3.3.6). □

3.3.2 Consequences of the curvature condition for $p \neq 1, n-1$

By substituting the relations between the Sasakian curvature tensor A and the Riemannian curvature tensor R of Lemma 2.2.3 into the curvature condition of Proposition 3.1.11 it turns out that the curvature condition in terms of A has exactly the same shape as before. In this section we discuss the subspace B^p of $\Omega^p(M)$ consisting of forms α that satisfy this curvature condition.

Definition 3.3.2. *Let (M, g, ξ) be a Sasakian manifold and $1 \leq p \leq n-1$. By B^p we denote the subspace of $\Omega^p(M)$ consisting of forms α that satisfy the **curvature condition***

$$\begin{aligned} A(X,Y)\alpha = &\tfrac{1}{p(n-p)} \left(Y^* \wedge X \lrcorner q(A)\alpha - X^* \wedge Y \lrcorner q(A)\alpha\right) \\ &+ \tfrac{1}{p} \left(Y \lrcorner A^+(X)\alpha - X \lrcorner A^+(Y)\alpha\right) \\ &+ \tfrac{1}{n-p} \left(Y^* \wedge A^-(X)\alpha - X^* \wedge A^-(Y)\right) \end{aligned} \tag{3.3.7}$$

for all vector fields X and Y on M. We say $\alpha \in B^p_+$ if

$$A(X,Y)\alpha = \tfrac{1}{p} \left(Y \lrcorner A^+(X)\alpha - X \lrcorner A^+(Y)\alpha\right),$$

and $\alpha \in B^p_-$ means

$$A(X,Y)\alpha = \tfrac{1}{n-p} \left(Y^* \wedge A^-(X)\alpha - X^* \wedge A^-(Y)\alpha\right).$$

Finally B^p_0 is the set consisting of all $\alpha \in \Omega^p(M)$ in the kernel of A, i.e.

$$A(X,Y)\alpha = 0$$

for all vector fields X and Y on M.

Note that we have $B^1 = B^1_+ = \Omega^1(M)$ and $B^{n-1} = B^{n-1}_- = \Omega^{n-1}(M)$ as well as $B^1_0 = V(A)$. The next proposition completely clarifies the relation between B^p and $B^p_\pm$.

Proposition 3.3.3. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold.*

(i) $B^p_\pm \subset B^p$

(ii) $B^p_+ \cap B^p_- = B^p_0$

(iii) If $p \leq m$, then $B_p = B^p_+$; if $p \geq m+1$, then $B_p = B^p_-$.

Proof. If $\alpha \in B^p_+$ then

$$\begin{aligned} A^-(X)\alpha &= \sum e_i \lrcorner A(X, e_i)\alpha \\ &= \tfrac{1}{p} \left(\sum e_i \lrcorner e_i \lrcorner A^+(X) - \sum e_i \lrcorner X \lrcorner A^+(e_i)\alpha\right) \\ &= -\tfrac{1}{p} X \lrcorner q(A)\alpha, \end{aligned}$$

which shows $\alpha \in B^p$, thus $B^p_+ \subset B^p$. Likewise we obtain $B^p_- \subset B^p$.

To show (ii) we assume $\alpha \in B^p_+ \cap B^p_-$, i.e. we have

$$A(X,Y)\alpha = \tfrac{1}{p} Y \lrcorner A^+(X)\alpha - \tfrac{1}{p} X \lrcorner A^+(Y)\alpha, \tag{3.3.8}$$

$$A(X,Y)\alpha = \tfrac{1}{n-p} Y^* \wedge A^-(X)\alpha - \tfrac{1}{n-p} X^* \wedge A^-(Y)\alpha. \tag{3.3.9}$$

In both equations we choose $Y = \xi$ and obtain

$$\begin{aligned} \xi \lrcorner A^+(X)\alpha &= 0, \\ \eta \wedge A^-(X)\alpha &= 0. \end{aligned}$$

We take the interior product of (3.3.8) with ξ and the exterior product of (3.3.9) with η and get

$$\begin{aligned} \xi \lrcorner A(X,Y)\alpha &= 0, \\ \eta \wedge A(X,Y)\alpha &= 0, \end{aligned}$$

thus $A(X,Y)\alpha = 0$.

Next we show (iii), which is a lengthy discussion of the curvature condition. The reason that $B^p = B^p_+$ only is guaranteed for $p \leq m$ is the injectivity of $L : \Omega^k(M) \longrightarrow \Omega^{k+2}$ for $k \leq m-1$ by Proposition 1.2.4. Let $\alpha \in B^p, 2 \leq p \leq m$, be any form that satisfies the curvature condition (3.3.7). First we choose $X = e_i$ and $Y = \phi e_i$ and sum up; with Proposition 2.2.5 this yields

$$\begin{aligned} 0 = \quad & -\sum A(e_i, \phi e_i)\alpha \\ & + \tfrac{1}{p(n-p)} \Big(\sum \phi e_i^* \wedge e_i \lrcorner q(A) - \sum e_i^* \wedge \phi e_i \lrcorner q(A)\alpha \Big) \\ & + \tfrac{1}{p} \Big(\sum \phi e_i^* \lrcorner A^+(e_i)\alpha - \sum e_i \lrcorner A^+(\phi e_i)\alpha \Big) \\ & + \tfrac{1}{n-p} \Big(\sum \phi e_i^* \wedge A^-(e_i)\alpha - \sum e_i^* \wedge A^-(\phi e_i)\alpha \Big) \\ = 2\rho^A \bullet \alpha & + \tfrac{2}{p(n-p)} \phi_{\mathrm{D}} q(A)\alpha - \tfrac{2}{p}\rho^A \bullet \alpha - \tfrac{2}{n-p}\rho^A \bullet \alpha, \end{aligned}$$

i.e.

$$\phi_{\mathrm{D}} q(A)\alpha = (n - p\,(n-p))\rho^A \bullet \alpha. \tag{3.3.10}$$

We keep that in mind and return to the curvature condition (3.3.7). With $Y = \xi$ the horizontal and vertical part of the resulting equation are given by

$$\eta \wedge \Big(A^-(X)\alpha + \tfrac{1}{p} X \lrcorner q(A)\alpha \Big) = \tfrac{1}{p}\eta(X)\eta \wedge \mathrm{v}(q(A)\alpha), \tag{3.3.11}$$

$$\xi \lrcorner \Big(A^+(X)\alpha + \tfrac{1}{n-p} X^* \wedge q(A)\alpha \Big) = \tfrac{1}{n-p}\eta(X)\mathrm{h}(q(A)\alpha). \tag{3.3.12}$$

We replace X by e_i, apply $\sum \phi e_i^* \wedge$ and $\sum \phi e_i \lrcorner$ to both equations and get

$$L(\xi \lrcorner q(A)\alpha) = 0, \tag{3.3.13}$$

$$\Lambda(\eta \wedge q(A)\alpha) = 0, \tag{3.3.14}$$

$$(p-1)\eta \wedge \rho^A \bullet \alpha = 0, \tag{3.3.15}$$

$$(n-p-1)\xi \lrcorner \rho^A \bullet \alpha = 0, \tag{3.3.16}$$

where we used (3.3.10). Since we assume $p \neq 1, n-1$, combining (3.3.10), (3.3.15) and (3.3.16) yields

$$\rho^A \bullet \alpha = 0, \tag{3.3.17}$$

$$\phi_{\mathrm{D}} q(A)\alpha = 0. \tag{3.3.18}$$

As $\deg(\xi \lrcorner q(A)\alpha) = p - 1 \leq m - 1$, equation (3.3.13) implies

$$\xi \lrcorner q(A)\alpha = 0. \tag{3.3.19}$$

With (3.3.19) we may rewrite (3.3.11) and (3.3.12) as

$$\eta \wedge \left(A^-(X)\alpha + \tfrac{1}{p}X \lrcorner q(A)\alpha\right) = 0, \tag{3.3.20}$$

$$\xi \lrcorner A^+(X)\alpha = 0. \tag{3.3.21}$$

Again we return to the curvature condition (3.3.7). This time we replace Y by e_i and apply $\sum \phi e_i^* \wedge$ to it:

$$\begin{aligned}
0 = \quad & -\sum \phi e_i^* \wedge A(X, e_i)\alpha \\
& + \tfrac{1}{p(n-p)} \left(\sum \phi e_i^* \wedge e_i^* \wedge X \lrcorner q(A) - \sum \phi e_i^* \wedge X^* \wedge e_i \lrcorner q(A)\alpha\right) \\
& + \tfrac{1}{p} \left(\sum \phi e_i^* \wedge e_i \lrcorner A^+(X)\alpha - \sum \phi e_i^* \wedge X \lrcorner A^+(e_i)\alpha\right) \\
& + \tfrac{1}{n-p} \left(\sum \phi e_i^* \wedge e_i^* \wedge A^-(X)\alpha - \sum \phi e_i^* \wedge X^* \wedge A^-(e_i)\alpha\right) \\
\\
= & -\tfrac{2}{n-p} L \left(A^-(X)\alpha + \tfrac{1}{p} X \lrcorner q(A)\alpha\right) \\
& + \tfrac{1}{p} \left(\phi_{\mathrm{D}} A^+(X)\alpha - (p-1)A^+(\phi X)\alpha\right),
\end{aligned}$$

i.e.

$$L \left(A^-(X)\alpha + \tfrac{1}{p} X \lrcorner q(A)\alpha\right) = \tfrac{n-p}{2p} \left(\phi_{\mathrm{D}} A^+(X)\alpha - (p-1)A^+(\phi X)\alpha\right),$$

where we used (3.3.17) and (3.3.18). We take the interior product with ξ and obtain

$$L \left(\xi \lrcorner \left(A^-(X)\alpha + \tfrac{1}{p} X \lrcorner q(A)\alpha\right)\right) = 0,$$

where we used (3.3.21). Because of

$$\deg \left(\xi \lrcorner \left(A^-(X)\alpha + \tfrac{1}{p} X \lrcorner q(A)\alpha\right)\right) = p - 2 < m - 1$$

we may use the injectivity of L again and get

$$\xi \lrcorner \left(A^-(X)\alpha + \tfrac{1}{p} X \lrcorner q(A)\alpha\right) = 0,$$

which together with (3.3.20) implies

$$A^-(X)\alpha + \tfrac{1}{p} X \lrcorner q(A)\alpha = 0.$$

Thus we have $\alpha \in B^p_+$.

The remaining statement of (iii) is proven in the same way, where we use (3.3.14) and the injectivity of $\Lambda : \Omega^k(M) \longrightarrow \Omega^{k-2}(M)$ for $k \geq m + 2$. $\square$

With Proposition 3.3.3 we can reduce our discussion of $\alpha \in B^p$ to $\alpha \in B^p_\pm$. The next two propositions are about these spaces.

Proposition 3.3.4. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and $\alpha \in B^p_+$.*

- *$A(X,Y)\alpha$ is horizontal and primitive.*
- *$\phi_{\mathrm{D}}^2 A(X,Y)\alpha = -(p-2)^2 A(X,Y)\alpha$ and $\phi_{\mathrm{D}} A^+(X)\alpha = (p-1)A^+(\phi X)\alpha$.*
- *If $p \geq 2$, then $\rho^A \bullet \alpha = \phi_{\mathrm{D}} q(A)\alpha = 0$.*
- *If $p \geq m+1$, then $A(X,Y)\alpha = 0$.*

Proof. Recall that $\alpha \in B^p_+$ means

$$A(X,Y)\alpha = \tfrac{1}{p}\left(Y \lrcorner A^+(X)\alpha - X \lrcorner A^+(Y)\alpha\right). \tag{3.3.22}$$

We manipulate this equation in the same way as we did it with the full curvature condition.

Starting with $Y = \xi$ in equation (3.3.22) yields $\xi \lrcorner A^+(X)\alpha = 0$. Taking the inner product of (3.3.22) with ξ shows that $A(X,Y)\alpha$ is horizontal. This automatically implies that $\rho^A \bullet \alpha$ and $\phi_{\mathrm{D}} q(A)\alpha$ are horizontal, too. By equation (3.3.15) we know that they are also vertical for $p \neq 1$.

Going back to (3.3.22), we replace Y by e_i and build the contraction $\sum \phi e_i^* \wedge$:

$$\phi_{\mathrm{D}} A^+(X)\alpha = (p-1)A^+(\phi X)\alpha. \tag{3.3.23}$$

We apply ϕ_{D} to this and get

$$\begin{aligned} \phi_{\mathrm{D}}^2 A^+(X)\alpha &= (p-1)\phi_{\mathrm{D}} A^+(\phi X)\alpha \\ &= (p-1)^2 A^+(\phi^2 X)\alpha \\ &= -(p-1)^2 A^+(X)\alpha. \end{aligned}$$

Applying ϕ_{D} twice to (3.3.22) and using (3.3.23) we obtain, since $A^+(X)\alpha$ is horizontal,

$$\begin{aligned} \phi_{\mathrm{D}}^2 A(X,Y)\alpha = \tfrac{1}{p}\big(&\phi^2 Y \lrcorner A^+(X)\alpha + 2\phi Y \lrcorner \phi_{\mathrm{D}} A^+(X)\alpha + Y \lrcorner \phi_{\mathrm{D}}^2 A^+(X)\alpha \\ &- \phi^2 X \lrcorner A^+(Y)\alpha - 2\phi X \lrcorner \phi_{\mathrm{D}} A^+(Y)\alpha - X \lrcorner \phi_{\mathrm{D}}^2 A^+(Y)\alpha\big) \\ = \tfrac{1}{p}\big(&-Y \lrcorner A^+(X)\alpha + 2(p-1)\phi Y \lrcorner A^+(\phi X)\alpha - (p-1)^2 Y \lrcorner A^+(X)\alpha \\ &+ X \lrcorner A^+(Y)\alpha - 2(p-1)\phi X \lrcorner A^+(\phi Y)\alpha + (p-1)^2 X \lrcorner A^+(Y)\alpha\big) \\ = -(&(p-1)^2+1)A(X,Y)\alpha + 2(p-1)A(\phi X, \phi Y)\alpha \\ = -(&p-2)^2 A(X,Y)\alpha. \end{aligned}$$

It remains to show $A(X,Y)\alpha = 0$ for $p \geq m+1$. But by Proposition 3.3.3 we have

$$B^p_+ \subset B^p = B^p_-$$

in this case, thus

$$B^p_+ = B^p_+ \cap B^p_- = B^p_0. \qquad \square$$

Of course there is an analogous result for B^p_-.

Proposition 3.3.5. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and $\alpha \in B^p_-$.*

- *$A(X,Y)\alpha$ is vertical and coprimitive.*
- *$\phi_{\mathrm{D}}^2 A(X,Y)\alpha = -(n-p-2)^2 A(X,Y)\alpha$ and $\phi_{\mathrm{D}} A^-(X)\alpha = (n-p-1)A^-(\phi X)\alpha$.*
- *If $p \leq n-2$, then $\rho^A \bullet \alpha = \phi_{\mathrm{D}} q(A)\alpha = 0$.*
- *If $p \leq m$, then $A(X,Y)\alpha = 0$.*

3.3.3 Application to conformal Killing forms

We apply the results of the previous section to conformal Killing forms and have immediate consequences. First we reformulate some results of Section 3.1 in terms of the sets B^p and $B^p_\pm$ and then apply the propositions of Section 3.3.2.

By Proposition 3.1.11 we have $\psi \in B^p$ for all $\psi \in \mathcal{CK}^p(M,g)$, and Corollary 3.1.14 can be reformulated as

$$\begin{aligned} \delta\psi \in \mathcal{K}^{p-1}(M,g) &\Longleftrightarrow \psi \in B^p_+, \\ d\psi \in *\mathcal{K}^{p+1}(M,g) &\Longleftrightarrow \psi \in B^p_-. \end{aligned}$$

Corollary 3.3.6. *Let (M, g, ξ) be a $n = (2m+1)$-dimensional Sasakian manifold and $\psi \in \mathcal{CK}^p(M,g)$.*

- *If $p \leq m$ or if $\delta\psi$ is a Killing form, we have*
$$A(X,Y)\psi = \tfrac{1}{p}\left(Y \lrcorner A^+(X)\psi - X \lrcorner A^+(Y)\psi\right).$$
- *If $p \geq m+1$ or if $d\psi$ is a $*$-Killing form, we have*
$$A(X,Y)\psi = \tfrac{1}{n-p}\left(Y^* \wedge A^-(X)\psi - X^* \wedge A^-(Y)\psi\right).$$

Let $p \leq m$. Then Proposition 3.3.3 and Corollary 3.1.14 imply that $\delta\psi$ is again a Killing $(p-1)$-form. This allows us to apply all results concerning conformal Killing forms not only on ψ but also on $\delta\psi$ as well. Thus $A(X,Y)(\delta\psi)$ is horizontal. We covariantly differentiate the identity $\xi \lrcorner A(X,Y)\psi = 0$ in direction of ξ, use the conformal Killing equation and obtain

$$\begin{aligned} 0 &= \nabla_\xi(\xi \lrcorner A(X,Y)\psi) \\ &= A(X,Y)(\xi \lrcorner \nabla_\xi\psi) \\ &= \tfrac{1}{n-p+1} A(X,Y)\left(\xi \lrcorner (\eta \wedge \delta\psi)\right) \\ &= \tfrac{1}{n-p+1} \xi \lrcorner (\eta \wedge A(X,Y)(\delta\psi)) \\ &= \tfrac{1}{n-p+1} A(X,Y)(\delta\psi). \end{aligned}$$

In particular we have $A^+(X)(\delta\psi) = 0$, i.e. $R^+(X)(\delta\psi) = -(p-1)X^* \wedge \delta\psi$. With Corollary 3.1.15 we get

$$\nabla_X(d\delta\psi) = \tfrac{p}{p-1} R^+(X)\psi = -p\, X^* \wedge \delta\psi$$

and see that $\delta\psi$ is a special Killing form. Thus we have proven the following theorem.

Theorem 3.3.7. *Let (M,g) be a $(2m+1)$-dimensional Riemannian manifold with $m>1$ admitting a Sasakian structure and let $\psi \in \mathcal{CK}^p(M,g)$. If $p \le m$, then $\delta\psi$ is a special Killing form, and if $p \ge m+1$, then $d\psi$ is a special $*$-Killing form.*

Proof. The remaining cases $p=1$ and $p=n-1$ are already covered in Section 3.3.1. □

We formulate another lemma which has several applications on Sasakian manifolds before we state the next main result.

