Evolution of Geometries with Torsion

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Sebastian Wolfgang Stock

aus Köln

Berichterstatter: Prof. Dr. Uwe Semmelmann

Prof. Dr. George Marinescu

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Kurzzusammenfassug

Wir reduzieren das Einbettungsproblem für $SU(2)$ und $SU(3)$-Strukturen auf das Einbettungsproblem für G_2-Strukturen. Der G_2-Fall wird mittels Automorphismen des Tangentialbündels untersucht und wir zeigen dass keine nicht-trivialen Langzeitlösungen des Einbettungsproblems existieren. Hitchins Flussgleichung für den G_2-Fall lässt sich zu einer Gleichung für die entsprechenden Automorphismen des Tangentialbündels verallgemeinern. Diese verallgemeinerte Flussgleichung beschreibt eine Deformation der Ausgangsstruktur mittels ihrer intrinsischen Torsion. Für reell-analytische Strukturen besitzt diese Flussgleichung stets eine eindeutige reell-analytische Lösung.

Wir erweitern den Kähler-Ricci Fluss auf $SU(n)$-Strukturen und untersuchen wann dieser gegen eine parallele $SU(n)$-Struktur konvergiert. Unser Ansatz ermöglicht zudem eine Erweiterung des Ricci Flusses auf G_2 und Spin_7-Strukturen. Für $SU(3)$-Strukturen auf sieben-dimensionalen Mannigfaltigkeiten beschreiben wir eine Gray-Hervella Klassifikation und definieren damit das G_2-Analogon zu Kähler $SU(3)$-Strukturen. Diese G_2-Strukturen besitzen eine Faserung, deren Fasern mittels des Ricci-Flusses deformiert werden können. Der faserweise Ricci-Fluss deformiert die ambiente G_2-Struktur zu einer Ricci-flachen G_2-Struktur.

Abstract

We reduce the embedding problem for hypo $SU(2)$ and $SU(3)$-structures to the embedding problem for hypo G_2-structures into parallel Spin(7)-manifolds. The latter will be described in terms of gauge deformations. This description involves the intrinsic torsion of the initial G_2-structure and allows us to prove that the evolution equations, for all of the above embedding problems, do not admit non-trivial longtime solutions. For G_2-structures we introduce a new flow, which generalizes Hitchin's flow equations. This intrinsic torsion flow admits unique solutions in the real analytic category.

We extend the Kähler-Ricci flow to $SU(n)$-structures and characterize under which conditions this flow converges to a parallel $SU(n)$-structure. This approach also yields an extension of the Ricci flow to G_2 and Spin_7-structures. For $SU(3)$-structures in dimension seven we derive the analogue of the Gray-Hervella classification. Based on this classification, we define a type of G_2-structure which can be regarded as the seven dimensional analogue of Kähler $SU(3)$-structures. This type of G_2-structures allow a fibrewise Ricci flow that converges to a Ricci flat G_2-structure.

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Table of Contents

Introduction

The Berger-Simons classification **[6]**, **[48]** of the possible holonomy groups of a Riemannian manifold leads to the question which of the Ricci flat holonomy groups can actually be realized as the holonomy group of a Riemannian metric on a compact manifold. A metric with holonomy group equal to $G \subset O(n)$ induces a G-structure, i.e. a reduction of the frame bundle to the structure group G. Conversely, such a reduction yields a metric with holonomy $G \subset O(n)$ if the reduction is compatible with the Levi-Civita connection of the induced $O(n)$-structure. This compatibility is measured by the intrinsic torsion of the G-structure, which takes values in the G-module

$$\mathbb{R}^{n*} \otimes \mathfrak{g}^{\perp}.$$

In the $U(n)$ and G_2-case, Gray et al. **[27]**, **[32]** decomposed this G-module into irreducible summands and classified structures according to the irreducible components of their intrinsic torsion. For certain torsion types many explicit examples of structures with the prescribed torsion type are known. For instance, Kähler structures with vanishing first real Chern class can be regarded as certain types of $SU(n)$-structures. Namely, the intrinsic torsion of a $SU(n)$-structure decomposes into a Kähler part and a component measuring the defect of the structure to give a further holonomy reduction to $SU(n) \subset U(n)$. Yau's student H. Cao proves in **[17]** that the Kähler-Ricci flow can actually be used to deform a Kähler $SU(n)$-structure into a Ricci flat structure. In other words, Cao studies a particular evolution of a geometry with torsion to prove the Calabi conjecture. In this thesis we discuss two approaches to construct manifolds with special holonomy via evolution of geometries with torsion:

I. Hitchin's flow equations

In **[37]** N. Hitchin introduced certain evolution equations for G_2 and $SU(3)$ structures on a manifold M, whose solutions are the gradient flow of a certain volume functional. A family of structures, evolving according to Hitchin's flow equations for time $t \in I \subset \mathbb{R}$, yields a parallel structure on the product $I \times M$. In this sense, a solution of the evolution equations embeds the initial structure into a manifold with a parallel structure and is therefore called a solution of the embedding problem for the initial structure.

Hitchin's flow equations were extended in **[22]** and **[28]** to $SU(2)$-structures in dimension five. Similar equations are known for embedding $SU(2)$-structures in dimension five and $SU(3)$-structures in dimension six into manifolds with a nearly parallel $SU(3)$ and G_2-structure, respectively, cf. **[21]**. This evolution equations lead to a huge variety of embedding problems for certain geometries.

Solving the embedding problem for a given structure has two different aspects. First

one has to establish the existence of a solution to the flow equations. Secondly, the particular solution has to satisfy certain compatibility conditions to actually define a family of G-structures. In the G_2-case no compatibility conditions occur, since the structure is described by a single stable 3-form. In contrast, the $SU(2)$ and $SU(3)$-case involve various compatibility conditions. Hitchin proves that the $SU(3)$ evolution equations already imply the desired compatibility conditions. A similar result holds for the embedding problem for $SU(3)$-structures into nearly parallel G_2-structures, cf. [**49**].
R. Bryant [**11**] shows that in the real analytic category, the embedding problem for hypo $SU(3)$ and G_2-structures can be solved. Bryant also provided counter-examples in the smooth category. The embedding problem for $SU(2)$-structures in dimension five was solved by D. Conti and S. Salamon in [**22**], cf. also [**21**].

II. Ricci flow for $SU(3)$ and G_2-structures

Yau's proof [**50**] of the Calabi conjecture [**16**] settled the existence of compact manifolds with holonomy equal to $SU(m)$. First mayor progress towards the exceptional cases was achieved by R. Bryant and S. Salamon [**13**], who established the first complete, but non compact, examples with holonomy equal to G_2 and Spin(7). It took until 1996 before D. Joyce [**40**], [**41**], [**42**] proved the existence of compact manifolds with holonomy equal to G_2 and Spin(7). Nevertheless, an a priori existence theorem for G_2 manifolds is still missing today.
Cao's work [**17**] on the Kähler-Ricci flow motivates the conjecture that a similar flow could deform G_2-structures with sufficiently small torsion into parallel structures. Recently there have been various approaches to define the analogue of a Kähler-Ricci flow for the G_2-case. Bryant [**10**] discusses the G_2-Laplacian evolution

$$\dot{\varphi} = \Delta_{\varphi}\varphi,$$

where $\varphi \in \Omega^3(M)$ is the structure tensor of the G_2-structure. Although this evolution seems to be quite natural, Bryant argues that one would not expect the Laplacian flow to converge for most G_2-structures. H. Weiß and F. Witt [**51**] describe the evolution of a G_2-structure under the gradient flow of a Dirichlet energy functional. The authors establish the short-time existence and uniqueness for this gradient flow.
However, it is still unclear what flow and what type of initial structure would be appropriate in the G_2-case. The attempt to deform the *whole* G_2-structure under a certain heat flow, seems to be symptomatic for all current approaches. In contrast, the Kähler-Ricci flow only deforms the ambient $U(n)$-structure, leaving the complex structure unchanged. This motivates the conjecture that Hamilton's Ricci flow should also be applicable to certain initial types of G_2-structures. A result due to R. Bryant [**10**], R. Cleyton and S. Ivanov [**20**] supports this conjecture. The authors prove that closed G_2-structures which are Einstein have to be parallel. This

indicates that the difference between a Ricci flat and a parallel G_2-structure is less drastic than it seems to be.

III. Methods

In the first and second chapter we develop certain methods to study general deformations of special geometries. Gauge deformations, i.e. automorphisms of the tangent bundle, provide a unifying approach to describe deformations of G-structures. In many cases the structure tensor φ of a given G-structure is stable in the sense that the orbit under the natural action of $GL(n)$ is open. Hence any smooth deformation φ_t of the structure tensor stays inside the open orbit and can be described by a family $[A_t] \in GL(n)/G$. Choosing a particular connection on $GL(n) \to GL(n)/G$, allows a description of the form $\varphi_t = A_t\varphi$, cf. Theorem 1.6. For a family of metrics g_t this is the familiar description $g_t = A_t g$, where A_t is symmetric and positive w.r.t. the initial metric g. Geometrically, the family of gauge deformations A_t describes the evolution of the principal G-reduction in vertical direction.
The evolution of the structure tensor $\varphi_t = A_t\varphi$ can be computed in terms of a G-equivariant map,

$$\dot{\varphi}_t = D_{\varphi_t}(\dot{A}_t A_t^{-1}),$$

cf. Lemma 1.16. This allows to translate the evolution equation for the structure tensors into a corresponding equation for the family of gauge deformations. The deformation of the underlying metric of the $G \subset O(n)$ structure is then obtained using polar decomposition to write $A_t = P_t Q_t \in S^2 \cdot O(n)$. In Theorem 1.19 we compute the change in the intrinsic torsion after deforming the initial structure by a gauge deformation. For a function $f : M \to \mathbb{R}$ and $A := f\mathrm{id}$, this yields the well-known formula for conformal changes, cf. **[2]**, **[43]**.

The space of gauge deformations $C^\infty(\mathrm{Aut}(TM))$ is an open subset of the Fréchet space $C^\infty(\mathrm{End}(TM))$. A solution $c(t)$ of a certain evolution equation can therefore be regarded as an integral curve of a vector field on a Fréchet space. In order for a solution to preserve some initial condition, we study in the second chapter the case where the vector field is tangent to the subspace determined by the initial condition, cf. Proposition 2.3. In contrast to finite dimensional geometry, the integral curve of a vector field tangent to some subspace does not have to stay inside the subspace. In the particular case where the solution can be developed in a power series of the form

$$c(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c^{(k)}(0),$$

we prove in Corollary 2.4 that the solution $c(t)$ actually stays inside the subspace. Hence Corollary 2.4 can be regarded as a conservation law for integral curves in Fréchet spaces.
The condition that the solution can be developed in a power series is quite restrictive. However, the Cauchy-Kowalevski Theorem proves, beyond the existence, that

the integral curves in question satisfy this condition. We translate the local version of the Cauchy-Kowalevski Theorem into a global version Theorem 2.11 for integral curves in Fréchet spaces of the form $C^\infty(V)$, where V is a vector bundle over a compact manifold.

IV. Applications

We prove that the embedding problems for $SU(2)$ and $SU(3)$-structures can be reduced to the G_2-case, which will be studied in terms of gauge deformations in chapter four. It seems to be coincidence, that in the G_2-case, the intrinsic torsion $\mathcal{T}$ takes values in the G_2-module $\mathfrak{gl}(7)$ and therefore can be regarded again as an (infinitesimal) gauge deformation. In Proposition 4.12 we show that the intrinsic torsion flow for G_2-structures

$$\dot{A}_t = \mathcal{T}_t \circ A_t$$

can be regarded as a generalization of Hitchin's flow equation, and hence as a generalization of the $SU(2)$, $SU(3)$ and G_2-embedding problem. We describe the evolution of the metric and the intrinsic torsion under the intrinsic torsion flow, cf. Theorem 4.13. As a consequence of the Cheeger-Gromoll Splitting Theorem, we prove in Theorem 4.14 and Corollary 4.15 that there are no nontrivial long-time solutions for the embedding problem. The Cauchy-Kowalevski Theorem and the conservation law Corollary 2.4 allow us to prove that the intrinsic torsion flow preserves certain compatibility conditions, which implies that for any real analytic hypo $SU(2)$, $SU(3)$ and G_2-structure on a compact manifold, the embedding problem admits a unique real analytic solution. Moreover, the solution can be described by a family of gauge deformations

$$A_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_0^{(k)},$$

where the series converges in the C^∞-topology on $C^\infty(\mathrm{End}(TM))$.

In chapter five we define a canonical extension of the Kähler-Ricci flow to $SU(n)$-structures via gauge deformations and characterize in Theorem 5.13 under which conditions this flow converges to a parallel $SU(n)$-structure. In Theorem 5.9 we prove that the canonical extension evolves under an equation that has a striking similarity with the evolution equation of the Kähler-Ricci flow. Below we list the evolution equations for the different types of Ricci flows, using the map D form Lemma 1.16:

Name	Structure Group	Evolution Equation
usual Ricci flow	$O(n)$	$\dot{g}_t = D_{g_t}(\mathrm{Ric}_t)$
Kähler-Ricci flow	$U(n)$	$(\dot{g}_t, \dot{\omega}_t) = D_{(g_t,\omega_t)}(\mathrm{Ric}_t)$
(special) Kähler-Ricci flow	$SU(n)$	$(\dot{g}_t, \dot{\omega}_t, \dot{\rho}_t) = D_{(g_t,\omega_t,\rho_t)}(\mathrm{Ric}_t)$

This motivates the conjecture that for a given G_2-structure φ on M with sufficiently small torsion, the flow

$$\dot{\varphi}_t = D_{\varphi_t}(\mathrm{Ric}_t)$$

should converge to a Ricci-flat G_2-structure. Essentially the same flow equation can be considered for Spin_7-structures, or more generally, for *any* $G \subset O(n)$ structures, described by certain structure tensors. We show that the family of metrics, corresponding to a family of structures φ_t, evolving according to $\dot{\varphi}_t = D_{\varphi_t}(\mathrm{Ric}_t)$, satisfies $\dot{g}_t = -2\mathrm{ric}_t$.
Moreover, we prove that all of the above evolution equations can be described in a unified way, using gauge deformations. Namely, a solution $A_t \in C^\infty(\mathrm{Aut}(TM))$ of

$$\begin{cases} \dot{A}_t = \mathrm{Ric}_t \circ A_t \\ A_0 = \mathrm{id} \end{cases}$$

yields a solution for *all* of the above Ricci flows. For instance, $\varphi_t := A_t\varphi$ solves $\dot{\varphi}_t = D_{\varphi_t}(\mathrm{Ric}_t)$, for any initial G_2-structure φ on M. We call A_t the universal Ricci flow for the initial metric g and prove in Theorem 5.14 that any compact Riemannian manifold admits a unique universal Ricci flow for some time $t \in [0, T)$.
The Spin_7-case reveals another advantage of working with families of gauge deformations. In contrast to the G_2-case, the orbit of the model tensor is not open in the Spin_7-case. Hence it is not obvious that a Spin_7-structure Ψ evolving according to $\dot{\Psi}_t = D_{\Psi_t}(\mathrm{Ric}_t)$ actually defines a whole family of Spin_7-structures. Describing the solution via a family of gauge deformation $\Psi_t = A_t\Psi$ completely circumvents this problem.

Based on the discussion of $SU(3)$-structures in chapter three, we define a type of G_2-structure which can be regarded as the seven dimensional analogue of Kähler $SU(3)$-structures. This type of G_2-structures allow a fibrewise Ricci flow. Using Cao's result for Kähler structures in dimension six, we prove in Theorem 5.22 that the fibrewise Ricci flow converges to a Ricci flat G_2-structure.

1. Deformations of Principal Bundles

In this chapter we study deformations of principal bundles via gauge deformations. A gauge deformation is an equivariant map $A : P \to G$, where P is some principal bundle with structure group G. Given such a map and a reduction Q of P to $H \subset G$, we obtain a new H-reduction by

$$QA := \{qA(q) \mid q \in Q\} \subset P.$$

Hence a gauge deformation can be regarded as a vertical deformation of the initial reduction Q. In contrast, a diffeomorphism of M induces a horizontal deformation of a reduction $Q \subset FM$, where FM is the frame bundle of some manifold M.
Many reductions can be described by certain tensors. For instance, a family of metrics g_t on M induces a family of $O(n)$-reductions $F^{g_t}M \subset FM$. Using polar decomposition, one can easily see that such a family of metric can be described by a family of gauge deformations via $g_t = A_t g$, where $g := g_0$ is the initial metric. In Theorem 1.6 we obtain a generalization of this description for certain families of tensors.
The compatibility of a given $G \subset O(n)$ reduction $P \subset F^gM$ with the Levi-Civita connection on F^gM is measured by the intrinsic torsion of $P \subset F^gM$. Deforming the initial structure by a gauge deformation effects the intrinsic torsion. In Theorem 1.19 we compute the change in the intrinsic torsion under a general gauge deformation. Using Theorem 1.6, we obtain in Corollary 1.21 a characterization of G-structures that can be deformed to torsion-free structures.

Stability

Let $\pi : P \to M$ be a principal G-bundle and $\varrho : G \to \mathrm{Aut}(V)$ be a real G-representation. We identify sections of the associated bundle $P \times_\varrho V$ with equivariant maps $\varphi : P \to V$, satisfying

$$\varphi(pg) = g^{-1}\varphi(p) := \varrho(g^{-1})\varphi(p),$$

for all $p \in P$ and $g \in G$. A G-structure $P \subset FM$ is a reduction of the frame bundle $\pi : FM \to M$ to a Lie subgroup $G \subset GL(n)$. A basis $p \in FM$ corresponds to an isomorphism $p : \mathbb{R}^n \to T_{\pi(p)}M$ which identifies the standard basis $(e_1, .., e_n)$ of $\mathbb{R}^n$ with the basis p of $T_{\pi(p)}M$. Hence an element $g \in GL(n)$ acts on FM by

$$pg := p \circ g : \mathbb{R}^n \longrightarrow T_{\pi(p)}M,$$

making FM into a principal $GL(n)$-bundle over M.

Definition 1.1. Let $\pi : P \to M$ be a principal G-bundle, $\varrho : G \to \mathrm{Aut}(V)$ a real G-representation, $\varphi_0 \in V$ and $\varphi : P \to V$ equivariant.

(1) φ_0 is stable if the orbit $G\varphi_0 := \varrho(G)\varphi_0 \subset V$ is open.

(2) φ is stable if $\varphi(p) \in V$ is stable, for all $p \in P$.

(3) φ is of type φ_0 if $\varphi(p) \in G\varphi_0 \subset V$, for all $p \in P$.

An equivariant map $\varphi : P \to V$ of type $\varphi_0 \in V$ defines a reduction of P to the isotropy group $\mathrm{Iso}_G(\varphi_0) \subset G$ via

$$P^\varphi := \{p \in P \mid \varphi(p) = \varphi_0\}.$$

Conversely, given such a reduction $Q \subset P$, we obtain an equivariant map $\varphi : P \to V$ of type φ_0 by extending the constant map $\varphi_{|Q} \equiv \varphi_0$ equivariantly to a map $P \to V$.

Proposition 1.2. Let $P \to M$ be a principal G-bundle, $\varrho : G \to \mathrm{Aut}(V)$ a real G-representation and $\varphi_0 \in V$. Then the reductions of P to the isotropy group $\mathrm{Iso}_G(\varphi_0) \subset G$ correspond to equivariant maps $\varphi : P \to V$ of type φ_0.

$\square$

One of the main motivations to study stability in relation with G-structures is the following

Proposition 1.3. Let $P \to M$ be a principal G-bundle over a connected manifold M, and $\varrho : G \to \mathrm{Aut}(V)$ a real G-representation. If $\varphi : P \to V$ is stable, then for any $p \in P$, the map φ is already of type $\varphi_0 := \varphi(p) \in V$. In particular, φ induces a reduction to the isotropy group of φ_0.

Proof: Since φ is stable,

$$W_0 := \{x \in M \mid \exists\, p \in P_x : \varphi(p) = \varphi_0\} = \pi(\varphi^{-1}(G\varphi_0)) \subset M$$

is open. For $x \in M \setminus W_0$ choose some $p \in P_x$. Then $\varphi_1 := \varphi(p) \notin G\varphi_0$ and $W_1 \subset M$ defines an open set containing x. If $W_1 \cap W_0 \neq \emptyset$, we find $p, q \in P$ with $\pi(p) = \pi(q)$, $\varphi(p) = \varphi_0$ and $\varphi(q) = \varphi_1$. Hence $q = pg$ for some $g \in G$ and

$$\varphi_1 = \varphi(q) = g^{-1}\varphi(p) = g^{-1}\varphi_0 \in G\varphi_0,$$

which contradicts $x \notin W_0$.

□

A similar result holds for whole families of stable maps.

Proposition 1.4. Suppose $P \to M$ is a principal G-bundle over a connected manifold M, $\varrho : G \to \mathrm{Aut}(V)$ is a real G-representation and that $\{\varphi_t : P \to V\}_{t \in I}$ is a family of stable tensors, where $I \subset \mathbb{R}$ is some interval containing zero. If $\varphi := \varphi_{t=0}$ is of type $\varphi_0 \in V$, then φ_t is of type $\varphi_0 \in V$, for all $t \in I$.

Proof: Since φ is of type φ_0, we find $p \in P$ such that $\varphi(p) = \varphi_0$. By Proposition 1.3 we have

$$0 \in J_0 := \{t \in I \mid \varphi_t \text{ is of type } \varphi_0\} = \{t \in I \mid \varphi_t(p) \in G\varphi_0\}.$$

Hence $J_0 = (t \mapsto \varphi_t(p))^{-1}(G\varphi_0) \subset I$ is open and non-empty. For $t \in I \setminus J_0$ we have $\varphi_1 := \varphi_t(p) \notin G\varphi_0$ and $J_1 \subset I$ is open and contains t. If $J_0 \cap J_1 \neq \emptyset$, we get $G\varphi_0 \cap G\varphi_1 \neq \emptyset$ and hence $\varphi_1 \in G\varphi_1 = G\varphi_0$, in contradiction to $t \notin J_0$.

□

Gauge Deformations

One way to deform a given G-structure $P \subset FM$ is to transform it by an element $F \in \mathrm{Diff}(M)$. Namely consider

$$F_*P := \{F_*p \mid p \in P\},$$

where $F_*p \in F_{F(\pi(p))}M$ is defined by $(F_*p)e_i := F_*(pe_i)$. Since $F_*(pg) = (F_*p)g$, we see that $F_*P \subset FM$ defines again a G-structure. Similarly we can deform P by an element $A \in C^\infty(\mathrm{Aut}(TM))$,

$$PA := \{pA(p) \mid p \in P\},$$

where $pA(P) \in F_{\pi(p)}M$ is defined by $(pA(p))e_i := p(A(p)e_i)$. The latter deformation is a vertical deformation in the sense that $\pi(pA(p)) = \pi(p)$, whereas $\pi(F_*p) \neq \pi(p)$, for $F \neq \mathrm{id}$.

Definition 1.5. Suppose P is a principal G-bundle over M and $\varrho : G \to \mathrm{Aut}(V)$ is a real G-representation.

(1) A gauge deformation is an equivariant map $P \to G$, where G acts on itself by conjugation. The set of gauge deformations is denoted by

$$G(P) := C^\infty(P \times_G G).$$

(2) An infinitesimal gauge deformation is an equivariant map $P \to \mathfrak{g}$, where G acts on $\mathfrak{g}$ by the adjoint representation. The set of infinitesimal gauge deformations is denoted by

$$\mathfrak{g}(P) := C^\infty(P \times_{\mathrm{Ad}} \mathfrak{g}).$$

(3) Using $\exp(\mathrm{Ad}(g)X) = g\exp(X)g^{-1}$, for the usual exponential map $\exp : \mathfrak{g} \to G$, we can define

$$\exp : \mathfrak{g}(P) \to G(P) \quad \text{by} \quad \exp(X)(p) := \exp(X(p)).$$

(4) For $A \in G(P)$ and $\varphi : P \to V$ equivariant, we define an equivariant map

$$\varrho(A)\varphi : P \to V \quad \text{by} \quad (\varrho(A)\varphi)(p) := \varrho(A(p))\varphi(p).$$

The following Theorem essentially states that families of H-reductions can be described by certain families gauge deformations.

Theorem 1.6. Let $\pi : P \to M$ be a principal G-bundle, $\varrho : G \to \mathrm{Aut}(V)$ a real G-representation, $\varphi_0 \in V$ with isotropy group $H \subset G$, $\pi : G \to G/H$ the canonical projection and $\{\varphi_t : P \to V\}_{t\in I}$ a family of equivariant maps which are all of type φ_0. Suppose that $\mathfrak{g}$ is equipped with some $\mathrm{Ad}(H)$-invariant inner product and denote by $\mathfrak{h}^\perp$ the orthogonal complement of $\mathfrak{h} \subset \mathfrak{g}$. Denote by $Q \subset P$ the H-reduction induced by $\varphi := \varphi_{t=0}$ and let

$$\mathfrak{h}^\perp(Q) := \{X \in \mathfrak{g}(P) \mid X_{|Q} : Q \to \mathfrak{h}^\perp\}.$$

(1) There exists a family of gauge deformations $A_t \in G(P)$, $t \in I$, such that

$$\varphi_t = \varrho(A_t)\varphi \quad \text{and} \quad A_0 = e$$

(2) If $\pi \circ \exp : \mathfrak{h}^\perp \to G/H$ is a covering map, then there exists a family of infinitesimal gauge deformations $X_t \in \mathfrak{h}^\perp(Q)$, $t \in I$, such that

$$\varphi_t = \varrho(\exp(X_t))\varphi \quad \text{and} \quad X_0 = 0.$$

(3) If H and M are compact, then there exists an open subinterval $J \subset I$ containing 0 such that the conclusion in (2) holds for J instead of I.

Proof: Fix $q \in Q = \{p \in P \mid \varphi(p) = \varphi_0 \in V\}$ and define

$$\bar{A}(q) : I \to G/H \quad \text{by} \quad \bar{A}_t(q) := \pi \circ A_t(q),$$

where $A_t(q) \in G$ satisfies $\varphi_t(q) = \varrho(A_t(q))\varphi(q)$. Note that such an element $A_t(q) \in G$ exists, since $\varphi_t(q) \in G\varphi_0 = G\varphi(q)$ is of type φ_0.

Proof of part (1): The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ induces a horizontal distribution on the principal H-bundle $\pi : G \to G/H$ by

$$H_g := R_{g*}\mathfrak{h}^\perp.$$

Hence there exists a unique horizontal lift $A_t(q)$ of $\bar{A}_t(q)$ with $A_0(q) = e$. From $\pi \circ A_t(q) = \bar{A}_t(q)$ we get $\varphi_t(q) = \varrho(A_t(q))\varphi(q)$ and it remains to show that $A_t : Q \to G$ is H-equivariant, i.e. for all $q \in Q$, $h \in H$ and $t \in I$ we have to show that

$$c(t) := A_t(qh) = h^{-1}A_t(q)h =: d(t)$$

holds. The curve $c(t) \in G$ is horizontal by definition, whereas the horizontality of $d(t)$ follows from

$$\dot{d}(t) = \mathrm{Ad}(h^{-1})\dot{A}_t(q) \in \mathrm{Ad}(h^{-1})R_{A_t(q)*}\mathfrak{h}^\perp = R_{d(t)}\mathrm{Ad}(h^{-1})\mathfrak{h}^\perp \subset R_{d(t)}\mathfrak{h}^\perp,$$

by $\mathrm{Ad}(H)$-invariance. Now

$$\varrho(d(t))\varphi(qh) = \varrho(h^{-1})\varrho(A_t(q))\varphi(q) = \varrho(h^{-1})\varphi_t(q) = \varphi_t(qh)$$

implies $\pi \circ d(t) = \bar{A}_t(qh) = \pi \circ c(t)$ and $c(0) = e = d(0)$ yields $c(t) = d(t)$.

Proof of part (2): If $\pi \circ \exp : \mathfrak{h}^\perp \to G/H$ is a covering map, we can lift the map $\bar{A}(q) : I \to G/H$ uniquely to a map $X(q) : I \to \mathfrak{h}^\perp$ with $X_0(q) = 0$. Hence

$$\pi \circ \exp(X_t(q)) = \bar{A}_t(q),$$

which yields $\varphi_t(q) = \varrho(\exp(X_t(q)))\varphi(q)$. It remains to show that $X_t : Q \to \mathfrak{h}^\perp$ is equivariant, i.e. for all $q \in Q$, $h \in H$ and $t \in I$ we have to show that

$$c(t) := X_t(qh) = \mathrm{Ad}(h^{-1})X_t(q) =: d(t)$$

holds. Since already $c(0) = 0 = d(0)$, it suffices to verify that $\pi \circ \exp \circ c(t) = \pi \circ \exp \circ d(t)$ holds. To see this, observe that

$$\begin{aligned}\varrho(\exp(\mathrm{Ad}(h^{-1})X_t(q)))\varphi(qh) &= \varrho(h^{-1})\varrho(\exp(X_t(q)))\varrho(h)\varrho(h^{-1})\varphi(q)\\ &= \varrho(h^{-1})\varphi_t(q) = \varphi_t(qh)\end{aligned}$$

implies

$$\pi \circ \exp \circ d(t) = \pi(\exp(\mathrm{Ad}(h^{-1})X_t(q))) = \bar{A}_t(qh) = \pi \circ \exp \circ c(t).$$

Proof of part (3): Choose an open neighborhood $U \subset \mathfrak{h}^\perp$ of 0 such that

$$F := \pi \circ \exp_{|U \subset \mathfrak{h}^\perp} : U \to F(U)$$

is a diffeomorphism. Since H is compact, we can choose U to be $\mathrm{Ad}(H)$-invariant. Now consider

$$J_q := \{t \in I \mid \bar{A}_t(q) \in F(U)\}.$$

Since Q is compact, if M and H are compact, we can assume that there exists an open interval J such that $0 \in J \subset J_q$ for all $q \in Q$. For $t \in J$ define $X_t : Q \to U \subset \mathfrak{h}^\perp$ by

$$X_t(q) := F^{-1}(\bar{A}_t(q)).$$

Then $X_0(q) = 0$ and $\pi \circ \exp(X_t(q)) = F \circ F^{-1}(\bar{A}_t(q))$ implies

$$\varphi_t(q) = \varrho(\exp(X_t(q)))\varphi(q).$$

Now we obtain like in part (1) the equation $\varrho(\exp(\mathrm{Ad}(h^{-1})X_t(q)))\varphi(qh) = \varphi_t(qh)$, which implies $\pi(\exp(\mathrm{Ad}(h^{-1})X_t(q))) = \bar{A}_t(qh)$ and

$$F\big(X_t(qh)\big) = \bar{A}_t(qh) = \pi(\exp(\mathrm{Ad}(h^{-1})X_t(q))) = F\big(\mathrm{Ad}(h^{-1})X_t(q)\big).$$

Since $X_t(q) \in U$ and U is $\mathrm{Ad}(H)$-invariant, it follows $X_t(qh) = \mathrm{Ad}(h^{-1})X_t(q)$.

$\square$

Example 1.7. Given two Riemannian metrics g and g_t on M, we can define a gauge deformation $B_t \in C^\infty(\mathrm{Aut}(TM))$ by $g_t = B_t \lrcorner g$. So B_t is symmetric and positive w.r.t. g and hence there is a unique square root A_t of B_t^{-1} w.r.t. g, i.e. $B_t = A_t^{-1}A_t^{-1}$, where A_t is again symmetric and positive w.r.t. g. This shows that any two metrics are gauge equivalent,

$$g_t = A_t g.$$

We can also apply Theorem 1.6 to a family of metrics g_t on M and obtain the same gauge deformation A_t. Here $H := O(n) \subset GL(n) =: G$ and the $\mathrm{Ad}(O(n))$-invariant inner product on $\mathfrak{gl}(n)$ is given by $\langle X, Y\rangle := \mathrm{tr}(XY^T)$.

Example 1.8. Suppose I is an almost complex structure on M and consider TM as a complex vector bundle via $iX := IX$, for $X \in TM$. A hermitian metric on (M, I) is a Riemannian metric g which satisfies $g(I., I.) = g$ and hence induces a 2-form $\omega := g(I., .)$. Then

$$h := g - i\omega$$

defines a hermitian structure on the complex vector bundle (TM, I), i.e. the map $h : TM \times TM \to \mathbb{C}$ is $\mathbb{C}$-linear in the first argument and satisfies $h(X, Y) = \overline{h(Y, X)}$ and $h(X, X) > 0$, for every $X \neq 0$. We regard h as a $\mathbb{C}$-anti-linear isomorphism

$$h : TM \to T^*M \qquad \text{via} \qquad X \mapsto h(., X).$$

Given two hermitian metrics g and g_t on (M, I), we define a gauge deformation B_t by $h_t(Y, X) = h(Y, B_t X)$. Since

$$B_t I = I B_t \quad \text{and} \quad h(Y, B_t X) = h(B_t Y, X) \quad \text{and} \quad h(B_t X, X) > 0, \text{ for } X \neq 0$$

we can find a unique square root of B_t^{-1} w.r.t. h, i.e. $B_t = A_t^{-1}A_t^{-1}$ and

$$A_t I = I A_t \quad \text{and} \quad h(Y, A_t X) = h(A_t Y, X) \quad \text{and} \quad h(A_t X, X) > 0, \text{ for } X \neq 0.$$

In particular, we obtain a gauge deformation with

$$h_t = A_t h.$$

Since $A_t I = I A_t$, A_t is hermitian w.r.t. h if and only if it is symmetric w.r.t. $g = \mathrm{Re}(h)$. Ignoring the almost complex structure, we can write $g_t = \widetilde{A}_t g$, where $\widetilde{A}_t$ is defined like in Example 1.7. So $\widetilde{A}_t g = A_t g$ and from the symmetry of A_t and $\widetilde{A}_t$ it follows $\widetilde{A}_t^2 = A_t^2$. But since $\widetilde{A}_t^2$ and A_t^2 are positive, they have a unique positive square root and hence $\widetilde{A}_t = A_t$. So the gauge deformation A_t is precisely the one that we obtained in Example 1.7, but satisfies in addition $A_t I = I A_t$.
We can also apply Theorem 1.6 to a family of hermitian structures $h_t = g_t - i\omega_t$ on (M, I) and obtain the same gauge deformation A_t. Now $H := U(n) \subset GL(n, \mathbb{C}) =: G$ and the $\mathrm{Ad}(U(n))$-invariant inner product on $\mathfrak{gl}(n, \mathbb{C})$ is given by $\langle X, Y\rangle := \mathrm{Re}(\mathrm{tr}_{\mathbb{C}}(XY^*))$.

Intrinsic Torsion

Given a reduction $P \subset FM$, we have a natural concept of integrability. Namely we may ask whether there exist local sections $s = (X_1, .., X_n)$ in P such that $[X_i, X_j] = 0$ holds. Equivalently, we may look for sections in P which are induced by the basis field of a local chart of M. So integrable $GL(n, \mathbb{C})$-structures are complex structures on M and integrable $Sp(n, \mathbb{R})$-structures correspond to symplectic structures on M.
As soon as we consider reductions to subgroups of $O(n)$, this integrability concept is to restrictive. In fact, an integrable $O(n)$-structure would yield a flat metric on M. From this point of view, curvature is the obstruction to the existence of an integrable $O(n)$-structure. To develop a weaker concept of integrability we have to substitute the reference group $GL(n)$ by $O(n)$. Instead of measuring the compatibility of a given $G \subset O(n)$ structure with the $GL(n)$-structure FM, we have to measure the compatibility with the metric structure $F^g M$. This compatibility is measured by the so-called intrinsic torsion of the G-structure $P \subset F^g M$.

Definition 1.9. **(1)** A connection on a principal G-bundle $\pi : P \to M$ is a 1-form $\mathcal{Z}$ on P with values in $\mathfrak{g}$, such that

$$\mathcal{Z}(R_{p*}X) = X \qquad \text{and} \qquad R_g^* \mathcal{Z} = \mathrm{Ad}(g^{-1})\mathcal{Z}$$

for all $X \in \mathfrak{g}$, $g \in G$ and $p \in P$. We say that a connection $\mathcal{Z}$ on P reduces to a principal H-bundle $Q \subset P$ if the restriction of $\mathcal{Z}$ to TQ takes values in $\mathfrak{h}$.

(2) Given a connection $\mathcal{Z}$ on P, we call $H_p := \ker(\mathcal{Z}_p)$ the horizontal distribution of $\mathcal{Z}$. This distribution is complementary to the vertical space $V_p := \ker(\pi_{*p})$, i.e.

$$T_pP = H_p \oplus V_p \qquad \text{and satisfies} \qquad H_{pg} = R_{g*}H_p.$$

Hence there exists for each $X \in T_{\pi(p)}M$ a unique

$$h_pX \in H_p \quad \text{such that} \quad \pi_* h_p X = X.$$

We call h_pX the horizontal lift of X to $p \in P$ w.r.t the connection $\mathcal{Z}$. Independent of any connection, we always have a vertical lift of elements $X \in \mathfrak{g}$, defined by

$$v_p(X) := \left.\frac{d}{dt}\right|_{t=0} p\exp(tX) = R_{p*}(X).$$

(3) The frame bundle $\pi : FM \to M$ admits a $\mathbb{R}^n$-valued 1-form

$$\theta : T_pFM \to \mathbb{R}^n \qquad \text{with} \qquad X \mapsto p^{-1}\pi_*X.$$

Given a connection $\mathcal{Z}$ on a principal G-bundle $P \subset FM$, we call

$$T : P \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n \quad \text{with} \quad T(p)(x,y) := d\theta(h_p(px), h_p(py))$$

the torsion of $\mathcal{Z}$.

(4) The curvature of a connection $\mathcal{Z}$ on a principal G-bundle $P \to M$ is the map

$$R : P \to \Lambda^2\mathbb{R}^{n*} \otimes \mathfrak{g} \quad \text{with} \quad R(p)(x,y) := d\mathcal{Z}(h_p(px), h_p(py)).$$

Given a connection, we can differentiate tensors in horizontal directions.

Proposition 1.10. Suppose $\varrho : G \to \mathrm{Aut}(V)$ is a real G-representation and $\pi : P \to M$ is a principal G-bundle, equipped with a connection.

(1) For $X \in C^\infty(TM)$ and $\varphi : P \to V$ the map

$$\nabla_X\varphi : P \to V \quad \text{with} \quad (\nabla_X\varphi)(p) := \varphi_*(h_pX)$$

is again equivariant, i.e. ∇ defines a map

$$\nabla : C^\infty(TM) \times C^\infty(P \times_\varrho V) \to C^\infty(P \times_\varrho V).$$

(2) For $X \in C^\infty(TM)$ and $A \in G(P)$ the map

$$\nabla_XA : P \to \mathfrak{g} \quad \text{with} \quad (\nabla_XA)(p) := (L_{A(p)^{-1}})_*A_*(h_pX)$$

is again equivariant, i.e. ∇ defines a map

$$\nabla : C^\infty(TM) \times G(P) \to \mathfrak{g}(P).$$

PROOF: If we write $h_pX = \dot c(0)$, for some curve $c(t) \in P$, the first part follows from

$$\begin{aligned}(\nabla_X\varphi)(pg) &= \varphi_*(R_{g*}h_pX) = \frac{d}{dt}\Big|_{t=0}\varphi(c(t)g) = \varrho(g^{-1})\,\frac{d}{dt}\Big|_{t=0}\varphi(c(t))\\ &= \varrho(g^{-1})(\nabla_X\varphi)(p).\end{aligned}$$

Similarly for the second part,

$$\begin{aligned}(\nabla_X A)(pg) &= L_{A(pg)^{-1}*}A_*(R_{g*}h_pX) = \frac{d}{dt}\Big|_{t=0} A(pg)^{-1}A(c(t)g)\\ &= \frac{d}{dt}\Big|_{t=0} g^{-1}A(p)^{-1}A(c(t))g = \mathrm{Ad}(g^{-1})(\nabla_X A)(p).\end{aligned}$$

□

Note that the above definitions of covariant derivatives are not compatible with the embedding $GL(n) \subset \mathfrak{gl}(n)$. Namely the covariant derivative of a gauge deformation $A : FM \to GL(n)$ from Proposition 1.10 (2) is not equal to the covariant derivative of

$$A : FM \to GL(n) \subset \mathfrak{gl}(n)$$

in the sense of Proposition 1.10 (1).

By definition of the curvature tensor we have $R(p)(x,y) = -\mathcal{Z}[h(x),h(y)]_p$ and hence R measures the integrability of the horizontal distribution $\ker(\mathcal{Z})$. More generally we have

LEMMA 1.11. Let $\mathcal{Z}$ be a connection on a principal G-bundle $\pi : P \to M$. Then for $X, Y \in C^\infty(TM)$ and $A, B \in \mathfrak{g}$

(1) $[h(X),h(Y)]_p = h_p[X,Y]_{\pi(p)} - v_p(R(X,Y)(p))$,

(2) $[v(A),h(X)]_p = 0$,

(3) $[v(A),v(B)]_p = v_p[A,B]$.

PROOF: The first equation follows from $\pi_*[h(X),h(Y)]_p = [X,Y]_{\pi(p)}$ and since $R(X,Y)(p) = d\mathcal{Z}(h_p(X),h_p(Y)) = -\mathcal{Z}[h(X),h(Y)]_p$. The flow $\Phi_t(p) := p\exp(tA)$ of $v(A)$ satisfies $\Phi_{t*}h_p(X) = h_{\Phi_t(p)}(X)$ and hence

$$[v(A),h(X)]_p = L_{v_p(A)}h(X) = \frac{d}{dt}\Big|_{t=0}\Phi_{-t*}h_{\Phi_t(p)}(X) = 0,$$

which proves the second equation. Finally,

$$\begin{aligned}\Phi_{-t*}v_{\Phi_t(p)}(B) &= \frac{d}{ds}\bigg|_{s=0} p\exp(tA)\exp(sB)\exp(-tA)\\ &= R_{p*}(\mathrm{Ad}_{\exp(tA)}(B)) = R_{p*}(e^{t[A,B]})\end{aligned}$$

and hence

$$[v(A),v(B)]_p = R_{p*}(\frac{d}{dt}\bigg|_{t=0} e^{t[A,B]}) = R_{p*}([A,B]) = v_p[A,B].$$

□

Given a connection $\mathcal{Z}$ on a G-structure $P \subset FM$ and an equivariant map $\xi : P \to \mathbb{R}^{n*} \otimes \mathfrak{g}$, we obtain a new connection on P by

$$\widetilde{\mathcal{Z}} := \mathcal{Z} + \xi \circ \theta.$$

The corresponding torsion tensors satisfy $\widetilde{T} = T + \delta \circ \xi$, where

$$\delta : \mathbb{R}^{n*} \otimes \mathfrak{gl}(n) \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n \quad \text{is given by} \quad (\delta F)(x,y) := F(x)y - F(y)x.$$

Since the restriction of δ to $\mathbb{R}^{n*} \otimes \mathfrak{so}(n)$ is an isomorphism of $O(n)$-modules, every $O(n)$-reduction $F^gM \subset FM$ admits a unique torsion-free connection; the Levi-Civita connection $\mathcal{Z}^g$ of the metric g.

Now let $G \subset O(n)$ and consider a G-structure $P \subset F^gM$. Decomposing $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ with respect to the inner product $\langle X, Y\rangle := \mathrm{tr}(XY^T)$ on $\mathfrak{so}(n)$, we obtain a corresponding decomposition

$$\mathcal{Z}^g_{|TP} = \mathcal{Z} + \mathcal{Z}^\perp \in \mathfrak{g} \oplus \mathfrak{g}^\perp = \mathfrak{so}(n).$$

The 1-form $\mathcal{Z}$ takes values in $\mathfrak{g}$ and defines a connection on P, the so-called characteristic connection of $P \subset F^gM$. By construction, $\mathcal{Z}^\perp$ measures the defect of the Levi-Civita connection to reduce to a connection on $P \subset F^gM$.

Definition 1.12. The intrinsic torsion τ of a G-structure $P \subset F^gM$ is the equivariant map

$$\tau : P \longrightarrow \mathbb{R}^{n*} \otimes \mathfrak{g}^\perp \quad \text{defined by} \quad \tau(p)(x) := \mathcal{Z}^\perp(h_p(px)),$$

where $h_p(px)$ denotes the horizontal lift w.r.t. the characteristic connection $\mathcal{Z}$ on P. For $X \in C^\infty(TM)$, we denote by $\tau(X)$ the corresponding infinitesimal gauge deformation

$$\tau(X) \in C^\infty(P \times_{\mathrm{Ad}} \mathfrak{g}^\perp) \subset \mathfrak{so}(F^gM).$$

By definition of the intrinsic torsion we have $\tau \circ \theta = \mathcal{Z}^{\perp}$. Hence the torsion $\mathcal{T}$ of the characteristic connection $\mathcal{Z}$ on P satisfies

$$\mathcal{T} + \delta \circ \tau = 0.$$

Since $\pi_*(h_pX - R_{p*}\tau(X)(p)) = X \circ \pi(p)$ and $\mathcal{Z}^g(h_pX - R_{p*}\tau(X)(p)) = \mathcal{Z}^{\perp}(h_pX) - \tau(X)(p) = 0$, we get

$$h^g_pX = h_pX - R_{p*}\tau(X)(p),$$

which yields

$$\begin{aligned}(\nabla^g_X Y - \nabla_X Y)(p) &= Y_*(h^g_pX - h_pX) = -Y_*R_{p*}\tau(X)(p) \\ &= -\left.\frac{d}{dt}\right|_{t=0} Y(p\exp(t\tau(X)(p))) = -\left.\frac{d}{dt}\right|_{t=0} \exp(-t\tau(X)(p))Y(p) \\ &= (\tau(X)Y)(p).\end{aligned}$$

We summarize the above formulas in the following

Lemma 1.13. The intrinsic torsion τ of a G-structure $P \subset F^gM$ satisfies

(1) $\mathcal{Z}^g = \mathcal{Z} + \tau \circ \theta$,

(2) $0 = \mathcal{T} + \delta \circ \tau$,

(3) $h^g(X) = h(X) - R_*\tau(X)$,

(4) $\tau(X)Y = \nabla^g_X Y - \nabla_X Y$.

The intrinsic torsion vanishes if and only if the Levi-Civita connection reduces to a connection on $P \subset F^gM$. The condition $\tau = 0$ is in general very restrictive. Decomposing the G-module $\mathbb{R}^{n*} \otimes \mathfrak{g}^{\perp}$ into irreducible submodules, we may consider structures with intrinsic torsion taking values only in some of these submodules. This approach yields a rough classification of arbitrary G-structures in terms of their intrinsic torsion. Many of these classes have a rich geometry, including examples of Einstein and Ricci-flat manifolds.

Lemma 1.14. Let $\varrho : G \to \mathrm{Aut}(V)$ be a real G-representation.

(1) The map

$$D : V \times \mathfrak{g} \to V \quad \text{with} \quad D_\varphi(X) := \left.\frac{d}{dt}\right|_{t=0} \varrho(\exp(tX))\varphi.$$

is G-equivariant, i.e. $D_{(\varrho(g)\varphi)}(\mathrm{Ad}(g)X) = \varrho(g)D_\varphi(X)$, for all $g \in G$ and $X \in \mathfrak{g}$.

(2) For fixed $\varphi \in V$ with isotropy group $H \subset G$, the map

$$D_\varphi : \mathfrak{g} \to V$$

is H-equivariant, i.e. $D_\varphi(\mathrm{Ad}(h)X) = \varrho(h)D_\varphi(X)$, for all $h \in H$ and $X \in \mathfrak{g}$. Moreover,

$$\ker(D_\varphi) = \mathfrak{h}.$$

Proof: Part (1) follows from

$$\begin{aligned} D_{(\varrho(g)\varphi)}(\mathrm{Ad}(g)X) &= \frac{d}{dt}\bigg|_{t=0} \varrho(\exp(t\mathrm{Ad}(g)X))\varrho(g)\varphi = \frac{d}{dt}\bigg|_{t=0} \varrho(g\exp(tX))\varphi \\ &= \frac{d}{dt}\bigg|_{t=0} \varrho(g)\varrho(\exp(tX))\varphi = \varrho(g)D_\varphi(X) \end{aligned}$$

and the equivariance in part (2) is a special case of (1).

Since $H \subset G$ is closed, H is actually a Lie subgroup and the exponential map of G, restricted to $\mathfrak{h} \subset \mathfrak{g}$, is the exponential map of H. Hence $\exp(tX) \in H$, for $X \in \mathfrak{h}$, and it follows $\mathfrak{h} \subset \ker(D_\varphi)$. Conversely, $X \in \ker(D_\varphi)$ satisfies

$$\begin{aligned} \frac{d}{dt}\bigg|_{t} \varrho(\exp(tX))\varphi &= \frac{d}{ds}\bigg|_{s=0} \varrho(\exp((t+s)X))\varphi = \frac{d}{ds}\bigg|_{s=0} \varrho(\exp(tX)\exp(sX))\varphi \\ &= \frac{d}{ds}\bigg|_{s=0} \varrho(\exp(tX))\varrho(\exp(sX))\varphi \\ &= \varrho(\exp(tX))D_\varphi(X) = 0. \end{aligned}$$

Hence $\varrho(\exp(tX))\varphi = \varphi$, i.e. $\exp(tX) \in H$ for all $t \in \mathbb{R}$, which yields $X \in \mathfrak{h}$, since $H \subset G$ is closed.