Lemma 3.3.8. *Let (M,g) be a Riemannian manifold and $\psi \in \mathcal{CK}^p(M,g)$.*

- *If $\delta\psi$ is a special Killing form with constant c, then $\psi + \frac{1}{c(n-p+1)} d\delta\psi$ is a Killing form.*
- *If $d\psi$ is a special $*$-Killing form with constant c, then $\psi - \frac{1}{c(p+1)}\delta d\psi$ is a $*$-Killing form.*

In both cases ψ is the sum of a Killing and a $*$-Killing form.

Proof. We only prove the first statement, the other one follows in the same way or by combining (3.3.8) with the Hodge $*$-operator. Assume that $\delta\psi$ satisfies

$$\nabla_X(d\delta\psi) = c\, X^* \wedge \delta\psi.$$

Then $d\delta\psi$ is a $*$-Killing form by Corollary 3.1.8 and $\psi + \frac{1}{c(n-p+1)} d\delta\psi$ is a Killing form by Proposition 3.1.7:

$$\begin{aligned} X \lrcorner \nabla_X \left(\psi + \tfrac{1}{c(n-p+1)} d\delta\psi\right) &= X \lrcorner \nabla_X\psi + \tfrac{1}{c(n-p+1)} X \lrcorner \nabla_X(d\delta\psi) \\ &= -\tfrac{1}{n-p+1} X \lrcorner (X^* \wedge \delta\psi) + \tfrac{c}{c(n-p+1)} X \lrcorner (X^* \wedge \delta\psi) \\ &= 0. \qquad \square \end{aligned}$$

Theorem 3.3.9. *Let (M,g) be a n-dimensional Riemannian manifold with $n>3$. If (M,g) admits a Sasakian structure then for all $p=1,\dots,n-1$ we have*

$$\begin{array}{ccccc} \mathcal{CK}^p(M,g) & = & \mathcal{K}^p(M,g) & \oplus & *\mathcal{K}^p(M,g), \\ \psi & = & \sigma & + & \tau. \end{array}$$

If $p \ge 2$, then $\sigma = \frac{1}{(p+1)(n-p)}\delta d\psi$, and if $p \le n-2$, then $\tau = \frac{1}{p(n-p+1)} d\delta\psi$.

Proof. The cases $p=1$ and $p=n-1$ are already contained in Theorem 3.2.5, thus we assume $2 \le p \le n-2$. If (M,g) admits a Sasakian structure, then n has to be odd, i.e. $n=2m+1$. It is sufficient to consider the case $p \le m$, otherwise use the Hodge $*$-operator. Let $\psi \in \mathcal{CK}^p(M,g)$. If $2 \le p \le m$ we know from Theorem 3.3.7 that $\delta\psi$ is a special Killing form with constant $-p$, thus by Lemma 3.3.8 ψ is the sum of a Killing and a $*$-Killing form, where $\tau := \frac{1}{p\,(n-p+1)} d\delta\psi$ is the $*$-Killing part while the Killing part is given by $\sigma := \psi - \tau = \psi - \frac{1}{p\,(n-p+1)} d\delta\psi$. Since only parallel forms can be both Killing and $*$-Killing and Sasakian manifolds do not admit any non-vanishing parallel forms by Corollary 2.1.7 we obtain

$$\mathcal{CK}^p(M,g) = \mathcal{K}^p(M,g) \oplus *\mathcal{K}^p(M,g).$$

To show the remaining claim $\sigma = \frac{1}{(p+1)(n-p)}\delta d\psi$ recall that we have

$$A(X,Y)d\sigma = \tfrac{1}{p+1}\left(Y \lrcorner A^+(X)d\sigma - X \lrcorner A^+(Y)d\sigma\right) - \sum e_i^* \wedge (\nabla_{e_i}A)(X,Y)\sigma.$$

for $p \neq 1$ by Proposition 3.1.17. We set $Y = \xi$ and obtain, using the Propositions 2.2.6 and 2.3.1,

$$\begin{aligned} 0 &= \tfrac{1}{p+1}\xi \lrcorner A^+(X)d\sigma - \sum e_i^* \wedge (\nabla_{e_i}A)(X,\xi)\sigma \\ &= -\tfrac{1}{p+1}A^+(X)(\xi \lrcorner d\sigma) - \sum e_i^* \wedge A(\phi X, e_i)\sigma \\ &= -A^+(X)\nabla_\xi\sigma - A^+(\phi X)\sigma, \end{aligned}$$

i.e.

$$A^+(\phi X)\sigma = -A^+(X)\nabla_\xi\sigma. \tag{3.3.24}$$

We replace X by e_i and compute

$$\begin{aligned} q(A)\sigma &= -\sum \phi e_i \lrcorner A^+(\phi e_i)\sigma \\ &= \sum \phi e_i \lrcorner A^+(e_i)\nabla_\xi\sigma \\ &= -\rho^A \bullet \nabla_\xi\sigma \\ &= -\nabla_\xi(\rho^A \bullet \sigma) + (\nabla_\xi\rho^A) \bullet \sigma \\ &= 0, \end{aligned}$$

where we used $\nabla_\xi\rho^A = 0$ by Corollary 2.3.2 and $\rho^A \bullet \sigma = 0$ for $p \geq 2$ by Proposition 3.3.4. From Proposition 3.1.6 we finally get

$$\psi = \tfrac{1}{p\,(n-p)}(q(R) - q(A))\psi = \tfrac{1}{p\,(n-p)}q(R)\psi = \tfrac{1}{(p+1)(n-p)}\delta d\psi + \tfrac{1}{p(n-p+1)}d\delta\psi. \qquad \square$$

From the previous proof we obtain another important property of conformal Killing forms on Sasakian manifolds.

Corollary 3.3.10. *Let (M,g) be a n-dimensional Riemannian manifold and $\psi \in \mathcal{CK}^p(M,g)$ with $2 \leq p \leq n-2$. If (M,g) admits a Sasakian structure, then*

$$q(R)\psi = p\,(n-p)\psi.$$

3.4 Killing and special Killing forms

Due to Proposition 3.1.5 and Theorem 3.3.9 we have the decomposition

$$\mathcal{CK}^p(M,g) = \mathcal{K}^p(M,g) \oplus *\mathcal{K}^p(M,g) = \mathcal{K}^p(M,g) \oplus *(\mathcal{K}^{n-p}(M,g)),$$

thus in order to understand the space $\mathcal{CK}^*(M,g)$ it completely suffices to investigate the space $\mathcal{K}^*(M,g)$. Since special Killing forms are completely classified on compact simply-connected Sasakian manifolds and all known examples of Killing forms on Sasakian manifolds are special, one main aspect of this section is to give criterions for a Killing form being special. As we will see, this problem is closely related to the question if for a given Killing p-form σ the $(p+1)$-form $d\sigma$ is a $*$-Killing form. For $p \geq m+1$ we already know this by Theorem 3.3.7, thus in most situations we will focus on $p \leq m$.

3.4.1 Killing vector fields

The question whether for a given Killing 1-form K^* the 2-form dK^* is a conformal Killing form is completely answered for compact manifolds in [M07]. Here we will point out a few interesting facts about Killing 1-forms on Sasakian manifolds before we turn to forms of higher degree.

Lemma 3.4.1. *Let (M, g, ξ) be a Sasakian manifold and $K^* \in \mathcal{K}^1(M, g)$. Then $\mathcal{L}_\xi K^*$ is a horizontal special Killing form and also an eigenform of $\mathcal{L}_\xi^2$ with eigenvalue -4:*

$$\mathcal{L}_\xi^3 K^* = -4\mathcal{L}_\xi K^*.$$

Proof. By Proposition 3.1.4 it is clear that $\mathcal{L}_\xi K^*$ is a Killing form. From Corollary 3.3.1 we know $A(X, Y)\mathcal{L}_\xi K^* = 0$, in particular $A^+(X)\mathcal{L}_\xi K^* = 0$. From Proposition 3.1.13 we thus obtain

$$\begin{aligned}\nabla_X(d\mathcal{L}_\xi K^*) &= 2R^+(X)\mathcal{L}_\xi K^* \\ &= 2A^+(X)\mathcal{L}_\xi K^* - 2X^* \wedge \mathcal{L}_\xi K^* \\ &= -2X^* \wedge \mathcal{L}_\xi K^*,\end{aligned}$$

i.e. $\mathcal{L}_\xi K^*$ is a special Killing form. Taking the interior product of $\mathcal{L}_\xi K^*$ with ξ yields

$$\xi \lrcorner \mathcal{L}_\xi K^* = \xi \lrcorner \nabla_\xi K^* - \xi \lrcorner \phi K^* = 0.$$

Since $\mathcal{L}_\xi K^*$ satisfies the Killing equation we have

$$\nabla_\xi \mathcal{L}_\xi K^* = \frac{1}{2}\xi \lrcorner d\mathcal{L}_\xi K^* = \frac{1}{2}\mathcal{L}_\xi \mathcal{L}_\xi K^* = \frac{1}{2}\nabla_\xi \mathcal{L}_\xi^* - \frac{1}{2}\phi\mathcal{L}_\xi K^*,$$

thus $\nabla_\xi \mathcal{L}_\xi^* = -\phi\mathcal{L}_\xi^*$. We obtain

$$\begin{aligned}\mathcal{L}_\xi^3 K^* &= \mathcal{L}_\xi^2 (\mathcal{L}_\xi K^*) \\ &= \nabla_\xi^2 (\mathcal{L}_\xi K^*) - 2\phi\nabla_\xi (\mathcal{L}_\xi K^*) + \phi^2 (\mathcal{L}_\xi K^*) \\ &= 4\phi^2 \mathcal{L}_\xi K^* \\ &= -4\mathcal{L}_\xi K^*. \qquad \square\end{aligned}$$

Proposition 3.4.2. *Let (M, g, ξ) be a Sasakian manifold and $K^* \in \mathcal{K}^1(M, g)$. Then K^* splits as*

$$K^* = K_S^* + K_0^*,$$

where $K_S^* := \frac{1}{4}\mathcal{L}_\xi^2 K$ is a horizontal special Killing 1-form with $\mathcal{L}_\xi^2 K_S^* = -4K_S^*$ and K_0^* is a Killing 1-form with $\mathcal{L}_\xi K_0^* = 0$.

Proof. By Lemma 3.4.1 we have

$$\mathcal{L}_\xi^2 K_S^* = \tfrac{1}{4}\mathcal{L}_\xi^4 K^* = \tfrac{1}{4}\mathcal{L}_\xi(\mathcal{L}_\xi^3 K^*) = -\mathcal{L}_\xi^2 K^* = -4K_S^*$$

and

$$\mathcal{L}_\xi^2 K_0^* = \mathcal{L}_\xi^2 K^* - \mathcal{L}_\xi^2 K_S^* = \mathcal{L}_\xi^2 K^* + 4K_S^* = 0.$$

We apply $\mathcal{L}_\xi$ to $\mathcal{L}_\xi^2 K_0^* = 0$ and obtain $0 = \mathcal{L}_\xi^3 K_0^* = -4\mathcal{L}_\xi K_0^*$. By Lemma 3.4.1 we know that K_S^* is a horizontal special Killing form, therefore $K_0^* = K^* - K_S^*$ is a Killing form. $\square$

To close this section we remark that the horizontal part of a Killing vector field determines the vertical part up to a constant: We split

$$K^* = \mathrm{h}(K)^* + \mathrm{v}(K)\eta$$

with $\mathrm{v}(K) = \eta(K) = \xi \lrcorner K^*$. We have $\xi(\mathrm{v}(K)) = \xi \lrcorner \nabla_\xi K^* = 0$, thus $\nabla_\xi K^* = \nabla_\xi \mathrm{h}(K)^*$. We compute $d(\mathrm{v}(K))$:

$$\begin{aligned} d(\mathrm{v}(K)) &= d(\xi \lrcorner K^*) \\ &= \mathcal{L}_\xi K^* - \xi \lrcorner dK^* \\ &= \nabla_\xi K^* - \phi K^* - 2\nabla_\xi K^* \\ &= -\nabla_\xi (\mathrm{h}(K))^* - \phi(\mathrm{h}(K))^* \end{aligned}$$

and we see that $\mathrm{h}(K)$ determines $\mathrm{v}(K)$ up to a constant.

3.4.2 Killing p-forms with $p \neq 1$

We already showed that for every conformal Killing p-form ψ with $p \leq m$ the $(p-1)$-form $\delta\psi$ is again a conformal Killing form. It is natural to ask if the same is true for the $(p+1)$-form $d\psi$. Since we have the decomposition $\psi = \sigma + \tau$ into a Killing form σ and a $*$-Killing form τ, we get $d\psi = d\sigma$ since τ is closed. Thus the above problem reduces to the question if for every Killing p-form σ with $p \leq m$ the form $d\sigma$ is again a conformal Killing $(p+1)$-form. We show that for $p \geq 2$ this is equivalent to σ being a special Killing form since in this case every Killing form on a Sasakian manifold is an eigenform of the Laplace operator of (M,g). We use the differentiated curvature condition of Proposition 3.1.17 to show that every Killing m-form has to be special on a compact Sasakian manifold, thus we may further restrict our discussion to the case $p \leq m-1$. The question whether σ is a special Killing form is equivalent to $A^+(X)\sigma = 0$, which allows us to show that on a compact manifold every Killing form is the sum of a special Killing form and an eigenform of the Lie derivative.

Corollary 3.4.3. *Let (M,g) be a n-dimensional Riemannian manifold with Laplace operator Δ. If (M,g) admits a Sasakian structure, then every Killing p-form with $p \geq 2$ is an eigenform of Δ with eigenvalue $(p+1)(n-p)$, and if $p \leq n-2$, then every $*$-Killing p-form is an eigenform of Δ with eigenvalue $p\,(n-p+1)$.*

Proof. This is an immediate consequence of Theorem 3.3.9. □

Analogue to Corollary 3.3.10 we have the following consequence of Proposition 3.1.6 and the Corollaries 3.1.14 and 3.4.3

Corollary 3.4.4. *Let (M,g) be Riemannian manifold. If (M,g) admits a Sasakian structure, then for every Killing form $\sigma \in \mathcal{K}^p(M,g)$ with $p \geq 2$ we have $R^-(X)\sigma = -(n-p)X \lrcorner \sigma$ and $q(R)\sigma = p\,(n-p)\sigma$. Likewise for every $*$-Killing form $\tau \in *\mathcal{K}^p(M,g)$ with $p \leq n-2$ we have $R^+(X)\tau = -p\,X^* \wedge \tau$ and $q(R)\tau = p\,(n-p)\tau$.*

Proof. For $p \geq 2$ we compute

$$q(R)\sigma = \tfrac{p}{p+1}\delta d\sigma = \tfrac{p}{p+1}\Delta\sigma = p\,(n-p)\sigma,$$

which yields

$$R^-(X)\sigma = -\tfrac{1}{p}X \lrcorner\, q(R)\sigma = -(n-p)X \lrcorner\, \sigma.$$

The results for τ are proven in the same way. $\square$

Lemma 3.4.5. *Let (M, g) be a n-dimensional Riemannian manifold.*

- *Let $\sigma \in \mathcal{K}^p(M, g) \setminus \{0\}$ be special with $\nabla_X(d\sigma) = c\, X^* \wedge \sigma$. If (M, g) admits a Sasakian structure, then $c = -(p+1)$.*
- *Let $\tau \in *\mathcal{K}^p(M, g) \setminus \{0\}$ be special with $\nabla_X(\delta\tau) = c\, X \lrcorner\, \tau$. If (M, g) admits a Sasakian structure, then $c = n - p + 1$.*

Proof. Let ξ be a Sasakian structure on (M, g). From the special Killing equation we get

$$-c(n-p)\sigma = \Delta\sigma = (p+1)(n-p)\sigma$$

for $p \geq 2$ by Corollary 3.4.3. If $p = 1$ we use Corollary 3.1.15 and obtain

$$0 = \nabla_\xi(d\sigma) - c\,\eta \wedge \sigma = 2R^+(\xi)\sigma - c\eta \wedge \sigma = -(2+c)\eta \wedge \sigma,$$

which yields $c = -2 = -(p+1)$ or $\eta \wedge \sigma = 0$. If $\eta \wedge \sigma = 0$ we have $\sigma = f\eta$ and obtain

$$-c(n-1)\sigma = \Delta\sigma = 2q(R)\sigma = 2f\,q(R)\eta = 2(n-1)f\eta = 2(n-1)\sigma,$$

thus $c = -2$ or $\sigma = 0$, which was excluded.

The statement about special $*$-Killing forms is proven in the same way. $\square$

We give criterions whether $d\sigma$ is a $*$-Killing form or not.

Proposition 3.4.6. *Let (M, g, ξ) be a Sasakian manifold and $\sigma \in \mathcal{K}^p(M, g)$ with $p \neq 0$. Then the following properties are equivalent:*

(i) σ is a special Killing form.

(ii) The form $d(r^{p+1}\sigma)$ is parallel on the Riemannian cone $C(M)$.

(iii) $A(X, Y)\sigma = 0$ for all vector fields X and Y on M.

(iv) $A^+(X)\sigma = 0$ for all vector fields X on M.

These conditions are sufficient for $d\sigma \in *\mathcal{K}^{p+1}(M, g)$ and also necessary if $p \geq 2$. If furthermore M is a compact manifold, the above conditions are equivalent to

$$q(A)d\sigma = 0.$$

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is given by Proposition 3.1.9, and from

$$\nabla_X(d\sigma) + (p+1)X^* \wedge \sigma = \tfrac{p+1}{p}A^+(X)\sigma$$

we obtain (i) $\Longleftrightarrow$ (iii) with Lemma 3.4.5. The simplified curvature condition of Corollary 3.3.6 yields (iv) $\Longrightarrow$ (iii), and (iii) $\Longrightarrow$ (iv) is trivial.