□

Using part (1) of Lemma 1.14, we can make the following

Definition 1.15. Suppose $\varrho : G \to \mathrm{Aut}(V)$ is a real G-representation and $\pi : P \to M$ is a principal G-bundle. We define

$$D : C^\infty(P \times_\varrho V) \times \mathfrak{g}(P) \to C^\infty(P \times_\varrho V)$$

by

$$D_\varphi(X) := \frac{d}{dt}\bigg|_{t=0} \varrho(\exp(tX))\varphi.$$

Lemma 1.16. Let $\varrho : G \to \mathrm{Aut}(V)$ be a real G-representation and $g_t \in G$ and $\varphi_t \in V$ smooth curves. Then

$$\begin{aligned} \frac{d}{dt}\big(\varrho(g_t)\varphi_t\big) &= D_{(\varrho(g_t)\varphi_t)}(R_{g_t^{-1}*}\dot{g}_t) + \varrho(g_t)\dot{\varphi}_t \\ &= \varrho(g_t)D_{\varphi_t}(L_{g_t^{-1}*}\dot{g}_t) + \varrho(g_t)\dot{\varphi}_t. \end{aligned}$$

In particular, for $A_t \in G \subset GL(n)$ and $\varphi_t := \varrho(A_t)\varphi$

$$\dot{\varphi}_t = D_{\varphi_t}(\dot{A}_t A_t^{-1}).$$

Proof: The first equation follows from

$$\begin{aligned}
\frac{d}{dt}\big(\varrho(g_t)\varphi_t\big) &= \frac{d}{ds}\Big|_{s=0}\varrho(g_{t+s})\varphi_{t+s} = \frac{d}{ds}\Big|_{s=0}\varrho(g_{t+s}g_t^{-1})\varrho(g_t)\varphi_{t+s}\\
&= \varrho_{*(e,\varrho(g_t)\varphi_t)}(R_{g_t^{-1}*}\dot{g}_t, \varrho(g_t)\dot{\varphi}_t)\\
&= \frac{d}{ds}\Big|_{s=0}\varrho(\exp(sR_{g_t^{-1}*}\dot{g}_t), \varrho(g_t)\varphi_t + s\varrho(g_t)\dot{\varphi}_t)\\
&= \frac{d}{ds}\Big|_{s=0}\varrho(\exp(sR_{g_t^{-1}*}\dot{g}_t))\varrho(g_t)\varphi_t + \frac{d}{ds}\Big|_{s=0} s\varrho(\exp(sR_{g_t^{-1}*}\dot{g}_t))\varrho(g_t)\dot{\varphi}_t\\
&= D_{(\varrho(g_t)\varphi_t)}(R_{g_t^{-1}*}\dot{g}_t) + \varrho(g_t)\dot{\varphi}_t.
\end{aligned}$$

Now Lemma 1.14 (1) implies the second equation,

$$D_{(\varrho(g_t)\varphi_t)}(R_{g_t^{-1}*}\dot{g}_t) = D_{(\varrho(g_t)\varphi_t)}(\mathrm{Ad}(g_t)L_{g_t^{-1}*}\dot{g}_t) = \varrho(g_t)D_{\varphi_t}(L_{g_t^{-1}*}\dot{g}_t).$$

□

Proposition 1.17. Suppose $P \to M$ is a principal G-bundle over M, equipped with a connection. Let $\varrho : G \to \mathrm{Aut}(V)$ be a real G-representation and $\varphi : P \to V$ equivariant. Then we have for any $A \in G(P)$ and $X \in C^\infty(TM)$

$$\nabla_X(\varrho(A)\varphi) = \varrho(A)\nabla_X\varphi + \varrho(A)D_\varphi(\nabla_X A).$$

Proof: From Definition of the covariant derivative in 1.10 and Lemma 1.16 we obtain for $p \in P$

$$\begin{aligned}
\nabla_X(\varrho(A)\varphi)(p) &= (\varrho(A)\varphi)_{*p}(h_p(X))\\
&= \frac{d}{ds}\Big|_{s=0}\varrho(A(c(s)))\varphi(c(s)), \quad \text{where } \dot{c}(0) = h_p(X)\\
&= \varrho(A(p))D_{\varphi(p)}(L_{A^{-1}(p)*}\frac{d}{ds}\Big|_{s=0}A(c(s))) + \varrho(A(p))\frac{d}{ds}\Big|_{s=0}\varphi(c(s))\\
&= \varrho(A(p))D_{\varphi(p)}(\nabla_X A)(p) + \varrho(A(p))(\nabla_X\varphi)(p)\\
&= \big(\varrho(A)D_\varphi(\nabla_X A) + \varrho(A)\nabla_X\varphi\big)(p)
\end{aligned}$$

□

Proposition 1.18. Let $\varrho : GL(n) \to \mathrm{Aut}(V)$ be a real $GL(n)$-representation and $\varphi_0 \in V$ with isotropy group $G \subset O(n)$. An equivariant map $\varphi : FM \to V$ of type φ_0 induces a reduction $P \subset F^gM$ with intrinsic torsion $\tau : P \to \mathbb{R}^{n*} \otimes \mathfrak{g}^\perp$. Then for $X \in C^\infty(TM)$

$$\nabla^g_X\varphi = D_\varphi(\tau(X)).$$

Proof: By Lemma 1.13 we have

$$h^g_p X = h_p X - R_{p*}\tau(X)(p).$$

Since φ is constant along the reduction $P \subset F^g M$, we have $\varphi_*(h_p X) = 0$. This yields

$$\begin{aligned}(\nabla^g_X \varphi)(p) &= \varphi_*(h^g_p X) = \varphi_*(h_p X) - \varphi_*(R_{p*}\tau(X)(p)) \\ &= -\left.\frac{d}{dt}\right|_{t=0} \varphi(p \exp(t\tau(X)(p))) \\ &= -\left.\frac{d}{dt}\right|_{t=0} \varrho(\exp(-t\tau(X)(p)))\varphi(p) \\ &= D_\varphi(\tau(X))(p).\end{aligned}$$

□

Theorem 1.19. Let $\varrho : GL(n) \to \mathrm{Aut}(V)$ be a real $GL(n)$-representation, $\varphi_0 \in V$ with isotropy group $G \subset O(n)$ and $\varphi : FM \to V$ an equivariant map of type φ_0. Then we have for any gauge deformation $A : FM \to GL(n)$ and $X \in C^\infty(TM)$

$$\nabla^{Ag}_X(\varrho(A)\varphi) = \varrho(A)\Big(\nabla^g_X\varphi + D_\varphi\big(\nabla^g_X A - \xi(X) - \mathrm{Ad}(A^{-1})\xi(X)\big)\Big),$$

where $\xi(X) \in C^\infty(\mathrm{End}(TM))$ is given by

$$2g(\xi(X)Y, Z) = g(X, (\nabla^g_Y B)Z) + g(Y, (\nabla^g_X B)Z) + g(B^{-1}(\nabla^g_{BZ} B^{-1})X, Y)$$

and $B := AA^T$ w.r.t. the metric g.

Proof: The difference between the Levi-Civita connections is given by an equivariant map $\xi : FM \to \mathbb{R}^{n*} \otimes \mathfrak{gl}(n)$ with

$$h^{Ag}_p X = h^g_p X + R_{p*}\xi(X).$$

This yields

$$\begin{aligned}(\nabla^{Ag}_X \varphi)(p) &= \varphi_*(h^g_p X + R_{p*}\xi(X)) = (\nabla^g_X\varphi)(p) + \varphi_*(R_{p*}\xi(X)) \\ &= (\nabla^g_X\varphi)(p) - D_\varphi(\xi(X))(p)\end{aligned}$$

and

$$\begin{aligned}(\nabla^{Ag}_X A)(p) &= L_{A^{-1}(p)*}A_*(h^g_p X + R_{p*}\xi(X)) = (\nabla^g_X A)(p) + L_{A^{-1}(p)*}A_*(R_{p*}\xi(X)) \\ &= (\nabla^g_X A)(p) - \mathrm{Ad}(A^{-1}(p))\xi(X).\end{aligned}$$

From Proposition 1.17 we obtain

$$\begin{aligned}\nabla^{Ag}_X(\varrho(A)\varphi) &= \varrho(A)\Big(\nabla^{Ag}_X\varphi + D_\varphi(\nabla^{Ag}_X A)\Big) \\ &= \varrho(A)\Big(\nabla^g_X\varphi - D_\varphi(\xi(X)) + D_\varphi(\nabla^g_X A) - D_\varphi(\mathrm{Ad}(A^{-1})\xi(X))\Big) \\ &= \varrho(A)\Big(\nabla^g_X\varphi + D_\varphi\big(\nabla^g_X A - \xi(X) - \mathrm{Ad}(A^{-1})\xi(X)\big)\Big).\end{aligned}$$

Now we compute for $X, Y \in C^\infty(TM)$

$$(\nabla^g_X Y - \nabla^{Ag}_X Y)(p) = Y_*(h^g_p X - h^{Ag}_p X) = -Y_* R_{p*}\xi(X) = -\left.\frac{d}{dt}\right|_{t=0} \exp(-t\xi(X))Y(p)$$
$$= (\xi(X)Y)(p)$$

and Koszul's formula yields

$$\begin{aligned}
2g(\nabla^{Ag}_X Y, Z) &= 2g(AA^{-1}\nabla^{Ag}_X Y, Z) = 2g(A^{-1}\nabla^{Ag}_X Y, A^{-1}AA^T Z) \\
&= 2(Ag)(\nabla^{Ag}_X Y, BZ) \\
&= 2g(\nabla^g_X Y, Z) + g(X, \nabla^g_Y Z) + g(Y, \nabla^g_X Z) \\
&\quad - g(\nabla^g_{BZ} B^{-1}X, Y) + g(B^{-1}Y, \nabla^g_{BZ} X) \\
&\quad - g(B^{-1}Y, \nabla^g_X BZ) - g(B^{-1}X, \nabla^g_Y BZ).
\end{aligned}$$

From Proposition 1.17 we obtain $\nabla^g_X(BY) = B\nabla^g_X Y + B(\nabla^g_X B)Y$ and hence

$$g(B^{-1}Z, \nabla^g_X BY) = g(Z, \nabla^g_X Y) + g(Z, (\nabla^g_X B)Y).$$

Now

$$\begin{aligned}
2g(\xi(X)Y, Z) &= 2g(\nabla^g_X Y, Z) - 2g(\nabla^{Ag}_X Y, Z) \\
&= g(X, (\nabla^g_Y B)Z) + g(Y, (\nabla^g_X B)Z) \\
&\quad + g(\nabla^g_{BZ} B^{-1}X, Y) - g(B^{-1}Y, \nabla^g_{BZ} X)
\end{aligned}$$

and

$$\begin{aligned}
g(\nabla^g_{BZ} B^{-1}X, Y) &= g(B^{-1}\nabla^g_{BZ} X + B^{-1}(\nabla^g_{BZ} B^{-1})X, Y) \\
&= g(\nabla^g_{BZ} X, B^{-1}Y) + g(B^{-1}(\nabla^g_{BZ} B^{-1})X, Y)
\end{aligned}$$

yields eventually

$$2g(\xi(X)Y, Z) = g(X, (\nabla^g_Y B)Z) + g(Y, (\nabla^g_X B)Z) + g(B^{-1}(\nabla^g_{BZ} B^{-1})X, Y).$$

□

Corollary 1.20. The gauge deformation $A : FM \to GL(n)$ from Theorem 1.19 yields a parallel structure $\nabla^{Ag} A\varphi = 0$ if and only if for all vector fields $X \in C^\infty(TM)$

$$\tau(X) = \mathrm{pr}_{\mathfrak{g}^\perp}\left(\xi(X) + \mathrm{Ad}(A^{-1})\xi(X) - \nabla^g_X A\right),$$

where τ is the intrinsic torsion of $P \subset F^g M$ and the projection is taken w.r.t. P.

Proof: By Proposition 1.18 we have $\nabla^g_X \varphi = D_\varphi(\tau(X))$ and $\ker D_\varphi = \mathfrak{g}$ by Lemma 1.14. Hence Theorem 1.19 shows that $\nabla^{Ag} A\varphi = 0$ if and only if

$$\mathrm{pr}_{\mathfrak{g}^\perp}\left(\tau(X) + \nabla^g_X A - \xi(X) - \mathrm{Ad}(A^{-1})\xi(X)\right) = 0.$$

Since $\tau(X)(p) \in \mathfrak{g}^\perp$, for $p \in P$, the corollary follows.

$\square$

Corollary 1.21. A G-structure $P \subset F^g M$ like in Theorem 1.19 with intrinsic torsion τ can be deformed to a torsion-free structure if and only if there exists a solution $A \in GL(FM)$ of

$$\tau(X) = \mathrm{pr}_{\mathfrak{g}^\perp}\left(\xi(X) + \mathrm{Ad}(A^{-1})\xi(X) - \nabla^g_X A\right),$$

where ξ is defined like in Theorem 1.19 and the projection is taken w.r.t. P.

Proof: By Theorem 1.6 (1) any deformation $P_t \subset FM$ of the initial structure P can be described by a family of gauge deformations A_t and the Corollary follows from Corollary 1.20.

$\square$

Holonomy

Suppose $\mathcal{Z}$ is a connection on a principal G-bundle P over M. Given a piecewise smooth curve $c : [0,1] \to M$ and a point $p \in P$, there is a unique horizontal lift $c_p : [0,1] \to P$ of c to P such that $c_p(0) = p$. Namely, c_p is the integral curve of the lifted vector field $\dot{c}$. The parallel translation along c is the map

$$\mathcal{Z}_c : P_{c(0)} \to P_{c(1)} \quad \text{with} \quad p \mapsto c_p(1).$$

For a fixed point $p \in P$ consider

$$\mathrm{Hol}(p, \mathcal{Z}) := \{g \in G \mid pg = \mathcal{Z}_c(p) \text{ for some } c : [0,1] \to M \text{ with } c(0) = c(1) = x\}.$$

By Theorem 4.2 in **[45]**, $\mathrm{Hol}(p, \mathcal{Z})$ defines in fact a Lie subgroup of G, called the holonomy group of the connection $\mathcal{Z}$. Note that changing the reference point $p \in P$ only changes the conjugacy class $\mathrm{Hol}(p, \mathcal{Z}) \subset G$, as long as M is connected.
The holonomy bundle $Q(p) \subset P$ consists of all points in P that can be joined with p by a horizontal curve. In fibre direction, $Q(p)$ is generated precisely by the action of $\mathrm{Hol}(p, \mathcal{Z})$ and hence gives a $\mathrm{Hol}(p, \mathcal{Z})$-reduction of P. Moreover, the connection $\mathcal{Z}$ on P reduces to $Q(p)$. To see this, consider $X \in H_q$ for $q \in Q(p)$. Then $X = \dot{c}_q(0)$ for the lift of some curve c in M with $\dot{c}(0) = \pi_* X$. Since $q \in Q(p)$, we have $Q(q) = Q(p)$ and eventually $X = \dot{c}_q(0) \in T_q Q(q) = T_q Q(p)$.
On the other hand, consider a reduction $Q \subset P$ to a Lie subgroup H of G which is compatible with the connection on P. Then any horizontal curve stays in Q and so the holonomy group is a subgroup of H. Hence P admits a reduction to $H \subset G$

that is compatible with the connection on P if and only if the holonomy group is contained in H.
Holonomy can be measured in terms of curvature. The curvature tensor $R : P \to \Lambda^2\mathbb{R}^{n*} \otimes \mathfrak{g}$ satisfies

$$R(p)(x, y) = d\mathcal{Z}(h_p(px), h_p(py)) = -\mathcal{Z}[h(x), h(y)]_p.$$

Moreover we have

$$[h(x), h(y)]_p = \frac{d}{dt}\bigg|_{t=0} \mathcal{Z}_{c_t}(p),$$

where c_t denotes the family of loops in M that corresponds to the family of quadrangles with vertices $\{0, tpx, tp(x + y), tpy\}$ in $T_{\pi(p)}M$. The corresponding 1-parameter family $g_t \in \mathrm{Hol}(p, \mathcal{Z})$ is then given by $\mathcal{Z}_{c_t}(p) = pg_t$ and hence

$$R(p)(x, y) = -\mathcal{Z}(R_{p*}\dot{g}(0)) = -\dot{g}(0) \in \mathfrak{hol}(p, \mathcal{Z}).$$

Since $\mathrm{Hol}(q, \mathcal{Z}) = \mathrm{Hol}(p, \mathcal{Z})$ holds for $q \in Q(p)$, we get

$$\mathfrak{h} := \{R(q)(x, y) \mid x, y \in \mathbb{R}^n \text{ and } q \in Q(p)\} \subset \mathfrak{hol}(p, \mathcal{Z}).$$

One can actually show that $\mathfrak{h}$ defines a Lie subalgebra of $\mathfrak{g}$ and hence the distribution $H \oplus v(\mathfrak{h})$ on $Q(p)$ is integrable by Lemma 1.11. Since the horizontal distribution is contained in $H \oplus v(\mathfrak{h})$, the holonomy bundle $Q(p)$ is contained in the maximal integral manifold through $p \in Q(p)$. This shows that also $\mathfrak{hol}(p, \mathcal{Z}) \subset \mathfrak{h}$ holds and proves the Ambrose-Singer Theorem,

$$\mathfrak{hol}(p, \mathcal{Z}) = \{R(q)(x, y) \mid x, y \in \mathbb{R}^n \text{ and } q \in Q(p)\} \subset \mathfrak{g}.$$

2. Integral Curves in Fréchet Spaces

In the previous chapter we described deformations of principal bundles via families of gauge deformations $A_t \in C^\infty(\mathrm{Aut}(TM)) \subset C^\infty(\mathrm{End}(TM))$. Since the space of sections $C^\infty(\mathrm{End}(TM))$ is a Fréchet space, these type of vector spaces naturally enter the scene when describing deformations of various structures. Indeed, R. Hamilton makes intensive use of the Nash-Moser inverse function theorem for Fréchet spaces in his fundamental work **[34]** on the Ricci flow. Natural deformations very often arise as the gradient flow of some functional. More generally, we may consider deformations that evolve under the flow of a certain vector field X on $C^\infty(\mathrm{Aut}(TM))$, i.e.

$$\dot{A}_t = X \circ A_t.$$

In contrast to finite dimensional geometry, there does not have to exist even a short-time solution of the above equation. In the real analytic category, the Cauchy-Kowalevski Theorem ensures the (local) existence of solutions for certain partial differential equations. In this chapter we translate the Cauchy-Kowalevski Theorem into a global version for integral curves in Fréchet spaces of the form $C^\infty(V)$, where $V \to M$ is a vector bundle over a compact manifold M, cf. Theorem 2.11.

Beyond the existence, we show that the particular solution can be developed in a (convergent) power series. This property is the crucial ingredient to prove that the solutions coming from the Cauchy-Kowalevski Theorem preserve certain initial conditions. In this sense, Corollary 2.4 can be regarded as a conservation law for integral curves in Fréchet spaces. The basic idea stems from finite dimensional geometry: If a vector field X is tangent to some submanifold N, then any integral curve of X, which lies initially in N, stays in N for all times. Although not true for arbitrary integral curves, this observation carries over to Fréchet spaces if the integral curve can be developed in a power series.

Fréchet Spaces

Hamilton **[34]** gives an introduction to Fréchet manifolds which goes far beyond of what we require for our purposes. Although Proposition 2.3 and Corollary 2.4 can be generalized to Fréchet manifolds, we focus on Fréchet spaces to keep the technical efforts at a minimum.

A locally convex topological vector space $\mathcal{F}$ is a vector space with a collection of seminorms, i.e. functions $\|.\|_n : \mathcal{F} \to \mathbb{R}$, $n \in N$, which satisfy

$$\|f\|_n \geq 0, \qquad \|f+g\|_n \leq \|f\|_n + \|g\|_n \quad \text{and} \quad \|\lambda f\|_n = |\lambda| \|f\|_n$$

for all $f, g \in \mathcal{F}$ and scalars λ. Such a family defines a unique topology which is metrizable if and only if N is countable. In this case the topology is characterized by the property

$$\lim_{k\to\infty} f_k = f \in \mathcal{F} \quad \Leftrightarrow \quad \forall n \in N: \lim_{k\to\infty} \|f_k - f\|_n = 0.$$

The topology is Hausdorff if and only if

$$\big(\forall n \in N: \ \|f\|_n = 0\big) \quad \Rightarrow \quad f = 0.$$

The space is sequentially complete if every Cauchy sequence converges, where f_k is a Cauchy sequence if it is a Cauchy sequence for every seminorm $\|.\|_n$. A Fréchet space is a locally convex topological vector space, which is in addition metrizable, Hausdorff and complete.

Example 2.1. Suppose $F \to M$ is a vector bundle over a compact manifold M. Then the vector space

$$\mathcal{F} := C^\infty(F)$$

of smooth sections of F is a Fréchet space, where the collection of seminorms

$$\|f\|_n := \sum_{j=0}^{n} \sup_{p\in M} |(\nabla^{(j)} f)(p)|$$

can be defined after choosing Riemannian metrics and connections on TM and F, cf. [**34**] Example 1.1.5. The induced topology is the C^∞-topology on $\mathcal{F}$.
Given an open subset $U \subset F$, we consider the subset of all sections in $\mathcal{F}$, whose image lies in U,

$$\mathcal{U} := \{f \in \mathcal{F} \mid f(M) \subset U\}.$$

For $f \in \mathcal{U}$ we can find $\varepsilon > 0$ such that

$$f \in B^0_\varepsilon(f) := \{\tilde{f} \in \mathcal{F} \mid \|\tilde{f} - f\|_0 < \varepsilon\} \subset \mathcal{U}.$$

Since $B^0_\varepsilon(f) \subset \mathcal{F}$ is open, $\mathcal{U}$ is an open subset of the Fréchet space $\mathcal{F}$.

Smooth maps between Fréchet spaces can be defined as follows: Let $U \subset \mathcal{F}$ be an open subset of a Fréchet space $\mathcal{F}$ and $P : \mathcal{U} \to \mathcal{E}$ a continuous and nonlinear map into another Fréchet space $\mathcal{E}$. We say that P is C^1 on $\mathcal{U}$ if for every $f \in \mathcal{U}$ and every $v \in \mathcal{F}$ the limit

$$DP(f)v := \lim_{t\to 0} \frac{1}{t}(P(f + tv) - P(f))$$

exists and the map $DP : \mathcal{U} \times \mathcal{F} \to \mathcal{E}$ is continuous. Consequently, we say that P is C^k on $\mathcal{U}$ if P is C^{k-1} and the limit

$$D^{(k)}P(f)\{v_1, .., v_k\} := \lim_{t\to 0} \frac{1}{t}\Big(D^{(k-1)}P(f + tv_n)\{v_1, .., v_{k-1}\} - D^{(k-1)}P(f)\{v_1, .., v_{k-1}\}\Big)$$

exists for all $f \in \mathcal{U}$ and $v_1, .., v_k \in \mathcal{F}$, and the map $D^{(k)}P : \mathcal{U} \times \mathcal{F} \times .. \times \mathcal{F} \to \mathcal{E}$ is continuous. We call P a smooth map on $\mathcal{U}$ if P is C^k for all $k \in \mathbb{N}$. We summarize Corollary 3.3.5 and Theorem 3.6.2 from [**34**] in the following

Theorem 2.2. **(1)** If $P : \mathcal{U} \subset \mathcal{F} \to \mathcal{E}$ is C^1 and $c(t) \in \mathcal{U} \subset \mathcal{F}$ is a parameterized C^1 curve, then $P \circ c(t)$ is a parameterized C^1 curve and

$$\frac{d}{dt}(P \circ c(t)) = DP(c(t))\dot{c}(t).$$

(2) If $P : \mathcal{U} \subset \mathcal{F} \to \mathcal{E}$ is C^k, then for every $f \in \mathcal{U}$

$$D^{(k)}P(f)\{v_1, .., v_k\}$$

is completely symmetric and linear separately in $v_1, .., v_k \in \mathcal{F}$.

In the following we will consider curves $c(t) \in \mathcal{F}$ in a Fréchet space $\mathcal{F}$, which are integral curves of a vector field that is tangent to some subspace $\mathcal{E} \subset \mathcal{F}$. In finite dimension we would expect that any such integral curve with $c(0) \in \mathcal{E}$ actually stays in the subspace for all times. This conclusion fails for Fréchet spaces, as was pointed out to us by Christian Bär: Consider $\mathcal{F} := C^\infty[1, 2]$ and $\mathcal{E} := \{0\} \subset \mathcal{M}$. Then

$$c_t(x) := \begin{cases} (4\pi t)^{-\frac{1}{2}} \exp(-\frac{x^2}{4t}), & \text{for } t > 0 \\ 0, & \text{for } t \leq 0 \end{cases}$$

solves $\dot{c}_t = \Delta c_t = \partial^2 c_t / \partial x^2$ and hence defines an integral curve of the vector field $X(c) := \Delta c$. Although X is tangent to $\mathcal{E}$, i.e. $X(0) = 0$, and $c_0 = 0 \in \mathcal{E}$, the curve does not stay in $\mathcal{E}$, since $c_t \neq 0$, for $t > 0$. Note also that $t \mapsto c_t(x)$ is not real analytic in $t = 0$.

Proposition 2.3. Suppose $\mathcal{E} \subset \mathcal{F}$ is a closed subspace of the Fréchet space $\mathcal{F}$ and that $X : \mathcal{U} \subset \mathcal{F} \to \mathcal{F}$ is a smooth map defined on some open subset $\mathcal{U} \subset \mathcal{F}$. Let $f \in \mathcal{F}$ and assume that

$$X_{|\mathcal{U} \cap \mathcal{E}_f} : \mathcal{U} \cap \mathcal{E}_f \to \mathcal{E},$$

where $\mathcal{E}_f := \{f\} + \mathcal{E}$. If a smooth curve $c : (-\varepsilon, \varepsilon) \to \mathcal{F}$ satisfies

$$c(0) \in \mathcal{U} \cap \mathcal{E}_f \qquad \text{and} \qquad X \circ c(t) = \dot{c}(t),$$

where $\dot{c} : (-\varepsilon, \varepsilon) \to \mathcal{F}$ is the derivative of $c(t)$ by t, then for all $k \geq 1$

$$c^{(k)}(0) \in \mathcal{E},$$

where $c^{(k)} : (-\varepsilon, \varepsilon) \to \mathcal{F}$ is the k^{th} derivative of $c(t)$ by t.

PROOF: First we prove by induction on k that the k^{th} differential $D^{(k)}X$ of $X : \mathcal{F} \to \mathcal{F}$ satisfies

$$D^{(k)}X_{|\mathcal{U}\cap\mathcal{E}_f\times\mathcal{E}\times..\times\mathcal{E}} : \mathcal{U}\cap\mathcal{E}_f \times \mathcal{E} \times .. \times \mathcal{E} \to \mathcal{E}. \tag{1}$$

For $k = 0$ this is just the assumption $X_{|\mathcal{U}\cap\mathcal{E}_f} : \mathcal{U}\cap\mathcal{E}_f \to \mathcal{E}$. For $v_0 \in \mathcal{U}\cap\mathcal{E}_f$ and $v_1, .., v_{k+1} \in \mathcal{E}$ we have by definition

$$\begin{aligned}
&D^{(k+1)}X(v_0)\{v_1, .., v_{k+1}\} \\
&= \lim_{s\to 0} \frac{1}{s} \underbrace{(D^{(k)}X(\underbrace{v_0 + sv_{k+1}}_{\in\mathcal{U}\cap\mathcal{E}_f \text{ for } s \text{ small}})\{v_1, .., v_k\} - D^{(k)}X(v_0)\{v_1, .., v_k\})}_{\in\mathcal{E} \text{ by induction hypothesis}}
\end{aligned}$$

and since $\mathcal{E}$ is closed, we conclude that (1) holds for $k + 1$. Next we show that for $k \geq 0$ and any choice of smooth curves $t \mapsto v_0(t) \in \mathcal{U}$ and $t \mapsto v_1(t), .., v_k(t) \in \mathcal{F}$

$$\begin{aligned}
\frac{d}{dt}D^{(k)}X(v_0(t))\{v_1(t), .., v_k(t)\} &= D^{(k+1)}X(v_0(t))\{v_1(t), .., v_k(t), \dot{v}_0(t)\} \\
&\quad + \sum_{j=1}^{k} D^{(k)}X(v_0(t))\{v_1(t), .., \dot{v}_j(t), .., v_k(t)\}
\end{aligned} \tag{2}$$

holds. Applying Theorem 2.2 (1) to the map $D^{(k)}X : \mathcal{U} \times \mathcal{F} \times .. \times \mathcal{F} \to \mathcal{F}$, we get

$$\begin{aligned}
&\frac{d}{dt}D^{(k)}X(v_0(t))\{v_1(t), .., v_k(t)\} \\
&= D(D^{(k)}X)(v_0(t), .., v_k(t))\{\dot{v}_0(t), .., \dot{v}_k(t)\} \\
&= \lim_{s\to 0} \frac{1}{s} \Big(D^{(k)}X(v_0(t) + s\dot{v}_0(t))\{v_1(t) + s\dot{v}_1(t), .., v_k(t) + s\dot{v}_k(t)\} \\
&\qquad - D^{(k)}X(v_0(t))\{v_1(t), .., v_k(t)\} \Big)
\end{aligned}$$

and (2) follows, since $D^{(k)}X$ is linear in the arguments in $\{...\}$, cf. Theorem 2.2 (2). We will now show by induction on k that $c^{(k)}(0) \in \mathcal{E}$ holds. For $k = 1$ we have $\dot{c}(0) = X \circ c(0) \in \mathcal{E}$ by assumption. Since $\dot{c}(t) = X \circ c(t) = D^{(0)}X(c(t))$ and $c(t) \in \mathcal{U}$ for sufficiently small t, we can apply (2) to see that $c^{(k+1)}(t)$, again for sufficiently small t, can be expressed as a linear combination of

$$D^{(j)}X(c(t))\{v_1(t), .., v_j(t)\},$$

where $j \in \{1, .., k+1\}$ and $v_1(t), .., v_j(t) \in \{c^{(l)}(t) \mid 1 \leq l \leq k\}$. Since $c(0) \in \mathcal{U}\cap\mathcal{E}_f$, we get from $c^{(1)}(0), .., c^{(k)}(0) \in \mathcal{E}$ and (1)

$$D^{(j)}X(c(0))\{v_1(0), .., v_j(0)\} \in \mathcal{E},$$

and hence $c^{(k+1)}(0) \in \mathcal{E}$.

□

The following corollary can be regarded as a conservation law for integral curves in Fréchet spaces.

Corollary 2.4. If the curve $c : (-\varepsilon, \varepsilon) \to \mathcal{F}$ from Proposition 2.3 satisfies for all $t \in (-\varepsilon, \varepsilon)$

$$c(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c^{(k)}(0) \in \mathcal{F},$$

where the series converges w.r.t. the Fréchet topology in $\mathcal{F}$, then

$$c(t) - c(0) \in \mathcal{E},$$

for all $t \in (-\varepsilon, \varepsilon)$.

Proof: From Proposition 2.3 we get $c^{(k)}(0) \in \mathcal{E}$ for all $k \geq 1$ and hence

$$c(t) - c(0) = \sum_{k=1}^{\infty} \frac{t^k}{k!} c^{(k)}(0) \in \mathcal{E},$$

since $\mathcal{E} \subset \mathcal{F}$ is closed and the series converges in $\mathcal{F}$.

□

Real Analyticity

A formal power series in $X = (X_1, .., X_n)$ with coefficients in $\mathbb{R}$ is an expression of the form

$$S(X) = \sum_{p \in \mathbb{N}^n} a_p X^p,$$

where $a_p \in \mathbb{R}$ and $X^p := X_1^{p_1} \cdot .. \cdot X_n^{p_n}$, for $p = (p_1, .., p_n) \in \mathbb{N}^n$. Given a formal power series $S(X)$, we define

$$\Gamma := \{r = (r_1, .., r_n) \mid r_i \geq 0 \text{ and } \sum_{p \in \mathbb{N}^n} |a_p| \, r^p < \infty\}$$

and denote by Δ the interior of Γ, called the domain of convergence of the series. Hence the series

$$S(x) = \sum_{p \in \mathbb{N}^n} a_p x^p$$

is for every $x = (x_1, .., x_n) \in \mathbb{R}^n$ with $|x| = (|x_1|, .., |x_n|) \in \Gamma$ absolute convergent. We recall the following result:

Proposition 2.5. Suppose $S(X)$ is a formal power series with domain of convergence Δ. For $\bar{x} = (\bar{x}_1, .., \bar{x}_n) \in \mathbb{R}^n$ with $|\bar{x}| \in \Delta$ and $r_1, .., r_n$ with $0 < r_i < |\bar{x}_i|$, define

$$K := \{(x_1, .., x_n) \in \mathbb{R}^n \mid |x_i| \leq r_i\}.$$

(1) For any subset $P \subset \mathbb{N}^n$, the series

$$S_P(x) := \sum_{p \in P} a_p x^p$$

converges absolutely for all $x \in K$. In particular, the series $S(x) := \sum_{p \in \mathbb{N}^n} a_p x^p$ converges absolutely for $x \in K$.

(2) Suppose that $P_N \subset \mathbb{N}^n$ is a family of subsets, $N \in \mathbb{N}$, such that $\lim_{N \to \infty} P_N = \mathbb{N}^n$. Then

$$S_N(x) := \sum_{p \in P_N} a_p x^p$$

converges uniformly on K to the function $S : K \to \mathbb{R}$, $x \mapsto S(x)$.

Proof: Since $|\bar{x}| \in \Delta$ we can find $C > 0$ such that

$$|a_p \bar{x}^p| \leq C, \quad \text{for all } p \in \mathbb{N}^n.$$

Hence for $x \in K$

$$|a_p x^p| = |a_p \, \bar{x}_1^{p_1} \cdot .. \cdot \bar{x}_n^{p_n}| \frac{|x_1^{p_1} \cdot .. \cdot x_n^{p_n}|}{|\bar{x}_1^{p_1} \cdot .. \cdot \bar{x}_n^{p_n}|} \leq C \left(\frac{r_1}{|\bar{x}_1|} \right)^{p_1} \cdot .. \cdot \left(\frac{r_n}{|\bar{x}_n|} \right)^{p_n}.$$

Since $r_i / |\bar{x}_i| < 1$, we can apply the method of majorants to see that $S_P(x)$ converges absolutely for $x \in K$. To prove uniform convergence consider

$$\begin{aligned} \sup_{x \in K} |S(x) - S_N(x)| &= \sup_{x \in K} | \sum_{p \in \mathbb{N}^n \setminus P_N} a_p x^p| \\ &\leq C \sum_{p \in \mathbb{N}^n \setminus P_N} \left(\frac{r_1}{|\bar{x}_1|} \right)^{p_1} \cdot .. \cdot \left(\frac{r_n}{|\bar{x}_n|} \right)^{p_n} \end{aligned}$$

Given $\varepsilon > 0$, we can choose M large, so that $\sum_{p_i = M+1}^{\infty} \left(\frac{r_i}{|\bar{x}_i|} \right)^{p_i} \leq \frac{\varepsilon}{nCC_i}$, for $i = 1, .., n$, where

$$C_i := \sum_{\substack{(p_1 .. \hat{p}_i .. p_n) \\ \in \mathbb{N}^{n-1}}} \left(\frac{r_1}{|\bar{x}_1|} \right)^{p_1} \cdot .. \cdot \widehat{\left(\frac{r_i}{|\bar{x}_i|} \right)^{p_1}} \cdot .. \cdot \left(\frac{r_n}{|\bar{x}_n|} \right)^{p_n} < \infty \quad \text{(geometric series).}$$

The notation $\widehat{.}$ indicates that the corresponding factor is omitted. Since $\lim_{N \to \infty} P_N = \mathbb{N}^n$, we can find $N = N(M)$, such that $\{0, .., M\}^n \subset P_N$. Hence

$$\begin{aligned} \sup_{x \in K} |S(x) - S_N(x)| &\leq C \sum_{p \in \mathbb{N}^n \setminus \{0..M\}^n} \left(\frac{r_1}{|\bar{x}_1|} \right)^{p_1} \cdot .. \cdot \left(\frac{r_n}{|\bar{x}_n|} \right)^{p_n} \\ &\leq C \sum_{i=1}^{n} \sum_{p_i = M+1}^{\infty} C_i \left(\frac{r_i}{|\bar{x}_i|} \right)^{p_i} \leq \varepsilon. \end{aligned}$$

□

Definition 2.6. Let $U \subset \mathbb{R}^n$ open and $x_0 \in U$.
(1) A function $f : U \to \mathbb{R}$ is called real analytic in $x_0 \in U$ if there exists a formal power series S with

$$f(x) = S(x - x_0),$$

for all x in a neighborhood of x_0.
(2) A function $f : U \to \mathbb{R}$ is called real analytic in U if f is real analytic for every $x_0 \in U$.
(3) A function $F = (f_1, .., f_m) : U \to \mathbb{R}^m$ is called real analytic in U if each component $f_i : U \to \mathbb{R}$ is real analytic in U.

Note that the coefficients of S can be computed in terms of partial derivatives, which shows that S is uniquely determined by the condition $f(x) = S(x - x_0)$. Moreover we have the following basic properties, cf. [**18**] p.123:

Lemma 2.7.

(1) If $f : U \to \mathbb{R}$ is real analytic in $x_0 \in U$, then it is differentiable in a neighborhood of x_0 and the derivatives are again real analytic functions in $x_0 \in U$.

(2) If f and g are real analytic in x_0, then the product fg is real analytic in x_0.

(3) If $f : U \to \mathbb{R}$ is real analytic, then $1/f$ is real analytic in all points $x \in U$, where $f(x) \neq 0$.

(4) Compositions of real analytic functions are again real analytic.

A manifold M is called real analytic if it admits an atlas with real analytic transition functions. Similarly to the smooth category one can define real analytic vector bundles over M.

The Cauchy-Kowalevski Theorem

In this section we will develop a global version of the Cauchy-Kowalevski Theorem, cf. [**12**], III. Theorem 2.1:

Theorem 2.8. Let t be a coordinate on $\mathbb{R}$, $x = (x_i)$ be coordinates on $\mathbb{R}^n$, $y = (y_j)$ be coordinates on $\mathbb{R}^s$ and let $z = (z_i^j)$ be coordinates on $\mathbb{R}^{ns}$. Let $D \subset \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^{ns}$ open, and let $G : D \to \mathbb{R}^s$ be a real analytic mapping. Let $D_0 \subset \mathbb{R}^n$ be open and $f : D_0 \to \mathbb{R}^s$ be a real analytic mapping with Jacobian $Df(x) \in \mathbb{R}^{ns}$, i.e. $z_i^j(Df(x)) = \partial f^j(x)/\partial x_i$, so that $\{(t_0, x, f(x), Df(x)) \mid x \in D_0\} \subset D$ for some $t_0 \in \mathbb{R}$.
Then there exists an open neighborhood $D_1 \subset \mathbb{R} \times D_0$ of $\{t_0\} \times D_0$ and a real analytic mapping $F : D_1 \to \mathbb{R}^s$ which satisfies

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) & = G(t, x, F(t,x), \frac{\partial F}{\partial x}(t,x)) \\ F(t_0, x) & = f(x) \quad \text{for all } x \in D_0. \end{cases}$$

F is unique in the sense that any other real analytic solution of the above initial value problem agrees with F in some neighborhood of $\{t_0\} \times D_0$.

Remark 2.9. Since the solution $F = (f_i, .., f_s) : D_1 \to \mathbb{R}^s$ from Theorem 2.8 is real analytic, we can develop each component in a convergent power series around $(t_0, x_0) = (0,0) \in D_1$, i.e.

$$f_i(t,x) = \sum_{k=0}^{\infty} \Big(\sum_{p\in\mathbb{N}^n} a_{ikp} x^p \Big) t^k = \sum_{k=0}^{\infty} \Big(\frac{1}{k!} f_i^{(k)}(0,x) \Big) t^k.$$

Applying Proposition 2.5 (2) with $P_N := \{0, .., N\} \times \mathbb{N}^n$ shows that

$$f_i^N(t,x) = \sum_{k=0}^{N} \Big(\sum_{p\in\mathbb{N}^n} a_{ikp} x^p \Big) t^k = \sum_{k=0}^{N} \frac{t^k}{k!} f_i^{(k)}(0,x)$$

converges locally uniformly to the function $f_i(t,x)$, for $N \to \infty$. The partial derivatives of a formal power series $S(X)$ are defined by,

$$\frac{\partial S}{\partial X_i} := \sum_{p\in\mathbb{N}^n} p_i a_p X_1^{p_1} \cdot ..X_i^{p_i - 1} .. \cdot X_n^{p_n}.$$

The formal power series $\frac{\partial S}{\partial X_i}$ has the same domain of convergence Δ as the formal power series S. Moreover, the function $\frac{\partial S}{\partial X_i} : \Delta \to \mathbb{R}$ is the partial derivative of the function $S : \Delta \to \mathbb{R}$ w.r.t. x_i, cf. Satz 3.2 in **[18]**. Hence we can apply again Proposition 2.5 (2) to see that all partial derivatives of the function $f_i^N(t,x)$ converge locally uniformly to the corresponding partial derivative of $f_i(t,x)$. In summary, the functions

$$F_N(t,x) := \sum_{k=0}^{N} \frac{t^k}{k!} F^{(k)}(0,x)$$

converge, as $N \to \infty$, locally in C^∞-topology to the solution $F(t,x)$ from Theorem 2.8.

Definition 2.10. Suppose M is a real analytic manifold and $\pi : V \to M$ is a rank s real analytic vector bundle. We call a map

$$X : C^\infty(V) \to C^\infty(V)$$

a real analytic first order differential operator if every point of M has a neighborhood $U \subset M$, which is the domain of a real analytic chart $u : U \to \mathbb{R}^n$, and there exists a real analytic trivialization $(\pi, v) : V_{|U} \cong U \times \mathbb{R}^s$, together with a real analytic function

$$G : D \subset \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns} \to \mathbb{R}^s,$$

such that for every local section $c : U \subset M \to V$

$$v(X \circ c) = G(u, v \circ c, \frac{\partial c_i}{\partial u_j})$$

holds, where c_i is the i^{th} component of $v \circ c : U \to \mathbb{R}^s$.

We can now prove the following global version of the Cauchy-Kowalevski Theorem,

Theorem 2.11. Suppose $\pi : V \to M$ is a real analytic rank s vector bundle over a compact real analytic manifold M. Let $X : C^\infty(V) \to C^\infty(V)$ be a real analytic first order differential operator and let $c_0 \in C^\infty(V)$ be a real analytic section. Then the initial value problem

$$\begin{cases} \dot{c}(t) = X \circ c(t) \\ c(0) = c_0 \end{cases}$$

has a unique real analytic solution $c : (-\varepsilon, \varepsilon) \to C^\infty(V)$, i.e. $c : (-\varepsilon, \varepsilon) \times M \to V$ is real analytic. Moreover, the solution $c(t)$ satisfies

$$c(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_0^{(k)},$$

where the series converges in the C^∞-topology on $C^\infty(V)$.

Proof: First we prove that local sections $c_t : U \subset M \to V$ exist, which solve the initial value problem locally. Secondly, we show that the compactness of M ensures the existence of a global solution. Eventually we will use the uniqueness part of the Cauchy-Kowalevski Theorem to prove the uniqueness statement of the Theorem.

By Definition 2.10 we can find a real analytic chart $u : U \subset M \to \mathbb{R}^n$ and a trivialization $(\pi, v) : V_{|U} \cong U \times \mathbb{R}^s$, such that for each local section $c : U \subset M \to V$

$$v(X \circ c) = G(u, v \circ c, \frac{\partial c_i}{\partial u_j}) \tag{1}$$

holds, where $G : D \subset \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns} \to \mathbb{R}^s$ is real analytic. The map

$$f : D_0 := u(U) \subset \mathbb{R}^n \to \mathbb{R}^s \quad \text{with} \quad f(x) := v \circ c_0 \circ u^{-1}(x)$$

is real analytic and hence we can find by the Cauchy-Kowalevski Theorem a real analytic solution $F : (-\varepsilon, \varepsilon) \times \widetilde{D}_0 \to \mathbb{R}^s$ of

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) & = G(x, F(t,x), \frac{\partial F}{\partial x}(t,x)) \\ F(t_0, x) & = f(x) \quad \text{for all } x \in D_0, \end{cases}$$

where $\widetilde{D}_0 \subset D_0$ is open. Let $\widetilde{U} := u^{-1}(\widetilde{D}_0) \subset U$ and define for $t \in (-\varepsilon, \varepsilon)$

$$c(t) : \widetilde{U} \subset M \to V \quad \text{by} \quad c(t,p) := v_p^{-1} \circ F(t, u(p)), \tag{2}$$

where $v_p : V_p \cong \mathbb{R}^s$ is the isomorphism induced by the local trivialization (π, v). By definition, the map $c : (-\varepsilon, \varepsilon) \times \widetilde{U} \subset M \to V$ is real analytic and satisfies

$$c(0,p) = v_p^{-1} \circ F(0, u(p)) = v_p^{-1} \circ f(u(p)) = c_0(p). \tag{3}$$

Now we have for $i = 1, .., s$ and $j = 1, .., n$

$$\begin{aligned} \frac{\partial (v_i \circ c_t)}{\partial u_j}(p) &= \frac{\partial}{\partial u_j}\Big|_p \cdot (v_i \circ c_t) = (u_*^{-1} \frac{\partial}{\partial x_j}\Big|_{u(p)}) \cdot (v_i \circ c_t) \\ &= \frac{\partial}{\partial x_j}\Big|_{u(p)} \cdot (v_i \circ c_t \circ u^{-1}) = \frac{\partial}{\partial x_j}\Big|_{u(p)} \cdot F_i(t, .) \\ &= \frac{\partial F_i}{\partial x_j}(t, u(p)). \end{aligned} \tag{4}$$

Since by definition $v \circ c(t,p) = F(t, u(p))$ holds, we get from (1), applied to c_t

$$\begin{aligned} \dot{c}(t,p) &= v_p^{-1} \circ G(u(p), F(t, u(p)), \frac{\partial F}{\partial x}(t, u(p))) \\ &= v_p^{-1} \circ G(u(p), v \circ c_t(p), \frac{\partial (v_i \circ c_t)}{\partial u_j}(p)) \\ &= v_p^{-1} \circ v_p(X \circ c(t,p)) = X \circ c(t,p), \end{aligned}$$

i.e. c_t is the desired local solution of the initial value problem. Moreover, we get by Remark 2.9

$$\begin{aligned} c(t,p) = v_p^{-1} \circ F(t, u(p)) &= v_p^{-1}\big(\lim_{N\to\infty} \sum_{k=0}^{N} \frac{t^k}{k!} F^{(k)}(0, u(p))\big) \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} \frac{t^k}{k!} v_p^{-1} \circ F^{(k)}(0, u(p)) = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{t^k}{k!} c^{(k)}(0,p), \end{aligned}$$

i.e.

$$c_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_0^{(k)}, \tag{5}$$

where the series converges locally in C^∞-topology. Suppose now we apply the above construction to obtain two local sections

$$c_1(t) : U_1 \subset M \to V \quad \text{and} \quad c_2(t) : U_2 \subset M \to V,$$

where $t \in (-\varepsilon, \varepsilon)$, $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and $U_1 \cap U_2 \neq \emptyset$. Since c_1 and c_2 both solve the initial value problem

$$\begin{cases} \dot{c}_i(t) = X \circ c_i(t) \\ c_i(0) = c_0, \end{cases}$$

$i = 1, 2$, we see that $c_1(0) = c_2(0)$ and $\dot{c}_1(0) = \dot{c}_2(0)$ on $U_1 \cap U_2$. Differentiating the equation $\dot{c}_1(t) = X \circ c_1(t)$, shows that $c_1^{(k+1)}(t)$ can be expressed as a linear combination of

$$D^{(j)}X(c_1(t))\{v_1(t), .., v_j(t)\},$$

where $j \in \{1, .., k+1\}$ and $v_1(t), .., v_j(t) \in \{c_1^{(l)}(t) \mid 1 \leq l \leq k\}$, cf. the proof of Proposition 2.3. Now we obtain by induction $c_1^{(k)}(0) = c_2^{(k)}(0)$ on $U_1 \cap U_2$, for all $k \in \mathbb{N}$. Hence (5) implies $c_1(t) = c_2(t)$ on $U_1 \cap U_2$. If M is compact, we can cover M by finitely many domains $U_1, .., U_N$ of local sections $c_i(t) : U_i \subset M \to V$, which yield a global section $c(t) : M \to V$, where $t \in (-\varepsilon, \varepsilon)$ and $\varepsilon := \min\{\varepsilon_1, .., \varepsilon_N\}$. From (4) we get

$$c(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_0^{(k)},$$

and since M is compact, the series converges in C^∞-topology.

To prove uniqueness, suppose that we have two real analytic solutions $c_1, c_2 : (-\varepsilon, \varepsilon) \times M \to V$ of the initial value problem. By (1) we have for $k = 1, 2$ and $x \in u(U) \subset \mathbb{R}^n$

$$v(X \circ c_k(t) \circ u^{-1}(x)) = G(x, v \circ c_k(t) \circ u^{-1}(x), \frac{\partial c_{ki}(t)}{\partial u_j} \circ u^{-1}(x)).$$

Now $F_k(t, x) := v \circ c_k(t) \circ u^{-1}(x)$ satisfies

$$\frac{\partial F_k}{\partial t}(t, x) = v \circ \dot{c}_k(t) \circ u^{-1}(x) = v \circ X \circ c_k(t) \circ u^{-1}(x)$$

and by (4)

$$\frac{\partial c_{ki}(t)}{\partial u_j} \circ u^{-1}(x) = \frac{\partial (v_i \circ c_k(t))}{\partial u_j}(u^{-1}(x)) = \frac{\partial F_{ki}}{\partial x_j}(t, x),$$

for $i = 1, .., s$ and $j = 1, .., n$. Hence we showed

$$\frac{\partial F_k}{\partial t}(t, x) = G(x, F_k(t, x), \frac{\partial F_{ki}}{\partial x_j}(t, x)).$$

Since F_1 and F_2 are both real analytic and satisfy

$$F_1(0, x) = v \circ c_1(0) \circ u^{-1}(x) = v \circ c_0 \circ u^{-1}(x) = v \circ c_2(0) \circ u^{-1}(x) = F_2(0, x),$$

the uniqueness part of the Cauchy-Kowalevski Theorem yields $F_1(t, x) = F_2(t, x)$, i.e. $c_1(t) = c_2(t)$.

□

3. Special Geometries

In this chapter we give a detailed description of various G-structures. Most of this structures are described by stable forms and we compensate the lack of examples encountered in the previous section on stability. All subsections are organized in a similar way. First we describe certain model forms $\varphi_0 \in \Lambda^k \mathbb{R}^{n*}$ with isotropy group $G \in \{SU(2), SU(3), G_2, \mathrm{Spin}(7)\}$. Most of this forms turn out to be stable and we associate to them a volume element ε_0. This volume element allows us to define complementary forms $\psi_0 \in \Lambda^{n-k}\mathbb{R}^{n*}$ which satisfy

$$\varphi_0 \wedge \psi_0 = \varepsilon_0.$$

This equation clearly indicates that the form ψ_0 equals the Hodge dual of φ_0. In fact these forms coincide in the cases that will be discussed here. The decisive difference is that we need to know a priori about the existence of a $G \subset SO(n)$ structure to define the Hodge dual of φ_0. For instance, if a G-structure is described by a pair of forms, these forms usually have to satisfy certain compatibility conditions to actually define the desired $G \subset SO(n)$ reduction. In contrast, the associated volume element can be defined solely in terms of the single stable form φ_0. As a consequence, no compatibility conditions are involved when defining the complementary form ψ_0. The definition of the associated volumes in the $SU(3)$ and G_2-case is due to Hitchin **[37]** and we develop the corresponding description for the $SU(2)$-scenario.
In each of the subsections we develop the analogue of the Gray-Hervella **[32]** (resp. Fernández-Gray **[27]**) classification for the respective structure. In this approach, G-structures are distinguished by the irreducible components of their intrinsic torsion

$$\tau \in \mathbb{R}^{n*} \otimes \mathfrak{g}^{\perp}.$$

Although our methods would allow to give a complete list for all possible torsion types, we only focus on a description of those classes that seem to be relevant for our work. However, we will take special account of $SU(3)$-structures. The first reason is that the description of $SU(3)$-structures in dimension seven is quite exceptional compared to the description of $SU(n)$-structures for $n \neq 3$. This is due to the fact that the G_2-isotropy group of a unit vector equals $SU(3)$. The second reason for the interest in $SU(3)$-structures stems from the ambition to find an analogue of (special) Kähler-structures in dimension seven. Usually Sasakian structures are considered as the odd-dimensional analogue of Kähler structures. In Theorem 3.46 we expose Sasakian structures as a certain torsion type and describe a generalized concept of odd-dimensional Kähler structures, cf. Remark 3.48.