Corollary 3.1.8 shows that σ being special implies $d\sigma \in *\mathcal{K}^{p+1}(M,g)$. If $p \geq 2$, then $\delta d\sigma = \Delta\sigma = (p+1)(n-p)\sigma$ by Corollary 3.4.3, thus

$$\nabla_X(d\sigma) = -\tfrac{1}{n-p}X^* \wedge \delta d\sigma = -(p+1)X^* \wedge \sigma.$$

Finally we assume that M is compact. Since $\Delta\sigma = (p+1)(n-p)\sigma$, we have

$$q(A)d\sigma = q(R)d\sigma - (p+1)(n-p-1)d\sigma = q(R)d\sigma - \tfrac{n-p-1}{n-p}d\delta d\sigma.$$

Thus the criterion of Proposition 3.1.6 yields that $d\sigma$ is $*$-Killing if and only if $q(A)d\sigma = 0$. $\square$

Before we further investigate the properties of the Killing form σ on M we switch to the cone and consider the form $\widehat{\sigma} := d(r^{p+1}r^{p+1}\left(\nabla_X(d\sigma) + (p+1)X^* \wedge \sigma\right)\sigma)$ on the cone. We show that it is closed with respect to all four Dolbeault operators $\partial_C, \bar{\partial}_C, \partial_C^*, \bar{\partial}_C^*$ on the cone.

Proposition 3.4.7. *Let (M,g,ξ) be a Sasakian manifold and $\sigma \in \mathcal{K}^p(M,g)$. Define $\widehat{\sigma} := d(r^{p+1}\sigma)$. If $p \geq 2$, then we have*

$$\partial_C\widehat{\sigma} = 0, \qquad \partial_C^*\widehat{\sigma} = 0,$$
$$\bar{\partial}_C\widehat{\sigma} = 0, \qquad \bar{\partial}_C^*\widehat{\sigma} = 0.$$

Proof. From the proof of Proposition 3.1.9 we have

$$\begin{aligned}
\nabla_X\widehat{\sigma} &= r^{p+1}\left(\nabla_X(d\sigma) + (p+1)X^* \wedge \sigma\right) \\
&\quad + (p+1)r^p dr \wedge \left(\nabla_X\sigma - \tfrac{1}{p+1}X \lrcorner\, d\sigma\right) \\
&= r^{p+1}\left(\nabla_X(d\sigma) + (p+1)X^* \wedge \sigma\right) \\
&= \tfrac{p+1}{p}r^{p+1}A^+(X)\sigma
\end{aligned}$$

The complex structure of the cone is given by $JX = \phi X - r\eta(X)\partial_r$ and $J\partial_r = \frac{1}{r}\xi$. If $\{e_i\}$ is a local orthonormal frame of M, then $\{\widehat{e}_i = \frac{1}{r}e_i, \partial_r\}$ is a local orthonormal frame of $C(M)$, and if $X^{\widehat{*}}$ denotes the g_C-dual of X, then $X^{\widehat{*}} = r^2X^*$. We obtain

$$\begin{aligned}
d_C^c\widehat{\sigma} &= \sum J\widehat{e}_i^{\widehat{*}} \wedge \nabla_{\widehat{e}_i}^C\widehat{\sigma} + J(dr) \wedge \nabla_{\partial_r}^C\widehat{\sigma} \\
&= \tfrac{1}{r^2}\sum (Je_i)^{\widehat{*}} \wedge \nabla_{e_i}^C\widehat{\sigma} \\
&= \tfrac{p+1}{p}r^{p-1}\sum (\phi e_i - r\eta(e_i)\partial_r)^{\widehat{*}} \wedge A^+(e_i)\sigma \\
&= \tfrac{p+1}{p}r^{p+1}\sum \phi e_i^* \wedge A^+(e_i)\sigma - \tfrac{p+1}{p}r^p dr \wedge A^+(\xi)\sigma \\
&= 0.
\end{aligned}$$

In the same manner we get

$$\begin{aligned}
\delta_C\widehat{\sigma} &= -\tfrac{p+1}{p}r^{p-1}\sum e_i \lrcorner\, A^+(e_i)\sigma \\
&= \tfrac{p+1}{p}r^{p-1}q(A)\sigma \\
&= 0,
\end{aligned}$$

$$\begin{aligned}\delta^c_C\widehat{\sigma} &= -\tfrac{p+1}{p}r^{p-1}\sum \phi e_i \lrcorner A^+(e_i)\sigma + \tfrac{p+1}{p}r^p A^+(\xi)\sigma\\ &= 0.\end{aligned}$$

As an exact form, $\widehat{\sigma}$ is closed and we obtain $d\widehat{\sigma} = d^c_C\widehat{\sigma} = \delta_C\widehat{\sigma} = \delta^c_C\widehat{\sigma} = 0$, which is equivalent to $\partial_C\widehat{\sigma} = \bar{\partial}_C\widehat{\sigma} = \partial^*_C\widehat{\sigma} = \bar{\partial}^*_C\widehat{\sigma} = 0$. $\square$

The commutator relations between L, Λ, ϕ and d, δ involve the operators d^c and δ^c, which are given by multiples of ϕd and Λd if applied to a Killing form. This leads us to the following result.

Lemma 3.4.8. *Let (M, g, ξ) be a Sasakian manifold and $\sigma \in \mathcal{K}^p(M, g)$. Then we have*

$$\begin{aligned}d^c\sigma &= \tfrac{1}{p+1}\phi_{\mathrm{D}}d\sigma,\\ \delta^c\sigma &= -\tfrac{2}{p+1}\Lambda d\sigma,\end{aligned}$$

which lead to

$$\begin{aligned}\xi \lrcorner \sigma &= \tfrac{1}{p-1}d\Lambda\sigma - \tfrac{1}{p+1}\Lambda d\sigma = \tfrac{2}{(p-1)(n-p+2)}d\Lambda\sigma - \tfrac{1}{n-p+2}\delta\phi_{\mathrm{D}}\sigma,\\ \eta\wedge\sigma &= \tfrac{1}{p}d\phi_{\mathrm{D}}\sigma - \tfrac{1}{p+1}\phi_{\mathrm{D}}d\sigma = \tfrac{1}{p\,(n-p)}d\phi_{\mathrm{D}}\sigma + \tfrac{1}{n-p}\delta L\sigma.\end{aligned}$$

Proof. From the Killing equation $\nabla_X\sigma = \frac{1}{p+1}X \lrcorner d\sigma$ we obtain

$$\begin{aligned}d^c\sigma &= \sum \phi e_i^* \wedge \nabla_{e_i}\sigma = \tfrac{1}{p+1}\sum \phi e_i^* \wedge e_i \lrcorner d\sigma = \tfrac{1}{p+1}\phi_{\mathrm{D}}d\sigma,\\ \delta^c\sigma &= \sum \phi e_i \lrcorner \nabla_{e_i}\sigma = \tfrac{1}{p+1}\sum \phi e_i \lrcorner e_i \lrcorner d\sigma = -\tfrac{2}{p+1}\Lambda d\sigma,\end{aligned}$$

which yield the claim with the commutator relations

$$\begin{aligned}&[\Lambda, d] = -\delta^c - (\deg -1)\xi \lrcorner\,, &&[\phi_{\mathrm{D}}, d] = d^c - \deg\eta\wedge,\\ &[L, \delta] = d^c - (n - \deg -1)\eta\wedge, &&[\phi_{\mathrm{D}}, \delta] = \delta^c + (n-\deg)\xi \lrcorner\,.\end{aligned}$$ $\square$

With Proposition 3.4.6 we show that all Killing and $*$-Killing forms on Sasakian manifolds give rise to special Killing and special $*$-Killing forms. This was already shown by S. Yamaguchi in [Y72b] for compact Sasakian manifolds under the restriction $p \leq m$.

Proposition 3.4.9. *Let (M, g, ξ) be a n-dimensional Sasakian manifold and let $\sigma \in \mathcal{K}^p(M, g)$. Then $\Lambda^s\sigma$ is a Killing form for all $s \geq 1$ and a special Killing form if $s \neq \frac{p}{2}$. Likewise, if $\tau \in *\mathcal{K}^p(M, g)$, then $L^s\tau$ is a $*$-Killing form for all $s \geq 1$ and a special $*$-Killing form if $s \neq \frac{n-p}{2}$. If p is even, then $d\Lambda^{p/2}\sigma$ still is a $*$-Killing 1-form, and if p is odd, then $\delta L^{(n-p)/2}\tau$ still is a Killing $(n-1)$-form.*

Proof. We prove this by induction on s. Obviously it is sufficient to consider only the case $s = 1$. Recall that

$$\nabla_X\omega = \eta\wedge X^*,$$

thus

$$\begin{aligned}\nabla_X(\Lambda\sigma) &= (\nabla_X\omega)\lrcorner\sigma + \omega\lrcorner\nabla_X\sigma\\ &= X\lrcorner\left(\xi\lrcorner\sigma + \tfrac{1}{p+1}\Lambda d\sigma\right).\end{aligned}$$

We obtain $X \lrcorner \nabla_X(\Lambda\sigma) = 0$, so with Proposition 3.1.7 we see $\Lambda\sigma \in \mathcal{K}^{p-2}(M, g)$. To show that $\Lambda\sigma$ is a special Killing form, we recall

$$A(X, Y)(\Lambda\sigma) = \Lambda A(X, Y)\sigma = 0$$

from Proposition 3.3.4. For $1 = s \neq \frac{p}{2}$ we have $\deg(\Lambda\sigma) = p - 2 \neq 0$, thus Proposition 3.4.6 shows that $\Lambda\sigma$ is a special Killing form. If $p = 2$ we use $d\Lambda\sigma = \frac{1}{3}\Lambda d\sigma + \xi \lrcorner \sigma$ from Lemma 3.4.8 and compute

$$\begin{aligned}
X^* \wedge \nabla_X(d\Lambda\sigma) &= \tfrac{1}{3}X^* \wedge (\nabla_X\omega) \lrcorner d\sigma + \tfrac{1}{3}X^* \wedge \Lambda\nabla_X(d\sigma) + X^* \wedge (\nabla_X\xi) \lrcorner \sigma \\
&\quad + X^* \wedge \xi \lrcorner \nabla_X\sigma \\
&= \tfrac{1}{3}X^* \wedge X \lrcorner \xi \lrcorner d\sigma + \tfrac{1}{3}X^* \wedge \Lambda\nabla_X(d\sigma) + X^* \wedge \phi X \lrcorner \sigma \\
&\quad + \tfrac{1}{3}X^* \wedge \xi \lrcorner X \lrcorner d\sigma \\
&= \tfrac{1}{3}X^* \wedge \Lambda\left(\tfrac{3}{2}A^+(X)\sigma + 3X^* \wedge \sigma\right) + X^* \wedge \phi X \lrcorner \sigma \\
&= \tfrac{1}{2}X^* \wedge \Lambda A^+(X)\sigma \\
&= 0,
\end{aligned}$$

which shows that $d\Lambda\sigma$ is a $*$-Killing form with the criterion from Proposition 3.1.7.

The statement on $*$-Killing forms τ is proven in the same way. $\square$

Because the vanishing of $q(A)d\sigma$ is a necessary condition for σ being special and because $q(A)$ is defined as

$$q(A) = -\sum e_i \lrcorner A^+(e_i) = -\sum e_i^* \wedge A^-(e_i) = -\sum e_i^* \wedge e_j \lrcorner A(e_i, e_j)$$

we further investigate the forms $A(X, Y)d\sigma$ and $A^\pm(X)d\sigma$.

Proposition 3.4.10. *Let (M, g, ξ) be a $(2m + 1)$-dimensional Sasakian manifold and $\sigma \in \mathcal{K}^p(M, g)$. If $2 \leq p \leq m$, then $A(X, Y)d\sigma$ is primitive and $A^-(X)d\sigma$, and $q(A)d\sigma$ are primitive and horizontal. If $p \geq m + 1$, then $A(X, Y)d\sigma = 0$*

Proof. If $p \geq m + 1$ then $d\sigma$ is a special $*$-Killing form by Theorem 3.3.7, which yields

$$\begin{aligned}
0 &= \nabla_X(\delta d\sigma) - (n - p)X \lrcorner d\sigma \\
&= -\tfrac{n-p}{n-p-1}R^-(X)d\sigma - (n - p)X \lrcorner d\sigma \\
&= -(n - p)A^-(X)d\sigma.
\end{aligned}$$

The curvature condition of Corollary 3.1.16 yields $A(X, Y)d\sigma = 0$.

From Lemma 3.4.8 we have $\Lambda d\sigma = \frac{p+1}{p-1}d\Lambda\sigma - (p+1)\xi \lrcorner \sigma$. By Proposition 3.4.9 $d\Lambda\sigma$ is a $*$-Killing $(p - 1)$-form and we obtain, since $A(X, Y)\sigma$ is horizontal,

$$A(X, Y)\Lambda d\sigma = \tfrac{p+1}{p-1}A(X, Y)d\Lambda\sigma - (p + 1)\xi \lrcorner A(X, Y)\sigma = 0.$$

Corollary 3.4.4 yields $A^-(X)\sigma = 0$. We take the interior product of $A^-(X)d\sigma$ with ξ and obtain

$$\begin{aligned}
\xi \lrcorner A^-(X)d\sigma &= -A^-(X)(\xi \lrcorner d\sigma) \\
&= -(p + 1)A^-(X)\nabla_\xi d\sigma
\end{aligned}$$

$$
\begin{aligned}
&= -(p+1)\nabla_\xi A^-(X)\sigma + (p+1)A^-(\nabla_\xi X)\sigma \\
&= 0.
\end{aligned}
$$

Finally $q(A)d\sigma$ is primitive and horizontal since $A^+(X)d\sigma$ is primitive and $A^-(X)d\sigma$ is horizontal. $\square$

We take the curvature condition for $d\sigma$ of Proposition 3.1.17 and combine it with the curvature properties of Sasakian manifolds as we did before with the curvature condition for σ.

Proposition 3.4.11. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and $\sigma \in \mathcal{K}^p(M, g)$. If $2 \leq p \leq m$, then we have*

$$
\begin{aligned}
\eta \wedge A^+(X)\sigma &= \tfrac{1}{p+1}\phi_{\mathrm{D}} A^+(X)d\sigma - \tfrac{p}{p+1}A^+(\phi X)d\sigma, \\
2A^-(X)d\sigma &= -X \lrcorner\, q(A)d\sigma - \phi X \lrcorner\, \rho^A \bullet d\sigma.
\end{aligned}
$$

Proof. From Proposition 3.1.17 we have

$$
A(X,Y)d\sigma = \tfrac{1}{p+1}\big(Y \lrcorner\, A^+(X)d\sigma - X \lrcorner\, A^+(Y)d\sigma\big) - \sum e_i^* \wedge (\nabla_{e_i}A)(X,Y)\sigma.
$$

We replace Y with e_i and build the contraction $\sum \phi e_i^* \wedge$, this yields

$$
\begin{aligned}
0 = \sum \phi e_i^* \wedge \Big(& A(X,e_i)d\sigma - \tfrac{1}{p+1} e_i \lrcorner\, A^+(X)d\sigma + \tfrac{1}{p+1} X \lrcorner\, A^+(e_i)d\sigma \\
&+ \sum e_j^* \wedge (\nabla_{e_j}A)(X,e_i)\sigma \Big) \\
= A^+(\phi X)&d\sigma - \tfrac{1}{p+1}\phi_{\mathrm{D}} A^+(X)d\sigma + \tfrac{1}{p+1}\sum \phi e_i^* \wedge X \lrcorner\, A^+(e_i)d\sigma \\
&- \sum e_i^* \wedge e_j^* \wedge (\nabla_{e_j}A)(X,\phi e_i)\sigma \\
= A^+(\phi X)&d\sigma - \tfrac{1}{p+1}\phi_{\mathrm{D}} A^+(X)d\sigma + \tfrac{1}{p+1}\sum g(\phi e_i, X) A^+(e_i)d\sigma \\
&+ \sum e_i^* \wedge e_j^* \wedge (\nabla_{e_j}A)(\phi X, e_i)\sigma - \sum e_i^* \wedge e_j^* \wedge A(\eta(e_i)X - \eta(X)e_i, e_j)\sigma \\
= A^+(\phi X)&d\sigma - \tfrac{1}{p+1}\phi_{\mathrm{D}} A^+(X)d\sigma - \tfrac{1}{p+1}A^+(\phi X)d\sigma - \eta \wedge A^+(X)\sigma,
\end{aligned}
$$

while the contractions $e_i \lrcorner$ and $\phi e_i \lrcorner$ lead to

$$
\begin{aligned}
0 = \sum e_i \lrcorner \Big(& A(X,e_i)d\sigma - \tfrac{1}{p+1} e_i \lrcorner\, A^+(X)d\sigma + \tfrac{1}{p+1} X \lrcorner\, A^+(e_i)d\sigma \\
&+ \sum e_j^* \wedge (\nabla_{e_j}A)(X,e_i)\sigma \Big) \\
= A^-(X)&d\sigma - \tfrac{1}{p+1} X \lrcorner \sum e_i \lrcorner\, A^+(e_i)d\sigma + \sum e_i \lrcorner\, e_j^* \wedge (\nabla_{e_j}A)(X,e_i)\sigma,
\end{aligned}
$$

i.e.

$$
\sum e_i \lrcorner\, e_j^* \wedge (\nabla_{e_j}A)(X,e_i)\sigma = -A^-(X)d\sigma - \tfrac{1}{p+1} X \lrcorner\, q(A)d\sigma,
$$

and

$$
0 = \sum \phi e_i \lrcorner \Big(A(X,e_i)d\sigma - \tfrac{1}{p+1} e_i \lrcorner\, A^+(X)d\sigma + \tfrac{1}{p+1} X \lrcorner\, A^+(e_i)d\sigma
$$

$$
\begin{aligned}
&\quad + \sum e_j^* \wedge (\nabla_{e_j} A)(X, e_i)\sigma\Big) \\
&= A^-(\phi X)d\sigma - \tfrac{2}{p+1}\Lambda A^+(X)d\sigma - \tfrac{1}{p+1} X \lrcorner \sum \phi e_i \lrcorner A^+(e_i)d\sigma \\
&\quad - \sum e_i \lrcorner e_j^* \wedge (\nabla_{e_j} A)(X, \phi e_i)\sigma \\
&= A^-(\phi X)d\sigma - \tfrac{2}{p+1}\Lambda A^+(X)d\sigma + \tfrac{1}{p+1} X \lrcorner \rho^A \bullet d\sigma \\
&\quad + \sum e_i \lrcorner e_j^* \wedge (\nabla_{e_j} A)(\phi X, e_i)\sigma + \sum e_i \lrcorner e_j^* \wedge A(\eta(e_i)X - \eta(X)e_i, e_j)\sigma \\
&= A^-(\phi X)d\sigma - \tfrac{2}{p+1}\Lambda A^+(X)d\sigma + \tfrac{1}{p+1} X \lrcorner \rho^A \bullet d\sigma \\
&\quad - A^-(\phi X)d\sigma - \tfrac{1}{p+1}\phi X \lrcorner q(A)d\sigma + \xi \lrcorner A^+(X)\sigma + \eta(X)q(A)\sigma \\
&= \tfrac{1}{p+1} X \lrcorner \rho^A \bullet d\sigma - \tfrac{1}{p+1}\phi X \lrcorner q(A)d\sigma - \tfrac{2}{p+1}\Lambda A^+(X)d\sigma \\
&= \tfrac{1}{p+1} X \lrcorner \rho^A \bullet d\sigma - \tfrac{1}{p+1}\phi X \lrcorner q(A)d\sigma - \tfrac{2}{p+1} A^-(\phi X)d\sigma - \tfrac{2}{p+1} A^+(X)\Lambda d\sigma \\
&= \tfrac{1}{p+1} X \lrcorner \rho^A \bullet d\sigma - \tfrac{1}{p+1}\phi X \lrcorner q(A)d\sigma - \tfrac{2}{p+1} A^-(\phi X)d\sigma. \qquad \square
\end{aligned}
$$

With Proposition 3.4.11 we extend our list of equivalent characterizations of special Killing forms and clarify the action of ϕ_{D} on $A^{\pm}(X)d\sigma, q(A)d\sigma$ and $\rho^A \bullet \sigma$.

Corollary 3.4.12. *Let (M, g, ξ) be a Riemannian manifold and $\sigma \in \mathcal{K}^p(M, g)$ with $p \neq 1$. If (M, g) admits a Sasakian structure, then σ is a special Killing form if and only if $A^+(X)d\sigma = 0$.*

Proof. If $A^+(X)d\sigma = 0$ then Proposition 3.4.11 yields $\eta \wedge A^+(X)\sigma = 0$, which implies $A^+(X)\sigma = 0$ since $A^+(X)\sigma$ is always horizontal. Conversely, if σ is a special Killing form, then $d\sigma$ is special $*$-Killing and we obtain $A^+(X)d\sigma = 0$. $\square$

Corollary 3.4.13. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and $\sigma \in \mathcal{K}^p(M, g)$ with $2 \leq p \leq m$. Then we have*

$$
\begin{aligned}
\phi_{\mathrm{D}} A^-(X)d\sigma &= p\, A^-(\phi X)d\sigma, \\
\phi_{\mathrm{D}}(\eta \wedge A^+(X)d\sigma) &= p\, \eta \wedge A^+(\phi X)d\sigma, \\
\phi_{\mathrm{D}}(\xi \lrcorner A^+(X)d\sigma) &= (p-1)\, \xi \lrcorner A^+(\phi X)d\sigma, \\
\phi_{\mathrm{D}} q(A)d\sigma &= -(p-1)\rho^A \bullet d\sigma, \\
\phi_{\mathrm{D}} \rho^A \bullet d\sigma &= (p-1)q(A)d\sigma.
\end{aligned}
$$

Proof. We contract $2A^-(e_i)d\sigma = -e_i \lrcorner q(A)d\sigma - \phi e_i \lrcorner \rho^A \bullet d\sigma$ with $e_i^* \wedge$ and $\phi e_i^* \wedge$ and obtain

$$
\begin{aligned}
-2q(A)d\sigma &= 2\sum e_i^* \wedge A^-(e_i)d\sigma \\
&= -\sum e_i^* \wedge e_i \lrcorner q(A)d\sigma - \sum e_i^* \wedge \phi e_i \lrcorner \rho^A \bullet d\sigma \\
&= -(p+1)q(A)d\sigma + \phi_{\mathrm{D}} \rho^A \bullet d\sigma,
\end{aligned}
$$

$$
\begin{aligned}
-2\rho^A \bullet d\sigma &= 2\sum \phi e_i^* \wedge A^-(e_i)d\sigma \\
&= -\sum \phi e_i^* \wedge e_i \lrcorner q(A)d\sigma - \sum \phi e_i^* \wedge \phi e_i \lrcorner \rho^A \bullet d\sigma \\
&= -\phi_{\mathrm{D}} q(A)d\sigma - (p+1)\rho^A \bullet d\sigma.
\end{aligned}
$$

We apply ϕ_{D} to $2A^-(X)d\sigma = -X \lrcorner q(A)d\sigma - \phi X \lrcorner \rho^A \bullet d\sigma$ and get

$$\begin{aligned}
2\phi_{\mathrm{D}} A^-(X)d\sigma &= -\phi X \lrcorner q(A)d\sigma - X \lrcorner \phi_{\mathrm{D}} q(A)d\sigma \\
&\quad - \phi^2 X \lrcorner \rho^A \bullet d\sigma - \phi X \lrcorner \rho^A \bullet d\sigma \\
&= -\phi X \lrcorner q(A)d\sigma + (p-1)X \lrcorner \rho^A \bullet d\sigma \\
&\quad + X \lrcorner \rho^A \bullet d\sigma - (p-1)\phi X \lrcorner q(A)d\sigma \\
&= p\,(-\phi X \lrcorner q(A)d\sigma + X \lrcorner \rho^A \bullet d\sigma) \\
&= 2p\, A^-(\phi X)d\sigma.
\end{aligned}$$

From $\eta \wedge A^+(X)\sigma = \frac{1}{p+1}\phi_{\mathrm{D}} A^+(X)d\sigma - \frac{p}{p+1}A^+(\phi X)d\sigma$ we obtain

$$\phi_{\mathrm{D}}(\eta \wedge A^+(X)d\sigma) = -p\,\eta \wedge A^+(X)d\sigma,$$

while $\phi_{\mathrm{D}} A^+(X)\sigma = (p-1)A^+(\phi X)\sigma$ and $A^+(X)\nabla_\xi \sigma = -A^+(\phi X)\sigma$ yield

$$\begin{aligned}
\phi_{\mathrm{D}}(\xi \lrcorner A^+(X)d\sigma) &= -\phi_{\mathrm{D}} A^+(X)(\xi \lrcorner d\sigma) \\
&= -(p+1)\phi_{\mathrm{D}} A^+(X)\nabla_\xi \sigma \\
&= -(p-1)(p+1)A^+(\phi X)\nabla_\xi \sigma \\
&= (p-1)\xi \lrcorner A^+(\phi X)d\sigma. \qquad \square
\end{aligned}$$

With Corollary 3.4.13 we show that every Killing m-form has to be special on a compact $(2m+1)$-dimensional Sasakian manifold.

Corollary 3.4.14. *Let (M, g) be a compact $(2m+1)$-dimensional Riemannian manifold. If (M, g) admits a Sasakian structure, then every Killing m-form is a special Killing form.*

Proof. Let $\sigma \in \mathcal{K}^m(M, g)$. By Corollary 3.4.13 we know

$$\phi_{\mathrm{D}}^2(\eta \wedge A^+(X)d\sigma) = -m^2 \eta \wedge A^+(X)d\sigma.$$

Proposition 1.2.13 yields the existence of $a, b \in \mathbb{Z}$ with $0 \leq a, b \leq m$ such that $a + b = m + 2$ and $(a - b)^2 = m^2$. We obtain $a = m + 1$ or $b = m + 1$, which is a contradiction. Thus we get $\eta \wedge A^+(X)d\sigma = 0$, which yields $\eta \wedge q(A)d\sigma = 0$. We already know that $q(A)d\sigma$ is horizontal and obtain

$$q(A)d\sigma = 0.$$

On compact Sasakian manifolds this means that σ has to be special. $\square$

3.4.3 Reduction to eigenforms of $\mathcal{L}_\xi$

Proposition 3.4.6 allows us to show that for a given Killing form σ the form $\mathcal{L}_\xi^2 \sigma + (p-1)^2 \sigma$ is a special Killing form. With that we are able to reduce the question if a given Killing form is special on compact Sasakian manifolds to eigenforms of the Lie derivative.

Proposition 3.4.15. *Let (M, g, ξ) be a n-dimensional Sasakian manifold. If $\sigma \in \mathcal{K}^p(M, g)$ with $p \neq 0$ then $\mathcal{L}_\xi^2 \sigma + (p-1)^2 \sigma$ is a special Killing form, and if $\tau \in *\mathcal{K}^p(M, g)$ with $p \neq n$ then $\mathcal{L}_\xi^2 \tau + (n-p-1)^2 \tau$ is a special $*$-Killing form.*

Proof. By Proposition 3.1.4 we have that $\mathcal{L}_\xi^2\sigma$ is again a Killing form. The claim follows if we show

$$A^+(X)\left(\mathcal{L}_\xi^2\sigma + (p-1)^2\sigma\right) = 0.$$

From the proof of Theorem 3.3.9 we know on one hand

$$A^+(X)\nabla_\xi\sigma = -A^+(\phi X)\sigma.$$

On the other hand we have

$$\phi_{\mathrm{D}} A^+(X)\sigma = (p-1)A^+(\phi X)\sigma$$

by Proposition 3.3.4. This yields

$$A^+(X)\phi_{\mathrm{D}}\sigma = \phi_{\mathrm{D}} A^+(X)\sigma - A^+(\phi X)\sigma = (p-2)A^+(\phi X)\sigma.$$

We conclude

$$A^+(X)\mathcal{L}_\xi\sigma = A^+(X)\nabla_\xi\sigma - A^+(X)\phi_{\mathrm{D}}\sigma = -(p-1)A^+(\phi_{\mathrm{D}} X)\sigma.$$

Since $\mathcal{L}_\xi\sigma$ is again a Killing form we may replace σ by $\mathcal{L}_\xi\sigma$ and obtain

$$A^+(X)\mathcal{L}_\xi^2\sigma = -(p-1)A^+(\phi_{\mathrm{D}} X)\mathcal{L}_\xi\sigma = (p-1)^2A^+(\phi^2 X)\sigma = -(p-1)^2A^+(X)\sigma. \square$$

In order to understand the consequences of Proposition 3.4.15 we show that every Killing p-form can be decomposed into a sum of Killing p-forms that are eigenforms of the Lie derivative $\mathcal{L}_\xi^2$, if the manifold is compact.

Lemma 3.4.16. *Let (M, g, ξ) be a compact Sasakian manifold. Then every Killing p-form σ can be decomposed as*

$$\sigma = \sum_{k=1}^{r} \sigma_k,$$

where the σ_k are Killing forms with $\mathcal{L}_\xi^2\sigma_k = \lambda_k\sigma_k$, where $\lambda_k \neq \lambda_l$ for $k \neq l$.

Proof. The manifold M is compact and ξ is a Killing vector field, so the Lie derivative $\mathcal{L}_\xi$ is skew-adjoint with respect to the L^2 inner product on $\Omega^*(M)$. By the Propositions 3.1.4 and 3.1.12

$$\mathcal{L}_\xi^2 : \mathcal{K}^p(M, g) \longrightarrow \mathcal{K}^p(M, g)$$

is a self-adjoint linear mapping of a finite dimensional Euclidean vector space and therefore diagonalizable. We may decompose $\sigma \in K^p(M, g)$ into

$$\sigma = \sum_{k=1}^{r} \sigma_k$$

with $\mathcal{L}_\xi^2\sigma_k = \lambda_k\sigma_k$ and $\lambda_k \neq \lambda_l$ for $k \neq l$. We fix a vector field $X \in \Gamma(TM)$ and apply Lemma C.0.13 of Appendix C with $V = \Omega^p(M), A = \mathcal{L}_\xi^2 : \Omega^p(M) \longrightarrow \Omega^p(M)$ and $B : \Omega^p(M) \longrightarrow \Omega^p(M)$, where

$$B\alpha = \nabla_X\alpha - \tfrac{1}{p+1} X \lrcorner\, d\alpha.$$

The lemma shows that all σ_k satisfy $B\sigma_k = 0$ and thus, since X was arbitrary, have to be Killing forms. $\square$

Theorem 3.4.17. *Let (M, g) be a compact Riemannian manifold. If (M, g) admits a Sasakian structure ξ then every Killing form $\sigma \in \mathcal{K}^p(M, g)$ with $p \neq 0$ can be decomposed into*

$$\sigma = \sigma_S + \sigma',$$

where σ_S is a special Killing form and σ' is a Killing form with $\mathcal{L}_\xi^2 \sigma' = -(p-1)^2 \sigma'$.

Proof. Since M is compact we use Lemma 3.4.16 and obtain

$$\sigma = \sum_{k=1}^{r} \sigma_k,$$

where the σ_k are Killing forms with $\mathcal{L}_\xi^2 \sigma_k = \lambda_k \sigma_k$ and $\lambda_k \neq \lambda_l$ for $k \neq l$. By Proposition 3.4.15 the forms $\mathcal{L}_\xi^2 \sigma_k + (p-1)^2 \sigma_k = (\lambda_k + (p-1)^2)\sigma_k$ are special, thus σ_k is special for $\lambda_k \neq -(p-1)^2$. If all λ_k are different from $-(p-1)^2$ the theorem is proven with $\sigma = \sigma_S, \sigma' = 0$. If one eigenvalue is equal to $-(p-1)^2$ we may without loss of generality assume that $\lambda_1 = -(p-1)^2$. Then $\sigma_2, \ldots, \sigma_r$ are special Killing forms. We set

$$\sigma_S := \sum_{k=2}^{r} \sigma_k,$$
$$\sigma' := \sigma_1$$

and obtain $\sigma = \sigma_S + \sigma'$ as claimed. $\square$

From now on we will stick to the case $\sigma = \sigma'$, i.e. we assume $\mathcal{L}_\xi^2 \sigma = -(p-1)^2 \sigma$. In order to further reduce the problem to eigenforms of $\mathcal{L}_\xi$ we have to extend our discussion to complex-valued forms. Since we extend all $\mathbb{R}$-linear operators $\Omega^*(M) \longrightarrow \Omega^*(M)$ to $\Omega^*_{\mathbb{C}}(M) \longrightarrow \Omega^*_{\mathbb{C}}(M)$ by $\mathbb{C}$-linearity, a complex form is a conformal Killing form if and only if its real and imaginary part are conformal Killing forms. If we set $\sigma_{\mathbb{C}} := \sigma - i\frac{1}{p-1}\mathcal{L}_\xi \sigma \in \Omega^p_{\mathbb{C}}(M)$, we obtain a complex-valued Killing form $\sigma_{\mathbb{C}}$ with $\mathcal{L}_\xi \sigma_{\mathbb{C}} = i(p-1)\sigma_{\mathbb{C}}$. To simplify notation we write $\sigma = \sigma_{\mathbb{C}}$, thus our situation is

$$\nabla_X \sigma = \tfrac{1}{p+1} X \lrcorner\, d\sigma \qquad \text{and} \qquad \mathcal{L}_\xi \sigma = i(p-1)\sigma. \tag{3.4.1}$$

Furthermore we split σ into its horizontal and vertical part,

$$\sigma = \beta + \eta \wedge \gamma = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$$

with $\beta \in \Omega^p_{\mathbb{C}}(H), \gamma \in \Omega^{p-1}_{\mathbb{C}}(H)$.

We start with combining the equations (3.4.1) in order to obtain the decomposition of σ into eigenforms of ϕ_{D}. Since $\mathcal{L}_\xi$ commutes with the horizontal and the vertical projection we have

$$\mathcal{L}_\xi \beta = i(p-1)\beta \qquad \text{and} \qquad \mathcal{L}_\xi \gamma = i(p-1)\gamma. \tag{3.4.2}$$

From (3.4.1) we get

$$0 = \xi \lrcorner \nabla_\xi \sigma = \nabla_\xi (\xi \lrcorner\, \sigma) = \nabla_\xi \gamma,$$

and since the Lie derivative $\mathcal{L}_\xi$ can be expressed as $\mathcal{L}_\xi = \nabla_\xi - \phi_{\mathrm{D}}$ we obtain

$$\phi_{\mathrm{D}} \gamma = \nabla_\xi \gamma - \mathcal{L}_\xi \gamma = -i(p-1)\gamma,$$

i.e.

$$\gamma \in \Omega^{(p-1,0)}_{\mathbb{C}}(H). \tag{3.4.3}$$

The exterior differential and the interior product split as

$$d = \begin{pmatrix} d_H & 2L \\ \mathcal{L}_\xi & -d_H \end{pmatrix}, \qquad X \lrcorner = \begin{pmatrix} X \lrcorner & \eta(X)\mathrm{id} \\ 0 & -X \lrcorner \end{pmatrix}$$

for every tangent vector field X. Using (3.4.1) again yields

$$\begin{aligned}
\begin{pmatrix} \mathcal{L}_\xi\beta + \phi_{\mathrm{D}}\beta \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla_\xi\beta \\ 0 \end{pmatrix} = \nabla_\xi \begin{pmatrix} \beta \\ \gamma \end{pmatrix} &= \nabla_\xi\sigma = \tfrac{1}{p+1}\xi \lrcorner d\sigma \\
&= \tfrac{1}{p+1}\begin{pmatrix} \xi \lrcorner & \mathrm{id} \\ 0 & -\xi \lrcorner \end{pmatrix}\begin{pmatrix} d_H & 2L \\ \mathcal{L}_\xi & -d_H \end{pmatrix}\begin{pmatrix} \beta \\ \gamma \end{pmatrix} \\
&= \tfrac{1}{p+1}\begin{pmatrix} \mathcal{L}_\xi\beta - d_H\gamma \\ 0 \end{pmatrix},
\end{aligned}$$

which is equivalent to

$$d_H\gamma = -ip\,(p-1)\beta - (p+1)\phi_{\mathrm{D}}\beta. \tag{3.4.4}$$

Since $\gamma \in \Omega^{(p-1,0)}_{\mathbb{C}}(H)$ we know $d_H\gamma = \partial\gamma + \bar{\partial}\gamma \in \Omega^{(p,0)}_{\mathbb{C}}(H) \oplus \Omega^{(p-1,1)}_{\mathbb{C}}(H)$. Furthermore let

$$\beta = \sum_{a=0}^{p} \beta_{p-a,a} \in \bigoplus_{a=0}^{p} \Omega^{(p-a,a)}_{\mathbb{C}}(H)$$

be the decomposition of β in forms of type $(p-a,a)$, i.e. $\phi_{\mathrm{D}}\beta_{p-a,a} = -i(p-2a)\beta_{p-a,a}$. Then (3.4.4) becomes

$$\begin{aligned}
\Omega^{(p,0)}_{\mathbb{C}}(H) \oplus \Omega^{(p-1,1)}_{\mathbb{C}}(H) \ni \partial\gamma + \bar{\partial}\gamma &= i\sum_{a=0}^{p}[(p+1)(p-2a) - p\,(p-1)]\beta_{p-a,a} \\
&= -2i\sum_{a=0}^{p}[(p+1)a - p]\beta_{p-a,a} \in \bigoplus_{a=0}^{p}\Omega^{(p-a,a)}_{\mathbb{C}}(H).
\end{aligned}$$