Throughout the chapter we use the following notation: Let $(e_1, .., e_n)$ be the canonical basis of $\mathbb{R}^n$ with dual basis $(e^1, .., e^n)$ and volume element

$$\varepsilon_0 := e^1 \wedge .. \wedge e^n = e^{1..n} \in \Lambda^n \mathbb{R}^{n*}.$$

The inner product on $\mathbb{R}^n$ is

$$g_0 := \sum_{i=1}^n e^i \otimes e^i$$

and for $X, Y \in \mathfrak{gl}(n)$

$$\langle X, Y \rangle := \mathrm{tr}(XY^T)$$

defines an inner product on $\mathfrak{gl}(n) = \mathrm{End}(\mathbb{R}^n)$, where the transpose is defined w.r.t. g_0. We denote by

$$\mathfrak{so}_n = \mathfrak{g} \oplus \mathfrak{g}^\perp$$

the decomposition of $\mathfrak{so}(n)$ into orthogonal subspaces w.r.t. this inner product. For $n = 2m$ define

$$\omega_0 := e^{12} + .. + e^{2m-1,2m} \in \Lambda^2 \mathbb{R}^{2m*}$$

and $I_0 \in \mathrm{End}(\mathbb{R}^{2m})$ by

$$\omega_0 = g_0(I_0., .).$$

Since $I_0^2 = -\mathrm{id}$, we obtain a decomposition of $\mathbb{R}^{2m} \otimes \mathbb{C}$ into

$$\begin{aligned} T^{(1,0)} &:= \{x - iI_0x \mid x \in \mathbb{R}^{2m}\} = \mathrm{Eig}(I_0, +i), \\ T^{(0,1)} &:= \{x + iI_0x \mid x \in \mathbb{R}^{2m}\} = \mathrm{Eig}(I_0, -i), \end{aligned}$$

and we define

$$\begin{aligned} T^{(1,0)*} :&= \{\alpha \in \Lambda^1 \mathbb{R}^{2m*} \otimes \mathbb{C} \mid \alpha(Z) = 0, \text{ for all } Z \in T^{(0,1)}\} \\ &= \{\alpha - i\alpha \circ I_0 \mid \alpha \in \mathbb{R}^{2m*}\}, \\ T^{(0,1)*} :&= \{\alpha \in \Lambda^1 \mathbb{R}^{2m*} \otimes \mathbb{C} \mid \alpha(Z) = 0, \text{ for all } Z \in T^{(1,0)}\} \\ &= \{\alpha + i\alpha \circ I_0 \mid \alpha \in \mathbb{R}^{2m*}\}. \end{aligned}$$

Denote by $\Lambda^{(p,0)}$, respectively $\Lambda^{(0,p)}$, the p^{th} exterior power of $T^{(1,0)*}$, respectively $T^{(0,1)*}$,

$$\begin{aligned} \Lambda^{(p,0)} &:= \Lambda^p T^{(1,0)*} \\ \Lambda^{(0,p)} &:= \Lambda^p T^{(0,1)*} \end{aligned}$$

and let $\Lambda^{(p,q)} := \Lambda^{(p,0)} \otimes \Lambda^{(0,q)}$, such that

$$\Lambda^k \mathbb{R}^{2m*} \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{(p,q)}.$$

Since $e^1 + ie^2 = e^1 - ie^1 \circ I_0$,

$$\Phi_0 := (e^1 + ie^2) \wedge .. \wedge (e^{2m-1} + ie^{2m}) \in \Lambda^{(m,0)}$$

defines a form of type $(m, 0)$. We identify $\mathbb{C}^m = \mathbb{R}^{2m}$ via $z = x + iy = (x, y)$ and

$$GL(m, \mathbb{C}) = \{A \in GL(2m) \mid AI_0 = I_0A\}.$$

Under this identification, the hermitian structure $h_0 : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$, with $h_0(z,w) := \sum z_j \bar{w}_j$, equals

$$h_0 = g_0 - i\omega_0.$$

The canonical action of $GL(n)$ on $\mathbb{R}^n$ extends to an action of $GL(n)$ on the space of tensors on $\mathbb{R}^n$. In the case of forms, this action is compatible with the wedge product, i.e.

$$A(\varphi \wedge \psi) = A\varphi \wedge A\psi,$$

for $A \in GL(n)$ and $\varphi, \psi \in \Lambda^* \mathbb{R}^{n*}$. Moreover,

$$A\varepsilon_0 = \det(A^{-1})\varepsilon_0,$$

for $A \in GL(n)$, and for $A \in GL(m, \mathbb{C})$

$$A\Phi_0 = \det_{\mathbb{C}}(A^{-1})\Phi_0.$$

The isotropy groups of the above model tensors are listed below

$$\begin{array}{ll} \mathrm{Iso}_{GL(n)}(\varepsilon_0) = SL(n), & \mathrm{Iso}_{GL(m,\mathbb{C})}(\Phi_0) = SL(n,\mathbb{C}), \\ \mathrm{Iso}_{GL(n)}(g_0) = O(n), & \mathrm{Iso}_{GL(m,\mathbb{C})}(h_0) = U(m), \\ \mathrm{Iso}_{GL(2m)}(I_0) = GL(m,\mathbb{C}), & \mathrm{Iso}_{GL(2m)}(\omega_0) = \mathrm{Sp}(2m,\mathbb{R}). \end{array}$$

Lemma 3.1. Consider $G \subset GL(n)$, acting on $\Lambda^k \mathbb{R}^{n*}$. For $\varphi \in \Lambda^k \mathbb{R}^{n*}$, the map $D_\varphi : \mathfrak{g} \to V$ from Lemma 1.14 is given by

$$D_\varphi(A) = -\sum_{i=1}^n e^i \wedge Ae_i \lrcorner \varphi$$

Proof: Define $\mathrm{pr} : \mathbb{R}^{n*} \otimes \Lambda^{k-1}\mathbb{R}^{n*} \to \Lambda^k \mathbb{R}^{n*}$ by $\mathrm{pr}(\alpha \otimes \omega) = \alpha \wedge \omega$. Then for $x_1, .., x_k \in \mathbb{R}^n$

$$\begin{aligned} \mathrm{pr}(\alpha \otimes \omega)(x_1, .., x_k) = (\alpha \wedge \omega)(x_1, .., x_k) &= \sum_{|I|=k-1} \omega(e_I)(\alpha \wedge e^I)(x_i, .., x_k) \\ &= \sum_{|I|=k-1} \sum_{j=1}^k \omega(e_I)\alpha(x_j)(-1)^{j+1} e^I(x_1, .., \widehat{x}_j, .., x_k) \\ &= \sum_{j=1}^k (-1)^{j+1} \alpha(x_j)\omega(x_1, .., \widehat{x}_j, .., x_k). \end{aligned}$$

Hence for $\varphi \in \Lambda^k \mathbb{R}^{n*}$ and $A \in \mathfrak{g} \subset \mathfrak{gl}(n)$

$$
\begin{aligned}
\mathrm{pr}(A \lrcorner \varphi)(x_1, .., x_k) &= \sum_{i=1}^{n} \mathrm{pr}(e^i \otimes Ae_i \lrcorner \varphi) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k} (-1)^{j+1} e^i(x_j) \varphi(Ae_i, x_1, .., \widehat{x}_j, .., x_k) \\
&= \sum_{j=1}^{k} (-1)^{j+1} \varphi(Ax_j, x_1, .., \widehat{x}_j, .., x_k) = \sum_{j=1}^{k} \varphi(x_1, .., Ax_j, .., x_k) \\
&= -D_\varphi(A)(x_1, .., x_k),
\end{aligned}
$$

i.e.

$$
D_\varphi(A) = -\mathrm{pr}(A \lrcorner \varphi) = -\sum_{i=1}^{n} e^i \wedge Ae_i \lrcorner \varphi.
$$

□

Proposition 3.2. Let $\varphi_0 \in \Lambda^k \mathbb{R}^{n*}$ with isotropy group $G \subset O(n)$. An equivariant map $\varphi : FM \to \Lambda^k \mathbb{R}^{n*}$ of type φ_0 induces a reduction $P \subset F^g M$ with intrinsic torsion $\tau : P \to \mathbb{R}^{n*} \otimes \mathfrak{g}^\perp$. Then for $X \in C^\infty(TM)$

$$
D_\varphi(\tau(X)) = \nabla^g_X \varphi = L_X \varphi + D_\varphi(\nabla^g X).
$$

Proof: The first equation is precisely Proposition 1.18. The second equation follows from Lemma 3.1 and

$$
\begin{aligned}
D_\varphi(\nabla^g X) &= -\sum_{i=1}^{n} E^i \wedge (\nabla^g_{E_i} X) \lrcorner \varphi \\
&= -\sum_{i=1}^{n} E^i \wedge \nabla^g_{E_i}(X \lrcorner \varphi) + \sum_{i=1}^{n} E^i \wedge X \lrcorner (\nabla^g_{E_i} \varphi) \\
&= -d(X \lrcorner \varphi) - X \lrcorner d\varphi + \nabla^g_X \varphi \\
&= -L_X \varphi + \nabla^g_X \varphi.
\end{aligned}
$$

□

Given two real G-representations V and W, there are canonical isomorphism of G-modules

$$
\begin{aligned}
(\Lambda^k V)^* &= \Lambda^k V^*, \\
\mathrm{Hom}(V, W) &= V^* \otimes W, \\
\Lambda^k V^* &= \Lambda^{n-k} V \otimes \Lambda^n V^*.
\end{aligned}
$$

If V and W have the same dimension n, we define

$$
\det : \mathrm{Hom}(V, W) \to \Lambda^n V^* \otimes \Lambda^n W
$$

by

$$\Lambda^n V^* \otimes \Lambda^n W \otimes \Lambda^n W^* \ni \det(K) \otimes \varepsilon = \varepsilon \circ K \in \Lambda^n V^*,$$

where $0 \neq \varepsilon \in \Lambda^n W^*$ and we identified $\Lambda^n W \otimes \Lambda^n W^* = \mathbb{R}$. This definition is clearly independent of the choice of ε. For $g \in G$ we have

$$\det(gKg^{-1}) \otimes \varepsilon = \varepsilon \circ (gKg^{-1}) = g((\varepsilon \circ g) \circ K) = g(\det K \otimes \varepsilon \circ g) = \det K \otimes \varepsilon,$$

i.e. $\det(gKg^{-1}) = \det K = g \det K$ is G-equivariant. In the following sections we will frequently use the above identifications for $V = W = \mathbb{R}^n$ and $G \subset GL(n)$.

$SU(2)$-Structures in Dimension Five

In this section we consider the following model forms on $\mathbb{R}^5$:

$$\begin{aligned} \alpha_0 &:= e^1, & \omega_1 &:= e^{23} + e^{45}, \\ \omega_2 &:= e^{24} - e^{35}, & \omega_3 &:= e^{25} + e^{34}, \\ \rho_2 &:= \alpha_0 \wedge \omega_2 = e^{124} - e^{135}, & \rho_3 &:= \alpha_0 \wedge \omega_3 = e^{125} + e^{134} \end{aligned}$$

and $g_0(I_i\cdot, .) := \omega_i$, for $i = 1, 2, 3$. They satisfy certain relations, which can be verified in a direct computation:

Lemma 3.3. For all $x, y \in \mathbb{R}^5$ and $\beta \in \Lambda^1 \mathbb{R}^{5*}$

(1) $\omega_i \wedge \omega_j = 2\delta_{ij} e^{2345}$.
(2) $\omega_2(x, y)\varepsilon_0 = -(x \lrcorner \omega_1) \wedge (y \lrcorner \omega_1) \wedge \rho_2$.
(3) $\omega_3(x, y)\varepsilon_0 = -(x \lrcorner \omega_1) \wedge (y \lrcorner \omega_1) \wedge \rho_3$.
(4) $2\alpha_0(x)\varepsilon_0 = (x \lrcorner \rho_2) \wedge \rho_2$.
(5) $g_0(x, y)\varepsilon_0 = \alpha_0(x)\alpha_0(y)\varepsilon_0 + \alpha_0 \wedge \omega_1 \wedge (x \lrcorner \omega_2) \wedge (y \lrcorner \omega_3)$.
(6) $\beta(I_1 x)\varepsilon_0 = \alpha_0 \wedge \beta \wedge (x \lrcorner \omega_2) \wedge \omega_3$.
(7) $\omega_2(I_1 x, y) = -\omega_3(x, y)$.
(8) $\omega_3(I_1 x, y) = \omega_2(x, y)$.
(9) $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -\mathrm{id}$, on $\ker(\alpha_0)$.
(10) $\beta \wedge \omega_1 = I_3 \beta \wedge \omega_2$, for $\beta \in \Lambda^1 \ker(\alpha_0)^*$.
(11) $\beta \wedge \omega_3 = -I_1 \beta \wedge \omega_2$, for $\beta \in \Lambda^1 \ker(\alpha_0)^*$.

□

Usually a $SU(2)$-structure on a five dimensional manifold is described by a quadruplet of forms $(\alpha, \omega_1, \omega_2, \omega_3)$ which is of model type $(\alpha_0, \omega_1, \omega_2, \omega_3)$. This definition of $SU(2)$-structures can for instance be found in [**22**], [**29**]. There is an alternative to the usual definition, which is justified by the last equation in the next Lemma.

Lemma 3.4.

$$\mathrm{Iso}_{GL(5)}(\alpha_0) = \{\begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} \mid A \in GL(4) \text{ and } x \in \mathbb{R}^4\}.$$

$$\mathrm{Iso}_{GL(5)}(\omega_1) = \{\begin{pmatrix} \lambda & y^T \\ 0 & A \end{pmatrix} \mid A \in \mathrm{Sp}(4,\mathbb{R}), y \in \mathbb{R}^4 \text{ and } \lambda \neq 0\}.$$

$$\mathrm{Iso}_{GL(5)}(\alpha_0, \omega_1, \omega_2, \omega_3) = \begin{pmatrix} 1 & 0 \\ 0 & SU(2) \end{pmatrix}.$$

$$\mathrm{Iso}_{GL^+(5)}(\omega_1, \rho_2, \rho_3) = \begin{pmatrix} 1 & 0 \\ 0 & SU(2) \end{pmatrix}.$$

In particular, the forms α_0, ω_1, ω_2 and ω_3 are stable.

Proof: Write $B \in GL(5)$ as

$$B = \begin{pmatrix} \lambda & y^T \\ x & A \end{pmatrix},$$

where $\lambda \in \mathbb{R}$, $x, y \in \mathbb{R}^4$ and $A \in \mathfrak{gl}(4)$. Then $\alpha(Be_1) = \lambda$ and $\alpha(Be_j) = y^T e_j$, for $j \in \{2, .., 5\}$. Hence the stabilizer of the 1-form $\alpha_0 := e^1 \in \Lambda^1\mathbb{R}^{5*}$ has the above form.

For $B \in \mathrm{Iso}_{GL(5)}(\omega_1)$ and $i, j \in \{2, .., 5\}$ we get $\omega_1(e_i, e_j) = \omega_1(Be_i, Be_j) = \omega_1(Ae_i, Ae_j)$, i.e. $A \in \mathrm{Sp}(4, \mathbb{R})$. This yields

$$0 = \omega_1(Be_1, Be_j) = \omega_1(\lambda e_1 + x, (y^T e_j)e_1 + Ae_j) = \omega_1(x, Ae_j) = \omega_1(A^{-1}x, e_j)$$

and the non-degeneracy of ω_1, as a form on $\mathbb{R}^4$, implies $x = 0$ and proves the second equation of the lemma.

Now the third equation follows, since $\omega_2 = \mathrm{Re}(\Phi_0)$ and $\omega_3 = \mathrm{Im}(\Phi_0)$, where $\Phi_0 = (e^2 + ie^3) \wedge (e^4 + ie^5)$, and $SU(2) = \mathrm{Sp}(4, \mathbb{R}) \cap SL(2, \mathbb{C})$.

To obtain the last equation, we compute for $B = \begin{pmatrix} \lambda & y^T \\ 0 & A \end{pmatrix} \in \mathrm{Iso}_{GL(5)}(\omega_1) \cap \mathrm{Iso}_{GL^+(5)}(\alpha_0 \wedge \omega_2)$ and $i, j \in \{2, .., 5\}$

$$\begin{aligned} \omega_2(e_i, e_j) &= (\alpha_0 \wedge \omega_2)(e_1, e_i, e_j) = (\alpha_0 \wedge \omega_2)(Be_1, Be_i, Be_j) \\ &= (\alpha_0 \wedge \omega_2)(\lambda e_1, (y^T e_i)e_1 + Ae_i, (y^T e_j)e_1 + Ae_j) \\ &= (\alpha_0 \wedge \omega_2)(\lambda e_1, Ae_i, Ae_j) \\ &= \lambda\omega_2(Ae_i, Ae_j). \end{aligned}$$

Since the volume element $\varepsilon_0 = e^{2345}$ on $\mathbb{R}^4$ satisfies

$$\varepsilon_0 = \frac{1}{2}\omega_1^2 = \frac{1}{2}\omega_2^2 = \frac{1}{2}\omega_3^2,$$

we obtain from $A \in \mathrm{Sp}(4, \mathbb{R}) = \mathrm{Iso}_{GL(4)}(\omega_1)$

$$\det(A)\varepsilon_0 = A^{-1}\varepsilon_0 = A^{-1}\frac{1}{2}\omega_1^2 = \varepsilon_0,$$

i.e. $\det(A) = 1$. Now $A^{-1}\omega_2 = \lambda^{-1}\omega_2$ yields

$$\varepsilon_0 = A^{-1}\frac{1}{2}\omega_1^2 = \lambda^{-2}\varepsilon_0$$

and since $B \in GL^+(5)$, we get $\lambda = 1$. Similarly we get $A\omega_3 = \omega_3$, which yields $A \in SU(2)$. Now

$$\begin{aligned} \alpha_0 \wedge \omega_2 &= B^{-1}(\alpha_0 \wedge \omega_2) = B^{-1}\alpha_0 \wedge B^{-1}\omega_2 \\ &= B^{-1}\alpha_0 \wedge A^{-1}\omega_2, \quad \text{ since } e_1 \lrcorner \omega_2 = 0 \\ &= (\alpha_0(Be_1)e^1 + \sum_{j=2}^{5} \alpha_0(Be_j)e^j) \wedge \omega_2 \\ &= (\alpha_0 + \sum_{j=2}^{5} y_j e^j) \wedge \omega_2 \end{aligned}$$

yields $\sum_{j=2}^5 y_j e^j \wedge \omega_2 = 0$, i.e. $y = 0$.

The stability of α_0 follows from

$$5 = \dim(\Lambda^1\mathbb{R}^*) = \dim(GL(5)) - \dim(\mathrm{Iso}_{GL(5)}(\alpha_0)) = 25 - 20.$$

Similarly for ω_1,

$$10 = \dim(\Lambda^2\mathbb{R}^*) = \dim(GL(5)) - \dim(\mathrm{Iso}_{GL(5)}(\omega_1)) = 25 - 15,$$

and since $\omega_2, \omega_3 \in GL(5)\omega_1$, the Lemma follows.

□

Since the $GL^+(5)$ stabilizer of the triple $(\omega_1, \rho_2, \rho_3)$ is equal to $\{1\} \times SU(2)$, we expect that, after fixing an orientation for $\mathbb{R}^5$, we can reconstruct the forms α_0, ω_2 and ω_3 solely from the triple $(\omega_1, \rho_2, \rho_3)$. The first step is to reconstruct the volume element ε_0. Then the forms α_0, ω_2 and ω_3, as well as the metric g_0 and the endomorphism I_1, can be obtained from the formulas in Lemma 3.3.

Lemma 3.5. After choosing an orientation for $V := \mathbb{R}^5$, there is a homomorphism

$$\varepsilon : \Lambda^2V^* \oplus \Lambda^3V^* \oplus \Lambda^3V^* \to \Lambda^5V^* \oplus i\Lambda^5V^*$$

of $GL^+(5)$-modules, such that for the model tensors and the canonical orientation $[\varepsilon_0]$ of $\mathbb{R}^5$

$$\varepsilon(\omega_1, \rho_2, \rho_3) = \varepsilon_0 \in \Lambda^5V^* \subset \Lambda^5V^* \oplus i\Lambda^5V^*.$$

Proof: Given an orientation $[\varepsilon_+]$ for V, represented by an element $\varepsilon_+ \in \Lambda^5 V^*$, we can define a map

$$\sqrt[4]{} : \Lambda^5 V^* \otimes \Lambda^5 V^* \otimes \Lambda^5 V^* \otimes \Lambda^5 V^* \to \Lambda^5 V^* \oplus i\Lambda^5 V^*$$

by $\sqrt[4]{\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3 \otimes \varepsilon_4} = \sqrt[4]{\lambda_1\lambda_2\lambda_3\lambda_4}\varepsilon_+$, where $\lambda_i \in \mathbb{R}$ is defined by $\varepsilon_i = \lambda_i \varepsilon_+$. This definition is independent of the choice of representative ε_+ and for $A \in GL^+(5)$ we have

$$\sqrt[4]{A\varepsilon_1 \otimes A\varepsilon_2 \otimes A\varepsilon_3 \otimes A\varepsilon_4} = \sqrt[4]{\det A^{-4}\lambda_1\lambda_2\lambda_3\lambda_4}\varepsilon_+ = \det A^{-1}\sqrt[4]{\lambda_1\lambda_2\lambda_3\lambda_4}\varepsilon_+.$$

Now consider the $GL(5)$-equivariant map

$$K : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \to (V^* \otimes V) \otimes (V^* \otimes V) \otimes \Lambda^5 V^* \otimes \Lambda^5 V^*$$

defined by

$$K(\omega_1, \rho_2, \rho_3)(x, a, y, b) := \big(\rho_2 \wedge a \wedge b\big) \otimes \big(\rho_3 \wedge (x \lrcorner \omega_1) \wedge (y \lrcorner \omega_1)\big),$$

where $x, y \in V$ and $a, b \in V^*$. For the model tensors ω_1, ρ_2, ρ_3 let $K_0 := K(\omega_1, \rho_2, \rho_3)$. Then

$$\begin{aligned}
K_0(x,a,y,b) =& \Big(e^{124} \wedge (a_3e^3 + a_5e^5) \wedge (b_3e^3 + b_5e^5) \\
& - e^{135} \wedge (a_2e^2 + a_4e^4) \wedge (b_2e^2 + b_4e^4)\Big) \\
& \otimes \Big(e^{134} \wedge (-x_3e^2 + x_4e_5) \wedge (-y_3e^2 + y_4e_5) \\
& + e^{125} \wedge (x_2e^3 - x_5e^4) \wedge (y_2e^3 - y_5e^4)\Big) \\
=& \Big(e^{124} \wedge (a_3b_5e^{35} - a_5b_3e^{35}) - e^{135} \wedge (a_2b_4e^{24} - a_4b_2e^{24})\Big) \\
& \otimes \Big(e^{134} \wedge (-x_3y_4e^{25} + x_4y_3e^{25}) + e^{125} \wedge (-x_2y_5e^{34} + x_5y_2e^{34})\Big) \\
=& (a_5b_3 - a_3b_5 + a_2b_4 - a_4b_2)(-x_3y_4 + x_4y_3 - x_2y_5 + x_5y_2) \otimes \varepsilon_0^2.
\end{aligned}$$

Taking the trace of the first factor $V^* \otimes V$, we obtain a map

$$L = \mathrm{tr}(K) : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \to (V^* \otimes V) \otimes \Lambda^5 V^* \otimes \Lambda^5 V^*$$

and for the model tensors we obtain

$$L_0(y, b) := \mathrm{tr}(K_0)(y, b) = (-b_4y_5 + b_5y_4 - b_2y_3 + b_3y_2) \otimes \varepsilon_0^2,$$

i.e. $L_0 = I_1 \otimes \varepsilon_0^2$. Identifying $V^* \otimes V = \mathrm{Hom}(V, V)$, we define

$$L^2 : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \to (V^* \otimes V) \otimes (\Lambda^5 V^*)^4$$

and so

$$L_0^2 = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{id}_{\mathbb{R}^4} \end{pmatrix} \otimes \varepsilon_0^4.$$

Taking again the trace, we obtain a map

$$\mathrm{tr}(L^2) : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \to (\Lambda^5 V^*)^4$$

with $\mathrm{tr}(L_0^2) = -4\varepsilon_0^4$. Hence

$$\varepsilon := \sqrt[4]{-\frac{1}{4}\mathrm{tr}(L^2)} : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \to \Lambda^5 V^* \oplus i\Lambda^5 V^*$$

is the desired equivariant map.

□

Definition 3.6. Suppose $V = \mathbb{R}^5$ is equipped with a fixed orientation. For $(\omega_1, \rho_2, \rho_3) \in \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^*$ we call

$$\varepsilon := \varepsilon(\omega_1, \rho_2, \rho_3) \in \Lambda^5 V^*$$

from Lemma 3.5 the associated volume element. Whenever $\varepsilon \neq 0$, we define

$$\begin{aligned} 2\alpha(x)\varepsilon &:= (x \lrcorner \rho_2) \wedge \rho_2, \\ \omega_2(x,y)\varepsilon &:= -(x \lrcorner \omega_1) \wedge (y \lrcorner \omega_1) \wedge \rho_2, \\ \omega_3(x,y)\varepsilon &:= -(x \lrcorner \omega_1) \wedge (y \lrcorner \omega_1) \wedge \rho_3, \\ g(x,y)\varepsilon &:= \alpha(x)\alpha(y)\varepsilon + \alpha \wedge \omega_1 \wedge (x \lrcorner \omega_2) \wedge (y \lrcorner \omega_3), \\ \beta(I_1 x)\varepsilon &:= \alpha \wedge \beta \wedge (x \lrcorner \omega_2) \wedge \omega_3. \end{aligned}$$

Proposition 3.7. Consider $V = \mathbb{R}^5$ with the canonical orientation and

$$(\omega_1, \rho_2, \rho_3) \in \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^*$$

with $\varepsilon \neq 0$. Then $(\omega_1, \rho_2, \rho_3)$ lies in the $GL^+(5)$ orbit of the model forms $(\omega_1, \rho_2, \rho_3)$ if and only if the tensors from Definition 3.6 satisfy $\alpha \wedge \omega_1^2 > 0$, $g(x,x) > 0$, for $x \neq 0$, and

(1) $\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_3 = \omega_2 \wedge \omega_3 = 0$,
(2) $\omega_1^2 = \omega_2^2 = \omega_3^2$,
(3) $\rho_2 = \alpha \wedge \omega_2$ and $\rho_3 = \alpha \wedge \omega_3$.

In this case, the associated volume is given by $2\varepsilon = \alpha \wedge \omega_1^2 > 0$.

Proof: The relations can be easily verified if $(\omega_1, \rho_2, \rho_3)$ lies in the $GL^+(5)$ orbit of the model forms. Conversely, condition (1) implies that g from Definition 3.6 is symmetric

$$\begin{aligned} \alpha \wedge \omega_1 \wedge (x \lrcorner \omega_2) \wedge (y \lrcorner \omega_3) &= x \lrcorner (\alpha \wedge \omega_1) \wedge \omega_2 \wedge (y \lrcorner \omega_3) \\ &= -x \lrcorner (\alpha \wedge \omega_1) \wedge \omega_3 \wedge (y \lrcorner \omega_2) \\ &= \alpha \wedge \omega_1 \wedge (y \lrcorner \omega_2) \wedge (x \lrcorner \omega_3). \end{aligned}$$

Conditions (1), (2) and (3) also yield

$$\begin{aligned}
\omega_2(I_1x,y)\varepsilon &= -\alpha\wedge(y\lrcorner\omega_2)\wedge(x\lrcorner\omega_2)\wedge\omega_3\\
&= \alpha\wedge(y\lrcorner\omega_2)\wedge\omega_2\wedge(x\lrcorner\omega_3)\\
&= \alpha\wedge(y\lrcorner\omega_1)\wedge\omega_1\wedge(x\lrcorner\omega_3)\\
&= -\alpha\wedge(y\lrcorner\omega_1)\wedge(x\lrcorner\omega_1)\wedge\omega_3\\
&= (x\lrcorner\omega_1)\wedge(y\lrcorner\omega_1)\wedge\alpha\wedge\omega_3\\
&= -\omega_3(x,y)\varepsilon.
\end{aligned}\tag{4}$$

and

$$\begin{aligned}
\omega_3(I_1x,y)\varepsilon &= -\alpha\wedge(y\lrcorner\omega_3)\wedge(x\lrcorner\omega_2)\wedge\omega_3\\
&= -\alpha\wedge(y\lrcorner\omega_1)\wedge\omega_1\wedge(x\lrcorner\omega_2)\\
&= \alpha\wedge(y\lrcorner\omega_1)\wedge(x\lrcorner\omega_1)\wedge\omega_2\\
&= -(x\lrcorner\omega_1)\wedge(y\lrcorner\omega_1)\wedge\alpha\wedge\omega_2\\
&= \omega_2(x,y).
\end{aligned}\tag{5}$$

By definition we have $\alpha\circ I_1=0$ and hence

$$\begin{aligned}
g(I_1x,I_1y)\varepsilon &= \alpha\wedge\omega_1\wedge(I_1x\lrcorner\omega_2)\wedge(I_1y\lrcorner\omega_3)\\
&= -\alpha\wedge\omega_1\wedge(x\lrcorner\omega_3)\wedge(y\lrcorner\omega_2)\\
&= g(y,x)\varepsilon-\alpha(x)\alpha(y)\varepsilon\\
&= (g(x,y)-\alpha(x)\alpha(y))\varepsilon,
\end{aligned}\tag{6}$$

Similarly,

$$\begin{aligned}
g(I_1^2x,y)\varepsilon &= \alpha\wedge\omega_1\wedge(I_1^2x\lrcorner\omega_2)\wedge(y\lrcorner\omega_3)\\
&= -\alpha\wedge\omega_1\wedge(x\lrcorner\omega_2)\wedge(y\lrcorner\omega_3)\\
&= (-g(x,y)+\alpha(x)\alpha(y))\varepsilon
\end{aligned}\tag{7}$$

and

$$\begin{aligned}
\omega_1(I_1x,y)\varepsilon &= -\alpha\wedge(y\lrcorner\omega_1)\wedge(x\lrcorner\omega_2)\wedge\omega_3\\
&= \alpha\wedge\omega_1\wedge(y\lrcorner\omega_3)\wedge(x\lrcorner\omega_2)\\
&= (-g(x,y)+\alpha(x)\alpha(y))\varepsilon.
\end{aligned}\tag{8}$$

By (6) and (7) we have $I_1^2x=-x$ and $g(I_1x,I_1y)=g(x,y)$, for $x,y\in\ker(\alpha)$. Hence we can find a g-orthonormal basis for $\ker(\alpha)$ of the form

$$(a_2,a_3=I_1a_2,a_4,a_5=I_1a_4).$$

Since $\alpha\neq 0$ and $\alpha\circ I_1=0$, we have $\ker(I_1)\neq\{0\}$. So we can find $0\neq a_1\in\ker(I_1)$ with $\alpha(a_1)=1$, since $\alpha(a_1)=0$ and (6) would imply $0=I_1^2a_1=-a_1$. By (4) and (5) we have $x\lrcorner\omega_2=x\lrcorner\omega_3=0$, for $x\in\ker(I_1)$, and so

$$(a_1,a_2,a_3=I_1a_2,a_4,a_5=I_1a_4)$$

is a g-orthonormal basis for $\mathbb{R}^5$. If we define $A \in GL(5)$ by

$$Aa_i = e_i, \text{ for } i = 1, .., 5,$$

we have

$$A\alpha = \alpha_0, \quad Ag = g_0 \quad \text{and} \quad AI_1A^{-1}e_i = I_{10}e_i := \begin{cases} 0 & , i = 1 \\ e_{i+1} & , i \text{ even} \\ -e_{i-1} & , i \text{ odd} \end{cases}$$

From (8) and $\alpha \circ I_1 = 0$ we get in addition

$$\begin{aligned} A\omega_1 &= -\sum_{i<j} \omega_1(I_1^2A^{-1}e_i, A^{-1}e_j)e^{ij} = \sum_{i<j} g(I_1A^{-1}e_i, A^{-1}e_j)e^{ij} \\ &= \sum_{i<j} g_0(I_{10}e_i, e_j)e^{ij} = e^{23} + e^{45}, \end{aligned}$$

i.e. $\omega_1 = a^{23} + a^{45}$. In particular,

$$\det(A^{-1})\alpha \wedge \omega_1^2 = A(\alpha \wedge \omega_1^2) = 2\varepsilon_0 > 0$$

shows that $A \in GL^+(5)$ holds, since $\alpha \wedge \omega_1^2 > 0$. Equation (4) and (5) imply

$$(x + iIx) \lrcorner (\omega_2 + i\omega_3) = x \lrcorner \omega_2 - ix \lrcorner \omega_3 + ix \lrcorner \omega_3 - x \lrcorner \omega_2 = 0,$$

i.e. $\omega_2 + i\omega_3 \in \Lambda^{(2,0)}\ker(\alpha)^*$ w.r.t. the almost complex structure I_1. So we can find $z \in \mathbb{C}$ with

$$\omega_2 + i\omega_3 = z\Phi,$$

where $\Phi = (a^2 + ia^3) \wedge (a^4 + ia^5)$. Since $\Phi \wedge \overline{\Phi} = 4a^{2345}$, we get from (2)

$$4|z|^2a^{1..5} = \alpha \wedge z\Phi \wedge \overline{z\Phi} = \alpha \wedge (\omega_2^2 + \omega_3^2) = 2\alpha \wedge \omega_1^2 = 4|z|^2a^{1..5}.$$

So $z \in S^1$ and for

$$B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \{1\} \times U(2)$$

we have $BA \in GL^+(5)$. Then $BA\alpha = Be^1 = e^1$, $BA\Phi = B\Phi_0 = z^{-1}\Phi_0$ and hence $BA(\omega_2 + i\omega_3) = \Phi_0$. This yields

$$BA\rho_2 = e^{124} - e^{135} \quad \text{and} \quad BA\rho_3 = e^{125} + e^{134}$$

and from $B \in \{1\} \times U(2)$, we have

$$BA\omega_1 = B(e^{23} + e^{45}) = e^{23} + e^{45}.$$

The $GL^+(5)$-equivariance of the map ε from Lemma 3.5 yields eventually

$$2\varepsilon(\omega_1, \rho_2, \rho_3) = 2(BA)^{-1}\varepsilon_0 = \alpha \wedge \omega_1^2.$$

□

From Proposition 1.2, Lemma 3.4 and Proposition 3.7 we obtain

Corollary 3.8. Suppose M is a five dimensional manifold with a fixed orientation. Then $SU(2)$-structures on M which are compatible with the given orientation correspond to triplets of forms

$$(\omega_1, \rho_2, \rho_3) \in \Omega^2(M) \oplus \Omega^3(M) \oplus \Omega^3(M)$$

with $\varepsilon \neq 0$, and for which the tensors from Definition 3.6 satisfy $\alpha \wedge \omega_1^2 > 0$, $g(X, X) > 0$, for $X \neq 0$, and

(1) $\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_3 = \omega_2 \wedge \omega_3 = 0$,
(2) $\omega_1^2 = \omega_2^2 = \omega_3^2$,
(3) $\rho_2 = \alpha \wedge \omega_2$ and $\rho_3 = \alpha \wedge \omega_3$.

In this case, the associated volume is given by $2\varepsilon = \alpha \wedge \omega_1^2 > 0$.

□

We will now describe the Lie algebra of $SU(2)$ and $U(2)$ as subgroups of $SO(5)$.

Lemma 3.9. $A = (a_{ij}) \in \mathfrak{so}_5$ is an element of $\mathfrak{u}_2 \subset \mathfrak{so}_5$ if and only if $a_{1j} = 0$ and

$$a_{25} + a_{34} = 0, \qquad a_{24} - a_{35} = 0.$$

Moreover, $A \in \mathfrak{su}_2$ if in addition

$$a_{23} + a_{45} = 0.$$

Equivalently,

$$\mathfrak{u}_2 = \{A \in \mathfrak{so}_5 \mid Ae_1 = 0 \text{ and } AI_1 = I_1 A\},$$
$$\mathfrak{su}_2 = \{A \in \mathfrak{so}_5 \mid Ae_1 = 0 \text{ and } AI_i = I_i A, \text{ for } i = 1, 2, 3\}.$$

The orthogonal complements in $\mathfrak{so}_5$ are given by

$$\mathfrak{u}_2^\perp = \{\begin{pmatrix} 0 & -x^T \\ x & A \end{pmatrix} \mid x \in \mathbb{R}^4 \text{ and } A \in \mathbb{R}I_2 \oplus \mathbb{R}I_3 = \mathfrak{u}_2^\perp \subset \mathfrak{so}_4\},$$
$$\mathfrak{su}_2^\perp = \{\begin{pmatrix} 0 & -x^T \\ x & A \end{pmatrix} \mid x \in \mathbb{R}^4 \text{ and } A \in \mathbb{R}I_1 \oplus \mathbb{R}I_2 \oplus \mathbb{R}I_3 = \mathfrak{su}_2^\perp \subset \mathfrak{so}_4\}.$$

Proof: Since $U(2) = GL(2, \mathbb{C}) \cap SO(4)$, we have

$$\mathfrak{u}(2) = \mathfrak{gl}(2, \mathbb{C}) \cap \mathfrak{so}(4) = \{A \in \mathfrak{so}(4) \mid AI_1 = I_1 A\},$$

which can easily be seen to be described by the first two relations. Now $A = B + iC \in \mathfrak{su}(2)$ if $\mathrm{tr}_{\mathbb{C}}(A) = 0$, i.e. $\mathrm{tr}(B) = \mathrm{tr}(C) = 0$. Using the embedding $\mathfrak{gl}(2, \mathbb{C}) \subset \mathfrak{gl}(4)$ which is induced by I_1, we have $\mathrm{tr}(B) = \frac{1}{2}\mathrm{tr}(A) = 0$, since A is skew-symmetric, and $\mathrm{tr}(C) = a_{23} + a_{45}$, yielding the third relation. The description of $\mathfrak{su}_2$ follows from

$$SL(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) \mid A(\omega_2 + i\omega_3) = \omega_2 + i\omega_3\}$$

and $\omega_i = g_0(I_i.,.)$. For $A \in \mathfrak{so}_4$ we compute

$$\begin{aligned} \mathrm{tr}(AI_1) &= -2(a_{23} + a_{45}), \\ \mathrm{tr}(AI_2) &= 2(a_{35} - a_{24}), \\ \mathrm{tr}(AI_3) &= -2(a_{25} + a_{34}), \end{aligned}$$

and the Lemma follows.

□

Proposition 3.10. The following decompositions of $SU(2)$-modules are irreducible:

$$\begin{aligned} \mathfrak{su}_2^\perp &= \mathbb{R}I_1 \oplus \mathbb{R}I_2 \oplus \mathbb{R}I_3, \\ \mathfrak{so}_4 &= \mathfrak{su}_2 \oplus \mathbb{R}I_1 \oplus \mathbb{R}I_2 \oplus \mathbb{R}I_3, \\ \mathrm{End}(\mathbb{R}^4) &= \big(\mathbb{R}\mathrm{id} \oplus I_1\mathfrak{su}(2) \oplus I_2\mathfrak{su}(2) \oplus I_3\mathfrak{su}(2)\big) \oplus \big(\mathfrak{su}_2 \oplus \mathbb{R}I_1 \oplus \mathbb{R}I_2 \oplus \mathbb{R}I_3\big). \end{aligned}$$

The $SU(2)$-module $\Lambda^2 := \Lambda^2\mathbb{R}^{4*}$ decomposes accordingly into

$$\begin{aligned} \Lambda^2 =& \Lambda^2_3 \oplus \mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2 \oplus \mathbb{R}\omega_3, \text{ where} \\ & \Lambda^2_3 = \{\omega \in \Lambda^2 \mid \omega \wedge \omega_i = 0, \text{ for } i = 1,2,3\} \cong \mathfrak{su}_2. \end{aligned}$$

Proof: The decomposition $\mathfrak{su}_2^\perp = \mathbb{R}I_1 \oplus \mathbb{R}I_2 \oplus \mathbb{R}I_3$ is clearly irreducible. Since $\mathfrak{su}_2$, and hence $I_j\mathfrak{su}_2$, is irreducible, we see that the decomposition of $\mathrm{End}(\mathbb{R}^4)$ is $SU(2)$-irreducible.

□

Lemma 3.11. The maps $D_{\omega_i} : \mathrm{End}(\mathbb{R}^4) \to \Lambda^2\mathbb{R}^{4*}$, $i = 1,2,3$, define isomorphisms between certain submodules of $\mathrm{End}(\mathbb{R}^4)$ and $\Lambda^2\mathbb{R}^{4*}$,

	$\mathbb{R}\mathrm{id}$	$I_1\mathfrak{su}(2)$	$I_2\mathfrak{su}(2)$	$I_3\mathfrak{su}(2)$	$\mathfrak{su}_2$	$\mathbb{R}I_1$	$\mathbb{R}I_2$	$\mathbb{R}I_3$
D_{ω_1}	$\mathbb{R}\omega_1$	Λ^2_3	0	0	0	0	$\mathbb{R}\omega_3$	$\mathbb{R}\omega_2$
D_{ω_2}	$\mathbb{R}\omega_2$	0	Λ^2_3	0	0	$\mathbb{R}\omega_3$	0	$\mathbb{R}\omega_1$
D_{ω_3}	$\mathbb{R}\omega_3$	0	0	Λ^2_3	0	$\mathbb{R}\omega_2$	$\mathbb{R}\omega_1$	0

Proof: By Lemma 3.1 we have for $A \in \mathrm{End}(\mathbb{R}^4)$

$$D_{\omega_i}(A) = -\sum_{i=2}^{5} e^i \wedge Ae_i \lrcorner \omega_i = -2\mathrm{pr}_{\Lambda^2}(I_iA).$$

With this formula we can compute $D_{\omega_i}(A)$, for $A \in \mathrm{End}(\mathbb{R}^4)$ in a particular submodule, and obtain the above table.

$\square$

Definition 3.12. Let M be a five dimensional oriented manifold equipped with a $SU(2)$-structure $(\omega_1, \rho_2, \rho_3)$ with intrinsic torsion $\tau : P \to \mathbb{R}^{5*} \otimes \mathfrak{su}(2)^\perp$. According to the decomposition

$$\mathbb{R}^{5*} \otimes \mathfrak{su}_2^\perp = (\mathbb{R}^{5*} \otimes \mathbb{R}^4) \oplus \mathbb{R}^{5*} I_1 \oplus \mathbb{R}^{5*} I_2 \oplus \mathbb{R}^{5*} I_3$$

we decompose τ into a linear map $F : TM \to \ker(\alpha)$ and three 1-forms η_1, η_2 and η_3, such that

$$\tau(X) = \alpha \otimes F(X) - F(X) \lrcorner g \otimes \xi + \eta_1(X) I_1 + \eta_2(X) I_2 + \eta_3(X) I_3,$$

where $\xi \lrcorner g := \alpha$. Explicitly,

$$\begin{aligned} F(X) &= \tau(X)\xi, \\ \eta_i(X) &= \frac{1}{4}\langle \tau(X), I_i \rangle, \text{ since } \langle I_i, I_i \rangle = 4. \end{aligned}$$

Proposition 3.13. Let M be a five dimensional oriented manifold equipped with a $SU(2)$-structure $(\omega_1, \rho_2, \rho_3)$ with intrinsic torsion $\tau \cong F + \eta_1 + \eta_2 + \eta_3$. Then

$$\begin{aligned} \nabla^g \xi &= F, \\ \nabla^g \omega_1 &= 2(\eta_3 \otimes \omega_2 - \eta_2 \otimes \omega_3) - \alpha \wedge (F \lrcorner \omega_1), \\ \nabla^g \omega_2 &= 2(\eta_1 \otimes \omega_3 - \eta_3 \otimes \omega_1) - \alpha \wedge (F \lrcorner \omega_2), \\ \nabla^g \omega_3 &= 2(\eta_2 \otimes \omega_1 - \eta_1 \otimes \omega_2) - \alpha \wedge (F \lrcorner \omega_3) \end{aligned}$$

and

$$\begin{aligned} d\alpha &= 2\mathrm{pr}_{\Lambda^2}(F), \\ d\omega_1 &= 2(\eta_3 \wedge \omega_2 - \eta_2 \wedge \omega_3) - \alpha \wedge D_{\omega_1}(F), \\ d\omega_2 &= 2(\eta_1 \wedge \omega_3 - \eta_3 \wedge \omega_1) - \alpha \wedge D_{\omega_2}(F), \\ d\omega_3 &= 2(\eta_2 \wedge \omega_1 - \eta_1 \wedge \omega_2) - \alpha \wedge D_{\omega_3}(F). \end{aligned}$$

Proof: By Proposition 1.18 we have

$$\nabla^g_X(\alpha, \omega_1, \omega_2, \omega_3, \rho_2, \rho_3) = D_{(\alpha, \omega_1, \omega_2, \omega_3, \rho_2, \rho_3)}(\tau(X)).$$

Since $SO(5)$ acts on each factor of $\Lambda^1 \oplus \Lambda^2 \oplus \Lambda^2 \oplus \Lambda^2 \oplus \Lambda^3 \oplus \Lambda^3$ separately, the corresponding equation holds for each of the forms $\alpha, \omega_1, \omega_2, \omega_3, \rho_2$ and ρ_3. Let $(E_1, ..E_5)$ be a local Cayley frame for the $SU(2)$-structure. Applying Lemma 3.1, we find

$$
\begin{aligned}
g(\nabla^g_X\xi, Y) &= (\nabla^g_X\alpha)Y = D_\alpha(\tau(X))Y = -(\sum_{i=1}^5 E^i\wedge\tau(X)E_i\lrcorner\alpha)Y\\
&= -\sum_{i=1}^5 \alpha(\tau(X)E_i)E^i(Y) = -\alpha(\tau(X)Y) = g(\tau(X)\xi, Y)\\
&= g(F(X), Y).
\end{aligned}
$$

Using $E^i\wedge(I_jE_i\lrcorner\omega_j) = -E^i\wedge E^i = 0$, we get

$$
\begin{aligned}
\nabla^g_X\omega_1 &= D_{\omega_1}(\tau(X)) = -\sum_{i=1}^5 E^i\wedge\tau(X)E_i\lrcorner\omega_1\\
&= -\alpha\wedge F(X)\lrcorner\omega_1 - \sum_{i=2}^5 E^i\wedge(\eta_2(X)I_2E_i + \eta_3(X)I_3E_i)\lrcorner\omega_1\\
&= -\alpha\wedge F(X)\lrcorner\omega_1 - \sum_{i=2}^5 E^i\wedge(\eta_2(X)E_i\lrcorner\omega_3 - \eta_3(X)E_i\lrcorner\omega_2)\\
&= -\alpha\wedge F(X)\lrcorner\omega_1 - 2(\eta_2(X)\omega_3 - \eta_3(X)\omega_2),
\end{aligned}
$$

and similarly the equations for $\nabla^g\omega_2$ and $\nabla^g\omega_3$. Now

$$
d\alpha = \sum_{i=1}^5 E^i\wedge\nabla^g_{E_i}\alpha = \sum_{i=1}^5 E^i\wedge F(E_i)\lrcorner g = 2\mathrm{pr}_{\Lambda^2}(F)
$$

and

$$
\begin{aligned}
d\omega_1 &= \sum_{i=1}^5 E^i\wedge\nabla^g_{E_i}\omega_1\\
&= \alpha\wedge(\sum_{i=1}^5 E^i\wedge F(E_i)\lrcorner\omega_1) + 2\sum_{i=1}^5 E^i\wedge(\eta_3(E_i)\omega_2 - \eta_2(E_i)\omega_3)\\
&= -\alpha\wedge D_{\omega_1}(F) + 2(\eta_3\wedge\omega_2 - \eta_2\wedge\omega_3).
\end{aligned}
$$

The remaining equations are obtained similarly.

□

By Proposition 3.10 we have the following decomposition into irreducible $SU(2)$-modules

$$
\begin{aligned}
\mathbb{R}^{5*}\otimes\mathfrak{su}_2^\perp &= \mathbb{R}^{5*}\otimes(\mathbb{R}^4\oplus\mathbb{R}I_1\oplus\mathbb{R}I_2\oplus\mathbb{R}I_3)\\
&= \mathbb{R}^4\oplus\mathrm{End}(\mathbb{R}^4)\oplus(\mathbb{R}\oplus\mathbb{R}^{4*})\oplus(\mathbb{R}\oplus\mathbb{R}^{4*})\oplus(\mathbb{R}\oplus\mathbb{R}^{4*})\\
&= \mathbb{R}^4\oplus\big(\mathbb{R}\mathrm{id}\oplus I_1\mathfrak{su}(2)\oplus I_2\mathfrak{su}(2)\oplus I_3\mathfrak{su}(2)\big)\\
&\quad\oplus\big(\mathfrak{su}_2\oplus\mathbb{R}I_1\oplus\mathbb{R}I_2\oplus\mathbb{R}I_3\big)\oplus(\mathbb{R}\oplus\mathbb{R}^{4*})\oplus(\mathbb{R}\oplus\mathbb{R}^{4*})\oplus(\mathbb{R}\oplus\mathbb{R}^{4*})
\end{aligned}
$$

and hence there are 2^{15} different types of $SU(2)$-structures in dimension five. We are only interested in two particular classes of $SU(2)$-structures:

Theorem 3.14. Let $(\omega_1, \rho_2, \rho_3)$ be a $SU(2)$-structure on M with intrinsic torsion $\tau \cong F + \eta_1 + \eta_2 + \eta_3$ and $0 \neq \lambda \in \mathbb{R}$. Furthermore, decompose $\eta_i = \eta_{i0} + \eta_i(\xi)\alpha$ and $F = F_0 + \alpha \otimes F(\xi)$. In the following table we list different types of $SU(2)$-structures, the related torsion types and the corresponding equations for the structure tensors.