Since $(p+1)a - p \neq 0$ never vanish for $a = 0, \ldots, p$, we may conclude $\beta_{p-a,a} = 0$ for all indices $a = 2, \ldots, p$. We obtain

$$\partial\gamma = 2ip\,\beta_{p,0}, \qquad \bar{\partial}\gamma = -2i\beta_{p-1,1}.$$

Thus the Killing form σ is completely determined by its vertical part γ via

$$\sigma = \begin{pmatrix} \beta_{p,0} + \beta_{p-1,1} \\ \gamma \end{pmatrix} = \begin{pmatrix} -\frac{i}{2p}\,\partial\gamma + \frac{i}{2}\,\bar{\partial}\gamma \\ \gamma \end{pmatrix}. \tag{3.4.5}$$

In order to rewrite the Killing equation for σ in terms of γ we recall

$$\nabla_U = \begin{pmatrix} D_U & \phi U^* \wedge \\ -\phi U \lrcorner & D_U \end{pmatrix}$$

for every horizontal vector field U. We decompose $U = U^+ + U^-$ into its $(1,0)$- and $(0,1)$-part, i.e.

$$U^+ = \tfrac{1}{2}\,(U - i\phi U)\,, \qquad U^- = \tfrac{1}{2}\,(U + i\phi U)\,,$$

and obtain

$$\nabla_U\sigma = \begin{pmatrix} D_U & \phi U^*\wedge \\ -\phi U\lrcorner & D_U \end{pmatrix} \begin{pmatrix} -\frac{i}{2p}\,\partial\gamma + \frac{i}{2}\,\bar\partial\gamma \\ \gamma \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{i}{2p}\,D_U(\partial\gamma) + \frac{i}{2}\,D_U(\bar\partial\gamma) + \phi U^*\wedge\gamma \\ \\ D_U\gamma + \frac{i}{2p}\,\phi U\lrcorner\,\partial\gamma - \frac{i}{2}\,\phi U\lrcorner\,\bar\partial\gamma \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{i}{2p}D_U\big(\partial\gamma\big) - i(U^-)^*\wedge\gamma \\ \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{i}{2}\,D_U\,\big(\bar\partial\gamma\big) + i(U^+)^*\wedge\gamma \\ \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \\ D_U\gamma - \frac{1}{2p}U^+\lrcorner\,\partial\gamma - \frac{1}{2}\,U^-\lrcorner\,\bar\partial\gamma \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \frac{1}{2}\,U^+\lrcorner\,\bar\partial\gamma \end{pmatrix}$$

$$\in \Omega^{(p,0)}_{\mathbb{C}}(H) \oplus \Omega^{(p-1,1)}_{\mathbb{C}}(H) \oplus \eta\wedge\Omega^{(p-1,0)}_{\mathbb{C}}(H) \oplus \eta\wedge\Omega^{(p-2,1)}_{\mathbb{C}}(H).$$

Decomposition of $U\lrcorner\, d\sigma$ in the same manner yields

$$U\lrcorner\, d\sigma = \begin{pmatrix} U\lrcorner & 0 \\ 0 & -U\lrcorner \end{pmatrix} \begin{pmatrix} d_H & 2L \\ \mathcal{L}_\xi & -d_H \end{pmatrix} \begin{pmatrix} -\frac{i}{2p}\,\partial\gamma + \frac{i}{2}\,\bar\partial\gamma \\ \gamma \end{pmatrix}$$

$$= \begin{pmatrix} U^-\lrcorner\,\Big(\frac{i}{2}\,\partial\bar\partial\gamma - \frac{i}{2p}\,\bar\partial\partial\gamma + 2L\gamma\Big) \\ \\ 0 \end{pmatrix} + \begin{pmatrix} U^+\lrcorner\,\Big(\frac{i}{2}\,\partial\bar\partial\gamma - \frac{i}{2p}\,\bar\partial\partial\gamma + 2L\gamma\Big) \\ \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \\ \frac{p+1}{2p}\,U^+\lrcorner\,\partial\gamma + \frac{p+1}{2}\,U^-\lrcorner\,\bar\partial\gamma \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \frac{p+1}{2}\,U^+\lrcorner\,\bar\partial\gamma \end{pmatrix}$$

$$\in \Omega^{(p,0)}_{\mathbb{C}}(H) \oplus \Omega^{(p-1,1)}_{\mathbb{C}}(H) \oplus \eta\wedge\Omega^{(p-1,0)}_{\mathbb{C}}(H) \oplus \eta\wedge\Omega^{(p-2,1)}_{\mathbb{C}}(H).$$

Thus the Killing equation $\nabla_U\sigma = \frac{1}{p+1}U\lrcorner\, d\sigma$ decomposes into three linearly independent parts. We can further simplify the formulas since we have the (anti-)commutator relations $\{\partial,\bar\partial\} = -2L\mathcal{L}_\xi$ and $[L, X\lrcorner] = -\phi X^*\wedge$:

$$\begin{aligned} -p\,\partial\bar\partial\gamma + \bar\partial\partial\gamma + 4ipL\gamma &= (p+1)\bar\partial\partial\gamma + 2pL\mathcal{L}_\xi\gamma + 4ipL\gamma \\ &= (p+1)\bar\partial\partial\gamma + 2ip\,(p+1)L\gamma. \end{aligned}$$

This leads to

$$\begin{aligned} D_U\big(\partial\gamma\big) &= U^-\lrcorner\,\bar\partial\partial\gamma + 2ip\,U^-\lrcorner\,L\gamma - 2p\,(U^-)^*\wedge\gamma \\ &= U^-\lrcorner\,\bar\partial\partial\gamma + 2ip\,L(U^-\lrcorner\,\gamma) \end{aligned}$$

$$= U^- \lrcorner \bar\partial\partial\gamma,$$

$$\begin{aligned} D_U\left(\bar\partial\gamma\right) &= -\tfrac{1}{p}U^+ \lrcorner \bar\partial\partial\gamma - 2iU^+ \lrcorner L\gamma + 2(U^+)^* \wedge \gamma \\ &= -\tfrac{1}{p}U^+ \lrcorner \bar\partial\partial\gamma - 2iL(U^+ \lrcorner \gamma). \end{aligned}$$

We arrive at the following result:

Proposition 3.4.18. *Let (M, g, ξ) be a Sasakian manifold and $\sigma \in \Omega^p_{\mathbb{C}}(M)$ with $\mathcal{L}_\xi\sigma = i(p-1)\sigma$. Set $\gamma := \xi \lrcorner \sigma$. Then σ is a Killing p-form if and only if*

$$\sigma = -\tfrac{i}{2p}\partial\gamma + \tfrac{i}{2}\bar\partial\gamma + \eta \wedge \gamma$$

and γ is a $(p-1,0)$-form that satisfies the following equations for every horizontal vector field U:

$$\begin{aligned} D_U\gamma &= \tfrac{1}{p}\,U^+ \lrcorner \partial\gamma + U^- \lrcorner \bar\partial\gamma, \\ D_U\left(\partial\gamma\right) &= U^- \lrcorner \bar\partial\partial\gamma, \\ D_U\left(\bar\partial\gamma\right) &= -\tfrac{1}{p}U^+ \lrcorner \bar\partial\partial\gamma - 2iL(U^+ \lrcorner \gamma). \end{aligned}$$

In this case the exterior differential $d\sigma$ is given by

$$d\sigma = -\tfrac{i}{2p}\bar\partial\partial\gamma + L\gamma + \eta \wedge (-\tfrac{1}{2p}\partial\gamma - \tfrac{1}{2}\bar\partial\gamma).$$

Proof. The remaining formula for $d\sigma$ follows from

$$\sigma = \begin{pmatrix} -\tfrac{i}{2p}\partial\gamma + \tfrac{i}{2}\bar\partial\gamma \\ \gamma \end{pmatrix}$$

and

$$d = \begin{pmatrix} d_H & 2L \\ \mathcal{L}_\xi & -d_H \end{pmatrix}. \qquad \square$$

Since σ is a special Killing form if and only if $A^+(X)\sigma$ or $A^+(X)d\sigma$ vanish for all X we further investigate $A(X,Y)\sigma$ and $A(X,Y)d\sigma$ in the special case $\mathcal{L}_\xi\sigma = i(p-1)\sigma$. We start with the (p,q)-decomposition of the related forms.

Proposition 3.4.19. *Let (M, g, ξ) be a Sasakian manifold and $\sigma \in \mathcal{K}^p(M, g)$ with $\mathcal{L}_\xi\sigma = i(p-1)\sigma$ and $p \neq 1$. If $\gamma := \xi \lrcorner \sigma$, we have*

$$\begin{aligned} \sigma &\in \Big(\Omega^{(p,0)}(H) \oplus \Omega^{(p-1,1)}(H)\Big) \oplus \eta \wedge \Omega^{(p-1,0)}(H), \\ d\sigma &\in \Omega^{(p,1)}(H) \oplus \eta \wedge \Big(\Omega^{(p,0)}(H) \oplus \Omega^{(p-1,1)}(H)\Big), \\[1em] A(X,Y)\sigma &\in \Omega^{(p-1,1)}(H), \\ A(X,Y)d\sigma &\in \Omega^{(p,1)}(H) \oplus \eta \wedge \Omega^{(p-1,1)}(H), \\[1em] A^+(X)\sigma &\in \Omega^{(p,1)}(H), \\ A^+(X)d\sigma &\in \Omega^{(p+1,1)}(H) \oplus \eta \wedge \Omega^{(p,1)}(H), \\ A^-(X)d\sigma &\in \Omega^{(p,0)}(H), \\[1em] q(A)d\sigma, \rho^A \bullet d\sigma &\in \Omega^{(p,1)}(H). \end{aligned}$$

Proof. We already know that γ is a $(p-1,0)$-form, thus

$$\begin{aligned}\sigma &= -\tfrac{i}{2p}\partial\gamma + \tfrac{i}{2}\bar{\partial}\gamma + \eta\wedge\gamma \\ &\quad \in \quad \Omega^{(p,0)}(H)\oplus\Omega^{(p-1,1)}(H)\oplus\eta\wedge\Omega^{(p-1,0)}(H), \\ d\sigma &= -\tfrac{i}{2p}\bar{\partial}\partial\gamma + L\gamma + \eta\wedge(-\tfrac{1}{2p}\partial\gamma - \tfrac{1}{2}\bar{\partial}\gamma) \\ &\quad \in \Omega^{(p,1)}(H)\oplus\eta\wedge\Omega^{(p,0)}(H)\oplus\eta\wedge\Omega^{(p-1,1)}(H).\end{aligned}$$

To show $A(X,Y)\sigma\in\Omega^{(p-1,1)}(H)$ we note two facts:

- Since $[\phi_{\mathrm{D}}, A(X,Y)] = 0$ we have $A(X,Y)\left(\Omega^{(a,b)}(H)\right)\subset\Omega^{(a,b)}$, thus

$$A(X,Y)\sigma \quad \in \quad \Omega^{(p,0)}(H)\oplus\Omega^{(p-1,1)}(H)\oplus\eta\wedge\Omega^{(p-1,0)}(H).$$

- From $\phi_{\mathrm{D}}^2 A(X,Y)\sigma = -(p-2)^2 A(X,Y)\sigma$ we get

$$A(X,Y)\sigma\in \quad \Omega^{(p-1,1)}(H)\oplus\Omega^{(1,p-1)}(H).$$

We obtain $A(X,Y)\gamma = 0, A(X,Y)\partial\gamma = 0$ and $A(X,Y)\sigma = \frac{i}{2}A(X,Y)\bar{\partial}\gamma$, which implies

$$\begin{aligned}\phi_{\mathrm{D}}A^+(X)\sigma &= \tfrac{i}{2}\phi_{\mathrm{D}}A^+(X)\bar{\partial}\gamma \\ &= \tfrac{i}{2}A^+(\phi X)\bar{\partial}\gamma + \tfrac{i}{2}A^+(X)\phi_{\mathrm{D}}\bar{\partial}\gamma \\ &= A^+(\phi X)\sigma - i(p-2)\tfrac{i}{2}A^+(X)\bar{\partial}\gamma \\ &= \tfrac{1}{p-1}\phi_{\mathrm{D}}A^+(X)\sigma - i(p-2)A^+(X)\sigma,\end{aligned}$$

i.e. $\phi_{\mathrm{D}}A^+(X)\sigma = -i(p-2)A^+(X)\sigma$. This yields the remaining claims with Corollary 3.4.13 since we have $[A(X,Y),L] = 0, [A^+(X),L] = 0$ and $[A^-(X),L] = A^+(\phi X)$. $\square$

We have the following corollary.

Corollary 3.4.20. *Let (M,g,ξ) be a Sasakian manifold and $\sigma\in\mathcal{K}^p(M,g)$ with $\mathcal{L}_\xi\sigma = i(p-1)\sigma$ and $p\neq 1$. For $\gamma = \xi\lrcorner\sigma$ we have*

$$A(X,Y)\sigma = \tfrac{i}{2}A(X,Y)\bar{\partial}\gamma, \qquad A(X,Y)\gamma = 0, \qquad A(X,Y)\partial\gamma = 0,$$

and

$$\mathrm{Ric}^A(X)\lrcorner\gamma = 0, \qquad \mathrm{Ric}^A(X)\lrcorner\partial\gamma = 0.$$

For all horizontal tangent vectors U we have

$$A^+(U^+)\sigma = 0, \qquad A^+(U^+)d\sigma = 0, \qquad A^-(U^+)d\sigma = 0,$$

and furthermore we have

$$\rho^A\bullet d\sigma = -iq(A)d\sigma.$$

Proof. The proof of the first part is contained in the proof of Proposition 3.4.19.

To show $\mathrm{Ric}^A(X)\lrcorner\gamma = 0$ we start with $A(X,Y)\gamma = 0$, which we write as

$$\sum A(X,Y)e_i^*\wedge e_i\lrcorner\gamma = 0.$$

We set $Y = e_j$, contract with $e_j \lrcorner$ and obtain, using the first Bianchi identity,

$$\begin{aligned}
\mathrm{Ric}^A(X) \lrcorner \gamma &= \sum A(X, e_i) e_i \lrcorner \gamma \\
&= -\sum g(A(X, e_i)e_j, e_i) e_j \lrcorner \gamma \\
&= -\sum A(X, e_j) e_i^* \wedge e_j \lrcorner e_i \lrcorner \gamma \\
&= \sum A(e_j, e_i) X^* \wedge e_j \lrcorner e_i \lrcorner \gamma \\
&\quad + \sum A(e_i, X) e_j^* \wedge e_j \lrcorner e_i \lrcorner \gamma \\
&= \sum A(e_j, e_i) X^* \wedge e_j \lrcorner e_i \lrcorner \gamma \\
&\quad - \mathrm{Ric}^A(X) \lrcorner \gamma.
\end{aligned}$$

We continue

$$\begin{aligned}
2\mathrm{Ric}^A(X) \lrcorner \gamma &= \sum A(e_j, e_i) X^* \wedge e_j \lrcorner e_i \lrcorner \gamma \\
&= \sum A(\phi e_j, \phi e_i) X^* \wedge \phi e_j \lrcorner \phi e_i \lrcorner \gamma \\
&= -\sum A(e_j, e_i) X^* \wedge e_j \lrcorner e_i \lrcorner \gamma \\
&= -2\mathrm{Ric}^A(X) \lrcorner \gamma,
\end{aligned}$$

where used $\phi X \lrcorner \phi Y \lrcorner \gamma = -X \lrcorner Y \lrcorner \gamma$, which holds since γ is a $(p-1, 0)$-form. In the same way we get $\mathrm{Ric}^A(X) \lrcorner \partial\gamma = 0$.

To prove the statements about $A^{\pm}(U^+)$ we start with

$$\begin{aligned}
2A^+(U^+)\sigma &= A^+(U)\sigma - iA^+(\phi U)\sigma \\
&= A^+(U)\sigma - \tfrac{i}{p-1} \phi_{\mathrm{D}} A^+(U)\sigma \\
&= A^+(U)\sigma - \tfrac{i}{p-1}(-i(p-1))A^+(U)\sigma \\
&= 0.
\end{aligned}$$

We differentiate $A^+(U^+)\sigma = 0$ covariantly in direction of ξ and obtain that $A^+(U^+)d\sigma$ is horizontal:

$$\begin{aligned}
0 &= \nabla_\xi (A^+(U^+)\sigma) \\
&= A^+(\nabla_\xi(U^+))\sigma + A^+(U^+)\nabla_\xi \sigma \\
&= A^+((\nabla_\xi U)^+)\sigma + (p+1)A^+(U^+)(\xi \lrcorner d\sigma) \\
&= -(p+1)\xi \lrcorner A^+(U^+) d\sigma.
\end{aligned}$$

From Corollary 3.4.13 we know $\phi_{\mathrm{D}}(\eta \wedge A^+(U^+)d\sigma) = p\,\eta \wedge A^+(U^+)d\sigma$. Taking the interior product with ξ leads to $\phi_{\mathrm{D}} A^+(U^+)d\sigma = p\,A^+(U^+)d\sigma$, thus

$$\begin{aligned}
2A^+(U^+)d\sigma &= A^+(U)d\sigma - iA^+(\phi U)d\sigma \\
&= A^+(U)d\sigma - i\tfrac{1}{p}\phi_{\mathrm{D}} A^+(U)d\sigma \\
&= A^+(U)d\sigma - i\tfrac{1}{p}(-ip)A^+(U)d\sigma \\
&= 0.
\end{aligned}$$

Similarly we calculate

$$2A^-(U^+)d\sigma = A^-(U)d\sigma - iA^-(\phi U)d\sigma$$

$$
\begin{aligned}
&= A^-(U)d\sigma - i\tfrac{1}{p}\phi_{\mathrm{D}}A^-(U)d\sigma \\
&= A^-(U)d\sigma - i\tfrac{1}{p}(-ip)A^-(U)d\sigma \\
&= 0.
\end{aligned}
$$

Finally from Corollary 3.4.13 we obtain

$$
\begin{aligned}
\rho^A \bullet d\sigma &= -\tfrac{1}{p-1}\phi_{\mathrm{D}}q(A)d\sigma \\
&= -\tfrac{1}{p-1}i(1-p)q(A)d\sigma \\
&= -iq(A)d\sigma. \qquad \square
\end{aligned}
$$

To conclude this section we rewrite the relation $\frac{p+1}{p}A^+(X)\sigma = \nabla_X(d\sigma)+(p+1)X^*\wedge\sigma$ from Corollary 3.1.15 in terms of γ. The exterior product is given by

$$
X^*\wedge = \begin{pmatrix} -\phi^2 X^*\wedge & 0 \\ \eta(X)\mathrm{id} & \phi^2 X^*\wedge \end{pmatrix},
$$

which yields

$$
\begin{aligned}
\tfrac{p+1}{p}\begin{pmatrix} A^+(U)\sigma \\ 0 \end{pmatrix} &= \tfrac{p+1}{p}A^+(U)\sigma \\
&= \nabla_U(d\sigma) + (p+1)U^*\wedge\sigma \\
&= (p+1)\begin{pmatrix} D_U & \phi U^*\wedge \\ -\phi U \lrcorner & D_U \end{pmatrix}\begin{pmatrix} -\frac{i}{2p}\bar\partial\partial\gamma + L\gamma \\ -\frac{1}{2p}\partial\gamma - \frac{1}{2}\bar\partial\gamma \end{pmatrix} \\
&\qquad + (p+1)\begin{pmatrix} U^*\wedge & 0 \\ 0 & -U^*\wedge \end{pmatrix}\begin{pmatrix} -\frac{i}{2p}\partial\gamma + \frac{i}{2}\bar\partial\gamma \\ \gamma \end{pmatrix} \\
&= (p+1)\begin{pmatrix} -\frac{i}{2p}D_U\left(\bar\partial\partial\gamma\right) + LD_U\gamma - \frac{i}{p}(U^+)^*\wedge\partial\gamma + i(U^-)^*\wedge\bar\partial\gamma \\ \\ -\frac{1}{2p}D_U\left(\partial\gamma\right) - \frac{1}{2}D_U\left(\bar\partial\gamma\right) + \frac{i}{2p}\phi U\lrcorner\bar\partial\partial\gamma - L(\phi U\lrcorner\gamma) \end{pmatrix} \\
&= (p+1)\begin{pmatrix} -\frac{i}{2p}D_U\left(\bar\partial\partial\gamma\right) + LD_U\gamma - \frac{i}{p}(U^+)^*\wedge\partial\gamma + i(U^-)^*\wedge\bar\partial\gamma \\ \\ 0 \end{pmatrix}
\end{aligned}
$$

Because of $A^+(U^+)\sigma = 0$ we obtain

$$
D_{U^+}\left(\bar\partial\partial\gamma\right) = -2iU^+\lrcorner L\partial\gamma - 4(U^+)^*\wedge\partial\gamma
$$

and

$$
A^+(U)\sigma = -\tfrac{i}{2}D_{U^-}\left(\bar\partial\partial\gamma\right) + p\,U^-\lrcorner L\bar\partial\gamma + 2ip\,(U^-)^*\wedge\bar\partial\gamma.
$$

We have proven the following result.

Proposition 3.4.21. *Let (M,g,ξ) be a Sasakian manifold and $\sigma\in\mathcal{K}^p_{\mathbb{C}}(M,g)$ with $\mathcal{L}_\xi\sigma = i(p-1)\sigma$. Set $\gamma := \xi\lrcorner\sigma$. Then σ is a special Killing form if and only if γ satisfies*

$$
D_{U^-}\left(\bar\partial\partial\gamma\right) = -2ip\,U^-\lrcorner L\bar\partial\gamma + 4p\,(U^-)^*\wedge\bar\partial\gamma.
$$

3.4.4 Reduction to primitive forms

In this section we only consider Killing forms on non-Einstein manifolds; the Einstein case is covered in Section 3.5.2. By Proposition 3.1.10, on compact simply connected Sasakian manifolds which are not Einstein all possible special Killing forms are given by $\eta \wedge \omega^k$. Since by Proposition 3.4.9 the form $\Lambda\sigma$ is a special Killing form for $p \neq 2$ we obtain $\Lambda\sigma = 0$ if p is even and $\Lambda\sigma = c \cdot \eta \wedge \omega^{\frac{p-3}{2}}$ if p is odd. Since

$$\Lambda\omega^k = \tfrac{1}{2}k(n-2k+1)\omega^{k-1}$$

for every k, the form

$$\sigma - \tfrac{4c}{(p-1)(n-p+2)}\eta \wedge \omega^{\frac{p-1}{2}}$$

is primitive. It is also a Killing form because both σ and $\eta \wedge \omega^{\frac{p-1}{2}}$ are Killing forms. This leads us to the following reduction result.

Corollary 3.4.22. *Let (M, g, ξ) be a compact simply connected Sasakian manifold that is not Einstein, and $\sigma \in \mathcal{K}^p(M, g)$.*

- *If $p \neq 2$ is even, then σ is primitive.*
- *If p is odd, then there exists a primitive Killing form $\sigma_0 \in \mathcal{K}^p(M, g)$ such that*

$$\sigma = \sigma_0 + \mathrm{const} \cdot \eta \wedge \omega^{\frac{p-1}{2}}.$$

3.5 Conformal Killing forms under additional assumptions

In this section we investigate horizontal and vertical conformal Killing forms, conformal Killing forms on Eta-Einstein and compact Einstein manifolds as well as normal conformal Killing forms. In all of these situations we completely classify conformal Killing forms at least up to special Killing and $*$-Killing forms.

3.5.1 Horizontal and vertical conformal Killing forms

Horizontal conformal Killing forms have already been studied by S. Yamaguchi. We will extend his result of Theorem 3.2.3 in this case and completely classify these forms up to special Killing forms that are eigenforms of ϕ_{D}^2 with eigenvalue $-p^2$, where we still follow his main ideas. The classification is as follows.

Theorem 3.5.1. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and $\psi \in \mathcal{CK}^p(M, g)$ horizontal.*

- *Let p be even. If $2 \leq p \leq m$, then $\psi = \mathrm{const} \cdot \omega^{\frac{p}{2}} + \sigma_S$, where σ_S is a horizontal special Killing form with $\phi_{\mathrm{D}}^2\sigma_S = -p^2\sigma_S$. If $p \geq m+1$, then $\psi = \mathrm{const} \cdot \omega^{\frac{p}{2}}$.*
- *Let p be odd. If $1 \leq p \leq m$, then $\psi = \sigma_S$ with $\phi_{\mathrm{D}}^2\sigma_S = -p^2\sigma_S$. If $p \geq m+1$, then $\psi = 0$.*

We summarize the situation for $p \neq 0$ in the following table:

$\psi \in \mathcal{CK}^p(M,g)$ horizontal	$2 \leq p \leq m$	$p \geq m+1$
p even	$\psi \quad = \quad \sigma_S + c \cdot \omega^{\frac{p}{2}}$	$\psi \quad = \quad c \cdot \omega^{\frac{p}{2}}$
p odd	$\psi \quad = \quad \sigma_S$	$\psi \quad = \quad 0$

Here σ_S denotes a horizontal special Killing form with $\phi_{\mathrm{D}}^2 \sigma_S = -p^2 \sigma_S$.

The key observation to this classification is, due to S. Yamaguchi in [Y72a], that for every horizontal conformal Killing form ψ the form $\delta\psi$ is a vertical Killing form, which makes it possible to determine $\delta\psi$ completely. But the method of S. Yamaguchi to determine $\delta\psi$ works for all vertical Killing forms, not only for those being the codifferential of a conformal Killing form. Once we know $\delta\psi$ we are able to determine the $*$-Killing part of ψ and are left with the Killing part of ψ, which turns out to be horizontal. Therefore we will start with investigating horizontal and vertical Killing forms and then turn to horizontal conformal Killing forms.

Note that the decomposition Theorem 3.3.9 does not allow us to reduce our discussion to horizontal Killing and $*$-Killing forms since a priori it is not clear that the Killing and $*$-Killing part of a horizontal conformal Killing form are again horizontal. However, after our discussion we see that this always holds true.

It was shown in [Y75] that on compact Kähler manifolds every Killing form has to be parallel. On Sasakian manifolds this statement has to be wrong because for each k the form $\eta \wedge \omega^k$ is a Killing form. We will show that all vertical Killing forms are of this type and that every horizontal Killing form is an eigenform of ϕ_{D}^2 with eigenvalue $-p^2$ and special for $p \neq 1$. Furthermore we will prove that there are no basic Killing forms.

Lemma 3.5.2. *Let (M, g, ξ) be a $(2m+1)$-dimensional Sasakian manifold and $\sigma \in \mathcal{K}^p(M,g)$ a horizontal Killing form. Then $\phi_{\mathrm{D}}^2 \sigma = -p^2$, in particular $\sigma = 0$ if $p \geq m+1$. If $p \neq 0$, then σ is a special Killing form.*

Proof. We first discuss the case $p = 1$, where we trivially have $\phi^2\sigma = -\sigma$. The Killing equation yields

$$\mathcal{L}_\xi \sigma + \phi\sigma = \nabla_\xi \sigma = \tfrac{1}{2} \xi \lrcorner d\sigma = \tfrac{1}{2} \mathcal{L}_\xi \sigma,$$

thus $\phi\sigma = -\frac{1}{2}\mathcal{L}_\xi\sigma$, which implies $\sigma = -\phi^2\sigma = -\frac{1}{4}\mathcal{L}_\xi^2\sigma$. Proposition 3.1.4 shows that $\mathcal{L}_\xi\sigma$ is a Killing form, and by applying Proposition 3.4.1 to the Killing form $\mathcal{L}_\xi\sigma$ we know that $\mathcal{L}_\xi^2\sigma$ is a special Killing form. Hence σ is special.

For $p \neq 1$ we differentiate $\xi \lrcorner \sigma = 0$ covariantly and obtain

$$\phi X \lrcorner \sigma + \xi \lrcorner \nabla_X \sigma = 0,$$

which yields

$$X \lrcorner \phi X \lrcorner \sigma = 0.$$

We replace X by $X+Y$ and see

$$X \lrcorner \phi Y \lrcorner \sigma = -Y \lrcorner \phi X \lrcorner \sigma,$$

which leads to

$$\begin{aligned}
X \lrcorner \phi_{\mathrm{D}}\sigma &= -X \lrcorner \sum e_i^* \wedge \phi e_i \lrcorner \sigma \\
&= -\sum g(e_i, X)\phi e_i \lrcorner \sigma + \sum e_i^* \wedge X \lrcorner \phi e_i \lrcorner \sigma \\
&= -\phi X \lrcorner \sigma - \sum e_i^* \wedge e_i \lrcorner \phi X \lrcorner \sigma \\
&= -p\, \phi X \lrcorner \sigma.
\end{aligned}$$

By applying ϕ_{D} to this we get

$$\begin{aligned}
X \lrcorner \phi_{\mathrm{D}}^2\sigma &= \phi_{\mathrm{D}}(X \lrcorner \phi_{\mathrm{D}}\sigma) - \phi X \lrcorner \phi_{\mathrm{D}}\sigma \\
&= -p\, \phi_{\mathrm{D}}(\phi X \lrcorner \sigma) + p\, \phi^2 X \lrcorner \sigma \\
&= -p\, \phi^2 X \lrcorner \sigma - p\, \phi X \lrcorner \phi_{\mathrm{D}}\sigma + p\, \phi^2 X \lrcorner \sigma \\
&= p^2 \phi^2 X \lrcorner \sigma \\
&= -p^2 X \lrcorner \sigma,
\end{aligned}$$

thus $\phi_{\mathrm{D}}^2\sigma = -p^2\sigma$. With Proposition 3.3.4 we obtain

$$-p^2 A(X,Y)\sigma = A(X,Y)\phi_{\mathrm{D}}^2\sigma = \phi_{\mathrm{D}}^2 A(X,Y)\sigma = -(p-2)^2 A(X,Y)\sigma,$$

thus $A(X,Y)\sigma = 0$ if $p \neq 1$. Proposition 3.4.6 then shows that σ is a special Killing form if $p \neq 0, 1$. □

Proposition 3.5.3. *Let (M, g, ξ) be a Sasakian manifold. Then every basic Killing p-form with $p \neq 0$ has to vanish.*

Proof. Let $\sigma \in \mathcal{K}^p(M, g)$ be basic. Then we obtain

$$0 = \mathcal{L}_\xi\sigma = \nabla_\xi\sigma - \phi_{\mathrm{D}}\sigma = (p+1)\mathcal{L}_\xi\sigma - \phi_{\mathrm{D}}\sigma = -\phi_{\mathrm{D}}\sigma,$$

thus $0 = \phi_{\mathrm{D}}^2\sigma = -p^2\sigma$. □

Next we turn to vertical Killing forms.

Lemma 3.5.4. *Let $\sigma \in \mathcal{K}^p(M, g)$ be vertical. If p is even, then $\sigma = 0$, and if p is odd, then $\sigma = \mathrm{const} \cdot \eta \wedge \omega^{\frac{p-1}{2}}$.*

Proof. We first discuss the cases $p = 0$ and $p = 1$. If $p = 0$, then trivially $\sigma = 0$. If $p = 1$, then there exists a function $f \in \mathcal{C}^\infty(M)$ such that $\sigma = f\eta$. The Killing equation for σ implies $\nabla_\xi(f\eta) = \frac{1}{2}\xi \lrcorner d(f\eta)$, which yields $df = -\xi(f)\eta$. Taking the inner product with ξ we obtain $\xi(f) = -\xi(f) = 0$, thus $df = 0$ and f is a constant, i.e. $\sigma = \mathrm{const} \cdot \eta$.

Assume $p \geq 2$. Since σ is both Killing and vertical, $\nabla_\xi\sigma$ is both horizontal and vertical:

$$\begin{aligned}
\xi \lrcorner \nabla_\xi\sigma &= \tfrac{1}{p+1}\xi \lrcorner \xi \lrcorner d\sigma = 0, \\
\eta \wedge \nabla_\xi\sigma &= \nabla_\xi(\eta \wedge \sigma) = 0
\end{aligned}$$

which implies $\nabla_\xi \sigma = 0$. We have the commutator relation

$$[\nabla_\xi, \delta] = [\mathcal{L}_\xi + \phi_{\mathrm{D}}, \delta] = \delta^c + (n - \deg)\xi \lrcorner .$$

Applying this to σ we obtain on one hand $\delta^c \sigma = -(n-p)\xi \lrcorner \sigma$. On the other hand we have

$$\delta^c \sigma = -\tfrac{2}{p+1} \Lambda d\sigma.$$

From $\eta \wedge \sigma = 0$ we conclude $0 = d(\eta \wedge \sigma) = 2L\sigma - \eta \wedge d\sigma$. Thus we obtain, using the commutator relation $[L, \Lambda]\sigma = (p - \frac{n-1}{2})\sigma - \eta \wedge \xi \lrcorner \sigma = -\frac{1}{2}(n - 2p + 1)\sigma$,

$$\begin{aligned} L\Lambda\sigma &= \Lambda L\sigma + [L, \Lambda]\sigma \\ &= \tfrac{1}{2}\eta \wedge \Lambda d\sigma - \tfrac{1}{2}(n - 2p + 1)\sigma \\ &= \tfrac{1}{4}(p+1)(n-p)\eta \wedge \xi \lrcorner \sigma - \tfrac{1}{2}(n - 2p + 1)\sigma \\ &= \tfrac{1}{4}(p-1)(n-p+2)\sigma. \end{aligned}$$

We will generalize this to

$$L^s \Lambda^s \sigma = c_s \sigma \tag{3.5.1}$$

with $c_s \in \mathbb{R} \setminus \{0\}$ for all $s = 1, \ldots, \lfloor \frac{p}{2} \rfloor$. This determines σ completely up to a constant: If p is even, we choose $s = \frac{p}{2}$. With $f := \Lambda^s \sigma \in C^\infty(M)$ we obtain

$$\sigma = \tfrac{1}{c_s} f \omega^s,$$

which implies $\sigma = 0$ since σ is vertical while ω^s is horizontal. If p is odd, we choose $s = \frac{p-1}{2}$. Since σ is vertical there exists a function $f \in C^\infty(M)$ such that $\Lambda^s \sigma = f\eta$, which shows

$$\sigma = \tfrac{1}{c_s} f \eta \wedge \omega^s.$$

From Proposition 3.4.9 we know that $\Lambda^s \sigma$ is a vertical Killing 1-form, which we already discussed in the beginning of this proof. We obtain $f = \text{const}$ and $\sigma = \text{const} \cdot \eta \wedge \omega^s$.