Name	Torsion	Characterization
nearly hypo	$F_0 + 2\lambda I_2 \in I_2\mathfrak{su}_2 \oplus I_3\mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R}I_1$	$d\rho_2 + 4\lambda\omega_1^2 = 0$
	$\eta_2 = \lambda\alpha$, $\eta_3 = 0$ and $2\eta_{10} = I_1(F(\xi) \lrcorner g)$	$d\omega_1 + 6\lambda\rho_3 = 0$
hypo	$F_0 \in I_2\mathfrak{su}_2 \oplus I_3\mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R}I_1$	$d\omega_1 = d\rho_2 = d\rho_3 = 0$
	$\eta_2 = \eta_3 = 0$ and $2\eta_{10} = I_1(F(\xi) \lrcorner g)$.	

Proof: Since $\alpha \wedge D_{\omega_1}(F) = \alpha \wedge D_{\omega_1}(F_0)$ and $2\mathrm{pr}_{\Lambda^2}(F) = 2\mathrm{pr}_{\Lambda^2}(F_0) + \alpha \wedge (F(\xi) \lrcorner g)$, we obtain from Proposition 3.13

$$\begin{aligned}
d\omega_1 &= 2(\eta_{30} \wedge \omega_2 - \eta_{20} \wedge \omega_3) - \alpha \wedge (D_{\omega_1}(F_0) - 2\eta_3(\xi)\omega_2 + 2\eta_2(\xi)\omega_3),\\
d\rho_2 &= 2\mathrm{pr}_{\Lambda^2}(F_0) \wedge \omega_2 - 2\alpha \wedge (\eta_{10} \wedge \omega_3 - \eta_{30} \wedge \omega_1 - \frac{1}{2}(F(\xi) \lrcorner g) \wedge \omega_2),\\
d\rho_3 &= 2\mathrm{pr}_{\Lambda^2}(F_0) \wedge \omega_3 - 2\alpha \wedge (\eta_{20} \wedge \omega_1 - \eta_{10} \wedge \omega_2 - \frac{1}{2}(F(\xi) \lrcorner g) \wedge \omega_3).
\end{aligned}$$

Hypo case: The conditions $d\omega_1 = d\rho_2 = d\rho_3 = 0$ are equivalent to

$$\begin{aligned}
&(1) \quad 0 = \eta_{30} \wedge \omega_2 - \eta_{20} \wedge \omega_3\\
&(2) \quad 0 = D_{\omega_1}(F_0) - 2\eta_3(\xi)\omega_2 + 2\eta_2(\xi)\omega_3\\
&(3) \quad 0 = \mathrm{pr}_{\Lambda^2}(F_0) \wedge \omega_2\\
&(4) \quad 0 = \eta_{10} \wedge \omega_3 - \eta_{30} \wedge \omega_1 - \frac{1}{2}(F(\xi) \lrcorner g) \wedge \omega_2\\
&(5) \quad 0 = \mathrm{pr}_{\Lambda^2}(F_0) \wedge \omega_3\\
&(6) \quad 0 = \eta_{20} \wedge \omega_1 - \eta_{10} \wedge \omega_2 - \frac{1}{2}(F(\xi) \lrcorner g) \wedge \omega_3.
\end{aligned}$$

With Lemma 3.3 and Lemma 3.11 we see that the conditions on the torsion components yield $d\omega_1 = d\rho_2 = d\rho_3 = 0$. Conversely, equation (3) and (5) imply $\mathrm{pr}_{\mathfrak{so}_4}(F_0) \in \mathfrak{su}_2 \oplus \mathbb{R}I_1$. Wedging equation (2) with ω_2, Lemma 3.11 yields $0 = \eta_3(\xi)\omega_2^2$, i.e. $\eta_3(\xi) = 0$. Similarly, wedging (2) with ω_3, yields $\eta_2(\xi) = 0$ and hence $D_{\omega_1}(F_0) = 0$, i.e.

$$F_0 \in I_2\mathfrak{su}_2 \oplus I_3\mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R}I_1.$$

With Lemma 3.3, equation (1) yields $0 = (\eta_{30} + I_1\eta_{20}) \wedge \omega_2$ and hence

$$\eta_{30} = -I_1\eta_{20}.$$

Similarly (4) and (6) become

$$\begin{aligned}
0 &= -I_1\eta_{10} + I_3I_1\eta_{20} - \frac{1}{2}F(\xi) \lrcorner g,\\
0 &= I_3\eta_{20} - \eta_{10} + \frac{1}{2}I_1(F(\xi) \lrcorner g).
\end{aligned}$$

So $\eta_{20} = \eta_{30} = 0$ and $2\eta_{10} = I_1(F(\xi)\lrcorner g)$.

Nearly hypo case: The conditions $d\rho_2 + 4\lambda\omega_1^2 = 0$ and $d\omega_1 + 6\lambda\rho_3 = 0$ are equivalent to (1), (4) and

$$\begin{aligned} &(7) \quad 0 = \mathrm{pr}_{\Lambda^2}(F_0) \wedge \omega_2 + 2\lambda\omega_1^2, \\ &(8) \quad 0 = D_{\omega_1}(F_0) - 2\eta_3(\xi)\omega_2 + 2\eta_2(\xi)\omega_3 - 6\lambda\omega_3. \end{aligned}$$

Hence the conditions on the torsion components imply $d\rho_2 + 4\lambda\omega_1^2 = 0$ and $d\omega_1 + 6\lambda\rho_3 = 0$. Conversely, we obtain from $\lambda \neq 0$ equations (5) and (6). From (5) we get $\mathrm{pr}_{\mathfrak{so}_4}(F_0) \in \mathfrak{su}_2 \oplus \mathbb{R}I_1 \oplus \mathbb{R}I_2$ and, wedging (8) with ω_2, yields together with Lemma 3.11, $\eta_3(\xi) = 0$. So

$$0 = D_{\omega_1}(F_0) + 2\eta_2(\xi)\omega_3 - 6\lambda\omega_3 = D_{\omega_1}(F_0) + D_{\omega_1}(3\lambda I_2 - \eta_2(\xi)I_2),$$

i.e.

$$F_0 + (3\lambda - \eta_2(\xi))I_2 \in I_2\mathfrak{su}_2 \oplus I_3\mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R}I_1$$

and hence $\lambda = \eta_2(\xi)$ by (7). Equations (1), (4) and (6) yield, like in the hypo case, $2\eta_{10} = I_1(F(\xi)\lrcorner g)$ and $\eta_{20} = \eta_{30} = 0$.

$\square$

$SU(3)$-Structures in Dimension Six

In this section we consider the following model forms on $\mathbb{R}^6$:

$$\begin{aligned} \omega_0 &:= e^{12} + e^{34} + e^{56}, & \sigma_0 &:= e^{1234} + e^{1256} + e^{3456}, \\ \rho_0 &:= e^{135} - e^{245} - e^{236} - e^{146}, & \widehat{\rho}_0 &:= e^{136} - e^{246} + e^{235} + e^{145} \end{aligned}$$

and $g_0(I_0.,.) := \omega_0$. They satisfy certain relations, which can be verified in a direct computation:

Lemma 3.15. For all $x, y \in \mathbb{R}^6$ and $\beta \in \Lambda^1\mathbb{R}^{6*}$

(1) $\omega_0 \wedge \rho_0 = \omega_0 \wedge \widehat{\rho}_0 = 0$.
(2) $\rho_0 \wedge \widehat{\rho}_0 = 4\varepsilon_0$.
(3) $2\sigma_0 = \omega_0^2$.
(4) $\omega_0^3 = 6\varepsilon_0$.
(5) $\rho_0 \wedge (x\lrcorner\rho_0) = \widehat{\rho}_0 \wedge (x\lrcorner\widehat{\rho}_0) = -2I_0x\lrcorner\varepsilon_0$.
(6) $2\beta(I_0x)\varepsilon_0 = \rho_0 \wedge (x\lrcorner\rho_0) \wedge \beta$.
(7) $2g_0(x, y)\varepsilon_0 = (x\lrcorner\rho_0) \wedge (y\lrcorner\rho_0) \wedge \omega_0$.
(8) $I_0\lrcorner\rho_0 = -\widehat{\rho}_0$.

(9) $(x\lrcorner\rho_0) \wedge (x\lrcorner\rho_0) = (x\lrcorner\widehat{\rho}_0) \wedge (x\lrcorner\widehat{\rho}_0) = 2x\lrcorner((x\lrcorner g) \wedge \sigma_0)$

$\square$

Lemma 3.16. After choosing an orientation for $V := \mathbb{R}^6$, there are homomorphisms

$$\varepsilon : \Lambda^4 V^* \to \Lambda^6 V^* \oplus i\Lambda^6 V^*$$

and

$$\varepsilon : \Lambda^3 V^* \to \Lambda^6 V^* \oplus i\Lambda^6 V^*$$

of $GL^+(6)$-modules, such that for the model tensors and the canonical orientation $[\varepsilon_0]$ of $\mathbb{R}^6$

$$\varepsilon(\sigma_0) = \varepsilon(\rho_0) = \varepsilon(\widehat{\rho}_0) = \varepsilon_0.$$

Proof: Given an orientation, we can define a $GL^+(6)$-equivariant map $\sqrt[2]{\cdot}$ like in Lemma 3.5. The wedge product yields a homomorphism of $GL(6)$-modules

$$h : \Lambda^4 V^* = \Lambda^2 V \otimes \Lambda^6 V^* \to \Lambda^6 V \otimes (\Lambda^6 V^*)^3 = (\Lambda^6 V^*)^2.$$

Hence

$$\Lambda^4 V^* \ni \sigma \mapsto \varepsilon(\sigma) := \sqrt[2]{\frac{1}{6} h(\sigma)} \in \Lambda^6 V^* \oplus i\Lambda^6 V^*$$

is $GL^+(6)$-equivariant and for the model tensor we compute

$$\begin{aligned} h(\sigma_0) &= h(\sum_{i<j} e_{ij} \otimes e^{ij} \wedge \sigma_0) = h((e_{12} + e_{34} + e_{56}) \otimes \varepsilon_0) \\ &= 6 e_{1..6} \otimes \varepsilon_0^3, \end{aligned}$$

so $\varepsilon(\sigma_0) = \varepsilon_0$. Now consider the $GL(6)$-equivariant map

$$K : \Lambda^3 V^* \to (V^* \otimes V) \otimes \Lambda^6 V^*$$

defined by

$$K(\rho)(x, \beta) = \rho \wedge (x\lrcorner\rho) \wedge \beta,$$

where $x \in V$ and $\beta \in V^*$. For the model tensors we obtain by Lemma 3.15

$$K(\rho_0)(x, \beta) = K(\widehat{\rho}_0)(x, \beta) = 2\beta(I_0 x)\varepsilon_0.$$

Hence

$$K^2 : \Lambda^3 V^* \to (V^* \otimes V) \otimes (\Lambda^6 V^*)^2$$

satisfies $K^2(\rho_0) = K^2(\widehat{\rho}_0) = -4\mathrm{id}_V \otimes \varepsilon_0^2$ and

$$\Lambda^3 V^* \ni \rho \mapsto \varepsilon(\rho) := \sqrt[2]{-\frac{1}{24}\mathrm{tr}(K^2)} \in \Lambda^6 V^* \oplus i\Lambda^6 V^*$$

is the desired map.

$\square$

Lemma 3.17.

$$\mathrm{Iso}_{GL^+(6)}(\rho_0) = \mathrm{Iso}_{GL^+(6)}(\widehat{\rho}_0) = SL(3,\mathbb{C}).$$
$$\mathrm{Iso}_{GL^+(6)}(\sigma_0) = Sp(6,\mathbb{R}).$$
$$\mathrm{Iso}_{GL^+(6)}(\rho_0,\sigma_0) = SU(3).$$

In particular, the forms ρ_0, $\widehat{\rho}_0$ and σ_0 are stable.

Proof: For $A \in \mathrm{Iso}_{GL^+(6)}(\rho_0)$ we obtain $A\varepsilon_0 = \varepsilon_0$, by Lemma 3.16. Hence the formula for I_0 from Lemma 3.15 yields $AI_0A^{-1} = I_0$, i.e. $A \in GL(3,\mathbb{C})$. Again by Lemma 3.15 we have $\widehat{\rho}_0 = I_0 \lrcorner \rho_0$ and hence $A(\rho_0 + i\widehat{\rho}_0) = (\rho_0 + i\widehat{\rho}_0)$, yielding $A \in SL(3,\mathbb{C})$. The same arguments hold for $\widehat{\rho}_0$.

For $A \in \mathrm{Iso}_{GL^+(6)}(\sigma_0)$ Lemma 3.16 gives $A\varepsilon_0 = \varepsilon_0$. Now

$$\sigma_0 = \sum_{i<j} e_{ij} \otimes (e^{ij} \wedge \sigma_0) = (e_{12} + e_{34} + e_{56}) \otimes \varepsilon_0$$

yields $A\omega_0 = \omega_0$ and hence $\mathrm{Iso}_{GL^+(6)}(\sigma_0) = Sp(6,\mathbb{R})$.

The last equation follows form $SU(3) = SL(3,\mathbb{C}) \cap Sp(6,\mathbb{R})$. To prove stability, we compute

$$\dim(GL(6)/SL(3,\mathbb{C})) = 36 - 16 = \dim(\Lambda^3\mathbb{R}^{6*}),$$
$$\dim(GL(6)/Sp(6,\mathbb{R})) = 36 - 21 = \dim(\Lambda^2\mathbb{R}^{6*}).$$

□

Definition 3.18. Suppose $V = \mathbb{R}^6$ is equipped with a fixed orientation. For $\sigma \in \Lambda^4V^*$ and $\rho \in \Lambda^3V^*$ we call $\varepsilon(\sigma)$ and $\varepsilon(\rho)$ from Lemma 3.16 the associated volume elements.

(1) Whenever $\varepsilon(\sigma) \neq 0$, we define

$$\omega := \frac{1}{2}\sigma(\omega^*) \in \Lambda^2V^*,$$

where $\omega^* \in \Lambda^2V$ is defined by $\sigma = \omega^* \otimes \varepsilon(\sigma) \in \Lambda^2V \otimes \Lambda^4V^*$ and σ is considered as an element $\sigma \in \Lambda^2V^* \otimes \Lambda^2V^* = \mathrm{Hom}(\Lambda^2V, \Lambda^2V^*)$.

(2) Whenever $\varepsilon(\rho) \neq 0$, we define $I(\rho) \in \mathrm{End}(V)$ by

$$2\beta(I(\rho)x)\varepsilon(\rho) := \rho \wedge (x \lrcorner \rho) \wedge \beta$$

and

$$\widehat{\rho} := -I(\rho)\lrcorner\rho.$$

(3) Whenever $\varepsilon(\sigma) \neq 0$ and $\varepsilon(\rho) \neq 0$, we define

$$g(x,y) := \omega(x, Iy).$$

Proposition 3.19. Consider $V = \mathbb{R}^6$ with the canonical orientation. For $\sigma \in GL^+(6)\sigma_0$ and $\rho \in GL^+(6)\rho_0$ we have $\varepsilon(\sigma), \varepsilon(\rho) > 0$ and $(\sigma, \rho) \in GL^+(6)(\sigma_0, \rho_0)$ holds if and only if the tensors from Definition 3.18 satisfy $g(x, x) > 0$, for $x \neq 0$, and

$$(1)\ \ \omega \wedge \rho = 0 \qquad \text{and} \qquad (2)\ \ \varepsilon(\sigma) = \varepsilon(\rho).$$

Proof: The above relations are clearly satisfied for $(\sigma, \rho) \in GL^+(6)(\sigma_0, \rho_0)$. Conversely, for $A \in GL^+(6)$ and $\sigma = A\sigma_0$, Definition 3.18 (1) yields $\omega(\sigma) = A\omega(\sigma_0) = A\omega_0$. Hence

$$\sigma = A\sigma_0 = \frac{1}{2}(A\omega_0)^2 = \frac{1}{2}\omega^2$$

and

$$\varepsilon(\sigma) = A\varepsilon_0 = \frac{1}{3}A(\omega_0 \wedge \sigma_0) = \frac{1}{3}\omega \wedge \sigma.$$

By definition of g and (1) we have

$$\begin{aligned}2g(x, y)\varepsilon(\rho) &= 2\omega(x, Iy)\varepsilon = \rho \wedge (y \lrcorner \rho) \wedge (x \lrcorner \omega)\\ &= \rho \wedge (x \lrcorner \rho) \wedge (y \lrcorner \omega) = 2g(y, x)\varepsilon(\rho).\end{aligned}$$

Hence g is symmetric and defines a metric on V, since $g(x, x) > 0$, for $x \neq 0$. By definition of $I(\rho)$ we have for $A \in GL^+(6)$

$$\begin{aligned}2\beta(I(A\rho_0)x)\varepsilon(A\rho_0) &= A\rho_0 \wedge (x \lrcorner A\rho_0) \wedge \beta = A(\rho_0 \wedge (A^{-1}x \lrcorner \rho_0) \wedge A^{-1}\beta)\\ &= 2(A^{-1}\beta)(I_0 A^{-1}x)A\varepsilon_0 = 2\beta(AI_0A^{-1}x)\varepsilon(A\rho_0),\end{aligned}$$

i.e. $I(A\rho_0) = AI_0A^{-1}$ and for $\rho = A\rho_0$

$$\begin{aligned}\varepsilon(\rho) = A\varepsilon_0 &= \frac{1}{4}A(\rho_0 \wedge \widehat{\rho}_0)\\ &= -\frac{1}{4}(\rho \wedge A(I_0 \lrcorner \rho_0) = -\frac{1}{4}(\rho \wedge (I \lrcorner \rho))\\ &= \frac{1}{4}\rho \wedge \widehat{\rho}.\end{aligned}$$

In particular, we have for $\rho \in GL^+(6)\rho_0$

$$I(\rho)^2 = -\mathrm{id}.$$

This yields

$$g(Ix, Iy) = -\omega(Ix, y) = g(y, x) = g(x, y)$$

and hence we can find an orthonormal basis for g of the form

$$(a_1, a_2 = Ia_1, .., a_5, a_6 = Ia_5).$$

If we define $A \in GL(6)$ by

$$Aa_i = e_i, \qquad \text{for } i = 1, .., 6,$$

we obtain

$$A\omega = \omega_0 \quad \text{and} \quad Ag = g_0.$$

Hence
$$A\sigma = \frac{1}{2}(A\omega)^2 = \sigma_0$$
and
$$\det(A^{-1})\varepsilon(\sigma) = A\varepsilon(\sigma) = \frac{1}{3}A(\omega\wedge\sigma) = \frac{1}{3}\omega_0\wedge\sigma_0 = \varepsilon_0 > 0$$
implies $A \in GL^+(6)$, since $\varepsilon(\sigma) > 0$. By definition we have $\widehat{\rho} = -I\lrcorner\rho$, and $I^2 = -\mathrm{id}$ yields $\rho = I\lrcorner\widehat{\rho}$. Hence
$$(x + iIx)\lrcorner(\rho + i\widehat{\rho}) = 0,$$
i.e. $\rho + i\widehat{\rho} \in \Lambda^{(3,0)}V^*$ w.r.t. the almost complex structure I. So we can find $z \in \mathbb{C}$ with
$$\rho + i\widehat{\rho} = z\Phi,$$
where $\Phi := (a^1 + ia^2)\wedge .. \wedge(a^5 + ia^6)$. Now (2) and $\Phi\wedge\overline{\Phi} = -8ia^{1..6}$ imply
$$8|z|^2\varepsilon(\rho) = 8|z|^2\varepsilon(\sigma) = 8|z|^2a^{1..6} = i|z|^2\Phi\wedge\overline{\Phi} = i\,z\Phi\wedge\overline{z\Phi} = 2\rho\wedge\widehat{\rho} = 8\varepsilon(\rho),$$
i.e. $|z| = 1$. So
$$B := \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(3)$$
satisfies $BA(\rho + i\widehat{\rho}) = zBA\Phi = zB\Phi_0 = \Phi_0$ and hence
$$BA\rho = \rho_0.$$
Since $B \in U(3)$, we have $BA\sigma = B\sigma_0 = \sigma_0$ and since $BA \in GL^+(6)$, the proposition follows.

□

From Proposition 1.2, Lemma 3.17 and Proposition 3.19 we obtain

Corollary 3.20. Suppose M is a six dimensional manifold with a fixed orientation. Then $SU(3)$-structures on M, which are compatible with the given orientation, correspond to forms $\sigma \in \Omega^4(M)$ and $\rho \in \Omega^3(M)$, of type σ_0 and ρ_0, respectively, such that the tensors from Definition 3.18 satisfy $g(X,X) > 0$, for $X \neq 0$, and
$$(1)\ \ \omega\wedge\rho = 0 \qquad \text{and} \qquad (2)\ \ \varepsilon(\sigma) = \varepsilon(\rho) > 0.$$

□

We will now describe the Lie algebra of $SU(3)$ and $U(3)$ as subgroups of $SO(6)$.

Lemma 3.21. $A = (a_{ij}) \in \mathfrak{so}_6$ is an element of $\mathfrak{u}_3 \subset \mathfrak{so}_6$ if and only if
$$\begin{aligned} &a_{35} - a_{46} = 0, \quad && a_{45} + a_{36} = 0, \quad && a_{26} - a_{15} = 0, \\ &a_{16} + a_{25} = 0, && a_{13} - a_{24} = 0, && a_{14} + a_{23} = 0. \end{aligned}$$

Moreover, $A \in \mathfrak{su}_3$ if and only if in addition

$$a_{12} + a_{34} + a_{56} = 0.$$

Equivalently,

$$\begin{aligned}
\mathfrak{u}_3 &= \{A \in \mathfrak{so}(6) \mid AI_0 = I_0 A\},\\
\mathfrak{su}_3 &= \{A \in \mathfrak{so}(6) \mid AI_0 = I_0 A \text{ and } (A \lrcorner g_0) \wedge \sigma_0 = 0\}.
\end{aligned}$$

The orthogonal complements in $\mathfrak{so}_6$ are given by

$$\begin{aligned}
\mathfrak{u}_3^{\perp} &= \{A \in \mathfrak{so}(6) \mid A \lrcorner g = x \lrcorner \rho_0, \text{ for some } x \in \mathbb{R}^6\},\\
\mathfrak{su}_3^{\perp} &= \mathfrak{u}_3^{\perp} \oplus \mathbb{R} I_0.
\end{aligned}$$

PROOF: Since $U(3) = GL(3,\mathbb{C}) \cap SO(6)$, we have

$$\mathfrak{u}(3) = \mathfrak{gl}(3,\mathbb{C}) \cap \mathfrak{so}(6) = \{A \in \mathfrak{so}(6) \mid AI_0 = I_0 A\},$$

which can easily be seen to be described by the first six relations. Now $A = B + iC \in \mathfrak{su}(3)$ if $\mathrm{tr}_{\mathbb{C}}(A) = 0$, i.e. $\mathrm{tr}(B) = \mathrm{tr}(C) = 0$. Using the embedding $\mathfrak{gl}(3,\mathbb{C}) \subset \mathfrak{gl}(6)$ which is induced by I_0, we have $\mathrm{tr}(B) = \frac{1}{2}\mathrm{tr}(A) = 0$, since A is skew-symmetric, and

$$\mathrm{tr}(C)\varepsilon_0 = (a_{12} + a_{34} + a_{56})\varepsilon_0 = (A \lrcorner g) \wedge \sigma_0,$$

yielding the last relation. Since for $A \in \mathfrak{u}_3$

$$\begin{aligned}
\rho_0(x, AI_0 y, I_0 y) &= \rho_0(x, I_0 A y, I_0 y) = \rho_0(I_0 A y, I_0 y, x)\\
&= -\widehat{\rho}_0(Ay, I_0 y, x) = \rho_0(y, Ay, x)\\
&= -\rho_0(x, Ay, y),
\end{aligned}$$

we obtain for $B \in \mathfrak{so}(6)$ with $B \lrcorner g = x \lrcorner \rho_0$, for some $x \in \mathbb{R}^6$,

$$\begin{aligned}
\mathrm{tr}(BA) &= \sum_{i=1,3,5} g_0(BAe_i, e_i) + g_0(BAI_0 e_i, I_0 e_i)\\
&= \sum_{i=1,3,5} \rho_0(x, Ae_i, e_i) + \rho_0(x, AI_0 e_i, I_0 e_i)\\
&= 0.
\end{aligned}$$

This proves

$$\{A \in \mathfrak{so}(6) \mid A \lrcorner g = x \lrcorner \rho_0, \text{ for some } x \in \mathbb{R}^6\} \subset \mathfrak{u}_3^{\perp}$$

and, counting dimensions, we see that those spaces coincide. The description of $\mathfrak{su}_3^{\perp}$ follows from

$$\mathrm{tr}(I_0 A) = -a_{12} - a_{34} - a_{56} = 0,$$

for $A \in \mathfrak{su}_3$.

$\square$

PROPOSITION 3.22. The following decompositions of $SU(3)$-modules are irreducible:

$$
\begin{aligned}
\mathfrak{su}_3^\perp &= \mathfrak{u}_3^\perp \oplus \mathbb{R}I_0 \\
\mathfrak{so}_6 &= \mathfrak{su}_3 \oplus \mathfrak{u}_3^\perp \oplus \mathbb{R}I_0 \\
\mathrm{End}(\mathbb{R}^6) &= \big(\mathbb{R}\mathrm{id} \oplus I_0\mathfrak{su}(3) \oplus S^2_{12}\big) \oplus \big(\mathfrak{su}_3 \oplus \mathfrak{u}_3^\perp \oplus \mathbb{R}I_0\big), \text{ where} \\
S^2_{12} &= \{A \in S^2 \mid I_0A + AI_0 = 0\}.
\end{aligned}
$$

The above decomposition of $\mathrm{End}(\mathbb{R}^6)$ is actually a decomposition into symmetric and skew-symmetric endomorphisms, refined by a further decomposition into endomorphisms which (anti)commute with I_0:

	I_0^+	I_0^-
$S^2(\mathbb{R}^6)$	$\mathbb{R}\mathrm{id} \oplus I_0\mathfrak{su}(3)$	S^2_{12}
$\mathfrak{so}_6$	$\mathbb{R}I_0 \oplus \mathfrak{su}(3)$	$\mathfrak{u}_3^\perp$

The $SU(3)$-modules $\Lambda^k := \Lambda^k\mathbb{R}^{6*}$ decompose into the following irreducible submodules, where the lower index denotes the dimension of the submodule:

$$
\begin{aligned}
&\Lambda^1 = \Lambda^1_6 \\
&\Lambda^2 = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8, \text{ where} \\
&\qquad \Lambda^2_1 = \mathbb{R}\omega_0, \\
&\qquad \Lambda^2_6 = \{x \lrcorner \rho_0 \mid x \in \mathbb{R}^6\}, \\
&\qquad \Lambda^2_8 = \{\alpha \in \Lambda^2 \mid \alpha \wedge \rho_0 = 0 \quad \text{and} \quad \alpha \wedge \sigma_0 = 0\} \cong \mathfrak{su}_3. \\
&\Lambda^3 = \Lambda^3_1 \oplus \Lambda^{\hat{3}}_1 \oplus \Lambda^3_6 \oplus \Lambda^3_{12}, \text{ where} \\
&\qquad \Lambda^3_1 = \mathbb{R}\rho_0, \\
&\qquad \Lambda^{\hat{3}}_1 = \mathbb{R}\widehat{\rho}_0, \\
&\qquad \Lambda^3_6 = \{\alpha \wedge \omega_0 \mid \alpha \in \Lambda^1\} = \{x \lrcorner \sigma_0 \mid x \in \mathbb{R}^6\}, \\
&\qquad \Lambda^3_{12} = \{\alpha \in \Lambda^3 \mid \omega_0 \wedge \alpha = \rho_0 \wedge \alpha = \widehat{\rho}_0 \wedge \alpha = 0\}. \\
&\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_6 \oplus \Lambda^4_8, \text{ where} \\
&\qquad \Lambda^4_1 = \mathbb{R}\sigma_0, \\
&\qquad \Lambda^4_6 = \{*(x \lrcorner \rho_0) \mid x \in \mathbb{R}^6\} = \{\alpha \wedge \widehat{\rho}_0 \mid \alpha \in \Lambda^1\} = \{\alpha \wedge \rho_0 \mid \alpha \in \Lambda^1\}, \\
&\qquad \Lambda^4_8 = \{\alpha \in \Lambda^4 \mid *\alpha \wedge \rho_0 = 0 \quad \text{and} \quad *\alpha \wedge \sigma_0 = 0\}. \\
&\Lambda^5 = \Lambda^5_6.
\end{aligned}
$$

PROOF: Since $SU(3)$ acts transitively on the unit sphere, we see that Λ^1_6, Λ^2_6, Λ^3_6 and Λ^5_6 are irreducible. Since $\Lambda^2_8 \cong \mathfrak{su}_3$, the irreducibility of Λ^2_8 follows. Hence we see that the decompositions of Λ^2 and Λ^4 are irreducible, using the Hodge operator. For the irreducibility of the submodule Λ^3_{12} see **[19]** formula (2) and table 1.
The map $D_{\rho_0} : \mathrm{End}(\mathbb{R}^6) \to \Lambda^3$ satisfies $\ker(D_{\rho_0}) = \mathfrak{sl}(3,\mathbb{C})$ by Lemma 1.14 and Lemma 3.17. Hence

$$S^2_{12} \cap \ker(D_{\rho_0}) = \{0\}$$

and, since the decomposition of Λ^3 is irreducible, it follows $0 \neq D_{\rho_0}(S^2_{12}) = \Lambda^3_{12}$. In particular, S^2_{12} is irreducible by the irreducibility of Λ^3_{12}. Since also $\mathfrak{su}_3$ and $\mathfrak{u}_3^\perp$ are irreducible, the Proposition follows.

□

Lemma 3.23. The maps $D_{\omega_0}, D_{\sigma_0}, D_{\rho_0}$ and $D_{\widehat{\rho}_0}$ define isomorphisms between certain submodules of $\mathrm{End}(\mathbb{R}^6)$ and $\Lambda^2\mathbb{R}^{6*}$, $\Lambda^4\mathbb{R}^{6*}$ and $\Lambda^3\mathbb{R}^{6*}$, respectively:

	$\mathbb{R}\mathrm{id}$	$I_0\mathfrak{su}(3)$	S^2_{12}	$\mathfrak{su}_3$	$\mathfrak{u}_3^\perp$	$\mathbb{R}I_0$
D_{ω_0}	Λ^2_1	Λ^2_8	0	0	Λ^2_6	0
D_{σ_0}	Λ^4_1	Λ^4_8	0	0	Λ^4_6	0
D_{ρ_0}	Λ^3_1	0	Λ^3_{12}	0	Λ^3_6	$\Lambda^{\hat{3}}_1$
$D_{\widehat{\rho}_0}$	$\Lambda^{\hat{3}}_1$	0	Λ^3_{12}	0	Λ^3_6	Λ^3_1

Proof: Using Lemma 3.1 we can easily compute the images of $\mathbb{R}\mathrm{id}$ and $\mathbb{R}I_0$. In the proof of Lemma 3.22 we have already seen that $D_{\rho_0}(S^2_{12}) = \Lambda^3_{12}$ holds. The same argument yields $D_{\widehat{\rho}_0}(S^2_{12}) = \Lambda^3_{12}$ and by Schur's Lemma we get $D_{\omega_0}(S^2_{12}) = 0 = D_{\sigma_0}(S^2_{12})$. Now Lemma 1.14 yields

$$\ker(D_{\omega_0}) = \ker(D_{\sigma_0}) = \mathfrak{sp}(6,\mathbb{R}) \quad \text{and} \quad \ker(D_{\rho_0}) = \ker(D_{\widehat{\rho}_0}) = \mathfrak{sl}(3,\mathbb{C}).$$

Since $\mathfrak{su}_3 \subset \mathfrak{sp}(6,\mathbb{R})$ and $\dim(\mathfrak{sp}(6,\mathbb{R})) = 21$, we get

$$\mathfrak{sp}(6,\mathbb{R}) = \mathbb{R}I_0 \oplus \mathfrak{su}_3 \oplus S^2_{12}$$

and, since $\dim(\mathfrak{sl}(3,\mathbb{C})) = 16$,

$$\mathfrak{sl}(3,\mathbb{C}) = I_0\mathfrak{su}(3) \oplus \mathfrak{su}(3).$$

□

Definition 3.24. Let M be a six dimensional oriented manifold equipped with a $SU(3)$-structure (σ,ρ) with intrinsic torsion $\tau : P \to \mathbb{R}^{6*} \otimes \mathfrak{su}_3^\perp$. According to the decomposition

$$\mathbb{R}^{6*} \otimes \mathfrak{su}_3^\perp = (\mathbb{R}^{6*} \otimes \mathfrak{u}_3^\perp) \oplus \mathbb{R}^{6*} I_0,$$

we decompose τ into $\mathcal{T} \in C^\infty(\mathrm{End}(TM))$ and $\eta \in \Omega^1(M)$, such that

$$g(\tau(X)Y, Z) = \rho(\mathcal{T}(X), Y, Z) + \eta(X)\omega(Y,Z).$$

If we choose a Cayley frame $(E_1, .., E_6)$ for the $SU(3)$-structure and let $A_i \lrcorner g := E_i \lrcorner \rho$, then

$$\mathcal{T}(X) = \frac{1}{4}\sum_{i=1}^{6} \langle \tau(X), A_i\rangle E_i, \text{ since } \langle A_i, A_i\rangle = 4.$$
$$\eta(X) = \frac{1}{6}\langle \tau(X), I\rangle, \text{ since } \langle I, I\rangle = 6.$$

Proposition 3.25. Let M be a six dimensional oriented manifold equipped with a $SU(3)$-structure (σ, ρ) with intrinsic torsion $\tau \cong \mathcal{T} + \eta$. Then

$$
\begin{aligned}
\nabla^g \omega &= -2\mathcal{T} \lrcorner \widehat{\rho}, & d\omega &= 2D_{\widehat{\rho}}(\mathcal{T}),\\
\nabla^g \sigma &= -2(\mathcal{T} \lrcorner \widehat{\rho}) \wedge \omega, & d\sigma &= 2D_{\widehat{\rho}}(\mathcal{T}) \wedge \omega,\\
\nabla^g \rho &= -2\mathcal{T} \lrcorner \sigma + 3\eta \otimes \widehat{\rho}, & d\rho &= 2D_{\sigma}(\mathcal{T}) + 3\eta \wedge \widehat{\rho},\\
\nabla^g \widehat{\rho} &= -2I\mathcal{T} \lrcorner \sigma - 3\eta \otimes \rho, & d\widehat{\rho} &= 2D_{\sigma}(I\mathcal{T}) - 3\eta \wedge \rho.
\end{aligned}
$$

Proof: By Proposition 1.18 we have for the intrinsic torsion $\tau : P \to \mathbb{R}^{6*} \otimes \mathfrak{su}_3^{\perp}$ of the $SU(3)$-structure

$$\nabla^g_X(\omega, \sigma, \rho, \widehat{\rho}) = D_{(\omega,\sigma,\rho,\widehat{\rho})}(\tau(X)).$$

Since $SO(6)$ acts on each factor $\Lambda^2 \times \Lambda^4 \times \Lambda^3 \times \Lambda^3$ separately, the corresponding equation hold for each of the tensors ω, σ, ρ and $\widehat{\rho}$. Let $(E_1, .., E_6)$ be a local Cayley frame for the $SU(3)$-structure. Applying Lemma 3.1, we find

$$
\begin{aligned}
\nabla^g_X \omega = D_\omega(\tau(X)) &= -\sum_{i=1}^{6} E^i \wedge \tau(X)E_i \lrcorner \omega = -\sum_{i,j=1}^{6} g(I\tau(X)E_i, E_j)E^{ij}\\
&= 2\sum_{i,j=1}^{6} g(\tau(X)E_i, IE_j)E^{ij} = \sum_{i,j=1}^{6} (\rho(\mathcal{T}(X), E_i, IE_j) + \eta(X)\omega(E_i, IE_j))E^{ij}\\
&= \sum_{i,j=1}^{6} \rho(\mathcal{T}(X), E_i, IE_j)E^{ij} = -\sum_{i=1}^{6} \widehat{\rho}(E_j, \mathcal{T}(X), E_i)E^{ij}\\
&= -2\sum_{i<j} \widehat{\rho}(\mathcal{T}(X), E_i, E_j)E^{ij} = -2\mathcal{T}(X) \lrcorner \widehat{\rho}.
\end{aligned}
$$

Using that $2\sigma = \omega^2$, the same computation yields

$$\nabla^g_X \sigma = -2(\mathcal{T}(X) \lrcorner \widehat{\rho}) \wedge \omega.$$

Similarly, with Lemma 3.15

$$
\begin{aligned}
\nabla^g_X \rho = D_\rho(\tau(X)) &= -\sum_{i=1}^{6} E^i \wedge \tau(X)E_i \lrcorner \rho = -\sum_{i,j=1}^{6} g(\tau(X)E_i, E_j)E^i \wedge E_j \lrcorner \rho\\
&= -\sum_{i,j=1}^{6} \big(\rho(\mathcal{T}(X), E_i, E_j) + \eta(X)\omega(E_i, E_j)\big)E^i \wedge E_j \lrcorner \rho\\
&= -\sum_{j=1}^{6} \rho(E_j, \mathcal{T}(X), .) \wedge E_j \lrcorner \rho + 3\eta(X)\widehat{\rho}\\
&= -\frac{1}{2}\mathcal{T}(X) \lrcorner \sum_{j=1}^{6} (E_j \lrcorner \rho \wedge E_j \lrcorner \rho) + 3\eta(X)\widehat{\rho}\\
&= -2\mathcal{T}(X) \lrcorner \sigma + 3\eta(X)\widehat{\rho}
\end{aligned}
$$

and

$$\begin{aligned}\nabla^g_X\widehat{\rho} &= -\sum_{j=1}^{6}\rho(E_j,\mathcal{T}(X),.)\wedge E_j\lrcorner\widehat{\rho} - 3\eta(X)\rho \\ &= -\sum_{j=1}^{6}\widehat{\rho}(E_j,I\mathcal{T}(X),.)\wedge E_j\lrcorner\widehat{\rho} - 3\eta(X)\rho \\ &= -\frac{1}{2}I\mathcal{T}(X)\lrcorner\sum_{j=1}^{6}(E_j\lrcorner\widehat{\rho}\wedge E_j\lrcorner\widehat{\rho}) - 3\eta(X)\rho \\ &= -2I\mathcal{T}(X)\lrcorner\sigma - 3\eta(X)\rho.\end{aligned}$$

Now the exterior derivatives are

$$\begin{aligned} d\omega &= \sum_{i=1}^{6}E^i\wedge\nabla^g_{E_i}\omega = -2\sum_{i=1}^{6}E^i\wedge\mathcal{T}(E_i)\lrcorner\widehat{\rho} = 2D_{\widehat{\rho}}(\mathcal{T}). \\ d\sigma &= 2D_{\widehat{\rho}}(\mathcal{T})\wedge\omega. \\ d\rho &= \sum_{i=1}^{6}E^i\wedge\nabla^g_{E_i}\rho = 2D_\sigma(\mathcal{T}) + 3\eta\wedge\widehat{\rho}. \\ d\widehat{\rho} &= \sum_{i=1}^{6}E^i\wedge\nabla^g_{E_i}\widehat{\rho} = 2D_\sigma(I\mathcal{T}) - 3\eta\wedge\rho.\end{aligned}$$

□

Definition 3.26. The Nijenhuis tensor of an almost complex structure I on M is defined by

$$N_I(X,Y) := [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY].$$

The Newlander-Nirenberg Theorem states that $N_I = 0$ is actually equivalent to the integrability of the almost complex structure I.

By Proposition 3.22 we have the following decomposition into irreducible $SU(3)$-modules

$$\begin{aligned}\mathbb{R}^{6*}\otimes\mathfrak{su}_3^\perp &= \mathbb{R}^{6*}\otimes(\mathfrak{u}_3^\perp\oplus\mathbb{R}I_0) = \mathrm{End}(\mathbb{R}^6)\oplus\mathbb{R}^{6*} \\ &= \big(\mathbb{R}\mathrm{id}\oplus I_0\mathfrak{su}(3)\oplus S^2_{12}\big)\oplus\big(\mathfrak{su}_3\oplus\mathfrak{u}_3^\perp\oplus\mathbb{R}I_0\big)\oplus\mathbb{R}^{6*}\end{aligned}$$

and hence there are 2^7 different types of $SU(3)$-structures in dimension six. Before we characterize some of these classes, we need

Lemma 3.27. For $A\in\mathrm{End}(\mathbb{R}^6)$ we have

$$\forall x,y\in\mathbb{R}^6\ \ \widehat{\rho}_0(Ax,x,y)=0 \quad\Leftrightarrow\quad \forall x,y\in\mathbb{R}^6\ \ \rho_0(Ax,x,y)=0 \quad\Leftrightarrow\quad A=\lambda\mathrm{id}+\mu I_0$$

PROOF: It suffices to prove the second equivalence, since $\widehat{\rho}_0(I_0.,.,.) = \rho_0$. If $A = \lambda\mathrm{id} + \mu I_0$, we clearly have

$$\rho_0(Ax, x, y) = \lambda\rho_0(x, x, y) - \mu\widehat{\rho}_0(x, x, y) = 0.$$

Conversely, suppose that $\rho_0(Ax, x, y) = 0$ holds for all $x, y \in \mathbb{R}^6$. For $y = e_1$ we get $0 = (e^{35} - e^{46})(Ax, x)$ and choosing $x \in \{e_3, e_4, e_5, e_6\}$ yields $0 = a_{35} = a_{53} = a_{46} = a_{64}$. Repeating this argument for $y = e_2, .., e_6$ shows that

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}, \text{ where } A_i := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Computing

$$\begin{aligned} 0 &= \rho_0(A(e_1 + e_3), e_1 + e_3, y) \\ &= a_1\rho_0(e_1, e_3, y) + c_1\rho_0(e_2, e_3, y) + a_2\rho_0(e_3, e_1, y) + c_2\rho_0(e_4, e_1, y), \end{aligned}$$

for $y = e_6$ and $y = e_5$, yields $c_1 = c_2$ and $a_1 = a_2$. Similarly we get eventually

$$A_1 = A_2 = A_3 = A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\begin{aligned} 0 &= \rho_0(A(e_1 + e_4), e_1 + e_4, y) \\ &= a\rho_0(e_1, e_4, y) + c\rho_0(e_2, e_4, y) + b\rho_0(e_3, e_1, y) + d\rho_0(e_4, e_1, y), \end{aligned}$$

yields $\lambda := a = d$, for $y = e_6$, and $\mu := c = -b$, for $y = e_5$.

□

We are interested in the following classes of $SU(3)$-structures:

THEOREM 3.28. Let (σ, ρ) be a $SU(3)$-structure on M with intrinsic torsion $\tau \cong \mathcal{T} + \eta$ and $0 \neq \lambda \in \mathbb{R}$. In the following table we list different types of $SU(3)$-structures, the related torsion types and the corresponding equations for the structure tensors.

Name	Torsion	Characterization
Nearly Hypo	$I\mathcal{T} + \frac{1}{8}\lambda I \in S^2$ $\eta = 0$	$d\rho = \lambda\sigma$
Hypo	$I\mathcal{T} \in S^2$ $\eta = 0$	$d\sigma = d\rho = 0$
Nearly Parallel (nearly Kähler)	$I\mathcal{T} = \lambda\mathrm{id}$ $\eta = 0$	$d\omega = 6\lambda\rho$ and $d\widehat{\rho} = -4\lambda\omega^2$. Equivalently: For all $X \in TM$ $(\nabla^g_X I)X = 0$ and $d\rho = 0$.
Parallel (Calabi-Yau)	$\mathcal{T} = 0$ $\eta = 0$	$d\omega = d\rho = d\widehat{\rho} = 0$

Complex	$\mathcal{T} \in S^2_{12} \oplus \mathfrak{u}_3^{\perp}$	$N_I = 0$
Kähler	$\mathcal{T} = 0$	$N_I = 0$ and $d\omega = 0$. Equivalently, $\nabla^g \omega = 0$.

PROOF: From Proposition 3.22 and Schur's Lemma we see that

$$\ker(\omega \wedge . : \Lambda^3 \to \Lambda^5) = \Lambda^3_1 \oplus \bar{\Lambda}^3_1 \oplus \Lambda^3_{12}.$$

Now Lemma 3.23 and Proposition 3.25 give

$$\begin{aligned}
d\omega = 0 \quad &\Leftrightarrow \quad \mathcal{T} \in I_0\mathfrak{su}_3 \oplus \mathfrak{su}_3.\\
d\sigma = 0 \quad &\Leftrightarrow \quad \mathcal{T} \in \mathbb{R}\mathrm{id} \oplus I_0\mathfrak{su}(3) \oplus S^2_{12} \oplus \mathfrak{su}_3 \oplus \mathbb{R}I_0.\\
d\rho = 0 \quad &\Leftrightarrow \quad \mathcal{T} \in S^2_{12} \oplus \mathfrak{su}_3 \oplus \mathfrak{u}_3^{\perp} \oplus \mathbb{R}I_0 \text{ and}\\
&\qquad 2D_\sigma(\mathrm{pr}_{\mathfrak{u}_3^{\perp}}(\mathcal{T})) + 3\eta \wedge \widehat{\rho} = 0.\\
d\widehat{\rho} = 0 \quad &\Leftrightarrow \quad I\mathcal{T} \in S^2_{12} \oplus \mathfrak{su}_3 \oplus \mathfrak{u}_3^{\perp} \oplus \mathbb{R}I_0 \text{ and}\\
&\qquad 2D_\sigma(\mathrm{pr}_{\mathfrak{u}_3^{\perp}}(I\mathcal{T})) - 3\eta \wedge \rho = 0.
\end{aligned}$$

With this characterizations, and the non-degeneracy of $\widehat{\rho}$, the description of hypo structures and parallel structures follows.

Nearly hypo case: By Proposition 3.25 the condition $d\rho = \lambda\sigma$ holds if and only if

$$\lambda\sigma = 2D_\sigma(\mathcal{T}) + 3\eta \wedge \widehat{\rho}. \tag{1}$$

Since $D_\sigma(\mathrm{id}) = -4\sigma$ by Lemma 3.1, we see that the condition on the torsion components imply $d\rho = \lambda\sigma$. Conversely, $\lambda \neq 0$ yields $d\sigma = 0$ and hence (1) is equivalent to $\eta = 0$ and

$$\mathcal{T} \in \mathbb{R}\mathrm{id} \oplus S^2_{12} \oplus \mathfrak{su}_3 \oplus \mathbb{R}I_0$$

and

$$\lambda\sigma = 2D_\sigma(\mathrm{pr}_{\mathbb{R}\mathrm{id}}\mathcal{T}) = 2D_\sigma(\frac{1}{6}\mathrm{tr}(\mathcal{T})\mathrm{id}) = -\frac{4}{3}\mathrm{tr}(\mathcal{T})\sigma.$$

So

$$S^2_{12} \oplus \mathfrak{su}_3 \oplus \mathbb{R}I_0 \ni \mathcal{T} - \frac{1}{6}\mathrm{tr}(\mathcal{T})\mathrm{id} = \mathcal{T} + \frac{1}{8}\lambda\mathrm{id}$$

and since $I(S^2_{12} \oplus \mathfrak{su}_3 \oplus \mathbb{R}I_0) = S^2$, the description of nearly hypo structures follows.

Complex case: Using that $[X, Y] = \nabla^g_X Y - \nabla^g_Y X$ and $(\nabla^g_X I)Y = \nabla^g_X IY - I(\nabla^g_X Y)$, we get

$$N_I(X, Y) = I(\nabla^g_X I)Y - (\nabla^g_{IX} I)Y - I(\nabla^g_Y I)X + (\nabla^g_{IY} I)X.$$

Since $\nabla^g_X \omega = (\nabla^g_X I) \lrcorner g$, we get from Proposition 3.25

$$g((\nabla^g_X I)Y, Z) = -2(\mathcal{T}X \lrcorner \widehat{\rho})(Y, Z) = -2\widehat{\rho}(\mathcal{T}X, Y, Z)$$

and hence

$$\begin{aligned}
g(N_I(X,Y),Z) &= -g((\nabla^g_X I)Y, IZ) - g((\nabla^g_{IX} I)Y, Z) \\
&\quad + g((\nabla^g_Y I)X, IZ) + g((\nabla^g_{IY} I)X, Z) \\
&= 2\widehat{\rho}(\mathcal{T}X, Y, IZ) + 2\widehat{\rho}(\mathcal{T}IX, Y, Z) \\
&\quad - 2\widehat{\rho}(\mathcal{T}Y, X, IZ) - 2\widehat{\rho}(\mathcal{T}IY, X, Z) \\
&= 2\rho(\mathcal{T}X, Y, Z) - 2\rho(I\mathcal{T}IX, Y, Z) \\
&\quad - 2\rho(\mathcal{T}Y, X, Z) + 2\rho(I\mathcal{T}IY, X, Z) \\
&= 2\big(\rho((\mathcal{T} - I\mathcal{T}I)X, Y, Z) - \rho((\mathcal{T} - I\mathcal{T}I)Y, X, Z)\big) \\
&= 4\big(\rho(\mathrm{pr}_{I+}(\mathcal{T})X, Y, Z) - \rho(\mathrm{pr}_{I+}(\mathcal{T})Y, X, Z)\big),
\end{aligned}$$

where pr_{I+} denotes the projection on the space of endomorphisms which commute with I. It follows immediately $N_I = 0$, for $\mathcal{T} \in S^2_{12} \oplus \mathfrak{u}_3^{\perp} = I^-$, cf. Proposition 3.22. Conversely, $N_I = 0$ yields

$$0 = g(N_I(X,Y),Y) = -4\rho(\mathrm{pr}_{I+}(\mathcal{T})Y, X, Y),$$

which is by Lemma 3.27 equivalent to $\mathrm{pr}_{I+}(\mathcal{T}) = \lambda\mathrm{id} + \mu I$, for some functions $\lambda, \mu : M \to \mathbb{R}$. Then $N_I = 0$ yields

$$\begin{aligned}
0 &= \lambda\rho(X,Y,Z) + \mu\rho(IX,Y,Z) - \lambda\rho(Y,X,Z) - \mu\rho(IY,X,Z) \\
&= \lambda\rho(X,Y,Z) - \mu\widehat{\rho}(X,Y,Z) - \lambda\rho(Y,X,Z) + \mu\widehat{\rho}(Y,X,Z) \\
&= 2(\lambda\rho - \mu\widehat{\rho})(X,Y,Z),
\end{aligned}$$

i.e. $\lambda = \mu = 0$ and so $\mathrm{pr}_{I+}(\mathcal{T}) = 0$, i.e. $\mathcal{T} \in S^2_{12} \oplus \mathfrak{u}_3^{\perp} = I^-$, cf. Proposition 3.22.