It remains to show (3.5.1), which we will do by induction on s. For $s = 1$ the statement is true with

$$c_1 = \tfrac{1}{4}(p-1)(n-p+2) \neq 0,$$

since $p \geq 2$. Now assume (3.5.1) holds for $s < \lfloor \frac{p}{2} \rfloor$. If we write $n = 2m+1$, we have the commutator rule

$$[L^k, \Lambda] = -k(m - \deg - k + 1)L^{k-1} - k\ \eta \wedge \xi \lrcorner L^{k-1},$$

which simplifies to

$$[L^k, \Lambda] = -k(m - \deg - k + 2)L^{k-1}$$

on vertical forms. Since σ and therefore $\Lambda^s \sigma$ is vertical, we get

$$\begin{aligned} L^{s+1}\Lambda^{s+1}\sigma &= LL^s \Lambda \Lambda^s \sigma \\ &= L\left(\Lambda L^s - s(m - \deg(\Lambda^s \sigma) - s + 2)L^{s-1}\right)\Lambda^s \sigma \\ &= L\Lambda L^s \Lambda^s \sigma - s(m - p + s + 2)L^s \Lambda^s \sigma \end{aligned}$$

$$= (c_1 - s(m-p+s+2))c_s\sigma$$
$$= c_{s+1}\sigma.$$

Since $c_s \neq 0$ by induction, it remains to show that $c_1 \neq s(m-p+s+3)$. Assume that this is false, i.e. that

$$(p-1)(2m-p+3) = 4s(m-p+s+2) \tag{3.5.2}$$

holds. We see that the left-hand side has to be even and that the right-hand side is positive. Thus p has to be odd and $(m-p+s+2) > 0$. We obtain $\lfloor \frac{p}{2} \rfloor = \frac{p-1}{2}$, i.e. we have $s < \frac{p-1}{2}$. This implies

$$\begin{aligned} 4s(m-p+s+2) &< 4 \cdot \tfrac{p-1}{2} \cdot (m-p+\tfrac{p-1}{2}+2) \\ &= (p-1)(2m-p+3), \end{aligned}$$

which is a contradiction to (3.5.2). $\square$

With Lemma 3.5.2 and Lemma 3.5.4 we are finally able to prove Theorem 3.5.1.

Proof (of Theorem 3.5.1). The conformal Killing equation with $X = \xi$ becomes

$$\nabla_\xi \psi = \tfrac{1}{p+1}\xi \lrcorner d\psi - \tfrac{1}{n-p+1}\eta \wedge \delta\psi.$$

Taking the interior product with ξ shows $\xi \lrcorner (\eta \wedge \delta\psi) = 0$, thus $\delta\psi$ is vertical. To see that $\delta\psi$ is a Killing form we recall equation (3.3.11), namely

$$\eta \wedge \Big(A^-(X)\psi + \tfrac{1}{p}X \lrcorner q(A)\psi\Big) = \tfrac{1}{p}\eta(X)\eta \wedge \mathrm{v}(q(A)\psi).$$

Since ψ is horizontal and $q(A)$ commutes with $\xi \lrcorner$, the right-hand side has to vanish and $A^-(X)\psi + \frac{1}{p}X \lrcorner q(A)\psi$ is vertical. But it is also horizontal because $A^-(X)$ commutes with $\xi \lrcorner$ as well. Thus we obtain

$$A^-(X)\psi + \tfrac{1}{p}X \lrcorner q(A)\psi = 0,$$

which by Corollary 3.1.14 shows that $\delta\psi$ is a Killing $(p-1)$-form. From Lemma 3.5.4 we know $\delta\psi = 0$ if p is odd or $\delta\psi = \mathrm{const} \cdot \eta \wedge \omega^{\frac{p-2}{2}}$ if p is even. If p is odd, then ψ is a horizontal Killing form and Lemma 3.5.2 yields the claim. If p is even, then by Proposition 3.2.1 $\delta\psi$ is a special Killing form with constant $-p$. From Lemma 3.3.8 we obtain that

$$\sigma := \psi - \tfrac{1}{p\,(n-p+1)}d\delta\psi$$

is a Killing form, thus $\psi = \sigma + \frac{1}{p\,(n-p+1)}d\delta\psi = \sigma + \mathrm{const} \cdot \omega^{\frac{p-1}{2}}$. Again Lemma 3.5.2 yields the remaining claim. $\square$

Theorem 3.5.1 yields a corresponding result on vertical conformal Killing forms:

Theorem 3.5.5. *Let (M, g, ξ) be a $n = (2m+1)$-dimensional Sasakian manifold and $\psi \in \mathcal{CK}^p(M, g)$ vertical.*

- *Let p be odd. If $m+1 \le p \le n-2$, then $\psi = \mathrm{const} \cdot \eta \wedge \omega^{\frac{p-1}{2}} + \tau_S$, where τ_S is a vertical special $*$-Killing form with $\phi_{\mathrm{D}}^2 \tau_S = -(n-p)^2 \tau_S$. If $p \le m$, then $\psi = \mathrm{const} \cdot \eta \wedge \omega^{\frac{p-1}{2}}$.*

- *Let p be even. If $m+1 \le p \le n-1$, then $\psi = \tau_S$ is a vertical special $*$-Killing form with $\phi_{\mathrm{D}}^2 \tau_S = -(n-p)^2 \tau_S$. If $p \le m$, then $\psi = 0$.*

We conclude this section with the discussion of conformal Killing forms ψ that satisfy the additional equation

$$\phi_{\mathrm{D}}^2 \psi = -\lambda^2 \psi.$$

The following result is an immediate consequence of the Theorems 3.5.1 and 3.5.5 since every form of this type is either horizontal or vertical, depending on the parity of $\lambda - p$.

Corollary 3.5.6. *Let (M, g, ξ) be a n-dimensional Sasakian manifold and $\psi \in \mathcal{CK}^p(M, g)$ with $\phi_{\mathrm{D}}^2 \psi = -\lambda^2 \psi$, where $\lambda \in \mathbb{N}$.*

- *If $\lambda \notin \{0, p, n-p\}$, then $\psi = 0$.*

- *If $\lambda = p$, then $\psi = \sigma_s$ is a horizontal special Killing form, and if $\lambda = n-p$, then $\psi = \tau_S$ is a vertical special $*$-Killing form.*

- *Assume $\lambda = 0$. If $p \neq 0$ is even, then $\psi = \mathrm{const} \cdot \omega^{\frac{p}{2}}$, and if $p \neq n$ is odd, then $\psi = \mathrm{const} \cdot \eta \wedge \omega^{\frac{p-1}{2}}$.*

3.5.2 Sasaki-Einstein and Eta-Einstein manifolds

Theorem 3.5.7. *Let (M, g) be an Einstein manifold and $\sigma \in \mathcal{K}^p(M, g)$ with $p \ge 3$. If (M, g) admits a Sasakian structure, then*

$$q(A) d\sigma = 0.$$

In particular, if M is compact, then σ is a special Killing form.

Proof. From Proposition 3.1.20 we know

$$d(\Delta\sigma) = -\tfrac{1}{p} \mathrm{Ric}_{\mathrm{D}}(d\sigma) + \tfrac{p-1}{p} q(R)(d\sigma) + \tfrac{p+1}{p} (\delta R)^+ \sigma.$$

Since (M, g) is Sasaki-Einstein we have $\mathrm{Ric}_{\mathrm{D}} = -\deg(n-1)\mathrm{id}$ and $\delta A = \delta R = 0$ by Lemma 2.1.11. From $\Delta\sigma = (p+1)(n-p)\sigma$ we obtain

$$q(R) d\sigma = (p+1)(n-p-1) d\sigma,$$

thus $q(A) d\sigma = 0$. If M is compact then by Proposition 3.4.6 we see that σ is a special Killing form. $\square$

Since 3-Sasakian manifolds are automatically Einstein we have an immediate corollary.

Corollary 3.5.8. *Let (M, g) be compact Riemannian manifold. If (M, g) admits a 3-Sasakian structure then every Killing p-form is special if $p \ge 2$, and every $*$-Killing p-form is special if $p \le n-2$.*

If the manifold is strictly Eta-Einstein then we can completely determine conformal Killing forms up to Killing vector fields.

Theorem 3.5.9. *Let (M, g, ξ) be a strictly Eta-Einstein manifold and $\psi \in \mathcal{CK}^p(M, g)$.*

- *If $p \neq 1$ is odd, then $\psi = \mathrm{const} \cdot \eta \wedge \omega^{\frac{p-1}{2}}$, and if $p \neq n-1$ is even, then $\psi = \mathrm{const} \cdot \omega^{\frac{p}{2}}$.*
- *If $p = 1$, then $\psi = K^* + \mathrm{const} \cdot \eta$, and if $p = n-1$, then $\psi = \mathrm{const} \cdot \omega^{\frac{n-1}{2}} + *(K^*)$, where in both cases K is a Killing vector field.*

Proof. Since (M, g, ξ) is a strictly Eta-Einstein manifold we have

$$\mathrm{Ric} = (n-1-c)\mathrm{id} + c\eta \otimes \xi$$

with $c \in \mathbb{R} \setminus \{0\}$, or equivalently

$$\mathrm{Ric}^A = c\phi^2.$$

Thus the Ricci-form ρ^A satisfies

$$\rho^A \bullet X = c\mathrm{Ric}^A(\phi X) = -c\phi X,$$

which implies that the action of ρ^A on $\Omega^p(M)$ is given by

$$\rho^A \bullet = -c\phi_{\mathrm{D}}.$$

With Theorem 3.3.9 we decompose ψ into the sum of a Killing form σ and a $*$-Killing form τ, i.e. $\psi = \sigma + \tau$. Using the Propositions 3.3.4 and 3.3.5 yields

$$-c\phi_{\mathrm{D}}\sigma = \rho^A \bullet \sigma = 0$$

for $p \neq 1$ and

$$-c\phi_{\mathrm{D}}\tau = \rho^A \bullet \tau = 0$$

for $p \neq n-1$. Then the claim follows from Corollary 3.5.6. $\square$

3.5.3 Normal conformal Killing forms

In [L04], F. Leitner introduces the notion of normal conformal Killing forms on pseudo-Riemannian manifolds. These are forms that satisfy the conditions

$$\nabla_X \psi = \tfrac{1}{p+1} X \lrcorner d\psi - \tfrac{1}{n-p+1} X^* \wedge \delta\psi, \tag{3.5.3}$$

$$\nabla_X (d\psi) = (p+1) K(X)^* \wedge \psi - (p+1) X^* \wedge \Box_p \psi, \tag{3.5.4}$$

$$\nabla_X (\delta\psi) = -(n-p+1) K(X) \lrcorner \psi - (n-p+1) X \lrcorner \Box_p \psi, \tag{3.5.5}$$

$$\nabla_X (\Box_p \psi) = -\tfrac{1}{p+1} K(X) \lrcorner d\psi - \tfrac{1}{n-p+1} K(X)^* \wedge \psi, \tag{3.5.6}$$

where $K \in \Gamma(\mathrm{End}(TM))$ and $\Box_p : \Omega^p(M) \longrightarrow \Omega^p(M)$ are defined by

$$\begin{aligned} K(X) &:= -\tfrac{1}{n-2}\left(\mathrm{Ric}(X) - \tfrac{\mathrm{scal}}{2(n-1)} X\right) \\ \Box_p &:= \tfrac{1}{n-2p}\left(\nabla^*\nabla - \tfrac{\mathrm{scal}}{2(n-1)}\right) && \text{for } 2p \neq n, \\ \Box_{\frac{n}{2}} &:= \tfrac{1}{n}\left(\tfrac{1}{p+1}(\delta d - d\delta) + 2K_{\mathrm{D}} - \tfrac{\mathrm{scal}}{2(n-1)}\right). \end{aligned}$$

In the remainder of this section we prove the following theorem:

Theorem 3.5.10. *Let (M, g, ξ) be a n-dimensional Sasakian manifold and $\psi \in \Omega^p(M)$ a normal conformal Killing p-form with $2 \leq p \leq n-2$. The set M_ψ of points where ψ does not vanish,*

$$M_\psi := \{x \in M \,|\, \psi_x \neq 0\},$$

is an open Einstein submanifold of M and ψ is the sum of a special Killing and a special $*$-Killing p-form.

Proof. We rewrite (3.5.4) and (3.5.5) in terms of the Sasakian curvature tensor A and use our results of the Sections 3.3 and 3.4 since (3.5.3) precisely means that ψ is a conformal Killing form. We decompose $\psi = \sigma + \tau \in K^p(M, g) \oplus *\mathcal{K}^p(M, g)$. Because of $2 \leq p \leq n-2$, the Weitzenböck formula $\nabla^*\nabla = \Delta - q(R)$ yields $\nabla^*\nabla\sigma = (n-p)\sigma$ and $\nabla^*\nabla\tau = p\,\tau$ and this gives

$$\begin{aligned}
\Box_p\psi &= \tfrac{1}{n-2p}\left(\nabla^*\nabla\sigma + \nabla^*\nabla\tau - \tfrac{\mathrm{scal}^A + n(n-1)}{2(n-1)}\sigma - \tfrac{\mathrm{scal}^A + n(n-1)}{2(n-1)}\tau\right)\\
&= -\left(\tfrac{\mathrm{scal}^A}{2(n-1)(n-2p)} - \tfrac{1}{2}\right)\sigma - \left(\tfrac{\mathrm{scal}^A}{2(n-1)(n-2p)} + \tfrac{1}{2}\right)\tau.
\end{aligned}$$

The endomorphism K becomes

$$\begin{aligned}
K(X) &= -\tfrac{1}{n-2}\left((\mathrm{Ric}^A + n - 1)X - \tfrac{\mathrm{scal}^A + n(n-1)}{2(n-1)}X\right)\\
&= -\tfrac{1}{n-2}\mathrm{Ric}^A(X) + \left(\tfrac{\mathrm{scal}^A}{2(n-1)(n-2)} - \tfrac{1}{2}\right)X.
\end{aligned}$$

From Proposition 3.1.15 we know $\nabla_X(d\psi) = \nabla_X(d\sigma) = -(p+1)X^* \wedge \sigma + \frac{p+1}{p}A^+(X)\sigma$ and $\nabla_X(\delta\psi) = \nabla_X(\delta\tau) = (n-p+1)X \lrcorner \tau - \frac{n-p+1}{n-p}A^-(X)\tau$. We obtain

$$\begin{aligned}
\tfrac{1}{p}A^+(X)\sigma &= \tfrac{1}{p+1}\left(\nabla_X(d\sigma) + (p+1)X^* \wedge \sigma\right)\\
&= K(X)^* \wedge \psi - X^* \wedge \Box_p\psi + X^* \wedge \sigma\\
&= -\tfrac{1}{n-2}\mathrm{Ric}^A(X)^* \wedge \psi + \tfrac{(n-p-1)\mathrm{scal}^A}{(n-1)(n-2)(n-2p)}X^* \wedge \psi,
\end{aligned} \tag{3.5.7}$$

$$\begin{aligned}
\tfrac{1}{n-p}A^-(X)\tau &= -\tfrac{1}{n-p+1}\left(\nabla_X(\delta\tau) - (n-p+1)X \lrcorner \tau\right)\\
&= K(X) \lrcorner \psi + X \lrcorner \Box_p\psi - X \lrcorner \tau\\
&= -\tfrac{1}{n-2}\mathrm{Ric}^A(X) \lrcorner \psi - \tfrac{(p-1)\mathrm{scal}^A}{(n-1)(n-2)(n-2p)}X \lrcorner \psi.
\end{aligned} \tag{3.5.8}$$

We set $X = \xi$ in 3.5.7 and 3.5.8 and obtain $\mathrm{scal}^A \cdot \eta \wedge \psi = 0$ and $\mathrm{scal}^A \cdot \xi \lrcorner \psi = 0$, thus

$$\mathrm{scal}^A \cdot \psi = 0.$$

With this we may simplify (3.5.7) and (3.5.8) to

$$A^+(X)\sigma = -\tfrac{p}{n-2}\mathrm{Ric}^A(X)^* \wedge \psi, \tag{3.5.9}$$

$$A^-(X)\tau = -\tfrac{n-p}{n-2}\mathrm{Ric}^A(X) \lrcorner \psi. \tag{3.5.10}$$

Since σ is a Killing and τ is a $*$-Killing p-form, we know $\sigma \in B^p_+$ and $\tau \in B^p_-$ (cf Definition 3.3.2). From the Propositions 3.3.4 and 3.3.5 we obtain

- $A^-(X)\tau = 0$ and $\xi \lrcorner A^+(X)\sigma = 0$ for $p \leq m$,

- $A^+(X)\sigma = 0$ and $\eta \wedge A^-(X)\tau = 0$ for $p \geq m+1$,

where we set $n = 2m+1$. We first treat the case $p \leq m$, where we obtain

$$\mathrm{Ric}^A(X) \lrcorner \psi = 0$$

from (3.5.10). Taking the inner product of (3.5.9) with ξ yields $\mathrm{Ric}^A(X)^* \wedge \xi \lrcorner \psi = 0$. We get

$$\begin{aligned} 0 &= \mathrm{Ric}^A(X) \lrcorner \left(\mathrm{Ric}^A(X)^* \wedge \xi \lrcorner \psi\right) \\ &= g(\mathrm{Ric}^A(X), \mathrm{Ric}^A(X))\xi \lrcorner \psi - \mathrm{Ric}^A(X)^* \wedge \mathrm{Ric}^A(X) \lrcorner \xi \lrcorner \psi \\ &= \|\mathrm{Ric}^A(X)\|^2 \xi \lrcorner \psi \end{aligned}$$

Consider the smooth function $M \longrightarrow \mathbb{R}, x \longmapsto \|\xi \lrcorner \psi\|_x$. Then we have

$$\|\mathrm{Ric}^A(X)\|_x \|\xi \lrcorner \psi\|_x = 0 \tag{3.5.11}$$

for all $x \in M$ and all $X \in T_xM$. We define

$$\begin{aligned} E &:= \{x \in M \mid \|\xi \lrcorner \psi\|_x = 0\}, \\ M_1 &:= \mathrm{int}(E), \\ M_2 &:= M \setminus E. \end{aligned}$$

Then E is a closed and M_1 and M_2 are open subsets of M and we have $M = M_1 \cup M_2 \cup \partial E$. Note that $M_1 \cup M_2$ cannot be empty since otherwise $M = \partial E \subset E \subset M$, which implies $E = M$ and $M = \partial E = \partial M = \emptyset$. So $M_1 \cup M_2$ is an open, dense submanifold of M.

- If $M_1 \neq \emptyset$, then M_1 is an open submanifold of M and ψ is a horizontal conformal Killing form on M_1. By Theorem 3.5.1 we know that σ is a special Killing form, which by Proposition 3.4.6 means $A^+(X)\sigma = 0$. The normal conformal Killing equation (3.5.9) implies $\mathrm{Ric}^A(X)^* \wedge \psi = 0$ on M_1. Taking the interior product with $\mathrm{Ric}^A(X)$ we obtain $\|\mathrm{Ric}^A(X)\|^2 \psi = 0$ on M_1.

- If $M_2 \neq \emptyset$, then M_2 is an open submanifold of M with $\|\xi \lrcorner \psi\|_x \neq 0$ for all $x \in M_2$. From (3.5.11) we obtain $\|\mathrm{Ric}^A(X)\|_x = 0$ for all $x \in M_2$ and $X \in T_xM_2 = T_xM$. Equation (3.5.9) shows that σ is a special Killing form.

Since $M_1 \cup M_2$ is dense in M we obtain

$$\|\mathrm{Ric}^A(X)\|_x \psi_x = 0$$

for all $x \in M$ and all $X \in T_xM$, which yields the claim.

The case $p \geq m+1$ is treated similarly. $\square$

A Lie derivative of the curvature tensor

Lemma A.0.11. *Let (M,g) be a Riemannian manifold and let $v \in \Gamma(TM)$. If we define $B(X,Y) := \nabla^2_{X,Y}v - R(X,v)Y$, then we have*

$$(\mathcal{L}_v R)(X,Y)Z = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z).$$

Proof. This is a straightforward calculation: Assuming that X, Y and Z are parallel at a point we obtain

$$\begin{aligned}
&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) - (\mathcal{L}_v R)(X,Y)Z \\
&= \nabla_X(\nabla^2_{Y,Z}v - R(Y,v)Z) - \nabla_Y(\nabla^2_{X,Z})v - R(X,v)Z) \\
&\quad - \mathcal{L}_v(R(X,Y)Z) + R(\mathcal{L}_v X, Y)Z + R(X, \mathcal{L}_v Y)Z + R(X,Y)\mathcal{L}_v Z \\
&= \nabla_X\nabla_Y\nabla_Z v - \nabla_X\nabla_{\nabla_Y Z}v - (\nabla_X R)(Y,v)Z \\
&\quad - \nabla_Y\nabla_X\nabla_Z v + \nabla_Y\nabla_{\nabla_X Z}v + (\nabla_Y R)(X,v)Z \\
&\quad - \mathcal{L}_v(R(X,Y)Z) - R(X,Y)\nabla_Z v \\
&= \nabla_X\nabla_Y\nabla_Z v - \nabla_Y\nabla_X\nabla_Z v - \nabla_X\nabla_{\nabla_Y Z}v + \nabla_Y\nabla_{\nabla_X Z}v \\
&\quad + \nabla_{R(X,Y)Z}v - R(X,Y)\nabla_Z v \\
&\quad + (\nabla_v R)(X,Y)Z - \nabla_v(R(X,Y)Z) \\
&= -\nabla_X\nabla_{\nabla_Y Z}v + \nabla_Y\nabla_{\nabla_X Z}v + \nabla_{\nabla_X\nabla_Y Z}v - \nabla_{\nabla_Y\nabla_X Z}v \\
&= -\nabla^2_{X,\nabla_Y Z}v + \nabla^2_{Y,\nabla_X Z}v \\
&= 0. \qquad \square
\end{aligned}$$

B Cone geometry

Let (M,g) be a Riemannian manifold. The Riemannian cone $(C(M), g_C)$ over M is defined by $C(M) := M \times \mathbb{R}_+$ and

$$g_C := r^2 g + dr^2.$$

In the following we identify every vector field on M with a vector field on $C(M)$ via $X_x \hat{=} X_{(x,r)} = X_x + 0 \cdot \partial_r \in T_x M \oplus T_r\mathbb{R}$.

It is well-known that the Levi-Civita connection ∇^C of the cone acts on 1-forms via

$$\begin{aligned}
\nabla^C_X Y^* &= \nabla_X Y^* - \tfrac{1}{r}g(X,Y)dr, \\
\nabla^C_{\partial_r} Y^* &= -\tfrac{1}{r}Y^*, \\
\nabla^C_X dr &= rX^*, \\
\nabla^C_{\partial_r} dr &= 0,
\end{aligned}$$

where X^* and Y^* are the g-duals of X and Y. With this the covariant derivative of p-forms $\alpha \in \Omega^p(M)$ is given by

$$\begin{aligned}
\nabla^C_X \alpha &= \nabla_X \alpha + \tfrac{1}{r} X \lrcorner (dr \wedge \alpha), \\
\nabla^C_{\partial_r} \alpha &= -\tfrac{p}{r}\alpha.
\end{aligned}$$

We use this in order to compute the action of $R_C(X,Y)$ on $\Omega^p(M)$.

Lemma B.0.12. *Let (M, g) be a Riemannian manifold. Then for all $X, Y \in TM$ and all $\alpha \in \Omega^p(M)$ we have*

$$R_C(X, Y)\alpha = R(X, Y)\alpha + (X^* \wedge Y^*) \bullet \alpha.$$

Proof.

$$\begin{aligned}
R_C(X, Y)\alpha &= \nabla^C_X \nabla^C_Y \alpha - \nabla^C_Y \nabla^C_X \alpha - \nabla^C_{[X,Y]}\alpha \\
&= \nabla^C_X \left(\nabla_Y \alpha + \tfrac{1}{r} Y \lrcorner (dr \wedge \alpha)\right) \\
&\qquad - \nabla^C_Y \left(\nabla_X \alpha + \tfrac{1}{r} X \lrcorner (dr \wedge \alpha)\right) \\
&\qquad\qquad - \nabla_{[X,Y]}\alpha + \tfrac{1}{r}[X, Y] \lrcorner (dr \wedge \alpha) \\
&= \nabla^C_X \nabla_Y \alpha + \tfrac{1}{r} Y \lrcorner (\nabla^C_X dr \wedge \alpha) + \tfrac{1}{r} Y \lrcorner (dr \wedge \nabla^C_X \alpha) \\
&\qquad - \nabla^C_Y \nabla_X \alpha - \tfrac{1}{r} X \lrcorner (\nabla^C_Y dr \wedge \alpha) - \tfrac{1}{r} X \lrcorner (dr \wedge \nabla^C_Y \alpha) \\
&\qquad\qquad - \nabla_{[X,Y]}\alpha \\
&= \nabla_X \nabla_Y \alpha - \nabla_Y \nabla_Y \alpha - \nabla_{[X,Y]}\alpha \\
&\qquad + Y \lrcorner (X^* \wedge \alpha) - X \lrcorner (Y^* \wedge \alpha) \\
&= R(X, Y)\alpha + (Y \lrcorner (X^* \wedge \alpha) - X \lrcorner (Y^* \wedge \alpha)) \\
&= R(X, Y)\alpha + (X^* \wedge Y^*) \bullet \alpha. \qquad \square
\end{aligned}$$

C Linear Algebra

Lemma C.0.13. *Let V be a vector space over a field K and $A, B \in \mathrm{End}_K(V)$ such that $\ker(B)$ is A-invariant. If there exist eigenvectors $v_1, \ldots, v_r \in V \setminus \{0\}$ of A with pairwise different eigenvalues such that $\sum v_k \in \ker(B)$, then $v_k \in \ker(B)$ for all k.*

Proof. (of Lemma C.0.13) Let λ_k be the eigenvalue of v_k, i.e. $Av_k = \lambda_k v_k$. Since $v \in \ker(B)$ and $A(\ker(B)) \subset \ker(B)$ we know $B(A^s v) = 0$ for all $s \geq 0$, in particular for all s with $0 \leq s \leq r - 1$. This yields $\sum \lambda_k^s B v_k = 0$. We obtain the following linear system of equations:

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \ldots & \lambda_r^{r-1} \end{pmatrix} \begin{pmatrix} Bv_1 \\ Bv_2 \\ Bv_3 \\ \vdots \\ Bv_r \end{pmatrix} = 0.$$

The matrix on the left is a Vandermonde matrix with Vandermonde determinant

$$\prod_{k>l} (\lambda_k - \lambda_l)$$

and is thus invertible since the eigenvalues are pairwise different. But then $Bv_k = 0$ for all k is the only possible solution. $\square$

D Commutator and anticommutator relations

This Appendix contains a list of commutator and anticommutator relations on Sasakian manifolds.

$$
\begin{aligned}
[L,\Lambda] &= (\deg - m)\mathrm{id} - \eta\wedge\xi\lrcorner & [\Lambda, L] &= (m-\deg) + \eta\wedge\xi\lrcorner \\
[L^k,\Lambda] &= k(\deg - m + k - 1)L^{k-1} & [\Lambda^k, L] &= k(m - \deg + k - 1)\Lambda^{k-1} \\
&\quad - k\,\eta\wedge\xi\lrcorner L^{k-1} & &\quad + k\,\eta\wedge\xi\lrcorner\Lambda^{k-1} \\
[L, X^*\wedge] &= 0 & [\Lambda, X^*\wedge] &= \phi X\lrcorner \\
[L, X\lrcorner] &= -\phi X^*\wedge & [\Lambda, X\lrcorner] &= 0 \\
[L,\phi] &= 0 & [\Lambda,\phi] &= 0 \\
[L,\nabla_X] &= -\eta\wedge\ X^*\wedge & [\Lambda,\nabla_X] &= \xi\lrcorner\ X\lrcorner \\
[L,d] &= 0 & [\Lambda, d] &= -\delta^c - (\deg - 1)\xi\lrcorner \\
[L,\delta] &= d^c - (n - \deg - 1)\eta\wedge & [\Lambda,\delta] &= 0 \\
[L,d^c] &= -2\eta\wedge L & [\Lambda, d^c] &= \delta + \xi\lrcorner\mathcal{L}_\xi \\
[L,\delta^c] &= -d + \eta\wedge\mathcal{L}_\xi & [\Lambda,\delta^c] &= 2\xi\lrcorner\Lambda \\
[L, A^+(X)] &= 0 & [\Lambda, A^+(X)] &= A^-(\phi X) \\
[L, R^+(X)] &= 2X^*\wedge L & [\Lambda, R^+(X)] &= R^-(\phi X) - 2X\wedge\Lambda \\
& & &\quad + (n - 2\deg)\phi X\lrcorner \\
[L, A^-(X)] &= -A^+(\phi X) & [\Lambda, A^-(X)] &= 0 \\
[L, R^-(X)] &= -R^+(\phi X) - 2X\lrcorner L & [\Lambda, R^-(X)] &= 2X\lrcorner\Lambda \\
&\quad + (n - 2\deg)\phi X^*\wedge & & \\
[L,\Delta] &= 2\eta\wedge d - 2(n - 2\deg - 1)L & [\Lambda,\Delta] &= 2\xi\lrcorner\delta + 2(n - 2\deg + 1)\Lambda \\
[L,\mathcal{L}_\xi] &= 0 & [\Lambda,\mathcal{L}_\xi] &= 0 \\
[L, q(A)] &= 0 & [\Lambda, q(A)] &= 0 \\
[L, q(R)] &= -2(n - 2\deg - 2)L & [\Lambda, q(R)] &= 2(n - 2\deg + 2)\Lambda
\end{aligned}
$$

$$
\begin{aligned}
\{\eta\wedge,\xi\lrcorner\} &= \mathrm{id} & \{\xi\lrcorner,\eta\} &= \mathrm{id} \\
[\eta\wedge,\phi] &= 0 & [\xi\lrcorner,\phi] &= 0 \\
[\eta\wedge,\nabla_X] &= -\phi X^*\wedge & [\xi\lrcorner,\nabla_X] &= -\phi X\lrcorner \\
\{\eta\wedge, d\} &= 2L & \{\xi\lrcorner, d\} &= \mathcal{L}_\xi \\
\{\eta\wedge,\delta\} &= -\mathcal{L}_\xi & \{\xi\lrcorner,\delta\} &= 2\Lambda \\
\{\eta\wedge, d^c\} &= 0 & \{\xi\lrcorner, d^c\} &= -\eta\wedge\xi\lrcorner\deg\mathrm{id} \\
\{\eta\wedge,\delta^c\} &= -(n - \deg - 1)\mathrm{id} - \eta\wedge\xi\lrcorner & \{\xi\lrcorner,\delta^c\} &= 0 \\
\{\eta\wedge, A^+(X)\} &= 0 & \{\xi\lrcorner, A^+(X)\} &= 0 \\
\{\eta\wedge, R^+(X)\} &= -X^*\wedge\eta\wedge & \{\xi\lrcorner, R^+(X)\} &= -\deg\eta(X)\mathrm{id} \\
& & &\quad + X^*\wedge\xi\lrcorner \\
\{\eta\wedge, A^-(X)\} &= 0 & \{\xi\lrcorner, A^-(X)\} &= 0 \\
\{\eta\wedge, R^-(X)\} &= -\eta\wedge X\lrcorner & \{\xi\lrcorner, R^-(X)\} &= -X\lrcorner\xi\lrcorner \\
&\quad + (\deg - 2m)\eta(X)\mathrm{id} & &
\end{aligned}
$$

$$
\begin{aligned}
[\eta\wedge,\Delta] &= 2d^c - 2(n-\deg -1)\eta\wedge & [\xi\lrcorner,\Delta] &= -2\delta^c - 2(\deg -1)\xi\lrcorner\\
[\eta\wedge,\mathcal{L}_\xi] &= 0 & [\xi\lrcorner,\mathcal{L}_\xi] &= 0\\
[\eta\wedge,q(A)] &= 0 & [\xi\lrcorner,q(A)] &= 0\\
[\eta\wedge,q(R)] &= -(n-2\deg -1)\eta\wedge & [\xi\lrcorner,q(R)] &= (n-2\deg +1)\xi\lrcorner
\end{aligned}
$$

$$
\begin{aligned}
[\phi_{\mathrm{D}},\nabla_X] &= -(\eta\wedge X^*)\bullet & [\nabla_X,\phi_{\mathrm{D}}] &= (\eta\wedge X^*)\bullet\\
[\phi_{\mathrm{D}},d] &= d^c - \deg\eta\wedge & [\nabla_\xi,d] &= d^c - \deg\eta\wedge\\
[\phi_{\mathrm{D}},\delta] &= \delta^c + (n-\deg)\xi\lrcorner & [\nabla_\xi,\delta] &= \delta^c + (n-\deg)\xi\lrcorner\\
[\phi_{\mathrm{D}},d^c] &= -d + 2\xi\lrcorner L + \eta\wedge\mathcal{L}_\xi & [\nabla_\xi,d^c] &= -d + 2\xi\lrcorner L + \eta\wedge\mathcal{L}_\xi\\
[\phi_{\mathrm{D}},\delta^c] &= -\delta + 2\eta\wedge\Lambda - \xi\lrcorner\mathcal{L}_\xi & [\nabla_\xi,\delta^c] &= -\delta + 2\eta\wedge\Lambda - \xi\lrcorner\mathcal{L}_\xi\\
[\phi_{\mathrm{D}},A^+(X)] &= A^+(\phi X) & [\nabla_\xi,A^+(X)] &= A^+(\nabla_\xi X)\\
[\phi_{\mathrm{D}},R^+(X)] &= R^+(\phi X) & [\nabla_\xi,R^+(X)] &= R^+(\nabla_\xi X)\\
[\phi_{\mathrm{D}},A^-(X)] &= A^-(\phi X) & [\nabla_\xi,A^-(X)] &= A^-(\nabla_\xi X)\\
[\phi_{\mathrm{D}},R^-(X)] &= R^-(\phi X) & [\nabla_\xi,R^-(X)] &= R^-(\nabla_\xi X)\\
[\phi_{\mathrm{D}},\Delta] &= 2\deg\mathcal{L}_\xi + 2\eta\wedge\delta - 2\xi\lrcorner d & [\nabla_\xi,\Delta] &= 2\deg\mathcal{L}_\xi + 2\eta\wedge\delta - 2\xi\lrcorner d\\
[\phi_{\mathrm{D}},q(A)] &= 0 & [\nabla_\xi,q(A)] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\{d,\delta\} &= \Delta & \{\delta,d\} &= \Delta\\
\{d,d^c\} &= \eta\wedge d + 2\deg L & \{\delta,d^c\} &= -(n-\deg -1)\mathcal{L}_\xi + \eta\wedge\delta\\
\{d,\delta^c\} &= -(p-1)\mathcal{L}_\xi - \xi\lrcorner d & \{\delta,\delta^c\} &= -\xi\lrcorner\delta - 2(n-\deg)\Lambda\\
[d,\Delta] &= 0 & [\delta,\Delta] &= 0\\
[d,\mathcal{L}_\xi] &= 0 & [\delta,\mathcal{L}_\xi] &= 0
\end{aligned}
$$

References

[B02] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, (2002).

[BG08] C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs. Oxford University Press, Oxford, (2008).

[FI02] T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. 6 (2002), no. 2, 303-335.

[FOW09] A. Futaki, H. Ono and G. Wang, *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds*, J. Differential Geom. 83 (2009), no. 3, 585-635.

[K71] T. Kashiwada, *A note on a Riemannian space with Sasakian* 3*-structure*, Natur. Sci. Rep. Ochanomizu Univ. 22 (1971), 1-2.

[L04] F. Leitner, *Normal conformal Killing forms*, preprint: arXiv:math/0406316v1 (2004) [05.05.2010].

[M07] A. Moroianu, *Killing vector fields with twistor derivative*, Journal of Differential Geometry 77, 1 (2007), 149-167.

[O62] M. Okumura, *On infinitesimal conformal and projective transformations of normal contact spaces*, Tohoku Math. J., 14 (1962), 398-412.

[S01] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Habilitationsschrift, LMU München, http://www.mi.uni-koeln.de/ semmelma/papers.html (2001) [05.05.2010]

[Y72a] S. Yamaguchi, *On a horizontal conformal Killing tensor of degree* p *in a Sasakian space*, Ann. Mat. Pura Appl. (4) 94 (1972), 217-230 .

[Y72b] S. Yamaguchi, *On a conformal Killing* p*-form in a compact Sasakian space*, Ann. Mat. Pura Appl. (4) 94 (1972), 231-245.

[Y75] S. Yamaguchi, *On a Killing* p*-form in a compact Kählerian manifold*, Tensor (N. S.) 29 (1975) , no. 3, 274-276.

Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass die Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen - noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt.

Die von mir vorgelegte Dissertation ist von Prof. U. Semmelmann betreut worden.

Köln, den 04.01.2011

Christian Stromenger