Kähler case: This follows immediately from the characterization of $N_I = 0$ and $d\omega = 0$, Proposition 3.25 and the fact that $\widehat{\rho}$ is non-degenerated.

Nearly Parallel case: The equations $d\omega = 6\lambda\rho$ and $d\widehat{\rho} = -4\lambda\omega^2$ translate into

$$-2D_\rho(I\mathcal{T}) = 6\lambda\rho, \tag{2}$$

$$2D_\sigma(I\mathcal{T}) - 3\eta \wedge \rho = -8\lambda\sigma. \tag{3}$$

Since $D_\rho(\mathrm{id}) = -3\rho$ and $D_\sigma(\mathrm{id}) = -4\sigma$, the conditions on the torsion imply $d\omega = 6\lambda\rho$ and $d\widehat{\rho} = -4\lambda\omega^2$. Conversely, $\lambda \neq 0$ yields $d\rho = 0$ and $d\sigma = 0$, which gives $\eta = 0$ and $\mathcal{T} \in S^2_{12} \oplus \mathfrak{su}_3 \oplus \mathbb{R}I_0$. Hence $I\mathcal{T} \in S^2$ and (2) yields

$$I\mathcal{T} \in \mathbb{R}\mathrm{id} \oplus I_0\mathfrak{su}_3 \quad \text{and} \quad -2D_\rho(\mathrm{pr}_{\mathbb{R}\mathrm{id}}(I\mathcal{T})) = 6\lambda\rho.$$

Now (3) gives

$$I\mathcal{T} \in \mathbb{R}\mathrm{id} \quad \text{and} \quad 2D_\sigma(\mathrm{pr}_{\mathbb{R}\mathrm{id}}(I\mathcal{T})) = -8\lambda\sigma.$$

Writing $I\mathcal{T} = f\mathrm{id}$, for some $f : M \to \mathbb{R}$, we get

$$6\lambda\rho = 6f\rho \quad \text{and} \quad -8\lambda\sigma = -8f\sigma,$$

i.e. $I\mathcal{T} = \lambda \mathrm{id}$. For the second characterization observe that by Proposition 3.25 and Lemma 3.27

$$(\nabla^g_X I)X = 0 \quad \Leftrightarrow \quad \widehat{\rho}(\mathcal{T}X, X, .) = 0 \quad \Leftrightarrow \quad \mathcal{T} = u\mathrm{id} + vI,$$

for some functions $u, v : M \to \mathbb{R}$. Hence it suffices to show that $\mathcal{T} = u\mathrm{id} + vI$ and $d\rho = 0$ imply $\eta = 0$ and $I\mathcal{T} = \lambda \mathrm{id}$, for some constant λ. First observe that $\mathcal{T} = u\mathrm{id} + vI$ yields $d\sigma = 0$. Next

$$0 = d\rho = -8u\sigma + 3\eta \wedge \widehat{\rho}$$

gives $u = 0$ and $\eta = 0$. So $d\widehat{\rho} = 8v\sigma$ and

$$0 = dv \wedge \sigma$$

shows that $dv = 0$, i.e. $\lambda := -v$ is constant and $I\mathcal{T} = \lambda \mathrm{id}$.

□

G_2-Structures in Dimension Seven

In this section we consider the following model forms on $\mathbb{R}^7$:

$$\varphi_0 = e^{246} - e^{356} - e^{347} - e^{257} + e^{123} + e^{145} + e^{167}.$$
$$\psi_0 = e^{2345} + e^{2367} + e^{4567} - e^{1247} + e^{1357} - e^{1346} - e^{1256}.$$

They satisfy certain relations, which can be verified in a direct computation:

Lemma 3.29. For all $x, y \in \mathbb{R}^7$

(1) $\varphi_0 \wedge \psi_0 = 7\varepsilon_0$.
(2) $6g_0(x, y)\varepsilon_0 = (x \lrcorner \varphi_0) \wedge (y \lrcorner \varphi_0) \wedge \varphi_0$.
(3) $(x \lrcorner \varphi_0) \wedge (x \lrcorner \varphi_0) = 2x \lrcorner ((x \lrcorner g_0) \wedge \psi_0)$.
(4) $(y \lrcorner x \lrcorner \varphi_0) \wedge (x \lrcorner \varphi_0) = -(y \lrcorner g_0) \wedge (x \lrcorner ((x \lrcorner g_0) \wedge \psi_0))$

□

Lemma 3.30. **(1)** For $V := \mathbb{R}^7$ there is a homomorphism

$$\varepsilon : \Lambda^3 V^* \to \Lambda^7 V^*$$

of $GL(7)$-modules, such that $\varepsilon(\varphi_0) = \varepsilon_0$.
(2) After choosing an orientation for V, there is a homomorphism

$$\varepsilon : \Lambda^4 V^* \to \Lambda^7 V^* \oplus i\Lambda^7 V^*$$

of $GL^+(7)$-modules, such that for the model tensor and the canonical orientation $\varepsilon(\psi_0) = \varepsilon_0$ holds.

Proof: For part (1) consider the $GL(7)$-equivariant map

$$K : \Lambda^3 V^* \to \mathrm{Hom}(V, V^* \otimes \Lambda^7 V^*) \quad \text{with} \quad K(\varphi)(x,y) := \frac{1}{6} x \lrcorner \varphi \wedge y \lrcorner \varphi \wedge \varphi.$$

Since

$$\begin{aligned}\det(K(\varphi)) &\in \Lambda^7 V^* \otimes \Lambda^7 (V^* \otimes \Lambda^7 V^*) \\ &= \Lambda^7 V^* \otimes \Lambda^7 V^* \otimes (\Lambda^7 V^*)^7 \\ &= (\Lambda^7 V^*)^9,\end{aligned}$$

we get an equivariant map

$$\det(K) : \Lambda^3 V^* \to (\Lambda^7 V^*)^9.$$

Even without a fixed orientation for V we can define

$$\varepsilon : \Lambda^3 V^* \to \Lambda^7 V^* \quad \text{by} \quad \varepsilon(\varphi) := \sqrt[9]{\det(K(\varphi))}$$

and for the model tensor we have by Lemma 3.29

$$\varepsilon(\varphi_0) = \sqrt[9]{\det(g_0 \otimes \varepsilon_0)} = \varepsilon_0.$$

For part (2) identify $\psi \in \Lambda^4 V^* = \Lambda^3 V \otimes \Lambda^7 V^*$ and consider the $GL(7)$-equivariant map

$$K : \Lambda^4 V^* \to \mathrm{Hom}(V^*, V \otimes (\Lambda^7 V^*)^2)$$

with

$$K(\psi)(\alpha, \beta) := \frac{1}{6}(\alpha \lrcorner \psi \wedge \beta \lrcorner \psi \wedge \psi) \in \Lambda^7 V \otimes (\Lambda^7 V^*)^3 = (\Lambda^7 V^*)^2.$$

Since

$$\begin{aligned}\det(K(\psi)) &\in \Lambda^7 V \otimes \Lambda^7 (V \otimes (\Lambda^7 V^*)^2) \\ &= \Lambda^7 V \otimes \Lambda^7 V \otimes (\Lambda^7 V^*)^{14} \\ &= (\Lambda^7 V^*)^{12},\end{aligned}$$

we get an equivariant map

$$\det(K) : \Lambda^4 V^* \to (\Lambda^7 V^*)^{12}.$$

Given an orientation for V, we can define

$$\varepsilon : \Lambda^4 V^* \to \Lambda^7 V^* \oplus i \Lambda^7 V^* \quad \text{by} \quad \varepsilon(\psi) := \sqrt[12]{\det(K(\psi))},$$

cf. Lemma 3.5. For the model tensor $\psi_0 = \varphi_0^* \otimes \varepsilon_0$ we compute

$$K(\psi_0)(\alpha, \beta) = g_0(\alpha, \beta) \varepsilon_0^2$$

and hence $\varepsilon(\psi_0) = \sqrt[12]{\varepsilon_0^{12}} = \varepsilon_0$.

$\square$

Lemma 3.31.

$$\mathrm{Iso}_{GL(7)}(\varphi_0) = G_2.$$
$$\mathrm{Iso}_{GL^+(7)}(\psi_0) = G_2.$$

In particular, the forms φ_0 and ψ_0 are stable.

Proof: A proof of the first statement can be found in [**47**], Lemma 11.1. For the second part observe that $A\varepsilon_0 = \varepsilon_0$ by Lemma 3.30, for $A \in \mathrm{Iso}_{GL^+(7)}(\psi_0)$. In addition, we have seen in the proof of Lemma 3.30 that

$$K(\psi_0)(\alpha, \beta) = g_0(\alpha, \beta)\varepsilon_0^2$$

holds. Hence $Ag_0 = g_0$, i.e. $A \in SO(7)$. Now observe that $\psi_0 = *_0\varphi_0$, where $*_0$ is the Hodge operator w.r.t. g_0 and the canonical orientation for $\mathbb{R}^7$. So

$$*_0\varphi_0 = A *_0 \varphi_0 = *_0 A\varphi_0$$

shows that $A \in \mathrm{Iso}_{GL(7)}(\varphi_0) = G_2$. Conversely, $A \in G_2 \subset SO(7)$ satisfies $A\psi_0 = *_0 A\varphi_0 = \psi_0$. Since $\dim(G_2) = 14$, stability follows from

$$\dim(GL(7)/G_2) = 49 - 14 = \dim(\Lambda^3\mathbb{R}^{7*}) = \dim(\Lambda^4\mathbb{R}^{7*}).$$

$\square$

Definition 3.32. Suppose $V = \mathbb{R}^7$ is equipped with a fixed orientation. For $\psi \in GL^+(7)\psi_0 \subset \Lambda^4V^*$ we call $\varepsilon(\psi) \in \Lambda^7V^*$ from Lemma 3.30 the associated volume element and define

$$\varphi = *_\psi\psi \in \Lambda^3V^*,$$

where $*_\psi$ is the Hodge operator associated to the volume $\varepsilon(\psi)$ and the metric given by $K(\psi)(\alpha, \beta) = g(\alpha, \beta)\varepsilon(\psi)^2$, cf. Lemma 3.30.

From Proposition 1.2 and Lemma 3.31 we obtain

Corollary 3.33. Suppose M is a seven dimensional manifold with a fixed orientation. Then G_2-structures on M, which are compatible with the given orientation, correspond to forms $\psi \in \Omega^4(M)$ of type ψ_0, such that $\varepsilon(\psi) > 0$.

$\square$

We will now describe the Lie algebra of $G_2 \subset SO(7)$.

Lemma 3.34. $A = (a_{ij}) \in \mathfrak{so}_7$ is an element of $\mathfrak{g}_2 \subset \mathfrak{so}_7$ if and only if

$$\begin{aligned} a_{23} + a_{45} + a_{67} &= 0, & a_{46} - a_{57} - a_{13} &= 0, & -a_{56} - a_{47} + a_{12} &= 0, \\ -a_{26} + a_{37} - a_{15} &= 0, & a_{36} + a_{27} + a_{14} &= 0, & a_{24} - a_{35} - a_{17} &= 0, \\ -a_{34} - a_{25} + a_{16} &= 0. \end{aligned}$$

Note that the i^{th} equation corresponds to $a_{kl}\varphi_{ikl} = 0$. The orthogonal complement in $\mathfrak{so}_7$ is given by

$$\mathfrak{g}_2^{\perp} = \{A \in \mathfrak{so}_7 \mid A \lrcorner g = x \lrcorner \varphi_0, \text{ for some } x \in \mathbb{R}^7\},$$

Proof: By Lemma 1.14 we have

$$\mathfrak{g}_2 = \ker(D_{\varphi_0} : \mathfrak{so}_7 \to \Lambda^3 \mathbb{R}^{7*})$$

and Lemma 3.1 yields $A \in \mathfrak{g}_2$ if and only if

$$0 = \sum_{i=1}^{7} e^i \wedge Ae_i \lrcorner \varphi_0 = \sum_{i,j=1}^{7} a_{ij} e^i \wedge e_j \lrcorner \varphi_0.$$

This system translates into the seven equations for the coefficients a_{ij}. Hence for $A \in \mathfrak{g}_2$

$$\sum_{j=1}^{7} \varphi(e_i, Ae_j, e_j) = 2 \sum_{j<k} a_{jk} \varphi(e_i, e_k, e_j) = 0$$

and we see that $B \lrcorner g := e_i \lrcorner \varphi_0$ defines an element $B \in \mathfrak{g}_2^{\perp}$. Since $\dim(\mathfrak{g}_2^{\perp}) = 7$, the Lemma follows.

$\square$

Proposition 3.35. The following decompositions of G_2-modules are irreducible:

$$\begin{aligned} \mathfrak{so}_7 &= \mathfrak{g}_2 \oplus \mathfrak{g}_2^{\perp} \\ \mathrm{End}(\mathbb{R}^7) &= \big(\mathbb{R}\mathrm{id} \oplus S_0^2\big) \oplus \big(\mathfrak{g}_2 \oplus \mathfrak{g}_2^{\perp}\big), \text{ where} \\ S_0^2 &= \{A \in S^2 \mid \mathrm{tr}(A) = 0\}. \end{aligned}$$

The G_2-modules $\Lambda^k := \Lambda^k \mathbb{R}^{7*}$ decompose into the following irreducible submodules, where the lower index denotes the dimension of the submodule:

$\Lambda^1 = \Lambda^1_7$

$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$, where

  $\Lambda^2_7 = \{x \lrcorner \varphi_0 \mid x \in \mathbb{R}^7\}$,

  $\Lambda^2_{14} = \{\omega \in \Lambda^2 \mid \psi_0 \wedge \omega = 0\} \cong \mathfrak{g}_2$,

$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where

  $\Lambda^3_1 = \mathbb{R}\varphi_0$,

$$\Lambda^3_7 = \{x \lrcorner \psi_0 \mid x \in \mathbb{R}^7\},$$
$$\Lambda^3_{27} = \{\omega \in \Lambda^3 \mid \varphi_0 \wedge \omega = 0 \text{ and } \psi_0 \wedge \omega = 0\},$$
$$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}, \text{ where}$$
$$\Lambda^4_1 = \mathbb{R}\psi_0,$$
$$\Lambda^4_7 = \{\alpha \wedge \varphi_0 \mid \alpha \in \Lambda^1\},$$
$$\Lambda^4_{27} = \{\omega \in \Lambda^4 \mid \varphi_0 \wedge \omega = 0 \text{ and } \varphi_0 \wedge *\omega = 0\},$$
$$\Lambda^5 = \Lambda^5_7 \oplus \Lambda^5_{14}, \text{ where}$$
$$\Lambda^5_7 = \{\alpha \wedge \psi_0 \mid \alpha \in \Lambda^1\},$$
$$\Lambda^5_{14} = \{\omega \in \Lambda^5 \mid \psi_0 \wedge *_0\omega = 0\},$$
$$\Lambda^6 = \Lambda^6_7.$$

PROOF: The decompositions of Λ^k into irreducible submodules can be found in [**43**], formulae 2.14-2.17 and 2.19-2.24. The map $D_{\varphi_0} : \mathrm{End}(\mathbb{R}^7) \to \Lambda^3$ satisfies $\ker(D_{\varphi_0}) = \mathfrak{g}_2$ by Lemma 1.14 and Lemma 3.31. Hence $\mathbb{R}\mathrm{id} \cap \ker(D_{\varphi_0}) = \{0\}$ and, by irreducibility, $D_{\varphi_0}(\mathbb{R}\mathrm{id}) = \Lambda^3_1$. This yields similarly $D_{\varphi_0}(\mathfrak{g}_2^\perp) = \Lambda^3_7$ and $D_{\varphi_0}(S^2_0) = \Lambda^3_{35}$. In particular, S^2_0 is irreducible by the irreducibility of Λ^3_{27}, which proves that the above decomposition of $\mathrm{End}(\mathbb{R}^7)$ is irreducible.

□

LEMMA 3.36. The maps $D_{\varphi_0} : \mathrm{End}(\mathbb{R}^7) \to \Lambda^3\mathbb{R}^{7*}$ and $D_{\psi_0} : \mathrm{End}(\mathbb{R}^7) \to \Lambda^4\mathbb{R}^{7*}$ define isomorphisms between certain submodules of $\mathrm{End}(\mathbb{R}^4)$ and $\Lambda^3\mathbb{R}^{7*}$, respectively $\Lambda^4\mathbb{R}^{7*}$:

	$\mathbb{R}\mathrm{id}$	S^2_0	$\mathfrak{g}_2$	$\mathfrak{g}_2^\perp$
D_{φ_0}	Λ^3_1	Λ^3_{27}	0	Λ^3_7
D_{ψ_0}	Λ^4_1	Λ^4_{27}	0	Λ^4_7

PROOF: We proved this already in Lemma 3.35 for D_{φ_0}. The proof for D_{ψ_0} is similar.

□

DEFINITION 3.37. Let M be a seven dimensional oriented manifold equipped with a G_2-structure ψ with intrinsic torsion $\tau : P \to \mathbb{R}^{7*} \otimes \mathfrak{g}_2^\perp$. We identify τ with an element $\mathcal{T} \in C^\infty(\mathrm{End}(TM))$, such that

$$g(\tau(X)Y, Z) = \varphi(\mathcal{T}(X), Y, Z).$$

If we choose a Cayley frame $(E_1, .., E_7)$ for the G_2-structure and let $A_i \lrcorner g := E_i \lrcorner \varphi$, then

$$\mathcal{T}(X) = \frac{1}{6}\sum_{i=1}^{7}\langle \tau(X), A_i\rangle E_i, \text{ since } \langle A_i, A_i\rangle = 6.$$

Proposition 3.38. Let M be a seven dimensional oriented manifold equipped with a G_2-structure ψ with intrinsic torsion $\tau \cong \mathcal{T}$. Then

$$\nabla^g \varphi = -3\mathcal{T} \lrcorner \psi, \qquad d\varphi = 3D_\psi(\mathcal{T}),$$
$$\nabla^g \psi = 3(\mathcal{T} \lrcorner g) \wedge \varphi, \qquad d\psi = 6\mathrm{pr}_{\Lambda^2}(\mathcal{T}) \wedge \varphi.$$

Proof: By Proposition 1.18 we have for the intrinsic torsion $\tau : P \to \mathbb{R}^{7*} \otimes \mathfrak{g}_2^\perp$ of the G_2-structure

$$\nabla^g_X(\varphi, \psi) = D_{(\varphi,\psi)}(\tau(X)).$$

Since $SO(7)$ acts on each factor $\Lambda^3 \times \Lambda^4$ separately, the corresponding equation hold for φ and ψ. Let $(E_1, .., E_7)$ be a local Cayley frame for the G_2-structure. Applying Lemma 3.1 and Lemma 3.29, we find

$$\begin{aligned}
\nabla^g_X \varphi = D_\varphi(\tau(X)) &= -\sum_{i=1}^{7} E^i \wedge \tau(X) E_i \lrcorner \varphi \\
&= -\sum_{i,j=1}^{7} g(\tau(X)E_i, E_j) E^i \wedge E_j \lrcorner \varphi \\
&= -\sum_{i,j=1}^{7} \varphi(\mathcal{T}(X), E_i, E_j) E^i \wedge E_j \lrcorner \varphi \\
&= -\sum_{j=1}^{7} \varphi(E_j, \mathcal{T}(X), .) \wedge E_j \lrcorner \varphi \\
&= -\frac{1}{2} \mathcal{T}(X) \lrcorner \sum_{j=1}^{7} E_j \lrcorner \varphi \wedge E_j \lrcorner \varphi \\
&= -3\mathcal{T}(X) \lrcorner \psi
\end{aligned}$$

and

$$\begin{aligned}
\nabla^g_X \psi &= -\sum_{j=1}^{7} \varphi(E_j, \mathcal{T}(X), .) \wedge E_j \lrcorner \psi \\
&= (\mathcal{T}(X) \lrcorner g) \wedge (\sum_{j=1}^{7} E_j \lrcorner (E^j \wedge \varphi)) \\
&= 3(\mathcal{T}(X) \lrcorner g) \wedge \varphi.
\end{aligned}$$

Hence the exterior derivatives are given by

$$\begin{aligned}
d\varphi &= \sum_{i=1}^{7} E^i \wedge \nabla^g_{E_i} \varphi = -3 \sum_{i=1}^{7} E^i \wedge \mathcal{T}(E_i) \lrcorner \psi \\
&= 3D_\psi(\mathcal{T})
\end{aligned}$$

and

$$\begin{aligned} d\psi &= \sum_{i=1}^{7} E^i \wedge \nabla^g_{E_i}\psi = 3\sum_{i=1}^{7} E^i \wedge (\mathcal{T}(E_i)\lrcorner g) \wedge \varphi \\ &= 6\mathrm{pr}_{\Lambda^2}(\mathcal{T}) \wedge \varphi. \end{aligned}$$

□

By Proposition 3.35 we have the following decomposition into irreducible G_2-modules

$$\begin{aligned} \mathbb{R}^{7*} \otimes \mathfrak{g}_2^\perp &= \mathbb{R}^{7*} \otimes \mathbb{R}^7 = \mathrm{End}(\mathbb{R}^7) \\ &= \left(\mathbb{R}\mathrm{id} \oplus S_0^2\right) \oplus \left(\mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp\right) \end{aligned}$$

and hence there are 2^4 different types of G_2-structures in dimension seven. We are interested in the following classes:

Theorem 3.39. Let ψ be a G_2-structure on M with intrinsic torsion $\tau \cong \mathcal{T}$ and $0 \neq \lambda \in \mathbb{R}$. In the following table we list different types of G_2-structures, the related torsion types and the corresponding equations for the structure tensors.

Name	Torsion	Characterization
Hypo	$\mathcal{T} \in S^2$	$d\psi = 0$
Nearly Parallel	$\mathcal{T} = -\frac{1}{12}\lambda\mathrm{id}$	$d\varphi = \lambda\psi$
Parallel	$\mathcal{T} = 0$	$d\psi = d\varphi = 0$

Proof: By Proposition 3.38 we have $d\psi = 0 \Leftrightarrow \mathrm{pr}_{\Lambda^2}(\mathcal{T}) \wedge \varphi = 0$. Since $\varphi \wedge . : \Lambda^2 \to \Lambda^5$ is an isomorphism, we actually have

$$d\psi = 0 \Leftrightarrow \mathrm{pr}_{\Lambda^2}(\mathcal{T}) = 0.$$

With Lemma 3.36 we see that $d\varphi = \lambda\psi$ is equivalent to

$$\mathcal{T} \in \mathbb{R}\mathrm{id} \oplus \mathfrak{g}_2 \quad \text{and} \quad 3D_\psi(\mathrm{pr}_{\mathbb{R}\mathrm{id}}) = \lambda\psi.$$

Since $D_\psi(\mathrm{id}) = -4\psi$, the condition on the torsion implies $d\varphi = \lambda\psi$. Conversely, $\lambda \neq 0$ yields $\mathcal{T} \in S^2$ and hence $\mathcal{T} = f\mathrm{id}$, for some function $f : M \to \mathbb{R}$. Then $\lambda\psi = 3D_\psi(f\mathrm{id}) = -12f\psi$ shows that

$$\mathcal{T} = -\frac{1}{12}\lambda\mathrm{id}.$$

By Proposition 3.38 and Lemma 3.36, $d\varphi = d\psi = 0$ is equivalent to $\mathrm{pr}_{\Lambda^2}(\mathcal{T}) = 0$ and $\mathcal{T} \in \mathfrak{g}_2$, i.e. $\mathcal{T} = 0$.

□

In this section we consider the following model tensors on $\mathbb{R}^7$:

$$\varphi_0 = e^{246} - e^{356} - e^{347} - e^{257} + e^{123} + e^{145} + e^{167},$$
$$\psi_0 = e^{2345} + e^{2367} + e^{4567} - e^{1247} + e^{1357} - e^{1346} - e^{1256},$$
$$\alpha_0 = e^1, \qquad \xi_0 = e_1$$

and on $\mathbb{R}^6 \cong \ker(\alpha_0)$

$$\omega_0 = \xi_0 \lrcorner \varphi_0 = e^{23} + e^{45} + e^{67},$$
$$\sigma_0 = \frac{1}{2}\omega_0^2 = e^{2345} + e^{2367} + e^{4567},$$
$$\rho_0 = \varphi_0 - \alpha_0 \wedge \omega_0 = e^{246} - e^{356} - e^{347} - e^{257},$$
$$\widehat{\rho}_0 = -\xi_0 \lrcorner \psi_0 = e^{247} - e^{357} + e^{346} + e^{256},$$

as well as $g_0(I_0.,.) := \omega_0$. Since the G_2-stabilizer of a unit vector in $\mathbb{R}^7$ equals $SU(3)$, the description of $SU(3)$-structures in dimension seven is much simpler than the description of $SU(2)$-structures in dimension five. Namely, Lemma 3.31 yields

Lemma 3.40.

$$\mathrm{Iso}_{GL(7)}(\alpha_0, \varphi_0) = SU(3).$$
$$\mathrm{Iso}_{GL^+(7)}(\alpha_0, \psi_0) = SU(3).$$

□

Proposition 3.41. Consider $V = \mathbb{R}^7$ with the canonical orientation and let $\psi \in GL^+(7)\psi_0$ and $\alpha \in \Lambda^1 V^*$. Then

$$(\psi, \alpha) \in GL^+(7)(\psi_0, \alpha_0) \quad \Leftrightarrow \quad g(\alpha, \alpha) = 1,$$

where g is the metric induced by ψ.

Proof: The Lemma follows immediately from the fact that the group G_2 acts transitively on $S^6 \subset \mathbb{R}^7$.

□

From Proposition 1.2, Lemma 3.40 and Proposition 3.41 we obtain

Corollary 3.42. Suppose M is a seven dimensional manifold with a fixed orientation. Then $SU(3)$-structures on M, which are compatible with the given orientation, correspond to pairs of forms $(\alpha,\psi)\in\Omega^1(M)\times\Omega^4(M)$, where ψ is of type ψ_0 with $\varepsilon(\psi)>0$ and α satisfies $g(\alpha,\alpha)=1$, w.r.t. the metric induced by ψ.

□

For a compact seven dimensional manifold M we have $\chi(M)=0$ and hence M admits a nowhere vanishing vector field. Hence any G_2-structure on M can be reduced to a $SU(3)$-structure by choosing a particular unit vector field.

We will now study the Lie algebra of $SU(3)\subset SO(7)$.

Lemma 3.43. $A=(a_{ij})\in\mathfrak{so}_7$ is an element of $\mathfrak{u}_3\subset\mathfrak{so}_7$ if and only if $Ae_1=0$ and

$$\begin{aligned} &a_{46}-a_{57}=0, && a_{56}+a_{47}=0, && a_{37}-a_{26}=0,\\ &a_{27}+a_{36}=0, && a_{24}-a_{35}=0, && a_{25}+a_{34}=0. \end{aligned}$$

Moreover, $A\in\mathfrak{su}_3$ if and only if in addition

$$a_{23}+a_{45}+a_{67}=0.$$

Equivalently,

$$\begin{aligned} \mathfrak{u}_3&=\{A\in\mathfrak{so}(7)\mid Ae_1=0\text{ and }AI_0=I_0A\},\\ \mathfrak{su}_3&=\{A\in\mathfrak{so}(7)\mid Ae_1=0\text{ and }AI_0=I_0A\text{ and }(A\lrcorner g_0)\wedge\sigma_0=0\}. \end{aligned}$$

The orthogonal complements in $\mathfrak{so}_7$ are given by

$$\begin{aligned} \mathfrak{u}_3^\perp&=\{\begin{pmatrix}0&-x^T\\x&A\end{pmatrix}\mid x\in\mathbb{R}^6\text{ and }A\in\mathfrak{u}_3^\perp\subset\mathfrak{so}_6\},\\ \mathfrak{su}_3^\perp&=\{\begin{pmatrix}0&-x^T\\x&A\end{pmatrix}\mid x\in\mathbb{R}^6\text{ and }A\in\mathfrak{su}_3^\perp\subset\mathfrak{so}_6\}. \end{aligned}$$

Proof: The Lemma follows immediately from Lemma 3.21.

□

Definition 3.44. Let M be a seven dimensional oriented manifold equipped with a $SU(3)$-structure (ψ,α) with intrinsic torsion $\tau:P\to\mathbb{R}^{7*}\otimes\mathfrak{su}_3^\perp$. According to the decomposition

$$\mathbb{R}^{7*}\otimes\mathfrak{su}_3^\perp=(\mathbb{R}^{7*}\otimes\mathbb{R}^6)\oplus(\mathbb{R}^{7*}\otimes\mathfrak{u}_3^\perp)\oplus\mathbb{R}^{7*}$$

we decompose τ into linear maps $F,\mathcal{T}:TM\to\ker(\alpha)$ and a 1-form η, such that

$$g(\tau(X)Y,Z)=\alpha(Y)g(F(X),Z)-\alpha(Z)g(F(X),Y)+\rho(\mathcal{T}(X),Y,Z)+\eta(X)\omega(Y,Z).$$

If we choose a Cayley frame $(E_1, .., E_7)$ for the G_2-structure and let $A_i \lrcorner g := E_i \lrcorner \rho$, for $i = 2, .., 7$, then

$$F(X) = \tau(X)\xi,$$
$$\mathcal{T}(X) = \frac{1}{6}\sum_{i=2}^{7}\langle \tau(X), A_i\rangle E_i, \text{ since } \langle A_i, A_i\rangle = 6,$$
$$\eta(X) = \frac{1}{6}\langle \tau(X), I\rangle, \text{ since } \langle I, I\rangle = 6.$$

Proposition 3.45. Let M be a seven dimensional oriented manifold equipped with a $SU(3)$-structure (ψ, α) with intrinsic torsion $\tau \cong F + \mathcal{T} + \eta$. Then

$$\nabla^g \alpha = F \lrcorner g,$$
$$\nabla^g \omega = -\alpha \wedge F \lrcorner \omega - 2\mathcal{T} \lrcorner \widehat{\rho},$$
$$\nabla^g \sigma = -\alpha \wedge (F \lrcorner \sigma) - 2(\mathcal{T} \lrcorner \widehat{\rho}) \wedge \omega,$$
$$\nabla^g \rho = -\alpha \wedge F \lrcorner \rho - 2\mathcal{T} \lrcorner \sigma + 3\eta \otimes \widehat{\rho},$$
$$\nabla^g \widehat{\rho} = -\alpha \wedge F \lrcorner \widehat{\rho} - 2I\mathcal{T} \lrcorner \sigma - 3\eta \otimes \rho$$

and

$$d\alpha = 2\mathrm{pr}_{\Lambda^2}(F),$$
$$d\omega = -\alpha \wedge D_\omega(F) + 2D_{\widehat{\rho}}(\mathcal{T}),$$
$$d\sigma = -\alpha \wedge D_\sigma(F) + 2D_{\widehat{\rho}}(\mathcal{T}) \wedge \omega,$$
$$d\rho = -\alpha \wedge D_\rho(F) + 2D_\sigma(\mathcal{T}) + 3\eta \wedge \widehat{\rho},$$
$$d\widehat{\rho} = -\alpha \wedge D_{\widehat{\rho}}(F) + 2D_\sigma(I\mathcal{T}) - 3\eta \wedge \rho.$$

Proof: By Proposition 1.18 we have for the intrinsic torsion $\tau : P \to \mathbb{R}^{7*} \otimes \mathfrak{su}_3^\perp$ of the $SU(3)$-structure

$$\nabla^g_X(\alpha, \omega, \sigma, \rho, \widehat{\rho}) = D_{(\alpha,\omega,\sigma,\rho,\widehat{\rho})}(\tau(X)).$$

Since $SO(7)$ acts on each factor separately, the corresponding equation hold for α, ω, σ, ρ and $\widehat{\rho}$. Let $(\xi = E_1, .., E_7)$ be a local Cayley frame for the $SU(3)$-structure. Applying Lemma 3.1, we find

$$\nabla^g_X \alpha = D_\alpha(\tau(X)) = -\sum_{i=1}^{7}\alpha(\tau(X)E_i)E^i = \sum_{i=2}^{7} g(\tau(X)\xi, E_i)E^i$$
$$= \sum_{i=1}^{7} g(F(X), E_i)E^i = F(X) \lrcorner g,$$

and, using that $g(F(X),\xi)=0$,

$$\begin{aligned}
\nabla^g_X\omega &= D_\omega(\tau(X)) = -\sum_{i=1}^7 E^i\wedge\tau(X)E_i\lrcorner\omega = \sum_{i,j=1}^7 g(\tau(X)E_i, IE_j)E^{ij}\\
&= \sum_{j=2}^7 g(\tau(X)\xi, IE_j)\alpha\wedge E^j + \sum_{i,j=2}^7 g(\tau(X)E_i, IE_j)E^{ij}\\
&= \sum_{j=2}^7 g(F(X), IE_j)\alpha\wedge E^j + \sum_{i,j=2}^7 (\rho(\mathcal{T}(X), E_i, IE_j) + \eta(X)\delta_{ij})E^{ij}\\
&= -\alpha\wedge F(X)\lrcorner\omega - \sum_{i,j=2}^7 \widehat{\rho}(\mathcal{T}(X), E_i, E_j)E^{ij}\\
&= -\alpha\wedge F(X)\lrcorner\omega - 2\mathcal{T}(X)\lrcorner\widehat{\rho}.
\end{aligned}$$

The same computation yields

$$\nabla^g_X\sigma = -\alpha\wedge(F(X)\lrcorner\omega)\wedge\omega - 2(\mathcal{T}(X)\lrcorner\widehat{\rho})\wedge\omega$$

and similarly we obtain

$$\begin{aligned}
\nabla^g_X\rho &= D_\rho(\tau(X)) = -\sum_{i,j=1}^7 g(\tau(X)E_i, E_j)E^i\wedge E_j\lrcorner\rho\\
&= -\sum_{j=2}^7 g(\tau(X)\xi, E_j)\alpha\wedge E_j\lrcorner\rho - \sum_{i,j=2}^7 g(\tau(X)E_i, E_j)E^i\wedge E_j\lrcorner\rho\\
&= -\alpha\wedge F(X)\lrcorner\rho - \sum_{i,j=2}^7 (\rho(\mathcal{T}(X), E_i, E_j) + \eta(X)\omega(E_i, E_j))E^i\wedge E_j\lrcorner\rho\\
&= -\alpha\wedge F(X)\lrcorner\rho - \frac{1}{2}\mathcal{T}(X)\lrcorner(\sum_{j=2}^7 E_j\lrcorner\rho\wedge E_j\lrcorner\rho) - \eta(X)\sum_{i=2}^7 E^i\wedge IE_i\lrcorner\rho\\
&= -\alpha\wedge F(X)\lrcorner\rho - 2\mathcal{T}(X)\lrcorner\sigma + 3\eta(X)\widehat{\rho}
\end{aligned}$$

and

$$\begin{aligned}
\nabla^g_X\widehat{\rho} &= -\alpha\wedge F(X)\lrcorner\widehat{\rho} - \sum_{i,j=2}^7 (\rho(\mathcal{T}(X), E_i, E_j) + \eta(X)\omega(E_i, E_j))E^i\wedge E_j\lrcorner\widehat{\rho}\\
&= -\alpha\wedge F(X)\lrcorner\widehat{\rho} - 3\eta(X)\rho - \sum_{i,j=2}^7 \widehat{\rho}(I\mathcal{T}(X), E_i, E_j)E^i\wedge E_j\lrcorner\widehat{\rho}\\
&= -\alpha\wedge F(X)\lrcorner\widehat{\rho} - 3\eta(X)\rho - \frac{1}{2}I\mathcal{T}(X)\lrcorner(\sum_{j=2}^7 E_j\lrcorner\widehat{\rho}\wedge E_j\lrcorner\widehat{\rho})\\
&= -\alpha\wedge F(X)\lrcorner\widehat{\rho} - 3\eta(X)\rho - 2I\mathcal{T}(X)\lrcorner\sigma.
\end{aligned}$$

Now the exterior derivatives are given by

$$d\alpha = \sum_{i=1}^7 E^i\wedge\nabla^g_{E_i}\alpha = 2\mathrm{pr}_{\Lambda^2}(F)$$

and

$$\begin{aligned} d\omega &= \sum_{i=1}^{7} E^i \wedge \nabla^g_{E_i}\omega = \sum_{i=1}^{7} E^i \wedge (-\alpha \wedge F(E_i) \lrcorner \omega - 2\mathcal{T}(E_i) \lrcorner \widehat{\rho}) \\ &= -\alpha \wedge D_\omega(F) + 2D_{\widehat{\rho}}(\mathcal{T}). \end{aligned}$$

Hence $d\sigma = -\alpha \wedge D_\sigma(F) + 2D_{\widehat{\rho}}(\mathcal{T}) \wedge \omega$, and similarly

$$\begin{aligned} d\rho &= \sum_{i=1}^{7} E^i \wedge \nabla^g_{E_i}\rho = \sum_{i=1}^{7} E^i \wedge (-\alpha \wedge F(E_i) \lrcorner \rho - 2\mathcal{T}(E_i) \lrcorner \sigma + 3\eta(E_i)\widehat{\rho}) \\ &= -\alpha \wedge D_\rho(F) + 2D_\sigma(\mathcal{T}) + 3\eta \wedge \widehat{\rho} \end{aligned}$$

and

$$\begin{aligned} d\widehat{\rho} &= \sum_{i=1}^{7} E^i \wedge \nabla^g_{E_i}\widehat{\rho} = \sum_{i=1}^{7} E^i \wedge (-\alpha \wedge F(E_i) \lrcorner \widehat{\rho} - 2I\mathcal{T}(E_i) \lrcorner \sigma - 3\eta(E_i)\rho) \\ &= -\alpha \wedge D_{\widehat{\rho}}(F) + 2D_\sigma(I\mathcal{T}) - 3\eta \wedge \rho. \end{aligned}$$

$\square$

Theorem 3.46. Let (ψ, α) be a $SU(3)$-structure on M with intrinsic torsion $\tau \cong F + \mathcal{T} + \eta$ and $0 \neq \lambda \in \mathbb{R}$. In the following table we list different types of $SU(3)$-structures, the related torsion types and the corresponding equations for the structure tensors. For this let

$$\mathcal{T}_0 := \mathcal{T}_{|\ker(\alpha)} \in \mathrm{End}(\ker(\alpha)) \quad \text{and} \quad \eta_0 := \eta_{|\ker(\alpha)} \in \Omega^1(\ker(\alpha)).$$

Name	Torsion	Characterization
Nearly Hypo	$I\mathcal{T}_0 + \frac{1}{8}\lambda I \in S^2$ $\eta_0 = 0$	$d\rho = \lambda\sigma$ on $\ker(\alpha)$.
Hypo	$I\mathcal{T}_0 \in S^2$ $\eta_0 = 0$	$d\sigma = d\rho = 0$ on $\ker(\alpha)$.
Nearly Parallel (nearly Kähler)	$I\mathcal{T}_0 = \lambda \mathrm{id}$ $\eta_0 = 0$	$d\omega = 6\lambda\rho$ on $\ker(\alpha)$ and $d\widehat{\rho} = -4\lambda\omega^2$ on $\ker(\alpha)$. Equivalently: For all $X \in \ker(\alpha)$ $(\nabla^g_X I)X = 0$ and $d\rho = 0$ on $\ker(\alpha)$.
Parallel (Calabi-Yau)	$\mathcal{T}_0 = 0$ $\eta_0 = 0$	$d\omega = d\rho = d\widehat{\rho} = 0$ on $\ker(\alpha)$.
Complex	$\mathcal{T}_0 \in S^2_{12} \oplus \mathfrak{u}_3^\perp$	$N_I = 0$ on $\ker(\alpha)$.
Kähler	$\mathcal{T}_0 = 0$	$N_I = 0$ and $d\omega = 0$ on $\ker(\alpha)$. Equivalently, $\nabla^g\omega = 0$ on $\ker(\alpha)$.

Sasakian	$\mathcal{T}=0$ $F=I$	$I=\nabla^g\xi$ and $(\nabla^g_X I)Y = g(\xi,Y)X - g(X,Y)\xi$, for all $X,Y\in TM$.

PROOF: The equations for the exterior derivatives of the structure tensors from Proposition 3.45 and Lemma 3.1 give

$$\begin{aligned} d\rho_{|\ker(\alpha)} &= 2D_\sigma(\mathcal{T})_{|\ker(\alpha)} + 3(\eta\wedge\widehat{\rho})_{|\ker(\alpha)} \\ &= -2\sum_{i=1}^{7}(E^i\wedge \mathcal{T}E_i\lrcorner\sigma)_{|\ker(\alpha)} + 3\eta_0\wedge\widehat{\rho}_{|\ker(\alpha)} \\ &= -2\sum_{i=2}^{7}E^i\wedge \mathcal{T}_0E_i\lrcorner\sigma_{|\ker(\alpha)} + 3\eta_0\wedge\widehat{\rho}_{|\ker(\alpha)},\ \text{ for } E^1=\alpha \\ &= 2D_{\sigma_{|\ker(\alpha)}}(\mathcal{T}_0) + 3\eta_0\wedge\widehat{\rho}_{|\ker(\alpha)}. \end{aligned}$$

Similarly we obtain formulae for the restriction of $d\omega$, $d\sigma$ and $d\widehat{\rho}$ to $\ker(\alpha)$. The resulting equations are precisely the equations from the six dimensional case, cf. Proposition 3.25. For the covariant derivatives we obtain again by Proposition 3.45

$$(\nabla^g\omega)_{|\ker(\alpha)} = -2\mathcal{T}_0\lrcorner\widehat{\rho}_{|\ker(\alpha)},$$

which was required in the proof of Theorem 3.28 to characterize the condition $N_I = 0$. With this observation, we can reduce all computations to $\ker(\alpha)$ and repeat arguments from the proof of Theorem 3.28.

For the description of Sasakian structures we use Proposition 3.45 to find $I = \nabla^g\xi = F$ and hence

$$\begin{aligned} (\nabla^g_X\omega)(Y,Z) &= (-\alpha\wedge F(X)\lrcorner\omega - 2\mathcal{T}(X)\lrcorner\widehat{\rho})(Y,Z) \\ &= (\alpha\wedge X\lrcorner g - 2\mathcal{T}(X)\lrcorner\widehat{\rho})(Y,Z) \\ &= g(\xi,Y)g(X,Z) - g(\xi,Z)g(X,Y) - 2\widehat{\rho}(\mathcal{T}(X),Y,Z). \end{aligned}$$

Now the characterization follows, since $\mathcal{T}(X)\in\ker(\alpha)$ and $\widehat{\rho}$ is non degenerated on $\ker(\alpha)$.

□

REMARK 3.47. M. Cabrera [**14**] studies $SU(3)$-structures on hyper surfaces which are induced by certain types of ambient G_2-structures. The only case where the induced structure is actually Kähler (i.e. of type $\mathcal{W}_5$ in the notation of [**14**]) occurs in Table 2 of [**14**]. Cabrera shows that in this case, the ambient G_2-structure must be parallel and the hyper surface has to be totally geodesic. However, Cabrera only studies ambient G_2-structures which are of one of the four canonical Gray-Hervella types. A generic $SU(3) \subset G_2$ structure with $\mathcal{T}_{\ker(\alpha)} = 0$ does not belong to one of these four types.

Remark 3.48. Sasakian structures in dimension $2n+1$ are $\{1\} \times U(n)$ structures which satisfy the integrability conditions

$$I = \nabla^g \xi \quad \text{and} \quad (\nabla^g_X I)Y = g(\xi, Y)X - g(X, Y)\xi.$$

Hence a $SU(3)$-structure on a seven dimensional manifold satisfies these conditions if and only if the underlying $\{1\} \times U(3)$ structure is a Sasakian structure, cf. [**8**]. Sasakian structures are usually considered as the odd-dimensional analogue of Kähler structures. From this point of view, Sasakian structures which are compatible with a topological G_2-structure can be regarded as the analogue of $SU(3)$-Kähler structures, i.e. Kähler structures with $c_1 = 0$. By Yau's proof of the Calabi conjecture, such a Kähler structure admits a unique Ricci flat Kähler structure within its cohomology class. But a Sasakian structure can neither be Ricci flat, nor allow parallel forms.

Another reason why the term 'analogue' should be used with caution, is the fact that it is used with respect to a certain embedding of the $\{1\} \times U(n)$ structure into an even dimensional space. In the Sasakian case the ambient space is the metric cone over the odd dimensional manifold M, but different choices for the ambient space will lead to different notions of what one should call an odd dimensional 'analogue' of Kähler structures.

To avoid the choice of an ambient space, it seems natural to call a $\{1\} \times U(n)$ structure Kähler if the $U(n)$-structure on $\ker(\alpha)$ is Kähler, i.e. $\mathcal{T} = 0$ on $\ker(\alpha)$. This notion can be refined by requiring the distributional part F to live in a certain submodule of

$$F \in \begin{pmatrix} 0 & 0 \\ \mathbb{R}^6 & \mathrm{End}(\mathbb{R}^6) \end{pmatrix},$$

where $\mathrm{End}(\mathbb{R}^6)$ decomposes as an $SU(3)$-module into

	I^+	I^-
$S^2(\mathbb{R}^6)$	$\mathbb{R}\mathrm{id} \oplus I_0\mathfrak{su}(3)$	S^2_{12}
$\mathfrak{so}_6$	$\mathbb{R}I_0 \oplus \mathfrak{su}(3)$	$\mathfrak{u}_3^\perp$

So the possible notions of 'analogue' Kähler structures are parameterized by the distributional part F. Theorem 3.46 states that $SU(3)$-Sasakian structures are precisely those types of Kähler structures for which $F = I \in \mathbb{R}I_0$ and $\mathcal{T}(\xi) = 0$ holds.

In this section we consider the following model form on $\mathbb{R}^8$:

$$
\begin{aligned}
\Psi_0 &= \psi_0 + e^1 \wedge \varphi_0 \\
&= e^{3456} + e^{3478} + e^{5678} - e^{2358} + e^{2468} - e^{2457} - e^{2367} \\
&\quad + e^{1357} - e^{1467} - e^{1458} - e^{1368} + e^{1234} + e^{1256} + e^{1278}.
\end{aligned}
$$

This form satisfies certain relations, which can be verified in a direct computation:

LEMMA 3.49. For all $x \in \mathbb{R}^8$

(1) $\Psi_0 \wedge \Psi_0 = 14\varepsilon_0$.
(2) $*_0\Psi_0 = \Psi_0$.
(3) $*_0((x \lrcorner g_0) \wedge \Psi_0) = x \lrcorner \Psi_0$.
(4) $*_0(x \lrcorner \Psi_0 \wedge \Psi_0) = 7x \lrcorner g_0$.

□

The isotropy group of Ψ_0 can be identified with the Lie group Spin(7), cf. **[47]** Lemma 12.2.

LEMMA 3.50.

$$\mathrm{Iso}_{GL(8)}(\Psi_0) = \mathrm{Spin}(7).$$

□

Since $\dim(GL(8)/\mathrm{Spin}(7)) = 64 - 21 < 70 = \dim(\Lambda^4\mathbb{R}^{8*})$, the form Ψ_0 is *not* stable. Nevertheless, we have by Proposition 1.2

COROLLARY 3.51. Spin(7)-structures on an eight dimensional manifold M correspond to forms $\Psi \in \Omega^4(M)$ of type Ψ_0.

□

We will now describe the Lie algebra of $\mathrm{Spin}(7) \subset SO(8)$.

Lemma 3.52. $A = (a_{ij}) \in \mathfrak{so}_8$ is an element of $\mathfrak{spin}_7 \subset \mathfrak{so}_8$ if and only if

$$\begin{aligned} a_{23} + a_{45} + a_{67} \qquad &= 0, & a_{46} - a_{57} - a_{13} + a_{12} &= 0, \\ -a_{56} - a_{47} + a_{12} + a_{13} &= 0, & -a_{26} + a_{37} - a_{15} + a_{14} &= 0, \\ +a_{36} + a_{27} + a_{14} + a_{15} &= 0, & a_{24} - a_{35} - a_{17} + a_{16} &= 0, \\ -a_{34} - a_{25} + a_{16} + a_{17} &= 0. \end{aligned}$$

Note that the i^{th} equation corresponds to $a_{kl}\varphi_{ikl} + a_{1i} = 0$. The orthogonal complement in $\mathfrak{so}_8$ is given by

$$\mathfrak{spin}_7^{\perp} = \{A \in \mathfrak{so}_8 \mid A \lrcorner g = x \lrcorner \varphi_0 + e^1 \wedge (x \lrcorner g_0), \text{ for some } x \in \mathbb{R}^8\}.$$

Proof: By Lemma 1.14 we have

$$\mathfrak{spin}_7 = \ker(D_{\Psi_0} : \mathfrak{so}_8 \to \Lambda^4\mathbb{R}^{7*})$$

and Lemma 3.1 yields $A \in \mathfrak{spin}_7$ if and only if

$$0 = \sum_{i=1}^{8} e^i \wedge Ae_i \lrcorner \psi_0 = \sum_{i,j=1}^{8} a_{ij} e^i \wedge e_j \lrcorner \psi_0.$$

This system translates into the seven equations for the coefficients a_{ij}. Hence for $A \in \mathfrak{spin}_7$ and $B_i \lrcorner g := e_i \lrcorner \varphi_0 + e^1 \wedge (e_i \lrcorner g)$

$$\begin{aligned} \operatorname{tr}(B_i A) &= \sum_{j=1}^{8} (\varphi_0(e_i, Ae_j, e_j) + a_{j1}\delta_{ij} - a_{ji}\delta_{j1}) \\ &= 2\sum_{j<k} a_{jk}\varphi_0(e_i, e_k, e_j) + a_{i1} - a_{1i} \\ &= 2a_{1i} - 2a_{1i} = 0 \end{aligned}$$

and we see that B_i defines an element $B_i \in \mathfrak{spin}_7^{\perp}$. Since $\dim(\mathfrak{spin}_7^{\perp}) = 7$ and $B_1 = 0$, the Lemma follows.

□

Proposition 3.53. The following decompositions of Spin(7)-modules are irreducible:

$$\begin{aligned} \mathfrak{so}_8 &= \mathfrak{spin}_7 \oplus \mathfrak{spin}_7^{\perp} \\ \operatorname{End}(\mathbb{R}^8) &= \left(\mathbb{R}\mathrm{id} \oplus S_0^2\right) \oplus \left(\mathfrak{spin}_7 \oplus \mathfrak{spin}_7^{\perp}\right), \text{ where} \\ S_0^2 &= \{A \in S^2 \mid \operatorname{tr}(A) = 0\}. \end{aligned}$$

The Spin(7)-modules $\Lambda^k := \Lambda^k\mathbb{R}^{8*}$ decompose into the following irreducible submodules, where the lower index denotes the dimension of the submodule:

$$\Lambda^1 = \Lambda^1_8$$

$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21}$, where

$\Lambda^2_7 = \{\omega \in \Lambda^2 \mid *_0(\Psi_0 \wedge \omega) = 3\omega\}$,

$\Lambda^2_{21} = \{\omega \in \Lambda^2 \mid *_0(\Psi_0 \wedge \omega) = -\omega\} \cong \mathfrak{spin}_7$,

$\Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48}$, where

$\Lambda^3_8 = \{x \lrcorner \Psi_0 \mid x \in \mathbb{R}^8\}$,

$\Lambda^3_{48} = \{\omega \in \Lambda^3 \mid \Psi_0 \wedge \omega = 0\}$,

$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}$, where

$\Lambda^4_1 = \mathbb{R}\Psi_0$,

$\Lambda^4_7 = \{\sum_{i=1}^8 e^i \wedge Ae_i \lrcorner \Psi_0 - (Ae_i \lrcorner g_0) \wedge (e_i \lrcorner \Psi_0) \mid A \in \mathfrak{spin}_7^\perp\}$,

$\Lambda^4_{27} = \{\omega \in \Lambda^4 \mid \omega = *_0\omega,\ \Psi_0 \wedge \omega = 0 \text{ and } \Lambda^4_7 \subset \ker(\omega \wedge\ .)\}$,

$\Lambda^4_{35} = \{\omega \in \Lambda^4 \mid *_0\omega = -\omega\}$,

$\Lambda^5 = \Lambda^5_8 \oplus \Lambda^5_{48}$, where

$\Lambda^5_8 = \{\alpha \wedge \Psi_0 \mid \alpha \in \Lambda^1\}$,

$\Lambda^5_{48} = \{\omega \in \Lambda^5 \mid \Psi_0 \wedge *_0\omega = 0\}$,

$\Lambda^6 = \Lambda^6_7 \oplus \Lambda^6_{21}$, where

$\Lambda^6_7 = \{\omega \in \Lambda^6 \mid \Psi_0 \wedge *_0\omega = 3\omega\}$,

$\Lambda^6_{21} = \{\omega \in \Lambda^6 \mid \Psi_0 \wedge *_0\omega = -\omega\}$,

$\Lambda^7 = \Lambda^7_8$.

Proof: The decompositions of Λ^k into irreducible submodules can be found in [**43**], formulae 4.7-4.10 and 4.12-4.19. The map $D_{\Psi_0} : \mathrm{End}(\mathbb{R}^8) \to \Lambda^4$ satisfies $\ker(D_{\Psi_0}) = \mathfrak{spin}_7$ by Lemma 1.14 and Lemma 3.50. Hence $\mathbb{R}\mathrm{id} \cap \ker(D_{\Psi_0}) = \{0\}$ and $D_{\Psi_0}(\mathbb{R}\mathrm{id})$ is a one dimensional submodule of Λ^4. By irreducibility, we see that $0 \neq D_{\Psi_0}(\mathbb{R}\mathrm{id}) = \Lambda^4_1$. This yields similarly $D_{\Psi_0}(\mathfrak{spin}_7^\perp) = \Lambda^4_7$ and $D_{\Psi_0}(S^2_0) = \Lambda^4_{35}$. In particular, S^2_0 is irreducible by the irreducibility of Λ^4_{35}, which proves that the above decomposition of $\mathrm{End}(\mathbb{R}^8)$ is irreducible.

□

Lemma 3.54. The map $D_{\Psi_0} : \mathrm{End}(\mathbb{R}^8) \to \Lambda^4\mathbb{R}^{8*}$ defines an isomorphism between certain submodules of $\mathrm{End}(\mathbb{R}^8)$ and $\Lambda^4\mathbb{R}^{8*}$:

	$\mathbb{R}\mathrm{id}$	S^2_0	$\mathfrak{spin}_7$	$\mathfrak{spin}_7^\perp$
D_{Ψ_0}	Λ^4_1	Λ^4_{35}	0	Λ^4_7

Proof: We proved this already in Lemma 3.53.

□

Consider the map

$$f_{\Psi_0} : \mathbb{R}^{8*} \otimes \mathfrak{spin}_7^\perp \to \Lambda^5\mathbb{R}^{8*} \quad \text{with} \quad \tau \mapsto \sum_{i=1}^8 e^i \wedge D_{\Psi_0}(\tau(e_i)).$$

This definition is independent of the choice of g_0-orthonormal basis $(e_1, .., e_8)$ and hence Lemma 1.14 shows that f_{Ψ_0} is Spin(7) $\subset SO(8)$ equivariant. From Lemma 3.1 we get

$$f_{\Psi_0}(\tau) = -\sum_{i,j,k} \tau_{ijk} e^{ij} \wedge e_k \lrcorner \Psi_0,$$

which can be used to show that f_{Ψ_0} is injective and hence an isomorphism. Since $\Lambda^5 = \Lambda^5_8 \oplus \Lambda^5_{48}$, we obtain a corresponding decomposition into irreducible Spin(7)-modules

$$\mathbb{R}^{8*} \otimes \mathfrak{spin}_7^\perp = \mathcal{W}_8 \oplus \mathcal{W}_{48}$$

and hence there are 2^2 different types of Spin(7)-structures in dimension eight:

Theorem 3.55. Let Ψ be a Spin(7)-structure on M with intrinsic torsion τ. In the following table we list different types of Spin(7)-structures, the related torsion types and the corresponding equations for the structure tensors.

Name	Torsion	Characterization
Balanced	$\tau \in \mathcal{W}_{48}$	$\Theta := *(*d\Psi \wedge \Psi) = 0$ (Lee form)
Locally conformal parallel	$\tau \in \mathcal{W}_8$	$d\Psi = \frac{1}{7}\Theta \wedge \Psi$
Parallel	$\tau = 0$	$d\Psi = 0$

Proof: From Proposition 1.18 we get $f_\Psi(\tau) = d\Psi$ and, since f_Ψ is an isomorphism, we have $d\Psi = 0 \ \Leftrightarrow \ \tau = 0$. By Proposition 3.53 we have

$$\tau \in \mathcal{W}_{48} \quad \Leftrightarrow \quad d\Psi \in \Lambda^5_{48} \quad \Leftrightarrow \quad \Theta = 0$$

and similarly

$$\tau \in \mathcal{W}_8 \quad \Leftrightarrow \quad d\Psi \in \Lambda^5_8 \quad \Leftrightarrow \quad d\Psi = \alpha \wedge \Psi,$$

for some 1-form α. Using Lemma 3.49, we get

$$\Theta = *(*d\Psi \wedge \Psi) = *(\xi \lrcorner \Psi \wedge \Psi) = 7\alpha,$$

where $\xi \lrcorner g := \alpha$.

$\square$

4. Embedding Theorems for Special Geometries

In **[36]** N. Hitchin introduces a flow equation for hypo G_2-structures on a manifold M, whose solutions yield parallel Spin(7)-structures on $I \times M$, for some interval $I \subset \mathbb{R}$. In this sense, a solution of the flow equation embeds the initial G_2-structure into a manifold with a parallel Spin(7)-structure and is therefore called a solution of the embedding problem for the initial structure. Similar equations are known for embedding $SU(2)$-structures in dimension five and $SU(3)$-structures in dimension six into manifolds with a parallel $SU(3)$ and G_2-structure, respectively, cf. **[21]**,**[22]**,**[23]**,**[28]**,**[29]**. R. Bryant shows in **[11]** that in the real analytic category, the embedding problem for hypo $SU(3)$ and G_2-structures can be solved. Bryant also provided counterexamples in the smooth category. The embedding problem for $SU(2)$-structures in dimension five was solved by D. Conti and S. Salamon in **[22]**, cf. also **[21]**.

In this section we describe a unifying approach to all of the above embedding problems. We reduce the $SU(2)$ and $SU(3)$ embedding problem to the G_2-case, which will be studied in terms of gauge deformations. Since the structure tensor $\varphi \in \Omega^3(M)$ of a G_2-structure is stable, any smooth deformation φ_t can be described by a family of gauge deformations $A_t \in C^\infty(\mathrm{Aut}(TM))$ via $\varphi_t = A_t\varphi$, cf. Theorem 1.6. In the G_2-case, the intrinsic torsion $\mathcal{T}$ takes values in the G_2-module $\mathfrak{gl}(7)$ and can therefore be regarded again as an (infinitesimal) gauge deformation. We prove that the intrinsic torsion flow for G_2-structures

$$\dot{A}_t = \mathcal{T}_t \circ A_t$$

can be regarded as a generalization of Hitchin's flow equation, and hence as a generalization of the $SU(2)$, $SU(3)$ and G_2-embedding problem, cf. Proposition 4.12. In Theorem 4.13 we determine the evolution of the metric and the intrinsic torsion under the intrinsic torsion flow. Using the Cheeger-Gromoll Splitting Theorem, we prove in Theorem 4.14 and Corollary 4.15 that there are no nontrivial longtime solutions for the embedding problem.

Generalized Cylinders

Let ξ be a unit vector field on (M, g), such that $d\alpha = 0$, where $\alpha := \xi \lrcorner g$. On the integral manifolds $i : N \hookrightarrow M$ of the distribution $\ker(\alpha)$, we denote by $g_N := i^*g$

the induced metric. Conversely, the collection of metrics on all integral manifolds determines the ambient metric via

$$g = \alpha \otimes \alpha + \{g_N\},$$

where $\{g_N\} := \mathrm{pr}^* g$ and $\mathrm{pr} : TM \to \ker(\alpha)$ is the projection $\mathrm{pr}(X) := X - \alpha(X)\xi$. The Weingarten map

$$\mathcal{W} := \nabla^g \xi$$

defines a symmetric endomorphism on (M, g) with

$$(L_\xi g)(X, Y) = 2g(\mathcal{W}X, Y)$$

and $W(\xi) = 0$. This shows that the integral curves of ξ are geodesics on (M, g) and that $\mathcal{W}$ reduces to a symmetric endomorphism $\mathcal{W}_N$ on each integral manifold $N \subset M$. We will now express the curvature quantities R, ric and scal on M in terms of the curvature quantities R_N, ric_N and scal_N on $N \subset M$.

Proposition 4.1. For $X, Y, U, V \in T_pN \subset T_pM$ we have

$$\begin{aligned}
g(R(U,X)Y,V) &= g_N(R_N(U,X)Y,V) \\
&\quad + g_N(\mathcal{W}_N U, Y) g_N(\mathcal{W}_N X, V) - g_N(\mathcal{W}_N X, Y) g_N(\mathcal{W}_N U, V) \\
g(R(\xi,X)Y,\xi) &= g_N(\mathcal{W}_N X, \mathcal{W}_N Y) - \frac{1}{2}(L_\xi^2 g)(X,Y) \\
g(R(U,\xi)Y,V) &= g_N((\nabla_V^{g_N} \mathcal{W}_N)Y, U) - g_N((\nabla_Y^{g_N} \mathcal{W}_N)V, U) \\
\mathrm{ric}(X,Y) &= \mathrm{ric}_N(X,Y) - \mathrm{tr}(\mathcal{W}_N) g_N(\mathcal{W}_N X, Y) \\
&\quad + 2g_N(\mathcal{W}_N X, \mathcal{W}_N Y) - \frac{1}{2}(L_\xi^2 g)(X,Y) \\
\mathrm{ric}(\xi,\xi) &= \mathrm{tr}(\mathcal{W}_N^2) - \frac{1}{2}\mathrm{tr}_g(L_\xi^2 g) \\
\mathrm{ric}(X,\xi) &= \mathrm{div}(\mathcal{W}_N)(X) - X \cdot \mathrm{tr}(\mathcal{W}_N) \\
\mathrm{scal} &= \mathrm{scal}_N + 3\mathrm{tr}(\mathcal{W}_N^2) - \mathrm{tr}(\mathcal{W}_N)^2 - \mathrm{tr}_g(L_\xi^2 g)
\end{aligned}$$

Proof: It suffices to consider vector fields X with $L_\xi X = 0$, so that $\nabla_\xi^g X = \mathcal{W}(X)$ and

$$\xi \cdot g(X, Y) = 2g(\mathcal{W}X, Y).$$

The first and third equation can be found in this form in [**46**], 4.2. Theorem 3 and 4, respectively. From [**46**] 4.2 Theorem 2 we get the second equation,

$$\begin{aligned}
R(\xi, X, Y, \xi) &= -g(\mathcal{W}X, \mathcal{W}Y) - g((\nabla_\xi^g \mathcal{W})X, Y) \\
&= -g(\mathcal{W}X, \mathcal{W}Y) - \xi \cdot g(\mathcal{W}X, Y) + g(\mathcal{W}X, \nabla_\xi^g Y) + g(\nabla_\xi^g X, \mathcal{W}Y) \\
&= g(\mathcal{W}X, \mathcal{W}Y) - \frac{1}{2}\xi \cdot (\xi \cdot g(X, Y)) \\
&= g(\mathcal{W}X, \mathcal{W}Y) - \frac{1}{2}(L_\xi^2 g)(X, Y).
\end{aligned}$$

Now let $(\xi, E_2, .., E_n)$ be a local orthonormal basis with $\nabla^g_X E_i = 0$ at a fixed point. Then at this point

$$\begin{aligned}
\operatorname{ric}(X,\xi) &= \sum_{i=2}^{n} R(E_i, X, \xi, E_i) = -\sum_{i=2}^{n} R(E_i, \xi, E_i, X) \\
&= \sum_{i=2}^{n} -g_N((\nabla^{g_N}_X \mathcal{W}_N)E_i, E_i) + g_N((\nabla^{g_N}_{E_i} \mathcal{W}_N)X, E_i) \\
&= \sum_{i=2}^{n} -X \cdot g_N(\mathcal{W}_N E_i, E_i) + \operatorname{div}(\mathcal{W}_N)(X) \\
&= -X \cdot \operatorname{tr}(\mathcal{W}_N) + \operatorname{div}(\mathcal{W}_N)(X).
\end{aligned}$$

The remaining equations are obtained similarly.

$\square$

Lemma 4.2. Suppose φ is a k-form on M such that $\xi \lrcorner \varphi = 0$. Then $\varphi = \{\varphi_N\}$ and

$$d\varphi = \{d\varphi_N\} + \alpha \wedge L_\xi \varphi,$$

where $d\varphi_N$ denotes the exterior derivative of φ_N on the integral manifold $N \subset M$.

Proof: Fix $N_0 \subset M$ and choose local coordinates $\{v_2, .., v_n\}$ for N_0. The flow Φ_t of ξ defines a diffeomorphism

$$\Phi_t : N_0 \to N_t,$$

where $N_t := \Phi_t(N_0)$ is again an integral manifold of $\ker(\alpha)$. For

$$q \in U := \bigcup_{t \in (-\varepsilon, \varepsilon)} \Phi_t(N_0)$$

exists a unique $p \in N_0$ and $t \in (-\varepsilon, \varepsilon)$ such that $q = \Phi_t(p)$. Now we obtain coordinates on U by

$$u_1(\Phi_t(p)) := t \quad \text{and} \quad u_i(\Phi_t(p)) := v_i(p), \text{ for } i \geq 2,$$

with

$$\left.\frac{\partial}{\partial u_1}\right|_{\Phi_t(p)} = \frac{d}{ds} u^{-1}(t+s, v(p)) = \frac{d}{ds}\Phi_{t+s}(p) = \xi \circ \Phi_t(p).$$

Hence $\frac{\partial}{\partial u_1} \lrcorner \varphi = 0$ and computing $d\varphi$ at $p \in N_0$ yields

$$\begin{aligned}
d\varphi(p) &= \sum_{1 \notin J} \sum_{i=2}^{7} \left.\frac{\partial}{\partial u_i}\right|_p \cdot \varphi(\frac{\partial}{\partial u_J})(du_i \wedge du_J)(p) + \sum_{1 \notin J} \left.\frac{\partial}{\partial u_1}\right|_p \cdot \varphi(\frac{\partial}{\partial u_J})(du_1 \wedge du_J)(p) \\
&= \sum_{1 \notin J} \sum_{i=2}^{7} \left.\frac{\partial}{\partial v_i}\right|_p \cdot \varphi_{N_0}(\frac{\partial}{\partial v_J})(dv_i \wedge dv_J)(p) + \alpha(p) \wedge \sum_{1 \notin J} \xi|_p \cdot \varphi(\frac{\partial}{\partial u_J}) du_J(p) \\
&= (d\varphi_{N_0})(p) + \alpha(p) \wedge \sum_{1 \notin J} \xi|_p \cdot \varphi(\frac{\partial}{\partial u_J}) du_J(p),
\end{aligned}$$

where we used that the flow of $\frac{\partial}{\partial u_i}$ stays in N_0, for $i \geq 2$. Now

$$L_\xi du_i = d\xi \lrcorner du_i = d\frac{\partial}{\partial u_1} \lrcorner du_i = 0$$

gives

$$L_\xi \varphi = L_\xi (\sum_{1 \notin J} \varphi(\frac{\partial}{\partial u_J}) du_J) = \sum_{1 \notin J} \xi \cdot \varphi(\frac{\partial}{\partial u_J}) du_J,$$

i.e.

$$d\varphi(p) = (d\varphi_{N_0})(p) + \alpha(p) \wedge (L_\xi \varphi)(p).$$

$\square$

Let $I \subset \mathbb{R}$ be an open interval, $\{g_t\}_{t \in I}$ a family of metrics on N, $M := I \times N$ and Φ_t the flow of the vector field $\xi := d/dt$ on M. We extend the family $\{g_t\}_{t \in I}$ to a $(2,0)$-tensor $\{g_t\}$ on M by

$$\xi \lrcorner \{g_t\} = 0 \quad \text{and} \quad \{g_t\}(\Phi_{t*}X, \Phi_{t*}Y) := g_t(X, Y),$$

for all $X, Y \in TN \subset TM$. With this definition,

$$\begin{aligned} \{\dot{g}_t\}(\Phi_{t*}X, \Phi_{t*}Y) = \dot{g}_t(X,Y) = \frac{d}{dt} g_t(X,Y) = \frac{d}{dt}\{g_t\}(\Phi_{t*}X, \Phi_{t*}Y) \\ = \frac{d}{ds}\bigg|_{s=0} \{g_t\}(\Phi_{t+s*}X, \Phi_{t+s*}Y) = \frac{d}{ds}\bigg|_{s=0} (\Phi_s^*\{g_t\})(\Phi_{t*}X, \Phi_{t*}Y) \\ = (L_\xi\{g_t\})(\Phi_{t*}X, \Phi_{t*}Y), \end{aligned}$$

i.e.

$$\{\dot{g}_t\} = L_\xi\{g_t\}.$$

The Riemannian manifold

$$M := I \times N \quad \text{with} \quad g := dt^2 + \{g_t\}$$

is called a generalized cylinder.

Lemma 4.3. Let $M = I \times N$ be a generalized cylinder with metric $g = dt^2 + \{g_t\}$. Then (M, g) is complete if and only if $I = \mathbb{R}$ and (N, g_t) is complete, for all $t \in I$.

Proof: We use the Hopf-Rinow Theorem and denote by (M, d_g) the metric space associated to (M, g). If M is complete, we get $I = \mathbb{R}$, since $c(t) = (t, p) \in M$ is a geodesic. To see that (N, g_t) is complete, it suffices to show that any closed and bounded subset is compact. If $A \subset (N, d_{g_t})$ is closed and bounded, A is also closed and bounded as a subset of (M, d_g). Hence $A \subset M$ is compact and since $N \subset M$ is closed, we see that $A \subset N$ is compact.

Conversely, let (t_n, p_n) be a Cauchy sequence in the metric space (M, d_g). Then t_n defines a Cauchy sequence in $I = \mathbb{R}$ and hence we can assume that $t_n \to t$, as

$n \to \infty$. Now (t, p_n) is still a Cauchy sequence in $(\{t\} \times N, d_{g_t})$ and hence we can find a convergent subsequence of (t, p_n).

$\square$

The Weingarten map $\mathcal{W} := \nabla^g \xi$ of a generalized cylinder $(M, g) = (I \times N, dt^2 + \{g_t\})$ induces a family of g_t-symmetric endomorphisms $\mathcal{W}_t$ on TN by

$$\mathcal{W}_t X := \mathrm{pr}_* \circ \mathcal{W} \circ \Phi_{t*} X,$$

where $\mathrm{pr} : I \times N \to N$ is the canonical projection.

Lemma 4.4. Let g_t be a family of metrics on N. Denote by ric_t the Ricci tensor of the metric g_t and by ric the Ricci tensor of the metric $g = dt^2 + \{g_t\}$ on $I \times N$. Then

$$\begin{aligned} \dot{g}_t(X, Y) &= 2g_t(\mathcal{W}_t X, Y), \\ g_t(\dot{\mathcal{W}}_t X, Y) &= \mathrm{ric}_t(X, Y) - \mathrm{ric}(\Phi_{t*}X, \Phi_{t*}Y) - \mathrm{tr}(\mathcal{W}_t) g_t(\mathcal{W}_t X, Y), \end{aligned}$$

for all $X, Y \in TN$.

Proof: For $X, Y \in TN$ we compute

$$\begin{aligned} \dot{g}_t(X, Y) &= (L_\xi \{g_t\})(\Phi_{t*}X, \Phi_{t*}Y) = (L_\xi g)(\Phi_{t*}X, \Phi_{t*}Y) \\ &= 2g(\mathcal{W}\Phi_{t*}X, \Phi_{t*}Y) = 2g_t(\mathcal{W}_t X, Y). \end{aligned}$$

Similarly, $(L_\xi^2 g)(\Phi_{t*}X, \Phi_{t*}Y) = \ddot{g}_t(X, Y)$ and by Proposition 4.1

$$\begin{aligned} \mathrm{ric}(\Phi_{t*}X, \Phi_{t*}Y) &= \mathrm{ric}_t(X, Y) - \mathrm{tr}(\mathcal{W}_t) g_t(\mathcal{W}_t X, Y) \\ &\quad + 2g_t(\mathcal{W}_t X, \mathcal{W}_t Y) - \frac{1}{2}\ddot{g}_t(X, Y). \end{aligned}$$

Now the second equation follows from

$$\begin{aligned} \frac{1}{2}\ddot{g}_t(X, Y) &= \frac{1}{2}\frac{d}{dt}\dot{g}_t(X, Y) = \frac{d}{dt} g_t(\mathcal{W}_t X, Y) \\ &= \dot{g}_t(\mathcal{W}_t X, Y) + g_t(\dot{\mathcal{W}}_t X, Y) \\ &= 2g_t(\mathcal{W}_t X, \mathcal{W}_t Y) + g_t(\dot{\mathcal{W}}_t X, Y). \end{aligned}$$

$\square$

Like for a family of metrics, we can lift a family of forms $\{\varphi_t\}_{t \in I}$ on N to a single form $\{\varphi_t\}$ on $M = I \times N$ with $L_\xi\{\varphi_t\} = \{\dot{\varphi}_t\}$. Hence Lemma 4.2 translates into

Lemma 4.5.

$$d\{\varphi_t\} = \{d\varphi_t\} + dt \wedge \{\dot{\varphi}_t\}.$$

$\square$

For notational reasons we define

$$G_k := \begin{cases} SU(2) & \text{for } k = 5. \\ SU(3) & \text{for } k = 6. \\ G_2 & \text{for } k = 7. \\ \mathrm{Spin}(7) & \text{for } k = 8. \end{cases}$$

The inclusions $G_k \subset G_{k+1}$, obtained by regarding G_k as the isotropy group of a unit vector under the natural action of G_{k+1}, allow us to lift any G_k structure on M^k to a G_{k+1} structure on $\mathbb{R} \times M^k$. More generally, we can lift whole families of structures on M^k, parameterized by $t \in I$, to a structure on

$$M^{k+1} := I \times M^k.$$

In order for the resulting structure to be (nearly) parallel, the underlying family has to be (nearly) hypo and must evolve according to certain evolution equations. In fact, (nearly) hypo structures are precisely those type of structures which are induced on hypersurfaces by (nearly) parallel structures on the ambient space.

For instance, a family of G_2-structures ψ_t on M^7 defines a 4-form $\Psi := \{\psi_t\} + dt \wedge \{\varphi_t\}$ of model tensor type Ψ_0, and hence a Spin(7)-structure on $M^8 := I \times M^7$. For notational simplicity we will suppress the bracket notation and call

$$\Psi := \psi_t + dt \wedge \varphi_t$$

the lift of ψ_t to $I \times M^7$. With Lemma 4.5 we get

$$d^8\Psi = d^7\psi_t + dt \wedge \dot{\psi}_t - dt \wedge d^7\varphi_t = d^7\psi_t + dt \wedge (\dot{\psi}_t - d^7\varphi_t),$$

where d^7, d^8 denotes the exterior derivative on M^7, M^8, respectively. Hence the Spin(7)-structure is parallel if and only if $d^7\psi_t = 0$ and $\dot{\psi}_t = d\varphi_t$. The second equation can be regarded as an evolution equation for the initial structure $\varphi := \varphi_{t=0}$. If the initial structure is hypo, the evolution equation guarantees that the hypo condition $d^7\psi_t = 0$ is preserved in time. In the following Proposition we list the lifting maps for the $SU(2)$, $SU(3)$ and G_2-case, the (nearly) hypo condition for the initial structure and the evolution equations to obtain (nearly) parallel structures on $I \times M^k$.

PROPOSITION 4.6. Let M^k be a manifold of dimension $k \in \{5, 6, 7\}$, equipped with a family of G_k-structures. Then the lift in the following table defines a G_{k+1}-structure on $M^{k+1} := I \times M^k$.

k	Lift	Initial Condition	Evolution
5	$\omega := \omega_1 + dt \wedge \alpha$ $\sigma := \frac{1}{2}\omega_1^2 + dt \wedge \alpha \wedge \omega_1$ $\rho := -\rho_3 + dt \wedge \omega_2$ $\widehat{\rho} := \rho_2 + dt \wedge \omega_3$	$0 = d\omega_1 + 6\lambda\rho_3$ $0 = d\rho_2 + 4\lambda\omega_1^2$ $0 = d\rho_3$	$\dot{\omega}_1 = d\alpha + 6\lambda\omega_2$ $\dot{\rho}_2 = d\omega_3 - 8\lambda\alpha \wedge \omega_1$ $\dot{\rho}_3 = -d\omega_2$
6	$\varphi := \rho + dt \wedge \omega$ $\psi := \sigma - dt \wedge \widehat{\rho}$	$0 = d\rho - \lambda\sigma$ $0 = d\sigma$	$\dot{\rho} = d\omega - \lambda\widehat{\rho}$ $\dot{\sigma} = -d\widehat{\rho}$
7	$\Psi := \psi + dt \wedge \varphi$	$0 = d\psi$	$\dot{\psi} = d\varphi$

(1) The structure on M^{k+1} is parallel if and only if the initial structure is hypo (i.e. $\lambda = 0$) and evolves according to the evolution equations from the table.

(2) The structure on M^{k+1} is nearly parallel if and only if the initial structure is nearly hypo (i.e. $\lambda \neq 0$) and evolves according to the evolution equations from the table.

(3) The metric of the G_{k+1}-structure on $I \times M^k$ is given by

$$g = dt^2 + g_t,$$

where g_t is the family of metrics induced by the family of G_k-structures on M^k.

PROOF: Choosing a Cayley frame $(E_2(t), .., E_k(t))$ for the family of G_k-structures, we obtain a Cayley frame for the lift by

$$(\frac{d}{dt}, E_2(t), .., E_k(t)).$$

This proves that the lift actually defines a G_{k+1}-structure and that the metric is given by the formula in (3). Since we already proved the case $k = 7$, we only have to consider the cases $k = 5$ and $k = 6$:

k=5: By Lemma 4.5 we have

$$d\omega = d\omega_1 + dt \wedge (\dot{\omega}_1 - d\alpha),$$
$$d\rho = -d\rho_3 - dt \wedge (\dot{\rho}_3 + d\omega_2),$$
$$d\widehat{\rho} = d\rho_2 + dt \wedge (\dot{\rho}_2 - d\omega_3),$$

and we see that the $SU(3)$-structure is parallel, i.e. $d\omega = d\rho = d\widehat{\rho} = 0$, if and only if the whole family of $SU(2)$-structures is hypo and satisfies the evolution equations from the table with $\lambda = 0$. Since the evolution equations preserve the hypo condition, it suffices to require the initial $SU(2)$-structure to be hypo. The $SU(3)$-structure is nearly parallel if and only if

$$d\omega = 6\lambda\rho = -6\lambda\rho_3 + 6\lambda dt \wedge \omega_2 \quad \text{and} \quad d\widehat{\rho} = -4\lambda\omega^2 = -4\lambda\omega_1^2 - 8\lambda dt \wedge \alpha \wedge \omega_1,$$

for some $\lambda \neq 0$. Hence it suffices to show that the evolution equations preserve $0 = d\omega_1 + 6\lambda\rho_3$ and $0 = d\rho_2 + 4\lambda\omega_1^2$. This follows from

$$\frac{d}{dt}(d\omega_1 + 6\lambda\rho_3) = 6\lambda d\omega_2 - 6\lambda\omega_2 = 0$$

and, using $\omega_1 \wedge \omega_2 = 0$,

$$\frac{d}{dt}(d\rho_2 + 4\lambda\omega_1^2) = -8\lambda d(\alpha \wedge \omega_1) + 8\lambda(\omega_1 \wedge d\alpha) = 8\lambda\alpha \wedge d\omega_1 = -64\lambda^2\alpha \wedge \rho_3 = 0.$$

k=6: By Lemma 4.5 we have

$$d\varphi = d\rho + dt \wedge (\dot{\rho} - d\omega),$$
$$d\psi = d\sigma + dt \wedge (\dot{\sigma} + d\widehat{\rho}).$$

Hence the G_2-structure is parallel, i.e. $d\varphi = d\psi = 0$, if and only if the whole family of $SU(3)$-structures is hypo and satisfies the evolution equations from the table with $\lambda = 0$. Since the evolution equations preserve the hypo condition, it suffices to require the initial $SU(3)$-structure to be hypo. The G_2-structure is nearly parallel if and only if

$$d\varphi = \lambda\psi = \lambda\sigma - \lambda dt \wedge \widehat{\rho},$$

for some $\lambda \neq 0$. Since the evolution equations imply

$$\frac{d}{dt}(d\rho - \lambda\sigma) = -\lambda d\widehat{\rho} + \lambda d\widehat{\rho} = 0,$$

the Proposition follows.

$\square$

Definition 4.7. Let M^k be a manifold of dimension $k \in \{5, 6, 7\}$, equipped with a (nearly) hypo G_k-structure. A family of G_k-structures which solves the evolution equations from Proposition 4.6 and equals the initial structure at $t = 0$ is called a solution of the embedding problem for the initial G_k-structure.

The Hypo Lift

The lift from Proposition 4.6 does not preserve the hypo condition. This motivates

Definition 4.8. Let M^k be a manifold of dimension $k \in \{5, 6\}$, equipped with a G_k-structure. We call

$k=5$	$k=6$
$\omega := \omega_3 + d\theta \wedge \alpha$	$\varphi := -\widehat{\rho} + d\theta \wedge \omega$
$\sigma := \frac{1}{2}\omega_3^2 + d\theta \wedge \rho_3$	$\psi := \sigma - d\theta \wedge \rho$
$\rho := \rho_2 - d\theta \wedge \omega_1$	
$\widehat{\rho} := -\alpha \wedge \omega_1 - d\theta \wedge \omega_2$	

the hypo lift of the G_k-structure to $S^1 \times M^k$. Conversely, given a G_{k+1}-structure on a manifold M^{k+1} of dimension $k+1$, we obtain a G_k-structure on every oriented hypersurface $i : M^k \hookrightarrow M^{k+1}$ by

$k=5$	$k=6$
$\omega_1 := -i^*(\frac{\partial}{\partial\theta} \lrcorner \rho)$	$\rho := -i^*(\frac{\partial}{\partial\theta} \lrcorner \psi)$
$\rho_2 := i^*\rho$	$\sigma := i^*\psi$
$\rho_3 := i^*(\frac{\partial}{\partial\theta} \lrcorner \sigma)$	

where $\frac{\partial}{\partial\theta}$ is a global vector field along $i : M^k \hookrightarrow M^{k+1}$, which is orthonormal to M^k. We call the G_k-structure the structure induced by the G_{k+1}-structure and $\frac{\partial}{\partial\theta}$.

Note that we just applied the lifts from Proposition 4.6 to the structures

$$(\alpha, \omega_3, -\omega_1, -\omega_2) = A(\alpha, \omega_1, \omega_2, \omega_3),$$

respectively,

$$(\omega, -\widehat{\rho}, \rho) = I(\omega, \rho, \widehat{\rho}),$$

where $A \in GL^+(5)$ is defines by

$$A(e_1, .., e_5) := (e_1, e_3, e_4, e_2, e_5).$$

Lemma 4.9. The hypo lift maps hypo structures to hypo structures.

Proof: In the $SU(2)$-case, we obtain $d\rho = 0$ if $d\omega_1 = d\rho_2 = 0$. The compatibility condition $\omega_3^2 = \omega_1^2$ and $d\rho_3 = 0$ imply $d\sigma = 0$. For a hypo $SU(3)$-structure we obtain immediately $d\psi = d\sigma + d\theta \wedge d\rho = 0$.

□

We will now study the compatibility of the hypo lift with the evolution equations from Proposition 4.6.

Lemma 4.10. **(1)** Suppose ψ is a family of G_2-structures on $M^7 = S^1 \times M^6$ which is the hypo lift of some family of $SU(3)$-structure (ρ, σ) on M^6. Then

$$\dot{\psi} = d\varphi \quad \Leftrightarrow \quad \begin{cases} \dot{\rho} = d\omega \\ \dot{\sigma} = -d\widehat{\rho} \end{cases}$$

(2) Suppose (ρ, σ) is a family of $SU(3)$-structures on $M^6 = S^1 \times M^5$ which is the hypo lift of some family of $SU(2)$-structure $(\omega_1, \rho_2, \rho_3)$ on M^5. Then

$$\left.\begin{array}{l} \dot{\rho} = d\omega \\ \dot{\sigma} = -d\widehat{\rho} \end{array}\right\} \quad \Leftrightarrow \quad \begin{cases} \dot{\omega}_1 = d\alpha \\ \dot{\rho}_2 = d\omega_3 \\ \dot{\rho}_3 = -d\omega_2 \\ (\frac{1}{2}\omega_3^2)^{\cdot} = d(\alpha \wedge \omega_1) \end{cases}$$

Proof: By assumption we have $\psi = \sigma - d\theta \wedge \rho$ and $\varphi = -\widehat{\rho} + d\theta \wedge \omega$. Hence

$$\dot{\psi} = \dot{\sigma} - d\theta \wedge \dot{\rho} \quad \text{and} \quad d\varphi = -d\widehat{\rho} - d\theta \wedge d\omega$$

and part (1) follows. Similarly for part (2),

$$\begin{aligned} \omega &= \omega_3 + d\theta \wedge \alpha, \qquad & \sigma &= \tfrac{1}{2}\omega_3^2 + d\theta \wedge \rho_3, \\ \rho &= \rho_2 - d\theta \wedge \omega_1, \qquad & \widehat{\rho} &= -\alpha \wedge \omega_1 - d\theta \wedge \omega_2 \end{aligned}$$

gives

$$\begin{aligned} \dot{\rho} &= \dot{\rho}_2 - d\theta \wedge \dot{\omega}_1, \\ d\omega &= d\omega_3 - d\theta \wedge d\alpha, \end{aligned}$$

and

$$\begin{aligned} \dot{\sigma} &= (\frac{1}{2}\omega_3^2)^{\cdot} + d\theta \wedge \dot{\rho}_3, \\ -d\widehat{\rho} &= d(\alpha \wedge \omega_1) - d\theta \wedge d\omega_2. \end{aligned}$$

□

Lemma 4.11. Let ψ be a G_2-structure on M^7 with metric g.
(1) If $M^7 = S^1 \times M^6$, then ψ is the hypo lift of some $SU(3)$-structure on M^6 if and only if

$$L_{\frac{\partial}{\partial\theta}}\psi = 0, \quad \frac{\partial}{\partial\theta} \perp_g TM^6 \quad \text{and} \quad g(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) = 1.$$

(2) If $M^7 = S^1_2 \times S^1_1 \times M^5$, then ψ is the hypo lift of some $SU(2)$-structure on M^5 if and only if

$$L_{\frac{\partial}{\partial\theta_i}}\psi = 0, \quad \frac{\partial}{\partial\theta_i} \perp_g TM^5 \quad \text{and} \quad g(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}) = \delta_{ij},$$

for $i, j = 1, 2$.

Proof: If ψ is the hypo lift of some $SU(2)$ or $SU(3)$-structure, we get $L_{\frac{\partial}{\partial\theta_i}}\psi = 0$ and the orthogonality condition on the S^1-directions. Conversely, we define forms σ and ρ on M^7 by

$$\psi = \underbrace{\frac{\partial}{\partial\theta}\lrcorner(d\theta\wedge\psi)}_{=:\sigma} + d\theta\wedge\underbrace{(\frac{\partial}{\partial\theta}\lrcorner\psi)}_{=:-\rho}.$$

Since $\frac{\partial}{\partial\theta}$ is orthonormal to M^6 and G_2 acts transitively on S^6, we can find a Caley frame for which σ and ρ are of type σ_0 and ρ_0. Hence (σ,ρ) defines a $SU(3)$-structure on each hypersurface $\{e^{i\theta}\}\times M^6$. Since

$$0 = L_{\frac{\partial}{\partial\theta}}\sigma - d\theta\wedge L_{\frac{\partial}{\partial\theta}}\rho$$

implies $L_{\frac{\partial}{\partial\theta}}\sigma = L_{\frac{\partial}{\partial\theta}}\rho = 0$, we see that σ and ρ are actually constant along the flow of $\frac{\partial}{\partial\theta}$. Part (2) of the Lemma follows similarly, using that G_2 acts transitively on pairs of orthonormal vectors.

□

The Model Case $G_2 \subset$ Spin(7)

Lemma 4.10 and 4.11 motivate the conjecture that the embedding problem for hypo $SU(2)$ and $SU(3)$-structures might be reduced to the embedding problem for G_2-structures. The reduction to the G_2 case has the advantage that no compatibility conditions are involved. To solve the embedding problem for hypo structures we consequently focus on studying the evolution equation

$$\dot{\psi}_t = d\varphi_t$$

on a compact seven dimensional manifold M^7. Motivated by Theorem 1.6 we try to find a solution of the form

$$\psi_t := A_t\psi,$$

where ψ is the initial hypo G_2-structure and $A_t \in C^\infty(\mathrm{Aut}(TM))$ is a family of gauge deformations with $A_0 = \mathrm{id}$. First we translate the above evolution equation into an equation for the family of gauge deformations.

Proposition 4.12. Suppose $\psi_t = A_t\psi$ is a family of G_2-structures on M^7, described by a family of gauge deformations $A_t \in C^\infty(\mathrm{Aut}(TM^7))$. If $\mathcal{T}_t = \mathcal{T}(A_t\psi)$ is the torsion endomorphism of ψ_t, then

$$\dot{\psi}_t = d\varphi_t \quad\Leftrightarrow\quad D_{\psi_t}(\dot{A}_t\circ A_t^{-1}) = 3D_{\psi_t}(\mathcal{T}_t).$$

Proof: By Lemma 1.16 and Proposition 3.38 we have

$$\dot{\psi}_t = D_{\psi_t}(\dot{A}_t \circ A_t^{-1}) \quad \text{and} \quad d\varphi_t = 3D_{\psi_t}(\mathcal{T}_t)$$

and the Proposition follows.

□

We can now compute the evolution of the metric and the torsion endomorphism.

Theorem 4.13. Let ψ_t be a family of hypo G_2-structures on M^7, which evolves under the flow $\dot{\psi}_t = d\varphi_t$. Then the evolution of the underlying metric g_t and the torsion endomorphism $\mathcal{T}_t$ are given by

$$\begin{aligned} \dot{g}_t(X,Y) &= -6g_t(\mathcal{T}_tX, Y), \\ \dot{\mathcal{T}}_tX &= -\frac{1}{3}\mathrm{Ric}_tX + 3\mathrm{tr}(\mathcal{T}_t)\mathcal{T}_tX, \end{aligned}$$

where $\mathrm{Ric}_t = \mathrm{Ric}(g_t)$ is the Ricci tensor of the metric g_t.

Proof: By Theorem 1.6 we can describe the evolution by a family of gauge deformations $\psi_t = A_t\psi$ and Proposition 4.12 yields $D_{\psi_t}(\dot{A}_t \circ A_t^{-1}) = 3D_{\psi_t}(\mathcal{T}_t)$. Since the evolution $\dot{\psi}_t = d\varphi_t$ preserves the hypo condition $d\psi_t = 0$, or equivalently $\mathcal{T}_t \in S^2$ w.r.t. g_t, we get from Lemma 3.36

$$\mathrm{pr}_{S^2}(\dot{A}_t \circ A_t^{-1}) = 3\mathcal{T}_t.$$

Now Lemma 1.16 gives

$$\dot{g}_t(X,Y) = D_{g_t}(\dot{A}_t \circ A_t^{-1})(X,Y) = -2g_t(\mathrm{pr}_{S^2}(\dot{A}_t \circ A_t^{-1})X, Y) = -6g_t(\mathcal{T}_tX, Y).$$

By Lemma 4.4 we see that $\mathcal{W}_t = -3\mathcal{T}_t$ and hence

$$-3g_t(\dot{\mathcal{T}}_tX, Y) = \mathrm{ric}_t(X,Y) - 9\mathrm{tr}(\mathcal{T}_t)g_t(\mathcal{T}_tX, Y),$$

where we used that the metric $g = dt^2 + g_t$ on $I \times M^7$ has holonomy contained in $\mathrm{Spin}(7)$ and hence is Ricci flat.

□

The following theorem shows that the flow will not produce complete metrics with special holonomy. In particular we can not expect to obtain periodic solutions which would lead to compact manifolds with special holonomy. In fact, the observation is that the length of the existence interval could be characteristic for the type of the initial structure.

Theorem 4.14. Suppose ψ is a hypo G_2-structures on a compact manifold M^7. Then the flow $\dot{\psi}_t = d\varphi_t$ is defined for all times $t \in \mathbb{R}$ if and only if the initial structure is already parallel.

PROOF: The metric on the product $M^8 := \mathbb{R} \times M^7$ has holonomy contained in Spin(7) and hence is Ricci flat. Since $g = dt^2 + g_t$, the first factor actually defines a line and M^8 is complete by Lemma 4.3. Now we can apply the Cheeger-Gromoll Splitting Theorem and see that M^8 splits as a Riemannian product. Note that the line, i.e. the first factor of M^8, is actually the one dimensional factor that splits off in the decomposition as a Riemannian product, cf. Lemma 6.86 in [**7**]. Hence $g_t = g_0$ is constant and Theorem 4.13 yields $\mathcal{T}_t = 0$.

$\square$

In Lemma 4.10 (1) we showed that a longtime solution of the $SU(3)$ embedding problem would yield a longtime solution for the G_2 embedding problem. Combining part (1) and (2) of Lemma 4.10, shows that a longtime solution of the $SU(2)$ embedding problem would also yield a longtime solution for the G_2 embedding problem if in addition the equation $(\frac{1}{2}\omega_3^2)^\cdot = d(\alpha \wedge \omega_1)$ is satisfied. If the initial $SU(2)$-structure is hypo, we have $d\omega_1 = 0$, for all times t. So

$$(\frac{1}{2}\omega_3^2)^\cdot = (\frac{1}{2}\omega_1^2)^\cdot = \omega_1 \wedge \dot{\omega}_1 = \omega_1 \wedge d\alpha = d(\alpha \wedge \omega_1)$$

and we obtain the following $SU(2)$ and $SU(3)$-analogue of Theorem 4.14.

COROLLARY 4.15. There are no non-trivial longtime solutions for the hypo $SU(2)$ and $SU(3)$ embedding problem on compact manifolds.

$\square$

In the nearly hypo case we can give a similar argument to show that there are no non-trivial longtime solutions of the embedding problem. Namely such a solution would yield a complete metric on the non-compact manifold $\mathbb{R} \times M$ with positive Ricci curvature, which contradicts Myer's Theorem.

In view of Proposition 4.12, the following theorem yields solutions of the G_2 embedding problem.

THEOREM 4.16. Let ψ be a real analytic hypo G_2-structure on the compact manifold M^7. Then the intrinsic torsion flow

$$\begin{cases} \dot{A}_t = 3\mathcal{T}_t \circ A_t \\ A_0 = \mathrm{id} \end{cases}$$

has a unique real analytic solution $A : (-\varepsilon, \varepsilon) \times M \to \mathrm{End}(TM)$. Moreover, the solution A_t is of the form

$$A_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_0^{(k)},$$

where the series converges in the C^∞-topology on $C^\infty(\mathrm{End}(TM))$.

PROOF: To apply Theorem 2.11 we have to show that the map

$$X : C^\infty(\mathrm{Aut}(TM)) \to C^\infty(\mathrm{End}(TM)) \quad \text{with} \quad X \circ A := 3\mathcal{T}(A\varphi) \circ A$$

is a real analytic first order differential operator in the sense of Definition 2.10. For this choose local coordinates $u : U \subset M \to \mathbb{R}^7$, for which φ is real analytic. These coordinates induce a local trivialization (π, v) of the bundle $\pi : \mathrm{End}(TM) \to M$ via

$$v(A) := \{a_{kl}\}_{k,l=1..7}, \quad \text{where} \quad A = \sum_{k,l=1}^{7} a_{kl} du_k \otimes \frac{\partial}{\partial u_l} \in \mathrm{End}(TM).$$

For a fixed local section $A = \sum a_{kl} du_k \otimes \frac{\partial}{\partial u_l} : U \to \mathrm{Aut}(TM)$ write

$$X \circ A = 3\mathcal{T}(A\varphi) \circ A = \sum_{a,b=1}^{7} f_{ab} du_a \otimes \frac{\partial}{\partial u_b}.$$

Now it suffices to find an expression

$$f_{ab} = G_{ab}(u, a_{kl}, \frac{\partial a_{kl}}{\partial u_j}) \tag{1}$$

for the coefficients $f_{ab} : U \to \mathbb{R}$, where $G_{ab} : D \subset \mathbb{R}^7 \times \mathbb{R}^{49} \times \mathbb{R}^{343} \to \mathbb{R}$ is real analytic. The formula

$$\nabla^{Ag} A\varphi = -3\mathcal{T}(A\varphi) \lrcorner (A\psi)$$

from Proposition 3.38 shows that the intrinsic torsion is a first order invariant of the G_2-structure and hence we can find an expression of the form (1) that is actually polynomial in a_{kl} and $\frac{\partial a_{kl}}{\partial u_j}$, and real analytic in u, since the initial structure is real analytic.

$\square$

LEMMA 4.17. Suppose ψ is a G_2-structure on M and $F \in \mathrm{Diff}(M)$. Then the intrinsic torsion satisfies

$$\mathcal{T}(F^*\psi) = F^*\mathcal{T}(\psi) = F_*^{-1}\mathcal{T}(\psi)F_*.$$

PROOF: By Koszul's formula we have

$$F_*(\nabla_X^{F^*g} Y) = \nabla_{F_*X}^{g} F_*Y.$$

Hence we get

$$
\begin{aligned}
&(\nabla_X^{F^*g} F^*\psi)(X_1, X_2, X_3, X_4)\\
&= X \cdot (F^*\psi)(X_1, X_2, X_3, X_4)\\
&\quad - (F^*\psi)(\nabla_X^{F^*g} X_1, X_2, X_3, X_4) - (F^*\psi)(X_1, \nabla_X^{F^*g} X_2, X_3, X_4)\\
&\quad - (F^*\psi)(X_1, X_2, \nabla_X^{F^*g} X_3, X_4) - (F^*\psi)(X_1, X_2, X_3, \nabla_X^{F^*g} X_4)\\
&= F_*X \cdot \psi(F_*X_1, F_*X_2, F_*X_3, F_*X_4)\\
&\quad - \psi(\nabla^g_{F_*X} F_*X_1, F_*X_2, F_*X_3, F_*X_4) - \psi(F_*X_1, \nabla^g_{F_*X} F_*X_2, F_*X_3, F_*X_4)\\
&\quad - \psi(F_*X_1, F_*X_2, \nabla^g_{F_*X} F_*X_3, F_*X_4) - \psi(F_*X_1, F_*X_2, F_*X_3, \nabla^g_{F_*X} F_*X_4)\\
&= (\nabla^g_{F_*X}\psi)(F_*X_1, F_*X_2, F_*X_3, F_*X_4)\\
&= F^*(\nabla^g_{F_*X}\psi)(X_1, X_2, X_3, X_4).
\end{aligned}
$$

From Proposition 3.38 we know that $\nabla^g_X\psi = 3(\mathcal{T}(\psi)\lrcorner g) \wedge \varphi$ holds, which gives

$$
\begin{aligned}
3(\mathcal{T}(F^*\psi)X\lrcorner F^*g) \wedge F^*\varphi = \nabla_X^{F^*g}(F^*\psi) &= F^*(\nabla^g_{F_*X}\psi)\\
&= 3F^*((\mathcal{T}(\psi)F_*X\lrcorner g) \wedge \varphi)\\
&= 3(F_*^{-1}\mathcal{T}(\psi)F_*X\lrcorner F^*g) \wedge F^*\varphi
\end{aligned}
$$

and the Lemma follows from the non-degeneracy of $F^*\varphi$.

□

Lemma 4.18. Suppose ψ is a G_2-structure on $M^7 = S^1 \times .. \times S^1 \times M^{7-k}$, which is the hypo lift of some $SU(4-k)$-structure on M^{7-k}. Then the Ricci tensor Ric of the metric $g = g(\psi)$ satisfies for each S^1-direction $\frac{\partial}{\partial\theta}$

$$L_{\frac{\partial}{\partial\theta}}\mathrm{Ric} = \mathrm{Ric}\frac{\partial}{\partial\theta} = d\theta \circ \mathrm{Ric} = 0.$$

The intrinsic torsion $\mathcal{T}$ satisfies

$$L_{\frac{\partial}{\partial\theta}}\mathcal{T} = \mathcal{T}\frac{\partial}{\partial\theta} = 0$$

and $d\theta \circ \mathcal{T} = 0$ if the structure is hypo.

Proof: If ψ is the hypo lift of some structure on M^{7-k}, then $g = d\theta_1^2 + .. + d\theta_k^2 + g_{7-k}$, for some metric g_{7-k} on M^{7-k}. Hence the Ricci tensor satisfies $\mathrm{Ric}\frac{\partial}{\partial\theta} = 0$,

$$d\theta \circ \mathrm{Ric} = g(\frac{\partial}{\partial\theta}, \mathrm{Ric}) = g(\mathrm{Ric}\frac{\partial}{\partial\theta}, .) = 0$$

and

$$L_{\frac{d}{d\theta}}\mathrm{Ric} = \frac{d}{ds}\bigg|_{s=0} \Phi_s^*\mathrm{Ric}(g) = \frac{d}{ds}\bigg|_{s=0} \mathrm{Ric}(\Phi_s^*g) = \frac{d}{ds}\bigg|_{s=0} \mathrm{Ric}(g) = 0.$$

From Proposition 3.2 and $L_{\frac{\partial}{\partial\theta}}\varphi = \nabla^g\frac{\partial}{\partial\theta} = 0$, we get $\tau(\frac{\partial}{\partial\theta}) = 0$, i.e. $\mathcal{T}\frac{\partial}{\partial\theta} = 0$. Lemma 4.17 and $L_{\frac{\partial}{\partial\theta}}\varphi = 0$ imply $L_{\frac{\partial}{\partial\theta}}\mathcal{T} = 0$. If the structure is hypo, i.e. $\mathcal{T}$ is symmetric, we get in addition

$$d\theta \circ \mathcal{T} = g(\frac{\partial}{\partial\theta}, \mathcal{T}) = g(\mathcal{T}\frac{\partial}{\partial\theta}, .) = 0.$$

$\square$

Lemma 4.19. Suppose ψ is a G_2-structure on $M^7 = S^1 \times .. \times S^1 \times M^{7-k}$, which is the hypo lift of some $SU(4-k)$-structure on M^{7-k}. If $A \in C^\infty(\mathrm{Aut}(TM))$ satisfies

$$A\frac{\partial}{\partial\theta_i} = \frac{\partial}{\partial\theta_i}, \quad d\theta_i \circ A = d\theta_i \quad \text{and} \quad L_{\frac{\partial}{\partial\theta_i}} A = 0,$$

then $A\psi$ is still the hypo lift of some $SU(4-k)$-structure.

Proof: By Lemma 4.11 we have $L_{\frac{\partial}{\partial\theta_i}}(A\psi) = 0$ and

$$(Ag)(\frac{\partial}{\partial\theta_i}, X) = g(\frac{\partial}{\partial\theta_i}, A^{-1}X) = d\theta_i(A^{-1}X) = d\theta_i(X) = g(\frac{\partial}{\partial\theta_i}, X).$$

Now the Lemma follows from Lemma 4.11.

$\square$

We can now state the main result of this section.

Theorem 4.20. Suppose ψ is a real analytic hypo G_2-structure on $M = S^1 \times .. \times S^1 \times M^{7-k}$, which is the hypo lift of some $SU(4-k)$-structure on M^{7-k}. Then the solution A_t of the intrinsic torsion flow from Theorem 4.16 satisfies

$$A_t\frac{\partial}{\partial\theta_i} = \frac{\partial}{\partial\theta_i}, \quad d\theta_i \circ A_t = d\theta_i \quad \text{and} \quad L_{\frac{\partial}{\partial\theta_i}} A_t = 0.$$

In particular, $A_t\psi$ is the hypo lift of some family of $SU(4-k)$-structures on M^{7-k}.

Proof: We apply Corollary 2.4 with the following dictionary,

(1) $\mathcal{F} := C^\infty(\mathrm{End}(TM)) \times C^\infty(\mathrm{End}(TM))$

(2) $\mathcal{U} := C^\infty(\mathrm{Aut}(TM)) \times C^\infty(\mathrm{End}(TM))$

(3) $\mathcal{E} := \{(B, \mathcal{T}) \in \mathcal{F} \mid 0 = L_{\frac{\partial}{\partial\theta_i}} B = L_{\frac{\partial}{\partial\theta_i}} \mathcal{T}$ and
$0 = B\frac{\partial}{\partial\theta_i} = \mathcal{T}\frac{\partial}{\partial\theta_i} = d\theta_i(B) = d\theta_i(\mathcal{T})\}$

(4) $X : \mathcal{U} \to \mathcal{F}$ is defined w.r.t. the initial metric g,
$X|_{(A,\mathcal{T})} := \big(3\mathcal{T} \circ A, -\frac{1}{3}\mathrm{Ric}(Ag) + 3\mathrm{tr}(\mathcal{T})\mathcal{T}\big)$.

(5) $c(t) := (A_t, \mathcal{T}_t)$.

Note that $\mathcal{U} \subset \mathcal{F}$ is open by Example 2.1. X is smooth and $\mathcal{E} \subset \mathcal{F}$ is closed, since differential operators are smooth by Example 3.6.6. in [**34**]. By Proposition 4.12, Theorem 4.13 and the definition of A_t, the curve $c(t)$ is an integral curve of the vector field X. From Lemma 4.18 we get $c(0) = (\mathrm{id}, \mathcal{T}_0) \in \mathcal{E}_f$, where $f := (\mathrm{id}, 0) \in \mathcal{F}$. Now it suffices to show that X is tangent to $\mathcal{U} \cap \mathcal{E}_f$, i.e.

$$X_{|\mathcal{U}\cap\mathcal{E}_f} : \mathcal{U} \cap \mathcal{E}_f \to \mathcal{E}.$$

For $(A = \mathrm{id} + B, \mathcal{T}) \in \mathcal{U} \cap \mathcal{E}_f$ we have

$$A\frac{\partial}{\partial\theta_i} = \frac{\partial}{\partial\theta_i}, \quad d\theta_i \circ A = d\theta_i \quad \text{and} \quad L_{\frac{\partial}{\partial\theta_i}} A = 0.$$

By Lemma 4.19 we see that $A\psi$ is still the hypo lift of some $SU(4-k)$-structure and Lemma 4.18 yields

$$L_{\frac{\partial}{\partial\theta_i}} \mathrm{Ric}(Ag) = \mathrm{Ric}(Ag)\frac{\partial}{\partial\theta_i} = d\theta_i \circ \mathrm{Ric}(Ag) = 0.$$

Now we can easily verify that $X(A, \mathcal{T}) \in \mathcal{E}$,

- $L_{\frac{\partial}{\partial\theta_i}}(\mathcal{T} \circ A) = 0$ and $L_{\frac{\partial}{\partial\theta_i}}(-\frac{1}{3}\mathrm{Ric}(Ag) + 3\mathrm{tr}(\mathcal{T})\mathcal{T}) = 0$,
- $\mathcal{T} \circ A\frac{\partial}{\partial\theta_i} = 0$ and $(-\frac{1}{3}\mathrm{Ric}(Ag) + 3\mathrm{tr}(\mathcal{T})\mathcal{T})\frac{\partial}{\partial\theta_i} = 0$,
- $d\theta_i(\mathcal{T} \circ A) = 0$ and $d\theta_i(-\frac{1}{3}\mathrm{Ric}(Ag) + 3\mathrm{tr}(\mathcal{T})\mathcal{T}) = 0$

and the Theorem follows.

□

Remark 4.21. The property $L_{\frac{\partial}{\partial\theta}} A_t = 0$ from Theorem 4.16 is a consequence of the diffeomorphism invariance of the evolution equation $\dot{A}_t = 3\mathcal{T}_t \circ A_t$. In fact, Lemma 4.17 shows that $B_t := \Phi_s^* A_t$ also solves $\dot{A}_t = 3\mathcal{T}_t \circ A_t$, where Φ_s is the flow of $\frac{\partial}{\partial\theta}$. Since Φ_s is real analytic, the uniqueness part of Theorem 4.16 yields $A_t = \Phi_s^* A_t$, i.e. $L_{\frac{\partial}{\partial\theta}} A_t = 0$.

We can now solve the embedding problem for real analytic hypo $SU(4-k)$-structures on M^{7-k} by reducing it to the embedding problem for real analytic hypo G_2-structures on $M = S^1 \times .. \times S^1 \times M^{7-k}$. Namely, the hypo lift of the initial $SU(4-k)$-structure yields a real analytic hypo G_2-structures on M. Theorem 4.16 yields a solution A_t of the intrinsic torsion flow. By Theorem 4.20 the family of G_2-structures $\psi_t = A_t\psi$ is still the hypo lift of some family of $SU(4-k)$-structures. Now Lemma 4.10 proves that the family of $SU(4-k)$-structures is a solution of the embedding problem.

Corollary 4.22. For any real analytic hypo $SU(2)$, $SU(3)$ and G_2-structure on a compact manifold, the embedding problem admits a unique real analytic solution.

Moreover, the solution can be described by a family of gauge deformations

$$A_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_0^{(k)},$$

where the series converges in the C^∞-topology on $C^\infty(\mathrm{End}(TM))$.

An alternative to solve the G_2-embedding problem is to apply the Cauchy Kowalevski Theorem 2.11 directly to the initial value problem $\dot{\psi}_t = d\varphi_t$ with $\psi_0 = \psi$. To obtain a solution for the $SU(2)$ and $SU(3)$ embedding problem, it suffices to prove that the family of metrics $g_t = g(\psi_t)$ leaves S^1-directions orthonormal, cf. Lemma 4.11. With Lemma 3.29 this condition can be translated into

$$\frac{\partial}{\partial \theta} \lrcorner \varphi_t \wedge X \lrcorner \varphi_t \wedge \varphi_t = \frac{6}{7} d\theta(X) \varphi_t \wedge \psi_t.$$

But this condition is nonlinear and hence we can not apply Corollary 2.4, which was tailor-made to prove that certain linear conditions are preserved. Considering instead the system 4.13 allows us to express the requirement on the S^1-directions in terms of the linear condition $\mathcal{T}_t \frac{\partial}{\partial \theta} = 0$.

The Nearly Hypo Case

In this section we study the embedding problem for nearly hypo $SU(2)$ and $SU(3)$-structures. Like in the hypo case, one would expect that the nearly hypo evolution equations for $SU(3)$-structures correspond under the hypo lift to the nearly hypo evolution equations for $SU(2)$-structures. A direct computation shows that this is not the case, which is due to the particular coefficients in the $SU(2)$ evolution equations, coming from the nearly Kähler condition. Due to this deficit we will treat both scenarios separately, starting with the $SU(3)$-case.

Theorem 4.23. For any real analytic nearly hypo $SU(3)$-structure (σ, ρ) on a compact manifold, the embedding problem admits a unique real analytic solution. Moreover, the solution is of the form

$$\sigma_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma_0^{(k)} \quad \text{and} \quad \rho_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \rho_0^{(k)},$$

where the series converge in the C^∞-topology.

PROOF: We can apply the Cauchy-Kowalevski Theorem 2.11 directly to the $SU(3)$-evolution equations from Proposition 4.6. To see this, note that the components of the tensors ω and $\widehat{\rho}$ can be computed as polynomials in the components of σ and ρ, and that in local coordinates, the exterior derivative can be expressed as polynomials in the directional derivatives. Now Theorem 2.5 from [**49**] shows that the evolution equations already guarantee the $SU(3)$-compatibility conditions.

□

Similarly, we can apply the Cauchy-Kowalevski Theorem 2.11 directly to the $SU(2)$-evolution equations from Proposition 4.6 and obtain

THEOREM 4.24. For any real analytic nearly hypo $SU(2)$-structure $(\omega_1, \rho_2, \rho_3)$ on a compact manifold, the evolution equations

$$\begin{aligned} \dot{\omega}_1 &= d\alpha + 6\lambda\omega_2, \\ \dot{\rho}_2 &= d\omega_3 - 8\lambda\alpha \wedge \omega_1, \\ \dot{\rho}_3 &= -d\omega_2 \end{aligned}$$

admit a unique real analytic solution. Moreover, the solution is of the form

$$\omega_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\omega_1^{(k)}(0) \quad \text{and} \quad \rho_{2/3}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\rho_{2/3}^{(k)}(0),$$

where the series converge in the C^∞-topology.

To solve the embedding problem for nearly hypo $SU(2)$-structures one has to show that the family of tensors from Theorem 4.24 actually defines a family of $SU(2)$-structures, i.e. $(\omega_1, \rho_2, \rho_3)$ has to satisfy the compatibility conditions from Proposition 3.7. Since this conditions are nonlinear, we can not apply Corollary 2.4. Nevertheless, one might ask whether there is an analogue of Theorem 2.5 [**49**] for the $SU(2)$-case. The main difference between the $SU(2)$ and $SU(3)$-case is that a reduction to $SL(3,\mathbb{C})$ in dimension six can be described by a single 3-form ρ and hence involves no compatibility conditions. In dimension five, reductions to $SL(2,\mathbb{C})$ correspond to triples $(\alpha, \omega_2, \omega_3)$ which have to satisfy certain compatibility conditions. Another $SU(3)$-specific ingredient in the proof of Theorem 2.5 [**49**] is that $\omega \wedge \rho = 0$ is actually equivalent to $\omega \wedge \widehat{\rho} = 0$. Therefore we can not expect that the evolution equations in the $SU(2)$-case imply all of the desired compatibility conditions.

5. Ricci flow for G_2-Structures

By Yau's proof of the Calabi conjecture, every Kähler structure (ω, g) on a complex manifold (M, I) with $c_1 = 0$, admits a unique Ricci flat Kähler structure $(\widetilde{\omega}, \widetilde{g})$ with $[\widetilde{\omega}] = [\omega]$. The restricted holonomy group of a Ricci flat Kähler structure is contained in $SU(n)$ and hence the Calabi-Yau theorem can be regarded as an existence result for manifolds with special holonomy. Cao [**17**] gave an alternative proof of the Calabi conjecture, using Hamilton's Ricci flow. We use gauge deformations to extend the Kähler-Ricci flow to a deformation of $SU(3)$-structures and characterize in Theorem 5.13 the conditions for the flow to converge to a parallel $SU(3)$-structure. Today a Calabi-Yau theorem is still missing for the G_2-case. The only result in this direction is a theorem due to Joyce, which is tailor made to prove the existence of parallel G_2-structures on certain resolutions of T^7/Γ, cf. [**39**] Thm. 11.6.1 and Chap. 12.

The condition $c_1 = 0$ is equivalent to the existence of a (topological) $SU(n)$-reduction of the Kähler structure. From this point of view, the candidates to apply the Calabi-Yau Theorem in dimension six are $SU(3)$-structures with intrinsic torsion $\tau \cong \eta$, i.e. the Kähler part $\mathcal{T}$ of the intrinsic torsion vanishes, cf. Theorem 3.28. This observation suggests that for G_2-structures the condition $c_1 = 0$ is already encoded in the topological reduction to the structure group G_2. So the actual task at hand is to find the analogue of Kähler $SU(3)$-structures for the G_2-case. Joyce calls Kähler $SU(3)$-structures almost Calabi-Yau structures, cf. [**38**] Def. 8.4.3. His proposal for a G_2-analogue are almost G_2-structures, satisfying $d\varphi = 0$, cf. [**38**] Def. 12.3.3. In our opinion, this is a disputable choice, since by Lemma 3.36 and Proposition 3.38 we have

$$d\varphi = 0 \quad \Leftrightarrow \quad \mathcal{T} \in \mathfrak{g}_2,$$

whereas for the $SU(3)$-case the Kähler condition becomes $\mathcal{T} = 0$. The proof of Theorem 3.28 actually shows that

$$d\omega = d\rho = 0 \;\Leftrightarrow\; \mathcal{T} \in \mathfrak{su}_3 \;\text{ and }\; \eta = 0,$$

which should therefore be regarded as the $SU(3)$-analogue of $d\varphi = 0$. A second glance at the intrinsic torsion shows that it is difficult to exhibit a Fernández-Gray class of G_2-structures that corresponds to Kähler $SU(3)$-structures. The reason for this is that the structure tensor φ of a G_2-structure contains information about the Kähler form ω, but as well about the complex volume element ρ. This is manifested in the formula $\varphi = \rho + d\theta \wedge \omega$ and suggests that it is not advisable to translate $d\omega = 0$ or $N_I = 0$ into conditions like $d\varphi = 0$ or $d\psi = 0$.

Another reason why G_2-structures with $d\varphi = 0$ are inappropriate candidates for Kähler G_2-structures is a result due to Bryant, Cleyton and Ivanov. Namely, any

Ricci flat G_2-structures with $d\varphi = 0$ is already parallel. In contrast, Ricci flat Kähler structures have only restricted holonomy contained in $SU(3)$. All this assures the suspicion that non of the Fernández-Gray types is a an appropriate candidate. Instead of searching a Kähler G_2-analogue, one can more generally ask for Kähler structures in dimension seven. In chapter four we already discussed that Sasakian structures are at least not a natural choice for Kähler structures in odd dimension. Sasakian G_2-structures are even less suitable as Kähler G_2-structures, since they do not allow parallel tensors or Ricci flat metrics. The only remaining candidates are Kähler $SU(3)$-structures from Theorem 3.46, which do not belong to a particular Fernández-Gray type.
In this chapter we find a unifying description for the Ricci flow, the Kähler-Ricci flow and the extension of the Kähler-Ricci flow to $SU(n)$-structures. This description extends naturally to G_2 and Spin_7-structures and allows us to define a universal Ricci flow. We prove existence and uniqueness of this flow. Another result is the description of a fibrewise Kähler-Ricci flow, whose limit metrics can be resembled to a Ricci flat metric on the ambient sevenfold.

Kähler Geometry

Let (M, I) be a $2n$-dimensional manifold, equipped with an almost complex structure I. If we extend I to an endomorphism of the complexified tangent bundle $TM \otimes \mathbb{C}$, we obtain a decomposition of $TM \otimes \mathbb{C}$ into eigenspaces of I:

$$T^{(1,0)}M := \{X - iIX \mid X \in TM\} = \mathrm{Eig}(I, i),$$
$$T^{(0,1)}M := \{X + iIX \mid X \in TM\} = \mathrm{Eig}(I, -i).$$

Moreover, we define

$$\begin{aligned} T^{(1,0)*}M :&= \{\alpha \in \Lambda^1 T^*M \otimes \mathbb{C} \mid \alpha(Z) = 0, \text{ for all } Z \in T^{(0,1)}M\} \\ &= \{\alpha - i\alpha \circ I \mid \alpha \in \Lambda^1 T^*M\}, \\ T^{(0,1)*}M :&= \{\alpha \in \Lambda^1 T^*M \otimes \mathbb{C} \mid \alpha(Z) = 0, \text{ for all } Z \in T^{(1,0)}M\} \\ &= \{\alpha + i\alpha \circ I \mid \alpha \in \Lambda^1 T^*M\}. \end{aligned}$$

Note that if we consider I as an endomorphism of T^*M via

$$I\alpha := -\alpha \circ I,$$

we have $T^{(1,0)*}M = \mathrm{Eig}(I, -i)$ and $T^{(0,1)*}M = \mathrm{Eig}(I, i)$. Denote by $\Lambda^{(p,0)}$, respectively $\Lambda^{(0,p)}$, the p^{th} exterior power of $T^{(1,0)*}M$, respectively $T^{(0,1)*}M$,

$$\Lambda^{(p,0)} := \Lambda^p T^{(1,0)*}M \quad \text{and} \quad \Lambda^{(0,p)} := \Lambda^p T^{(0,1)*}M$$

and let $\Lambda^{(p,q)} := \Lambda^{(p,0)} \otimes \Lambda^{(0,q)}$, such that

$$\Lambda^k T^* M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{(p,q)}.$$

Sections of $\Lambda^{(p,q)}$ are called (p,q)-forms and the bundle

$$K := \Lambda^{(n,0)}$$

is called the canonical line bundle of (M,I).

We now turn to the case where the almost complex structure is integrable, i.e. $N_I = 0$. By the Newlander-Nirenberg theorem, the condition $N_I = 0$ is equivalent to the existence of an atlas of complex charts with holomorphic transition functions. Given such a chart $z = x + iy : U \to \mathbb{C}^n$, defined on some open domain $U \subset M$, we define:

$$g_{j\bar{k}} := g(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}),$$

where

$$Z_j := \frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \in T^{(1,0)}M,$$
$$\bar{Z}_j := \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}) \in T^{(0,1)}M,$$

since

$$I\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} \quad \text{and} \quad I\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}.$$

$N_I = 0$ is also equivalent to the condition that the exterior derivative defines a map

$$d : C^\infty(\Lambda^{(p,q)}M) \to C^\infty(\Lambda^{(p+1,q)}M \oplus \Lambda^{(p,q+1)}M),$$

for all $0 \le p, q \le n$. The projection onto the $(p+1,q)$ (resp. $(p,q+1)$) component defines operators ∂ (resp. $\bar{\partial}$) such that

$$d = \partial + \bar{\partial}.$$

Moreover, we define the operator

$$d^c := i(\bar{\partial} - \partial),$$

which is actually a real operator, i.e. $d^c\alpha$ is a real $(k+1)$-form if α is a real k-form. The following formulas are an easy consequence of $d^2 = 0$ and the above definitions,

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0,$$
$$(d^c)^2 = dd^c + d^c d = 0,$$
$$\partial = \frac{1}{2}(d + id^c), \quad \bar{\partial} = \frac{1}{2}(d - id^c) \quad \text{and} \quad dd^c = 2i\partial\bar{\partial}.$$

Note also that for $f : M \to \mathbb{R}$ always

$$Idf = I(\partial f + \bar{\partial} f) = -i\partial f + i\bar{\partial} f = d^c f$$

holds.

Lemma 5.1. Let η be any 1-form on the complex manifold (M, I). Then $d^c\eta = 0$ if and only if $d(I\eta) = 0$. In particular, d^c-closed 1-forms are locally d^c-exact.

Proof: Write $\eta = \eta_{10} + \eta_{01}$ according to the decomposition $\Lambda^1 T^*M = \Lambda^{(1,0)}T^*M \oplus \Lambda^{(0,1)}T^*M$. Then $d^c\eta = i(\bar{\partial} - \partial)\eta = 0$ if and only if

$$\partial\eta_{10} = \bar{\partial}\eta_{01} = \partial\eta_{01} - \bar{\partial}\eta_{10} = 0. \tag{1}$$

Since $I\eta = -i\eta_{10} + i\eta_{01}$, we see that $dI\eta = (\partial + \bar{\partial})I\eta = 0$ is also equivalent to (1) and the first part of the Lemma follows. Now for $d^c\eta = 0$, we have by the Poincaré Lemma $I\eta = du$, for some local function $u : U \subset M \to \mathbb{R}$. Hence

$$\eta = -Idu = d^c(-u).$$

□

A proof of the next lemma can for instance be found in [**3**].

Lemma 5.2. Let (M, I) be a complex manifold and ω a real $(1,1)$-form on M.

(i) ω is closed, if and only if each point in M has an open neighborhood U, such that

$$\omega_{|U} = i\partial\bar{\partial}u,$$

for some real function $u : U \to \mathbb{R}$.

(ii) Suppose that M is compact. Then ω is exact, if and only if

$$\omega = i\partial\bar{\partial}u$$

for some real function $u : M \to \mathbb{R}$.

For a compact Kähler manifold M, the equation $\partial\bar{\partial}u = 0$ implies that u is constant. Hence the second part of Lemma 5.2 states that the Kähler metrics on a compact complex manifold (M, I), within a fixed Kähler class, are parameterized by smooth real valued functions on M.

From $\nabla^g I = 0$, we see that the curvature operator of a Kähler structure satisfies

$$R(X,Y)IZ = IR(X,Y)Z,$$

and hence

$$R(IX,IY,Z,U)=R(X,Y,Z,U)=R(X,Y,IZ,IU),$$

for all $X,Y,Z,U\in C^\infty(TM)$. Then the Ricci tensor $\mathrm{ric}(X,Y)=\sum_{j=1}^{2n}R(E_i,X,Y,E_i)$ satisfies

$$\mathrm{ric}(IX,IY)=\mathrm{ric}(X,Y)\quad\text{ and }\quad \mathrm{Ric}\circ I=I\circ\mathrm{Ric},$$

which shows that

$$\varrho(X,Y):=\mathrm{ric}(IX,Y)$$

defines a 2-form on M, which is called the Ricci form of the Kähler structure.

Proposition 5.3. Let (g,ω,I) be a Kähler structure on M with Ricci form ϱ. In local coordinates $z:U\to\mathbb{C}^n$ we have

$$\varrho=-i\partial\bar\partial\ln\,\det(g_{j\bar k})\qquad\text{and}\qquad \mathrm{ric}=-\partial\bar\partial\ln\,\det(g_{j\bar k}).$$

Moreover, $d\varrho=0$ and $[\varrho/2\pi]=c_1(TM)$ equals the first real Chern class of M.

Proof: $\nabla^g I=0$ implies

$$\nabla^g_{Z_j}\bar Z_k=\nabla^g_{\bar Z_j}Z_k=0$$

and the unmixed Christoffel symbols are defined by

$$\nabla^g_{Z_j}Z_k=\sum_l\Gamma^l_{jk}Z_l\quad\text{ and }\quad\nabla^g_{\bar Z_j}\bar Z_k=\sum_{\bar l}\Gamma^{\bar l}_{\bar j\bar k}\bar Z_l.$$

Since

$$R_{j\bar k l}=R(Z_j,\bar Z_k,Z_l)=-\nabla^g_{\bar Z_k}\nabla^g_{Z_j}Z_l=-\sum_s\big(\frac{\partial}{\partial\bar z_k}\cdot\Gamma^s_{lj}\big)Z_s,$$

we have

$$R^j_{j\bar k l}=-\frac{\partial}{\partial\bar z_k}\cdot\Gamma^j_{lj}. \tag{1}$$

Writing $G:=(g_{j\bar k})$, we compute

$$\frac{\partial}{\partial z_j}\ln\,\det(G)=\mathrm{tr}(\frac{\partial G}{\partial z_j}G^{-1})$$

and

$$\big(\frac{\partial G}{\partial z_j}G^{-1}\big)_{kl}=\sum_r(Z_j\cdot g_{k\bar r})G^{-1}_{rl}=\sum_r g(\nabla^g_{Z_j}Z_k,\bar Z_r)G^{-1}_{rl}=\sum_{r,s}\Gamma^s_{jk}G_{sr}G^{-1}_{rl}=\Gamma^l_{jk},$$

i.e.

$$\frac{\partial}{\partial z_j}\ln\,\det(G)=\sum_l\Gamma^l_{jl}. \tag{2}$$

Putting together (1) and (2) we obtain

$$\mathrm{ric}_{j\bar k}=\mathrm{ric}_{\bar k j}=\sum_l R^l_{l\bar k j}=-\sum_l\frac{\partial}{\partial\bar z_k}\cdot\Gamma^l_{jl}=-\frac{\partial}{\partial\bar z_k}\frac{\partial}{\partial z_j}\ln\,\det(G). \tag{3}$$

Since $\mathrm{ric}(I.,I.) = \mathrm{ric}$, we see that $\mathrm{ric}_{jk} = \mathrm{ric}_{\bar{j}\bar{k}} = 0$ and hence

$$\begin{aligned}
\varrho &= i\sum_{jk} \mathrm{ric}_{j\bar{k}} dz_j \wedge d\bar{z}_k \\
&= -i\sum_{jk} \frac{\partial}{\partial \bar{z}_k}\frac{\partial}{\partial z_j} \ln\, \det(G) dz_j \wedge d\bar{z}_k \\
&= i\bar{\partial}\partial \ln\, \det(G) \\
&= -i\partial\bar{\partial} \ln\, \det(G).
\end{aligned}$$

The identities for the operators ∂ and $\bar{\partial}$ show that in particular $d\varrho = 0$ holds. Since $\mathrm{tr}(A + iB) = i\mathrm{tr}(B)$, for an element $A + iB \in \mathfrak{u}(n)$, we obtain for the curvature, regarded as a $\mathfrak{u}(n)$ valued 2-form,

$$\mathrm{tr}(R(X,Y)) = i\sum_{j=1}^{n} g(R(X,Y)E_j, IE_j).$$

Now the first Bianchi identity gives

$$\begin{aligned}
\varrho(X,Y) = \mathrm{ric}(IX,Y) &= \sum_{j=1}^{n} g(R(E_j,IX)Y,E_j) + g(R(IE_j,IX)Y,IE_j) \\
&= \sum_{j=1}^{n} g(R(E_j,IX)IY,IE_j) - g(R(IE_j,IX)IY,E_j) \\
&= \sum_{j=1}^{n} g(R(IX,E_j)IE_j,IY) + g(R(IE_j,IX)E_j,IY) \\
&= -\sum_{j=1}^{n} g(R(E_j,IE_j)IX,IY) = -\sum_{j=1}^{n} g(R(E_j,IE_j)X,Y) \\
&= -\sum_{j=1}^{n} g(R(X,Y)E_j,IE_j) = i\mathrm{tr}(R(X,Y))
\end{aligned}$$

and hence

$$c_1(TM) = [\frac{i}{2\pi}\mathrm{tr}(R)] = [\frac{1}{2\pi}\varrho].$$

$\square$

Proposition 5.4. Let (g,ω,I) be a Kähler structure on M with canonical bundle K. The curvature of the Levi-Civita connection on K satisfies

$$R(X,Y)\Phi = i\varrho(X,Y)\Phi,$$

for all sections $\Phi \in C^\infty(K)$ and vector fields $X, Y \in C^\infty(TM)$. In particular,

$$c_1(K) = [\frac{i}{2\pi} i\varrho] = [-\frac{1}{2\pi}\varrho].$$

Proof: The Levi-Civita connection induces a connection on $\Lambda^n T^*M \otimes \mathbb{C}$ and, since $\nabla^g I = 0$, a connection on $K \subset \Lambda^n T^*M \otimes \mathbb{C}$. The curvature R of the Levi-Civita connection on $\Lambda^k T^*M \otimes \mathbb{C}$ acts by derivation, i.e.

$$R(X,Y)\omega = \frac{d}{dt}\bigg|_{t=0} \exp(tR(X,Y))\omega,$$

for $\omega \in C^\infty(\Lambda^k T^*M \otimes \mathbb{C})$. Hence we get for a complex volume form $\Phi \in C^\infty(K)$

$$\begin{aligned} R(X,Y)\Phi &= \frac{d}{dt}\bigg|_{t=0} \exp(tR(X,Y))\Phi \\ &= \frac{d}{dt}\bigg|_{t=0} \det_{\mathbb{C}}(\exp(-tR(X,Y))\Phi \\ &= -\operatorname{tr}(R(X,Y))\Phi \\ &= i\varrho(X,Y)\Phi, \end{aligned}$$

as we have seen in the proof of Proposition 5.3.

□

Proposition 5.5. Suppose (g,ω,I) is a Kähler structure on M with Ricci form ϱ and first real Chern class $c_1(TM)$. Then

(*i*) $c_1(TM) = 0 \quad \Leftrightarrow \quad [\varrho] = 0 \quad \Leftrightarrow \quad (g,\omega,I)$ is a Kähler $SU(n)$-structure

(*ii*) $\mathrm{ric} = 0 \quad \Leftrightarrow \quad \varrho = 0 \quad \Leftrightarrow \quad \mathrm{Hol}_0(g) \subset SU(n)$

Proof: By Proposition 5.3 and 5.4, we have $c_1(TM) = 0 \Leftrightarrow c_1(K) \Leftrightarrow [\varrho] = 0$. Since the first Chern class is a complete invariant for complex line bundles, i.e. the first Chern class $c_1 \in H^2(M;\mathbb{Z})$ classifies the line bundle up to isomorphism, we see that $c_1(K) = 0$ is equivalent to the existence of a global section $\Phi = \rho + i\widehat{\rho} \in C^\infty(K)$ of unit length. Such a section corresponds to a further reduction of the Kähler structure to a (topological) $SU(n)$-structure.
For the second equivalence in (ii) recall that $i\varrho$ is the curvature of K by Proposition 5.4 and that the vanishing of the curvature is equivalent to the existence of local parallel sections in K. This can be seen as follows: Fix $p \in M$ and an element $\Phi_0 \in K_p$ of unit length. For $U \subset M$ open and simply connected, we define a local section $\Phi : U \to K$ as follows: For $q \in U$ choose a curve $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$. Parallel translation of $\Phi_0 \in K_p$ along c gives an element

$$\Phi(q) \in K_q.$$

Since $\mathrm{ric} = 0$, the canonical bundle has $\mathrm{Hol}_0 = \{1\}$ by Proposition 5.3 and the Ambrose-Singer Theorem. Therefore $\Phi(q) \in K_q$ is independent of the choice of c and we obtain a well-defined section $\Phi : U \to K$, which is of unit length and parallel.

□

Let (g, ω) be a Kähler structure on a compact complex manifold (M, I) with Ricci form ϱ and first real Chern class $c_1(TM)$. Hamilton shows in **[35]** that the initial metric can be evolved under the Ricci flow $\dot{g}_t = -2\mathrm{ric}_t$ for a short time $t \in [0, T)$. Hamilton **[35]** also mentions that the solution of the Ricci flow actually yields a whole family of *Kähler* metrics $\{g_t\}$ on (M, I). In order to prove the Calabi-Conjecture, Cao **[17]** considers the following Kähler-Ricci flow on the complex manifold (M, I)

$$\begin{aligned} \dot{g}_t &= -2\mathrm{ric}_t - 2T(I.,.), \\ \dot{\omega}_t &= -2\varrho_t + 2T, \end{aligned} \tag{1}$$

where T is a real $(1,1)$-form such that $[T/2\pi] = c_1(TM) = [\varrho/2\pi]$. By Lemma 5.2 (ii) we can find $f : M \to \mathbb{R}$ such that

$$T - \varrho = i\partial\bar{\partial}f. \tag{2}$$

To solve (1), we try to find a solution of the form

$$g_t := g - i\partial\bar{\partial}u_t(I.,.), \tag{3}$$

or equivalently,

$$\omega_t := \omega + i\partial\bar{\partial}u_t, \tag{4}$$

where $u_t : M \to \mathbb{R}$ is a smooth family of functions on M. Note that $2i\partial\bar{\partial}u_t = dd^c u_t$ is actually a *real* (1,1)-form and that $\omega_t - \omega$ is exact, i.e. $[\omega_t] = [\omega]$. In local coordinates we have

$$\varrho_t := -i\partial\bar{\partial}\ln\,\det(g_t(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k})) \tag{5}$$

and we see that $\dot{\omega}_t = -2\varrho_t + 2T$ becomes

$$\partial\bar{\partial}\dot{u}_t = 2\partial\bar{\partial}\ln\,\det(g_{j\bar{k}} + \frac{\partial}{\partial z_j}\cdot\frac{\partial}{\partial \bar{z}_k}\cdot u_t) - 2\partial\bar{\partial}\ln\,\det(g_{j\bar{k}}) + 2\partial\bar{\partial}f.$$

Equivalently, by the maximum principle,

$$\dot{u}_t = 2\ln\,\det(g_{j\bar{k}} + \frac{\partial}{\partial z_j}\cdot\frac{\partial}{\partial \bar{z}_k}\cdot u_t) - 2\ln\,\det(g_{j\bar{k}}) + 2f. \tag{6}$$

So (1) can be reduced to the scalar equation (6), which is actually a complex Monge-Ampère equation. Cao studies this equation in **[17]** and his main result can be summarized in the following

THEOREM 5.6. Suppose (g, ω) is a Kähler structure on the complex manifold (M, I) and that $T/2\pi$ is a closed real $(1,1)$-form which represents the first real Chern class $c_1(TM)$ of M. Then the solution of

$$\dot{g}_t = -2\mathrm{ric}_t - 2T(I.,.)$$

exists for all times $t \in [0, \infty)$. As $t \to \infty$, the solution g_t converges in the C^∞-topology to a Kähler metric g_∞ within the same Kähler class as the initial metric. Moreover, $\dot{g}_t$ converges in the C^∞-topology to zero.

Remark 5.7. Since g_t converges in C^∞-topology to the metric g_∞, the Ricci tensor $\mathrm{ric}_t = \mathrm{ric}(g_t)$ converges to the Ricci tensor of the metric g_∞. Taking the limit of $\dot{g}_t = -2\mathrm{ric}_t - 2T(I.,.)$, we conclude that the Ricci form of the metric g_∞ is equal to T. Hence Cao's Theorem can be used to prove the existence statement of the Calabi conjecture.

An Extension of Cao's Result to $SU(n)$-Structures

From Proposition 5.5 we know that Kähler $SU(n)$-structures are precisely the Kähler structures with vanishing first Chern class. In this case Cao's Theorem yields a Ricci flat Kähler metric. On the other hand, Proposition 5.5 also tells us that Ricci flat Kähler structures are precisely the Kähler structures with local holonomy contained in $SU(n)$. This brings up the question whether the Ricci flow for $U(n)$-structures in Theorem 5.6 can be extended to a deformation of $SU(n)$ structures, such that the limit structure has holonomy contained in $SU(n)$. In the following we will discuss an approach to extend the Kähler-Ricci flow to $SU(n)$ structures.

Definition 5.8. **(i)** Suppose that (g_t, ω_t) is the solution of the Kähler-Ricci flow on (M, I) with initial data (g, ω). By Example 1.8 we can find a gauge deformation A_t which is symmetric and positive w.r.t. g and satisfies

$$(g_t, \omega_t, I) = A_t(g, \omega, I).$$

We call A_t the corresponding gauge deformation for the Ricci flow (g_t, ω_t, I).

(ii) If the initial structure is a Kähler $SU(n)$-structure (g, ω, ρ), we obtain a 1-parameter family of Kähler $SU(n)$-structures by

$$\rho_t := A_t \rho.$$

We call (g_t, ω_t, ρ_t) the canonical extension of the Kähler-Ricci flow to the $SU(n)$-Kähler structure (g, ω, ρ). From $A_t I = I A_t$ and $\rho \in \Lambda^{(3,0)}$ w.r.t. I, we get

$$\rho_t = A_t \rho = \det_{\mathbb{C}}(A_t^{-1})\rho,$$

where $\det_{\mathbb{C}}(A_t^{-1}) \in \mathbb{R}$, since A_t is hermitian.

An alternative to the canonical extension of the Kähler-Ricci flow is motivated by the following observation: Since

$$D_\omega(\mathrm{Ric}) = -\sum_{i=1}^{2n} E^i \wedge \mathrm{Ric}(E_i) \lrcorner \omega = -2\mathrm{pr}_{\Lambda^2}(I \circ \mathrm{Ric}) = -2\varrho,$$

the evolution equation $\dot{\omega}_t = -2\varrho_t$ can be reformulated as

$$\dot{\omega}_t = D_{\omega_t}(\mathrm{Ric}_t).$$

Similarly, $D_{g_t}(\mathrm{Ric}_t) = -2\mathrm{ric}_t$ gives

$$\dot{g}_t = D_{g_t}(\mathrm{Ric}_t).$$

Hence any initial $SU(3)$-Kähler structure (g, ω, ρ) should evolve according to

$$(\dot{g}_t, \dot{\omega}_t, \dot{\rho}_t) = (D_{g_t}(\mathrm{Ric}_t), D_{\omega_t}(\mathrm{Ric}_t), D_{\rho_t}(\mathrm{Ric}_t)) =: D_{(g_t,\omega_t,\rho_t)}(\mathrm{Ric}_t)$$

and indeed we have

Theorem 5.9. The canonical extension (g_t, ω_t, ρ_t) of the Kähler-Ricci flow to $SU(3)$-structures satisfies

$$(\dot{g}_t, \dot{\omega}_t, \dot{\rho}_t) = D_{(g_t,\omega_t,\rho_t)}(\mathrm{Ric}_t).$$

Proof: We have already seen that the equations $\dot{g}_t = D_{g_t}(\mathrm{Ric}_t)$ and $\dot{\omega}_t = D_{\omega_t}(\mathrm{Ric}_t)$ hold for the canonical extension $(g_t, \omega_t, \rho_t) = A_t(g, \omega, \rho)$ of the Kähler-Ricci flow. By Lemma 1.14, Lemma 1.16 and Lemma 3.23 we have

$$\begin{aligned}
\dot{g}_t = D_{g_t}(\mathrm{Ric}_t) \quad &\Leftrightarrow \quad D_{g_t}(\dot{A}_t A_t^{-1}) = D_{g_t}(\mathrm{Ric}_t) \\
&\Leftrightarrow \quad \mathrm{pr}_{S^2}(\dot{A}_t A_t^{-1}) = \mathrm{Ric}_t, \\
\dot{\omega}_t = D_{\omega_t}(\mathrm{Ric}_t) \quad &\Leftrightarrow \quad D_{\omega_t}(\dot{A}_t A_t^{-1}) = D_{\omega_t}(\mathrm{Ric}_t) \\
&\Leftrightarrow \quad \mathrm{pr}_{\mathfrak{u}_3^\perp}(\dot{A}_t A_t^{-1}) = 0 \text{ and } \mathrm{pr}_{\mathbb{R}\mathrm{id} \oplus I_0 \mathfrak{su}_3}(\dot{A}_t A_t^{-1}) = \mathrm{pr}_{\mathbb{R}\mathrm{id} \oplus I_0 \mathfrak{su}_3}(\mathrm{Ric}_t),
\end{aligned}$$

where all projections are taken w.r.t. the structure (g_t, ω_t, ρ_t). Similarly, the equation $\dot{\rho}_t = D_{\rho_t}(\mathrm{Ric}_t)$ is equivalent to

$$\text{(1)} \qquad \mathrm{pr}_{\mathfrak{u}_3^\perp \oplus \mathbb{R}I_0}(\dot{A}_t A_t^{-1}) = 0 \quad \text{and} \quad \mathrm{pr}_{\mathbb{R}\mathrm{id} \oplus S^2_{12}}(\dot{A}_t A_t^{-1}) = \mathrm{pr}_{\mathbb{R}\mathrm{id} \oplus S^2_{12}}(\mathrm{Ric}_t).$$

By Example 1.8 we have $A_t I = I A_t$ and $A_t^T = A_t$ w.r.t. the initial metric g. Since the complex structure is preserved, we get

$$\begin{aligned}
\mathrm{pr}_{\mathbb{R}I_0}(\dot{A}_t A_t^{-1}) &= \langle \dot{A}_t A_t^{-1}, I \rangle_t = -\mathrm{tr}(\dot{A}_t A_t^{-1} I) = -\mathrm{tr}((\dot{A}_t A_t^{-1} I)^T) \\
&= \mathrm{tr}(I A_t^{-1} \dot{A}_t) = \mathrm{tr}(A_t^{-1} I \dot{A}_t) = \mathrm{tr}(\dot{A}_t A_t^{-1} I) \\
&= -\mathrm{pr}_{\mathbb{R}I_0}(\dot{A}_t A_t^{-1}).
\end{aligned}$$

So $\mathrm{pr}_{\mathbb{R}I_0}(\dot{A}_t A_t^{-1}) = 0$ and since already $\mathrm{pr}_{S^2}(\dot{A}_t A_t^{-1}) = \mathrm{Ric}_t$ and $\mathrm{pr}_{\mathfrak{u}_3^\perp}(\dot{A}_t A_t^{-1}) = 0$ holds, the evolution equation $\dot{\rho}_t = D_{\rho_t}(\mathrm{Ric}_t)$ follows from (1).

□

Lemma 5.10. Let A_t be the gauge deformation corresponding to the Ricci flow (g_t, ω_t) on (M, I). Then for local holomorphic coordinates z on M

$$\det_{\mathbb{C}}(g_t(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k})) = \det_{\mathbb{C}}(A_t^{-2}) \det_{\mathbb{C}}(g(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k})).$$

Proof: First observe that

$$\alpha_{t,k} := g_t(., \frac{\partial}{\partial \bar{z}_k}) = \frac{1}{2}\left(g_t(., \frac{\partial}{\partial x_k}) - i g_t(., \frac{\partial}{\partial x_k}) \circ I\right) \in T^{*(1,0)}M$$

and that for $\Phi_t := \alpha_{t,1} \wedge .. \wedge \alpha_{t,n} \in \Lambda^{(n,0)}T^*M$

$$\Phi_t(\frac{\partial}{\partial z_1}, .., \frac{\partial}{\partial z_n}) = \det_{\mathbb{C}}(\alpha_{t,k}(\frac{\partial}{\partial z_j})) = \det_{\mathbb{C}}(g_t(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}))$$

holds. From $g_t = A_t g$ and $A_t \in S^2$ w.r.t. g, we get

$$\alpha_{t,k} = g(A_t^{-1}., A_t^{-1}\frac{\partial}{\partial \bar{z}_k}) = g(A_t^{-2}., \frac{\partial}{\partial \bar{z}_k}) = A_t^2 \alpha_k,$$

i.e. $\Phi_t = A_t^2 \Phi = \det_{\mathbb{C}}(A_t^{-2})\Phi$ and hence

$$\det_{\mathbb{C}}(g_t(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k})) = \det_{\mathbb{C}}(A_t^{-2})\Phi(\frac{\partial}{\partial z_1}, .., \frac{\partial}{\partial z_n}) = \det_{\mathbb{C}}(A_t^{-2})\det_{\mathbb{C}}(g(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k})).$$

□

Lemma 5.11. Let A_t be the gauge deformation corresponding to the Ricci flow (g_t, ω_t) on (M, I). Then the Ricci form ϱ_t of the metric g_t satisfies

$$\varrho - \varrho_t = dd^c \ln \det_{\mathbb{C}}(A_t^{-1}).$$

Proof: By Proposition 5.3 and Lemma 5.10 we have in local coordinates

$$\begin{aligned}
\varrho_t &= -i\partial\bar{\partial}\ln \det_{\mathbb{C}}(g_{t,j\bar{k}}) \\
&= -i\partial\bar{\partial}\ln \left(\det_{\mathbb{C}}(A_t^{-2}) \det_{\mathbb{C}}(g_{j\bar{k}})\right) \\
&= -2i\partial\bar{\partial}\ln \left(\det_{\mathbb{C}}(A_t^{-1})\right) - i\partial\bar{\partial}\ln \det_{\mathbb{C}}(g_{j\bar{k}}) \\
&= -dd^c\ln \det_{\mathbb{C}}(A_t^{-1}) + \varrho.
\end{aligned}$$

□

Lemma 5.12. Suppose (g, ω, ρ) is a Kähler $SU(3)$-structure on (M, I) with intrinsic torsion η, cf. Definition 3.24. Then the Ricci form satisfies

$$\varrho = -3d\eta.$$

Proof: The Lemma is a special case of Lemma 3.3 in **[15]**. The Kähler condition $\xi = 0$ (in the notation of **[15]**) yields Ric $= -3d\hat{\eta}$, where $\hat{\eta} \cong -\eta$ in our notation, since the intrinsic torsion is defined in **[15]** by $\bar{\nabla} = \nabla + \eta + \xi$, where $\bar{\nabla}$ is the covariant derivative of the characteristic $SU(3)$-connection and ∇ is the Levi-Civita connection. Since Cabrera and Swann use the opposite sign convention for the curvature tensor, we obtain in our notation

$$\varrho(X,Y) = \mathrm{ric}(IX,Y) = -\mathrm{ric}(X,IY) = 3d\eta(X,I^2Y) = -3d\eta(X,Y).$$

$\square$

We can now describe the condition under which the canonical extension of the Kähler-Ricci flow yields a parallel $SU(3)$ structure:

Theorem 5.13. Suppose (g,ω,ρ) is a $SU(3)$-Kähler structure on (M,I) with intrinsic torsion η, cf. Definition 3.24. Then the canonical extension of the Kähler-Ricci flow converges to a parallel $SU(3)$-structure on M if and only if $d^c\eta = 0$.

Proof: By Cao's Theorem the Kähler-Ricci flow converges to a Ricci flat Kähler structure $(g_\infty, \omega_\infty)$ on (M,I). If A_∞ denotes the corresponding gauge deformation, we have

$$\rho_\infty = \det_{\mathbb{C}}(A_\infty^{-1})\rho.$$

The $SU(3)$-structure is parallel if and only if $d\rho_\infty = 0$ holds. By Proposition 3.25 we have

$$\begin{aligned} d\rho_\infty &= d(\det_{\mathbb{C}}(A_\infty^{-1})) \wedge \rho + \det_{\mathbb{C}}(A_\infty^{-1})d\rho \\ &= d(\det_{\mathbb{C}}(A_\infty^{-1})) \wedge \rho + 3\det_{\mathbb{C}}(A_\infty^{-1})\eta \wedge \widehat{\rho} \\ &= d(\det_{\mathbb{C}}(A_\infty^{-1})) \wedge \rho - 3\det_{\mathbb{C}}(A_\infty^{-1})I\eta \wedge \rho. \end{aligned}$$

From the non-degeneracy of ρ we see that $d\rho_\infty = 0$ is equivalent to

$$\begin{aligned} & 0 = d(\det_{\mathbb{C}}(A_\infty^{-1})) - 3\det_{\mathbb{C}}(A_\infty^{-1})I\eta \\ \Leftrightarrow\ & 0 = d^c(\det_{\mathbb{C}}(A_\infty^{-1})) + 3\det_{\mathbb{C}}(A_\infty^{-1})\eta \qquad (1) \\ \Leftrightarrow\ & 0 = d^c\ln(\det_{\mathbb{C}}(A_\infty^{-1})) + 3\eta. \end{aligned}$$

So $d\rho_\infty = 0$ implies $d^c\eta = 0$. If conversely $d^c\eta = 0$ holds, we can find by the Poincaré Lemma 5.1 a local function $u : U \subset M \to \mathbb{R}$ such that $\eta = d^c u$. By construction, the metric g_∞ is Ricci flat and hence we obtain form Lemma 5.11

$$\varrho = dd^c\ln\ \det_{\mathbb{C}}(A_\infty^{-1}).$$

By Lemma 5.12 we have $\varrho = -3d\eta$ and so

$$dd^c u = d\eta = -\frac{1}{3}\varrho = -\frac{1}{3}dd^c\ln\ \det_{\mathbb{C}}(A_\infty^{-1}).$$

Hence $-3u = \ln\ \det_{\mathbb{C}}(A_\infty^{-1}) + c$, for some constant $c \in \mathbb{R}$. So $-3\eta = -3d^c u = d^c\ln\ \det_{\mathbb{C}}(A_\infty^{-1})$, which yields (1).

$\square$

In the previous section we have seen that the Ricci flow for $O(n)$ and $U(n)$-structures, as well as the canonical extension to $SU(n)$-structures, can be described in a unified way, using the map from Lemma 1.14:

$$\begin{aligned}
O(n): &\qquad \dot{g}_t = D_{g_t}(\mathrm{Ric}_t)\\
U(n): &\qquad (\dot{g}_t, \dot{\omega}_t) = D_{(g_t,\omega_t)}(\mathrm{Ric}_t)\\
SU(n): &\qquad (\dot{g}_t, \dot{\omega}_t, \dot{\rho}_t) = D_{(g_t,\omega_t,\rho_t)}(\mathrm{Ric}_t)
\end{aligned}$$

This motivates the conjecture that for a given G_2-structure φ on M with sufficiently small torsion, the flow

$$\dot{\varphi}_t = D_{\varphi_t}(\mathrm{Ric}_t) \tag{1}$$

should converge to a Ricci-flat G_2-structure. Similar flow equations can be considered for Spin_7-structures, or more generally, for *any* $G \subset O(n)$ structures, described by certain structure tensors. Like in the proof of Theorem 4.13, we see that the metric of the underlying structure evolves according to the Ricci flow $\dot{g}_t = -2\mathrm{ric}_t$. In contrast to the G_2-case, the orbit of the model tensor is not open in the Spin_7-case. Hence it is not obvious that a Spin_7-structure evolving according to (1) actually defines a whole family of Spin_7-structures. To avoid this problem, we can translate the above flow equation into an equation for a corresponding family of gauge deformations. By Lemma 1.16 we have $D_{\varphi_t}(\dot{A}_t A_t^{-1}) = \dot{\varphi}_t$, for $\varphi_t = A_t\varphi$. Hence a solution of $\dot{A}_t = \mathrm{Ric}_t \circ A_t$ yields a solution $\varphi_t = A_t\varphi$ of (1).

Theorem 5.14. Let (M, g) be a compact Riemannian manifold. Then there exists a unique solution $A_t \in C^\infty(\mathrm{Aut}(TM))$, $t \in [0, T)$, of the initial value problem

$$\begin{cases} \dot{A}_t = \mathrm{Ric}_t \circ A_t \\ A_0 = \mathrm{id} \end{cases}$$

Proof: Since M is compact we can find a solution g_t, $t \in [0, T)$, of the usual Ricci flow

$$\dot{g}_t = -2\mathrm{ric}_t \quad \text{with} \quad g_{t=0} = g_0. \tag{1}$$

Given an orthonormal basis $p = (E_1, .., E_n)$ for g, we can solve the linear ODE

$$\dot{E}_i(t) = \mathrm{Ric}_t \circ E_i(t) \quad \text{with} \quad E_i(0) = E_i, \tag{2}$$

for $i = 1, .., n$ and $t \in [0, T)$. From (1) and (2) we get

$$\begin{aligned}
\frac{d}{dt}\big(g_t(E_i(t), E_j(t))\big) &= -2\mathrm{ric}_t(E_i(t), E_j(t)) + \mathrm{ric}_t(E_i(t), E_j(t)) + \mathrm{ric}_t(E_i(t), E_j(t))\\
&= 0,
\end{aligned}$$

i.e. $p_t := (E_1(t), .., E_n(t))$ is actually an orthonormal basis w.r.t. the metric g_t. Modifying the initial basis p by an element $B \in O(n)$ yields a new basis pB given by

$$\tilde{E}_i := pBe_i = \sum_{j=1}^{n} b_{ij} E_j.$$

Hence $\tilde{E}_i(t) := \sum_{j=1}^{n} b_{ij} E_j(t)$ satisfies $\tilde{E}_i(0) = \tilde{E}_i$ and

$$\frac{d}{dt}\tilde{E}_i = \sum_{j=1}^{n} b_{ij} \mathrm{Ric}_t \circ E_i(t) = \mathrm{Ric}_t \circ \tilde{E}_i(t).$$

Since the solution of (2) is unique, we get

$$(pB)_t = p_t B, \tag{3}$$

for all $B \in O(n)$. For $t \in [0, T)$ define

$$A_t : F^g M \to \mathrm{Aut}(\mathbb{R}^n) \quad \text{by} \quad A_t(p) := p^{-1} \circ p_t.$$

Equation (3) shows that A_t is equivariant,

$$A_t(pB) = (pB)^{-1} \circ p_t \circ B = B^{-1} \circ p^{-1} \circ p_t \circ B = B^{-1} A_t(p)$$

and hence corresponds to an element $A_t \in \mathrm{Aut}(TM)$, given by

$$A_t E_i = E_i(t).$$

Now

$$\begin{aligned}(A_t g_0)(E_i(t), E_j(t)) &= g_0(A_t^{-1} E_i(t), A_t^{-1} E_j(t)) \\ &= g_0(E_i, E_j) \\ &= g_t(E_i(t), E_j(t))\end{aligned}$$

shows that

$$A_t g_0 = g_t \tag{4}$$

holds. From (3) we get $\mathrm{Ric}_t = \mathrm{Ric}(g_t) = \mathrm{Ric}(A_t g_0)$ and (2) becomes

$$\dot{E}_i(t) = \mathrm{Ric}(A_t g_0) \circ E_i(t) \quad \Leftrightarrow \quad \dot{A}_t E_i = \mathrm{Ric}(A_t g_0) \circ A_t E_i,$$

i.e. $\dot{A}_t = \mathrm{Ric}(A_t g_0) \circ A_t$.

□

Definition 5.15. Let (M, g) be a compact Riemannian manifold of dimension n and let A_t be the unique solution of

$$\begin{cases} \dot{A}_t = \mathrm{Ric}_t \circ A_t \\ A_0 = \mathrm{id} \end{cases}$$

from Theorem 5.14.

(1) We call A_t the universal Ricci flow for (M, g).

(2) If $n = 7$ and $g = g(\varphi)$, where φ is a G_2-structure on M, then we call $\varphi_t := A_t\varphi$ the Ricci flow for φ.

(3) If $n = 8$ and $g = g(\Psi)$, where Ψ is a Spin_7-structure on M, then we call $\Psi_t := A_t\Psi$ the Ricci flow for Ψ.

Note that the Ricci flow satisfies by Lemma 1.16

$$\begin{array}{ll} G_2 : & \dot{\varphi}_t = D_{\varphi_t}(\mathrm{Ric}_t), \\ \mathrm{Spin}_7 : & \dot{\Psi}_t = D_{\Psi_t}(\mathrm{Ric}_t). \end{array}$$

In contrast to the usual Ricci flow equation, the equation $\dot{A}_t = \mathrm{Ric}_t \circ A_t$ is not invariant under the full diffeomorphism group of M. Nevertheless, $\dot{A}_t = \mathrm{Ric}_t \circ A_t$ is invariant under the group $\mathrm{Isom}(M, g)$ and hence any isometry of the initial metric is preserved under the flow.

Remark 5.16. We proved in Theorem 5.9 that the canonical extension of the Kähler-Ricci flow already satisfies the evolution equation

$$(\dot{g}_t, \dot{\omega}_t, \dot{\rho}_t) = D_{(g_t,\omega_t,\rho_t)}(\mathrm{Ric}_t).$$

This brings up the question whether the Ricci flow for a metric g, coming from some G_2-structure φ on M, can be extended canonically to a solution φ_t of

$$\dot{\varphi}_t = D_{\varphi_t}(\mathrm{Ric}_t).$$

Like in the $SU(3)$-case we can write $g_t = A_t g$ for the solution of the Ricci flow. Here A_t is symmetric and positive w.r.t. the initial metric g. Then the canonical extension of the Ricci flow to the whole G_2-structure would be $\varphi_t := A_t\varphi$. However, the proof of Theorem 5.9 does not carry over to the G_2-case. One critical ingredient in the proof of Theorem 5.9 was the fact that the corresponding gauge deformation preserves the complex structure. This property stems from the assumption that the initial structure is actually Kähler. For a generic G_2-structure, the family of gauge deformations A_t, describing the Ricci flow, do not necessarily contain enough symmetries to reproduce the proof of Theorem 5.9.
As a consequence, a G_2-structure for which the Ricci flow converges to a Ricci flat G_2-structure, should have the property that the canonical extension yields a solution of $\dot{\varphi}_t = D_{\varphi_t}(\mathrm{Ric}_t)$.

Given a $SU(3)$ structure (α, φ) on a compact seven dimensional manifold M with $d\alpha = 0$, we obtain a fibration of M into compact integral manifolds of $\ker(\alpha)$. On a fixed integral manifold $i : N \hookrightarrow M$, the G_2-structure φ induces a $SU(3)$-structure by

$$g_N := i^* g, \quad \omega_N := i^* \omega \quad \text{and} \quad \rho_N := i^* \rho.$$

Conversely, the collection of metrics $\{g_N\}$ on all integral manifolds $N \subset M$ determines the metric on M by

$$g = \{g_N\} + \alpha \otimes \alpha.$$

Similarly we have

$$\omega = \{\omega_N\} \quad \text{and} \quad \varphi = \{\rho_N\} + \alpha \wedge \{\omega_N\}.$$

In this section we will evolve the induced $SU(3)$-structures under the Ricci flow and resemble the evolved structures to a $SU(3)$-structure on M. To obtain again a smooth structure on M, we need the following Lemma, which states that the Ricci flow depends smoothly on the initial metric.

Lemma 5.17. Let M be a compact manifold and g_s a smooth 1-parameter family of metrics on M. Denote by $g_s(t)$ the unique solution of

$$\begin{cases} \frac{d}{dt} g_s(t) = -2\mathrm{ric}(g_s(t)) \\ g_s(0) = g_s, \end{cases}$$

for $t \in [0, T_s)$ and $T_s := T(g^s) > 0$. Then $g_s(t)$ depends smoothly on s.

Proof: Let F be the vector bundle over M, whose fibres consist out of symmetric bilinear maps $T_pM \times T_pM \to \mathbb{R}$ and denote by $U \subset F$ the subset of positive symmetric bilinear maps. Then

$$\mathcal{U} := C^\infty(M \times [0,1], U) \subset C^\infty(M \times [0,1], F) =: \mathcal{F}$$

is an open subset of the Fréchet space $\mathcal{F}$, cf. Example 2.1. Hamilton applies the Nash-Moser inverse function theorem to the operator

$$\begin{aligned} \mathcal{E} : \mathcal{U} \subset \mathcal{F} &\to \mathcal{F} \times C^\infty(M, F) \\ f &\mapsto \left(\frac{df}{dt} - E(f), f_{|\{t=0\}}\right), \end{aligned}$$

where $E(f) := -2\mathrm{ric}(f)$, cf. the proof of Theorem 5.1, p.263 in **[35]**. The Nash-Moser inverse function theorem states that $\mathcal{E}$ is locally invertible and each (local) inverse is a smooth tame map, cf. **[35]** III Theorem 1.1.1. Now the solution for the Ricci flow with initial data $f(0) \in C^\infty(M, F)$ is given by $f := \mathcal{E}^{-1}(0, f(0))$, where

$\mathcal{E}^{-1}$ is the local inverse of $\mathcal{E}$, defined in some neighborhood of $(0, f(0))$. Since $\mathcal{E}^{-1}$ is smooth, we see that a smooth variation $s \mapsto f_s(0)$ of the initial value yields a solution $f_s := \mathcal{E}^{-1}(0, f_s(0))$ that depends smoothly on s.

$\square$

On a fixed (compact) integral manifold $N \subset M$ we can evolve the metric g_N under the Ricci flow for some time $t \in [0, T_N) \subset \mathbb{R}$. Since M is compact, we can find $0 < T \leq \infty$ such that the Ricci flow exists on each integral manifold for at least time T. Lemma 5.17 can be used to show that the solutions of the Ricci flow on each integral manifold can be resembled to a *smooth* tensor on M.

Lemma 5.18. Suppose that the Ricci flow $g_N(t)$ exists on each integral manifold $N \subset M$ for at least time $t \in [0, T)$, $0 < T \leq \infty$.
(1) The tensor

$$g_t := \{g_N(t)\} + \alpha \otimes \alpha$$

defines a family of smooth metrics on M.

(2) Let $A_N(t)$ be the gauge deformation from Example 1.7 such that $g_N(t) = A_N(t) g_N$. Then

$$A_t := \{A_N(t)\} + \alpha \otimes \xi$$

is smooth and satisfies $g_t = A_t g$.

Proof: We first prove the smoothness of g_t: Fix an integral manifold $N \subset M$ and let $N_s := \Phi_s(N)$, where Φ_s is the flow of ξ. Extending $X, Y \in C^\infty(TN)$ under the flow Φ_s via

$$\tilde{X}\Big|_{(s,p)} := \Phi_{s*}\, X|_p \quad \text{and} \quad \tilde{Y}\Big|_{(s,p)} := \Phi_{s*}\, Y|_p$$

yields smooth local vector fields on M. We will show that

$$M \ni (s, p) \longmapsto \{g_N(t)\}(\tilde{X}\Big|_{(s,p)}, \tilde{Y}\Big|_{(s,p)}) \in \mathbb{R}$$

is smooth. To see this, observe that

$$g_s(t) := \Phi_s^* g_{N_s}(t)$$

defines a family of metrics on N which satisfies

$$\begin{cases} \frac{d}{dt} g_s(t) = -2\mathrm{ric}(g_s(t)) \\ g_s(0) = \Phi_s^* g_{N_s}. \end{cases}$$

So

$$(s, p) \longmapsto \{g_N(t)\}(\tilde{X}\Big|_{(s,p)}, \tilde{Y}\Big|_{(s,p)}) = g_{N_s}(t)(\Phi_{s*}\, X|_p, \Phi_{s*}\, Y|_p) = g_s(t)(X|_p, Y|_p)$$

is smooth in s by Lemma 5.17, and smooth in p, since $g_s(t)$ is a smooth metric on N. Choosing local coordinates like in the proof of Lemma 4.2, we see that $\{g_N(t)\}$ is smooth, which implies the smoothness of g_t.

Now we prove that A_t is smooth: Since A_t is symmetric w.r.t. g, we have

$$A_t^{-1} \circ A_t^{-1} = g^{-1} \circ (A_t g) = g^{-1} \circ g_t : TM \to TM.$$

Now A_t^{-2} is smooth, since $g^{-1} : T^*M \to TM$ is smooth by assumption and $g_t : TM \to T^*M$ is smooth as we have just seen. Since A_t^{-2} is positive w.r.t. g, we see that $A_t = \exp(-\frac{1}{2}\ln(A_t^{-2}))$ is smooth, where the logarithm is defined w.r.t. g. Since clearly

$$g_t = \{A_N(t)g_N\} + \alpha \otimes \alpha = A_t(\{g_N\} + \alpha \otimes \alpha) = A_t g$$

holds, the Lemma follows.

□

Similar to Definition 5.8, the gauge deformations $A_N(t)$ from Example 1.7 can be used to define a whole family of $SU(3)$-structures on each integral manifold $N \subset M$:

$$g_N(t) = A_N(t)g_N, \quad \omega_N(t) := A_N(t)\omega_N \quad \text{and} \quad \rho_N(t) := A_N(t)\rho_N.$$

This families can again be resembled to a family of G_2-structures on M:

Proposition 5.19. For any $t \in [0, T)$,

$$\varphi_t := \{\rho_N(t)\} + \alpha \wedge \{\omega_N(t)\}$$

defines a G_2-structure on M with metric $g_t = \{g_N(t)\} + \alpha \otimes \alpha$ and dual $\psi_t = \{\sigma_N(t)\} - \alpha \wedge \{\widehat{\rho}_N(t)\}$.

Proof: From Lemma 5.18 we see that

$$A_t = \{A_N(t)\} + \alpha \otimes \xi \in C^\infty(\mathrm{Aut}(TM))$$

for any $t \in [0, T)$. Hence

$$\begin{aligned}
\varphi_t = A_t\varphi &= A_t(\{\rho_N\} + \alpha \wedge \{\omega_N\}) \\
&= A_t\{\rho_N\} + \alpha \wedge A_t\{\omega_N\} \\
&= \{A_N(t)\rho_N\} + \alpha \wedge \{A_N(t)\omega_N\} \\
&= \{\rho_N(t)\} + \alpha \wedge \{\omega_N(t)\}
\end{aligned}$$

defines a G_2-structure with metric $g_t = A_t g = \{g_N(t)\} + \alpha \otimes \alpha$ and dual $\psi_t = A_t\psi = \{\sigma_N(t)\} - \alpha \wedge \{\widehat{\rho}_N(t)\}$.

□

We now turn to the case where the initial structure (α, ψ) is Kähler, i.e. the induced $SU(3)$-structures on each integral manifold are Kähler, cf. Theorem 3.46. In this case the Kähler-Ricci flow on N can be described by a gauge deformation $A_N(t)$ with

$$(g_N(t), \omega_N(t), I) = A_N(t)(g_N, \omega_N, I).$$

Note that $A_N(t)$ is actually the same gauge deformation used in Lemma 5.18 and Proposition 5.19, but satisfies in addition $A_N(t)I_N = I_N A_N(t)$, cf. Example 1.8.

Remark 5.20. If the initial structure (α, ψ) is Kähler, Cao's theorem states that the Ricci flow converges on each integral manifold N to a Ricci flat metric g_N^∞ in C^∞-topology. In Lemma 5.18 we have seen that for finite time t, the metrics $g_N(t)$ can be resembled to a metric g_t on the ambient space M. The convergency of $g_N(t)$ in C^∞-topology would still guarantee the smoothness of g_∞ in fibre direction, but it seems difficult to ensure the smoothness of the limit metric g_∞ transverse to the fibres.

Definition 5.21. Let φ be a G_2-structure on M and ξ a unit vector field with dual $\alpha := \xi \lrcorner g$ and flow Φ_s. We say that the vector field ξ is a Kähler field for the G_2-structure φ if

(1) $d\alpha = 0$ and $\nabla^g(\xi \lrcorner \varphi) = 0$ on $\ker(\alpha)$.

(2) For all integral manifolds $N \subset M$ of $\ker(\alpha)$

$$[\omega_N] = [\Phi_s^* \omega_{N_s}] \quad \text{and} \quad I_N = \Phi_s^* I_{N_s},$$

where $N_s := \Phi_s(N)$.

Note that Theorem 3.46 ensures that the induced $SU(3)$-structures on the integral manifolds are Kähler. Hence $d\omega_N = 0$ and condition (2) states that the flow of the vector field ξ preserves the cohomology class and the complex structure.

The next result is essentially due to the uniqueness part of the Calabi-Yau theorem and solves in particular the problem encountered in Remark 5.20.

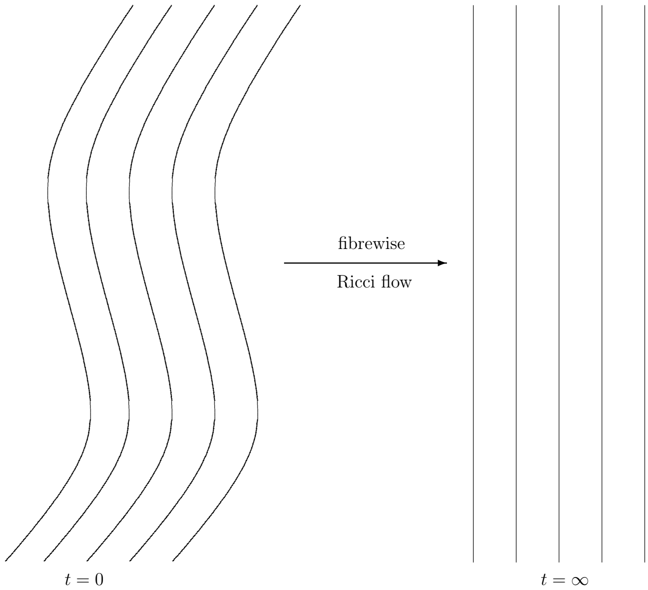
Theorem 5.22. Suppose ξ is a Kähler vector field for the G_2-structure φ. Then the Ricci flow limit metrics and Kähler forms satisfy

$$g_N(\infty) = \Phi_s^* g_{N_s}(\infty) \quad \text{and} \quad \omega_N(\infty) = \Phi_s^* \omega_{N_s}(\infty),$$

for each integral manifold $N \subset M$. In particular, $g_\infty = \alpha \otimes \alpha + \{g_N(\infty)\}$ defines a smooth metric on M with

$$L_\xi g_\infty = 0 \quad \text{and} \quad \nabla^{g_\infty} \xi = 0.$$

So the fibrewise Ricci flow tightens the fibres $N \subset M$:



PROOF: On a fixed integral manifold $N \subset M$ we have the Kähler structure

$$(g_N(\infty), \omega_N(\infty), I_N)$$

obtained by the Ricci flow and the Kähler structure

$$(\Phi_s^* g_{N_s}(\infty), \Phi_s^* \omega_{N_s}(\infty), \Phi_s^* I_{N_s} = I_N).$$

By Cao's theorem both structures are Ricci flat,

$$\mathrm{Ric}(\Phi_s^* g_{N_s}(\infty)) = \Phi_s^* \mathrm{Ric}(g_{N_s}(\infty)) = 0.$$

Since the Ricci flow and the flow of ξ preserve cohomology classes, we get

$$[\omega_N(\infty)] = [\omega_N] = [\Phi_s^* \omega_{N_s}] = \Phi_s^*[\omega_{N_s}] = \Phi_s^*[\omega_{N_s}(\infty)] = [\Phi_s^* \omega_{N_s}(\infty)].$$

Then the uniqueness part of the Calabi-Yau theorem states that the two structures coincide. For the smoothness of g_∞ choose local vector fields $X, Y \in C^\infty(TN)$ and extend them to local vector fields on M by

$$\left.\tilde{X}\right|_{(s,p)} := \Phi_{s*} \left.X\right|_p \quad \text{and} \quad \left.\tilde{Y}\right|_{(s,p)} := \Phi_{s*} \left.Y\right|_p.$$

Now observe that

$$\{g_N(\infty)\}(\left.\tilde{X}\right|_{(s,p)}, \left.\tilde{Y}\right|_{(s,p)}) = g_{N_s}(\infty)(\Phi_{s*} \left.X\right|_p, \Phi_{s*} \left.Y\right|_p) = g_N(\infty)(\left.X\right|_p, \left.Y\right|_p)$$

is constant and hence smooth in s. Let $\mathrm{pr} : TM \to \ker(\alpha)$ be the map $X \mapsto X - \alpha(X)\xi$. Then $L_\xi \alpha = 0$ yields for $X \in T_pM$

$$\mathrm{pr}(\Phi_{s*}X) = \Phi_{s*}X - \alpha(\Phi_{s*}X) \left.\xi\right|_{\Phi_s(p)} = \Phi_{s*}X - \alpha(X)\Phi_{s*} \left.\xi\right|_p = \Phi_{s*}(\mathrm{pr}X).$$

Hence for $X, Y \in T_pM$

$$\begin{aligned}\Phi_s^*\{g_N(\infty)\}(X,Y) &= g_{N_s}(\infty)(\mathrm{pr}(\Phi_{s*}X), \mathrm{pr}(\Phi_{s*}Y)) = g_{N_s}(\infty)(\Phi_{s*}(\mathrm{pr}X), \Phi_{s*}(\mathrm{pr}Y)) \\ &= (\Phi_s^\infty g_{N_s})(\mathrm{pr}X, \mathrm{pr}Y) = \{\Phi_s^\infty g_{N_s}\}(X,Y) \\ &= \{g_N(\infty)\}(X,Y),\end{aligned}$$

i.e. $L_\xi\{g_N(\infty)\} = 0$ and so $L_\xi g_\infty = 0$. Since also $d(\xi \lrcorner g_\infty) = d\alpha = 0$, it follows $\nabla^{g_\infty}\xi = 0$.

□

Corollary 5.23. The metric obtained by fibrewise Ricci flow in Theorem 5.22 is Ricci flat,

$$\mathrm{Ric}(g_\infty) = 0.$$

Proof: By Cao's theorem, the metrics $g_N(\infty)$ on the fibres are Ricci flat. Hence Proposition 4.1 with $g_\infty = \alpha \otimes \alpha + \{g_N(\infty)\}$ and $\mathcal{W}_\infty = L_\xi g_\infty = 0$ yields $\mathrm{Ric}(g_\infty) = 0$.

□

The fibrewise Ricci flow can be extended to a deformation of the ambient G_2-structure φ. For this choose a gauge deformation A_∞ like in Example 1.7 such that $g_\infty = A_\infty g$, where $g = g(\varphi)$ is the initial metric and g_∞ is the metric obtained by fibrewise Ricci flow from Theorem 5.22. Then we have

Corollary 5.24. Suppose ξ is a Kähler vector field for the G_2-structure φ on M. Then the fibrewise Ricci flow yields a Ricci flat G_2-structure $\varphi_\infty := A_\infty\varphi$ on M.

Corollary 5.25. The metric obtained by fibrewise Ricci flow in Theorem 5.22 satisfies

$$\mathrm{Hol}_0(g_\infty) \subset \{1\} \times SU(3) \subset G_2.$$

In particular, the full holonomy group $\mathrm{Hol}(g_\infty)$ is always a proper subgroup of G_2.

Proof: Since $\nabla^{g_\infty}\xi = 0$, we get from the DeRham splitting theorem

$$\mathrm{Hol}_0(g_\infty) = \{1\} \times \mathrm{Hol}_0(g_N(\infty)).$$

By Proposition 5.5, the restricted holonomy of the integral manifold $N \subset M$ is contained in $SU(3)$ and the first part of the corollary follows. Since the restricted holonomy group is the identity component of the full holonomy group, $\mathrm{Hol}(g_\infty) = G_2$ would imply $\mathrm{Hol}_0(g_\infty) = G_2$, which is impossible as we have just seen.

□

Corollary 5.26. Suppose ξ is a Kähler vector field for the G_2-structure φ on M. Then the fundamental group of M is infinite.

Proof: Let $(\hat{M}, \hat{g}_\infty)$ be the universal cover of (M, g_∞). By Corollary 5.25 we have

$$\mathrm{Hol}(\hat{g}_\infty) = \mathrm{Hol}_0(g_\infty) \subset \{1\} \times SU(3) \subset G_2.$$

If π_1 were finite, the universal cover would be compact. But a compact manifold with holonomy contained in G_2 and finite fundamental group has holonomy group equal to G_2, cf. **[39]** Prop. 10.2.2.

$\square$

6. Examples

In this chapter we describe classes of manifolds, admitting certain types of $SU(2)$, $SU(3)$ and G_2-structures. Typically this manifolds are the total space of some bundle over a base manifold that carries an additional structure. In quite a few cases the structure is real analytic and can be embedded into a space with a parallel structure. We do not describe any explicit solutions for the embedding problems, but some can for instance be found in **[9]**,**[24]**,**[25]**.
Of special interest is a new construction of R. Albuquerque **[1]**. Albuquerque constructs a G_2-structure on the unit tangent bundle T^1M^4. This structure is hypo if and only if the underlying metric on M^4 is Einstein. We extend Albuquerque's approach to construct a family of Spin(7)-structures Ψ_λ on $TM^4 \setminus \{0\}$ and show that this structure is balanced if and only if the underlying metric is Einstein with $\mathrm{ric} = \lambda g$.

G_2 and Spin(7)-Structures on TM^4

Given an oriented four dimensional manifold (M, g), we describe a construction due to Albuquerque **[1]**, which yields a G_2-structure on the unit tangent bundle T^1M. It turns out that the Einstein condition for M is encoded in the Lee form Θ of the G_2-structure. Although the Lee form corresponds in general only to the vectorial part $\mathfrak{g}_2^\perp$ of the intrinsic torsion, the vanishing of Θ implies in this particular case that the G_2-structure is actually hypo, i.e. the $\mathfrak{g}_2^\perp \oplus \mathfrak{g}_2$ component of the intrinsic torsion vanishes. Let ε_M be the induced volume form on (M, g) and $\pi : TM \to M$ be the tangent bundle. For every $u \in TM$, the Levi-Civita connection induces a splitting

$$T_uTM = V_u \oplus H_u$$

of the tangent space of TM into a vertical space $V_u = \ker(\pi_{*u})$ and a horizontal space $H_u = \pi^* T_{\pi(u)}M$. In particular, we may consider the vertical and horizontal lift of $X \in T_{\pi(u)}M$ to $u \in TM$, denoted respectively by

$$v_u(X) \in V_u \subset T_uTM \qquad \text{and} \qquad h_u(X) \in H_u \subset T_uTM.$$

The connection map

$$K : T_uTM \to T_{\pi(u)}M \qquad \text{is now given by} \qquad K(X) := J_u(X^v),$$

where $J_u : V_u \to T_{\pi(u)}M$ is the inverse of the vertical lift v_u. The Sasakian metric on TM is defined via

$$\widehat{g}(X,Y) := g(KX, KY) + g(\pi_* X, \pi_* Y).$$

It is well known that the curvature tensor of the Levi-Civita connection on M measures the integrability of the horizontal distribution. More generally we have by Lemma 2 in **[26]**

Lemma 6.1. Let R be the Riemannian curvature tensor of the Levi-Civita connection ∇^g on M. For any vector fields X, Y on M and $u \in TM$ we have

(i) $[h(X), h(Y)]_u = h_u([X,Y]) - v_u(R(X,Y)u)$,
(ii) $[v(X), v(Y)]_u = 0$,
(iii) $[h(X), v(Y)]_u = v_u(\nabla^g_X Y)$.

□

In order to establish a G_2-structure on the unit tangent bundle T^1M, we make the following

Definition 6.2. **(i)** For notational reasons we introduce the map

$$r : TM \setminus \{0\} \to \mathbb{R} \qquad \text{by} \qquad u \mapsto \sqrt{g(u,u)} = \|u\|.$$

(ii) The map

$$\theta : T_uTM \to V_u \qquad \text{with} \qquad X \mapsto v_u(\pi_* X)$$

rotates the horizontal onto the vertical space and annihilates vertical vectors. As a map $\theta : TTM \to TTM$, we may ask for the adjoint of θ with respect to $\widehat{g}$ and find

$$\theta^T = h \circ K,$$

i.e. for $X \in T_uTM$ we have $\theta^T(X) = h_u(KX)$. Hence $\theta^T \circ \theta = \mathrm{id}_H$ and $\theta \circ \theta^T = \mathrm{id}_V$.
(iii) The decomposition $TTM = TM \oplus TM$ equips TM with a natural symplectic structure ω. In terms of the map θ, we have

$$\omega(X,Y) := \widehat{g}(IX, Y),$$

where

$$I := \theta^T - \theta$$

satisfies $I^2 = -\mathrm{id}$. It is well known that ω is actually a closed 2-form on TM.
(iv) The $\widehat{g}$-gradient of r is given by

$$N^v : TM \setminus \{0\} \to V \qquad \text{with} \qquad u \mapsto \frac{1}{r(u)} v_u(u).$$

Using the map I from (iii), we may define the horizontal counterpart of N^v by

$$N^h : TM \setminus \{0\} \to H, \quad u \mapsto IN^v(u) = \frac{1}{r(u)} h_u(u).$$

We denote the dual 1-forms of N^v and N^h respectively by

$$\mu^v(X) := \widehat{g}(N^v, X) \qquad \text{and} \qquad \mu^h(X) := \widehat{g}(N^h, X).$$

Note that N^v is the outer normal vector field on each sphere bundle $T^rM \subset TM$, for $r > 0$.

(v) The volume element ε_M on M lifts to a volume element $K^*\varepsilon_M$ on the vertical distribution and a volume element $\pi^*\varepsilon_M$ on the horizontal distribution. Contracting this pull-backs, we obtain forms

$$\begin{aligned} \alpha(X, Y, Z) &:= \varepsilon_M(KN^v, KX, KY, KZ), \\ \beta(X, Y, Z) &:= \varepsilon_M(\pi_* N^h, \pi_* X, \pi_* Y, \pi_* Z), \end{aligned}$$

for $X, Y, Z \in T_uTM$. Define additional 3-forms on TM by

$$\begin{aligned} \rho(X, Y, Z) &:= \alpha(X, Y, Z) - \alpha(\theta X, \theta Y, Z) - \alpha(\theta Y, \theta Z, X) - \alpha(\theta Z, \theta X, Y), \\ \widehat{\rho}(X, Y, Z) &:= \alpha(\theta X, Y, Z) + \alpha(\theta Y, Z, X) + \alpha(\theta Z, X, Y) - \beta(X, Y, Z). \end{aligned}$$

(vi) In the following we construct a local frame field $(E_1, .., E_8)$ on $TM \setminus \{0\}$. First we have two globally defined vector fields on $TM \setminus \{0\}$

$$E_1 := N^v \qquad \text{and} \qquad E_2 := IE_1 = N^h.$$

The remaining vector fields will be defined only locally. Choose a local positive orthonormal basis $\{e_1, .., e_4\}$ of TM and denote by $v(e_i)$ the vertical lift to $V \subset TTM$, $i = 1, .., 4$. For $0 \neq u \in TM$ we write $\lambda_i(u) := g(u, e_i) \in \mathbb{R}$, such that $u = \sum \lambda_i(u) e_i$ holds. This yields

$$E_1(u) = N^v(u) = \frac{1}{r(u)} v_u(u) = \frac{1}{r(u)} \sum_i \lambda_i(u) v_u(e_i).$$

Now we define an orthonormal basis E_1, E_3, E_5, E_7 for V by

$$\begin{aligned} E_3(u) &:= \frac{1}{r(u)}(-\lambda_2(u)v_u(e_1) + \lambda_1(u)v_u(e_2) - \lambda_4(u)v_u(e_3) + \lambda_3(u)v_u(e_4)), \\ E_5(u) &:= \frac{1}{r(u)}(-\lambda_3(u)v_u(e_1) + \lambda_4(u)v_u(e_2) + \lambda_1(u)v_u(e_3) - \lambda_2(u)v_u(e_4)), \\ E_7(u) &:= \frac{1}{r(u)}(-\lambda_4(u)v_u(e_1) - \lambda_3(u)v_u(e_2) + \lambda_2(u)v_u(e_3) + \lambda_1(u)v_u(e_4)), \end{aligned}$$

and complete it to an orthonormal basis for TTM via

$$E_4 = IE_3, \quad E_6 = IE_5, \quad E_8 = IE_7.$$

In particular, we may choose $e_1, .., e_4 \in T_pM$ such that $e_1 = \frac{u}{\|u\|}$ holds, for a fixed $u \in T_pM \setminus \{0\}$. This yields

$$E_{2i}(u) = h_u(e_i) \qquad \text{and} \qquad E_{2i-1}(u) = v_u(e_i).$$

By choosing local normal coordinates around p, we may extend $e_1, .., e_4$ via parallel translation to a local basis field. Then

$$(\nabla^g_{e_i} e_j)_p = 0 \quad \text{ and hence } \quad [e_i, e_j]_p = 0$$

hold at p. We refer to the corresponding basis field $(E_1, .., E_8)$ as an adapted frame at u.

Lemma 6.3. In terms of the dual basis $(E^1, .., E^8)$ of $(E_1, .., E_8)$, the forms from Definition 6.2 are locally given by

(1) $\omega = E^{12} + E^{34} + E^{56} + E^{78}$,
(2) $\alpha = E^{357}$,
(3) $\beta = E^{468}$,
(4) $\rho = E^{357} - E^{368} - E^{467} - E^{458}$,
(5) $\widehat{\rho} = E^{367} + E^{358} + E^{457} - E^{468}$,
(6) $\mu^v = E^1$ and $\mu^h = E^2$.

□

Definition 6.4. From the previous Lemma we see immediately that the restriction of the forms

$$\varphi := \rho + \mu^h \wedge \omega \quad \text{ and } \quad \psi := \frac{1}{2}\omega^2 - \mu^v \wedge \mu^h \wedge \omega - \mu^h \wedge \widehat{\rho}$$

to T^1M defines a G_2-structure on T^1M. Moreover, we obtain a Spin(7)-structure on $TM \setminus \{0\}$ via

$$\Psi := \psi + \mu^v \wedge \varphi = \frac{1}{2}\omega^2 - \mu^h \wedge \widehat{\rho} + \mu^v \wedge \rho.$$

To study the type of structure that is induced by the forms φ and Ψ, we compute the exterior derivative of the dual 1-forms E^k of an adapted frame at u.

Lemma 6.5. For the horizontal 1-forms we have at u

$$\begin{aligned} dE^2 &= \frac{1}{r}(E^{34} + E^{56} + E^{78}), \\ dE^4 &= \frac{1}{r}(E^{23} + E^{58} + E^{67}), \\ dE^6 &= \frac{1}{r}(E^{25} - E^{38} - E^{47}), \\ dE^8 &= \frac{1}{r}(E^{27} + E^{36} + E^{45}). \end{aligned}$$

The analogue for the vertical forms involves the curvature R of (M, g). Let

$$\begin{aligned} \Omega_{2k-1} :=&\ R_{121k}E^{24} + R_{131k}E^{26} + R_{141k}E^{28} \\ &+ R_{231k}E^{46} + R_{241k}E^{48} + R_{341k}E^{68}, \end{aligned}$$

then we have at u

$$\begin{aligned}
dE^1 &= 0,\\
dE^3 &= r\Omega_3 + \frac{1}{r}E^{13} + \frac{2}{r}E^{57},\\
dE^5 &= r\Omega_5 + \frac{1}{r}E^{15} - \frac{2}{r}E^{37},\\
dE^7 &= r\Omega_7 + \frac{1}{r}E^{17} + \frac{2}{r}E^{35}.
\end{aligned}$$

PROOF: First note that $dE^k(E_i, E_j) = -E^k[E_i, E_j]$ holds. For E^k horizontal we find the following:

(i) By integrability of the vertical distribution we get immediately $dE^k(E_i, E_j) = 0$, for E_i, E_j vertical.

(ii) Suppose E_i, E_j are both horizontal. Recall that $\lambda_j(u) = g(u, e_j)$ holds by Definition 6.2 (vi). By construction of the local vector fields e_j, we see that λ_j is invariant under parallel transport, which yields

$$h_u(e_i) \cdot \lambda_j = 0. \tag{1}$$

Similarly, r is invariant under parallel transport, hence $h_u(e_i) \cdot r = 0$. Now extending the Lie-bracket $[E_i, E_j]$ by using the linear combination for E_i, E_j from Definition 6.2 (vi), we obtain essentially summands of the form $[h(e_i), h(e_j)]_u = h_u([e_i, e_j]) - v_u(R(e_i, e_j)u) = -v_u(R(e_i, e_j)u)$. Here we used Lemma 6.1 and that $[e_i, e_j] = 0$ at $p := \pi(u)$. Then the horizontality of E^k yields again $dE^k(E_i, E_j) = 0$.

(iii) Now let E_i be horizontal and E_j be vertical. First observe that

$$v_u(e_l) \cdot \lambda_k = \left.\frac{d}{dt}\right|_{t=0} g(u + te_l, e_k) = \delta_{lk}$$

holds. Since $u = r(u)e_1$, we get

$$v_u(e_l) \cdot r = \left.\frac{d}{dt}\right|_{t=0} \sqrt{g(u + te_l, u + te_l)} = \delta_{1l},$$

yielding

$$v_u(e_l) \cdot \frac{\lambda_k}{r} = \frac{1}{r}(\delta_{lk} - \delta_{1l}\delta_{1k}). \tag{2}$$

Using (2) and Lemma 6.1 together with $(\nabla^g_{e_i} e_j)_p = 0$, we computes at u,

$$\begin{array}{llll}
[E_1, E_2] = 0, & [E_1, E_4] = 0, & [E_1, E_6] = 0, & [E_1, E_8] = 0,\\
[E_2, E_3] = -\frac{1}{r}E_4, & [E_2, E_5] = -\frac{1}{r}E_6, & [E_2, E_7] = -\frac{1}{r}E_8, & [E_3, E_4] = -\frac{1}{r}E_2,\\
[E_3, E_6] = -\frac{1}{r}E_8, & [E_3, E_8] = \frac{1}{r}E_6, & [E_4, E_5] = -\frac{1}{r}E_8, & [E_4, E_7] = \frac{1}{r}E_6,\\
[E_5, E_6] = -\frac{1}{r}E_2, & [E_5, E_8] = -\frac{1}{r}E_4, & [E_6, E_7] = -\frac{1}{r}E_4, & [E_7, E_8] = -\frac{1}{r}E_2,
\end{array}$$

and obtain the above formulas for the horizontal forms.

Now consider the case where E^k is vertical.

(iv) Applying Lemma 6.1 we get for horizontal $E_{2i}(u) = h_u(e_i)$ and $E_{2j}(u) =$

$h_u(e_j)$

$$\begin{aligned} dE^{2k-1}(E_{2i}(u), E_{2j}(u)) &= h_u(e_i)\cdot E^{2k-1}(h(e_j)) - h_u(e_j)\cdot E^{2k-1}(h(e_i)) \\ &\quad - E^{2k-1}[h(e_i), h(e_j)]_u \\ &= -E^{2k-1}[h(e_i), h(e_j)]_u = E^{2k-1}v_u(R(e_i,e_j)u) \\ &= r(u)R_{ij1k}, \end{aligned}$$

where we used that $E_{2k-1}(u) = v_u(e_k)$ and $u = r(u)e_1$.

(v) The mixed terms are

$$\begin{aligned} dE^k(E_{2i}(u), E_{2j-1}(u)) &= dE^k(h_u(e_i), v_u(e_j)) \\ &= h_u(e_i)\cdot E^k(v(e_j)) - v_u(e_j)\cdot E^k(h_{e_i}) - E^k[h(e_i), v(e_j)]_u \\ &= 0, \end{aligned}$$

since $E^k(v(e_j))$ is horizontally constant by (1), $E^k(h_{e_i}) = 0$ and $[h(e_i), v(e_j)]_u = 0$ by Lemma 6.1.

(vi) Vertical terms can be computed by formula (2) and

$$dE^{2k-1}(E_{2i-1}(u), E_{2j-1}(u)) = v_u(e_i)\cdot E^{2k-1}(v(e_j)) - v_u(e_j)\cdot E^{2k-1}(v(e_i)).$$

The values for $dE^{2k-1}(E_{2i-1}(u), E_{2j-1}(u))$ are listed in the following table:

E_{2i-1}	E_{2j-1}	dE^1	dE^3	dE^5	dE^7
E_1	E_3	0	$\frac{1}{r(u)}$	0	0
E_1	E_5	0	0	$\frac{1}{r(u)}$	0
E_1	E_7	0	0	0	$\frac{1}{r(u)}$
E_3	E_5	0	0	0	$\frac{2}{r(u)}$
E_3	E_7	0	0	$\frac{-2}{r(u)}$	0
E_5	E_7	0	$\frac{2}{r(u)}$	0	0

Now we can easily verify the above formulas for the vertical forms.

□

Corollary 6.6. In an adapted frame we compute at $u \in TM \setminus \{0\}$

$$\begin{aligned} d\mu^v &= 0, \text{ moreover } \mu^v = dr. \\ d\omega &= 0, \\ d\rho &= r(\Omega_3 \wedge E^{57} - \Omega_5 \wedge E^{37} + \Omega_7 \wedge E^{35}) + r\mathrm{ric}_{11}\mu^h \wedge \beta \\ &\quad + \frac{2}{r}\mu^v \wedge \alpha - \frac{2}{r}\mu^h \wedge \beta + \frac{1}{r}\mu^v \wedge \rho - \frac{2}{r}\mu^h \wedge \widehat{\rho}, \\ d\widehat{\rho} &= \frac{1}{r}\mu^h \wedge (\rho - \alpha) + \frac{2}{r}\mu^v \wedge (\widehat{\rho} + \beta) + \frac{3}{r}\mu^h \wedge \alpha \\ &\quad + r(\mathrm{ric}_{12}E^3 + \mathrm{ric}_{13}E^5 + \mathrm{ric}_{14}E^7) \wedge \beta + r\mu^h \wedge \Omega, \end{aligned}$$

where

$$\Omega := R_{1212}(E^{467}+E^{458}) + R_{1213}(E^{348}-E^{568}) + R_{1214}(E^{678}-E^{346})$$
$$+ R_{1313}(E^{368}+E^{467}) + R_{1314}(E^{456}-E^{478}) + R_{1414}(E^{368}+E^{458}),$$

and ric is the Ricci tensor of (M,g). Moreover, the Lee form of the Spin(7)-structure is given by

$$\Theta := *(*d\Psi \wedge \Psi) = 2r(\mathrm{ric}_{11}E^1 + \mathrm{ric}_{12}E^3 + \mathrm{ric}_{13}E^5 + \mathrm{ric}_{14}E^7).$$

Proof: The equation $\mu^v = dr$ is easily verified, and implies $d\mu^v = 0$, which corresponds to $dE^1 = 0$ in Lemma 6.5. The formula for dE^2 may be rewritten as $d\mu^h = \frac{1}{r}(\omega - \mu^v \wedge \mu^h)$, which yields

$$d\omega = dr \wedge d\mu^h - \mu^v \wedge d\mu^h = 0.$$

The formulas for $d\rho$, $d\widehat{\rho}$ and Θ are verified in a direct computation, using Lemma 6.5 and the local form for ρ, $\widehat{\rho}$ and Ψ from Lemma 6.3.

□

In particular we found an interpretation of Ricci flatness in terms of special geometries. Namely, the vanishing of the Lee form Θ yields $\mathrm{ric}_{1i} = 0$ for *any* orthonormal basis $\{e_1, .., e_4\}$, and hence $\mathrm{ric} = 0$. Therefore the Spin(7)-structure from Definition 6.4 on $TM \setminus \{0\}$ is balanced, i.e. $\Theta = 0$, if and only if (M,g) is Ricci flat.

We can modify this result to give a characterization of Einstein manifolds (M,g) with arbitrary Einstein constant $\lambda \in \mathbb{R}$. First observe that changing the Cayley frame to $E_i(\lambda) = e^{-\frac{\lambda}{4}r^2}E_i$, corresponds to changing the structure tensors and Hodge operator into

$$\Psi_\lambda = e^{\lambda r^2}\Psi, \qquad \widehat{g}_\lambda = e^{\frac{\lambda}{2}r^2}\widehat{g} \qquad \text{and} \qquad *_\lambda = e^{\frac{7\lambda}{4}r^2} *.$$

Then $d\Psi_\lambda = 2\lambda r e^{\lambda r^2}\mu^v \wedge \Psi + e^{\lambda r^2} d\Psi = e^{\lambda r^2}(2\lambda r\mu^v \wedge \psi + d\Psi)$ and $*_\lambda d\Psi_\lambda = e^{\frac{11\lambda}{4}r^2}(\frac{2\lambda}{7}r\varphi + *d\Psi)$, since $\psi \wedge \varphi = 7E^{2345678}$. Now the Lee form satisfies

$$\begin{aligned}\Theta_\lambda = *_\lambda(*_\lambda d\Psi_\lambda \wedge \Psi_\lambda) &= e^{\frac{11\lambda}{2}r^2}(\frac{2\lambda}{7}r * (\varphi \wedge \psi) + *(*d\Psi \wedge \Psi))\\ &= e^{\frac{11\lambda}{2}r^2}(-2\lambda r\mu^v + \Theta)\\ &= 2re^{\frac{11\lambda}{2}r^2}(-\lambda\mu^v + \mathrm{ric}_{11}E^1 + \mathrm{ric}_{12}E^3 + \mathrm{ric}_{13}E^5 + \mathrm{ric}_{14}E^7)\end{aligned}$$

and we proved:

Theorem 6.7. (M,g) is Einstein with $\mathrm{ric} = \lambda g$ if and only if the Lee form of the Spin(7)-structure $\Psi_\lambda := e^{\lambda r^2}\Psi$ on $TM \setminus \{0\}$ vanishes. In particular, (M,g) is Ricci flat if and only if the Lee form of Ψ vanishes.

□

For the rest of this section we will study the induced G_2-structure φ on the unit tangent bundle $i : T^1M \subset TM$. Since $\psi = i^*\Psi$, the Lee form of the G_2 structure is given by Corollary 6.6 by the formula

$$\theta := *(*d\psi \wedge \psi) = i^*\Theta = 2(\mathrm{ric}_{12}E^3 + \mathrm{ric}_{13}E^5 + \mathrm{ric}_{14}E^7).$$

Hence the G_2-structure has vanishing Lee form if and only if (M, g) is Einstein. Surprisingly, $\theta = 0$ automatically implies the vanishing of the $\mathfrak{g}_2$-component of the intrinsic torsion. In fact we get from Corollary 6.6 and since $d\omega = 0$ and $d\mu^h \wedge \widehat{\rho} = 0$

$$\begin{aligned} d\psi &= d(\frac{1}{2}i^*\omega^2 - \mu^h \wedge \widehat{\rho}) = -d\mu^h \wedge \widehat{\rho} + \mu^h \wedge d\widehat{\rho} \\ &= \mu^h \wedge (\mathrm{ric}_{12}E^3 + \mathrm{ric}_{13}E^5 + \mathrm{ric}_{14}E^7) \wedge \beta \\ &= \frac{1}{2}\mu^h \wedge \theta \wedge \beta. \end{aligned}$$

Therefore (M, g) is Einstein if and only if the G_2-structure is hypo. Computing

$$\begin{aligned} d\varphi &= d\rho + d\mu^h \wedge i^*\omega = d\rho + i^*\omega^2 \\ &= \Omega_3 \wedge E^{57} - \Omega_5 \wedge E^{37} + \Omega_7 \wedge E^{35} + \mathrm{ric}_{11}\mu^h \wedge \beta \\ &\quad - 2\mu^h \wedge \beta - 2\mu^h \wedge \widehat{\rho} + i^*\omega^2 \\ &= \Omega_3 \wedge E^{57} - \Omega_5 \wedge E^{37} + \Omega_7 \wedge E^{35} + (\mathrm{ric}_{11} - 2)\mu^h \wedge \beta + 2\psi, \end{aligned}$$

shows that neither $d\varphi = 0$ nor $d\varphi = \lambda\psi$ is possible. The $\mathbb{R}\mathrm{id}$-component of the G_2-structure corresponds to $d\varphi \wedge \varphi$. To see this, observe that $d\varphi = 3D_\psi(\mathcal{T})$ by Proposition 3.38 and that the $\mathbb{R}\mathrm{id}$-component is mapped to Λ^4_1, which is identified via $\varphi \wedge\ .\ : \Lambda^4 \to \Lambda^7$ with Λ^7 by Schur's Lemma. Now The first Bianchi identity yields

$$\begin{aligned} d\varphi \wedge \varphi &= (\Omega_3 \wedge E^{57} - \Omega_5 \wedge E^{37} + \Omega_7 \wedge E^{35}) \wedge \varphi + (\mathrm{ric}_{11} + 12)E^{234567} \\ &= (R_{1221} + R_{1331} + R_{1441} + R_{3421} + R_{4231} + R_{2341} + \mathrm{ric}_{11} + 12)E^{234567} \\ &= 2(\mathrm{ric}_{11} + 6)E^{234567}. \end{aligned}$$

In summary we have, cf. **[1]** Thm. 3.3,

Theorem 6.8. The G_2-structure φ on T^1M with intrinsic torsion $\mathcal{T}$ satisfies

$$\begin{aligned} \varphi \text{ is hypo} \quad &\Leftrightarrow \quad \mathcal{T} \in \mathbb{R}\mathrm{id} \oplus S^2_0 \\ &\Leftrightarrow \quad \mathcal{T} \in \mathbb{R}\mathrm{id} \oplus S^2_0 \oplus \mathfrak{g}_2 \\ &\Leftrightarrow \quad (M, g) \text{ is Einstein.} \end{aligned}$$

Moreover, $\mathcal{T} \in S^2_0$ if and only if (M, g) is Einstein with $\lambda = -6$. The structure is never parallel or nearly parallel.

□

If (M,g) is Einstein, then there exists an atlas for M with real analytic transition functions, so that the metric g is real analytic in each chart, cf. Theorem 5.26 in [**7**]. Hence any adapted frame from Definition 6.2 (vi) is real analytic, which proves that the hypo G_2-structure φ on T^1M is real analytic. Now Corollary 4.22 yields

Theorem 6.9. Every compact Einstein manifold (M^4, g) admits a parallel Spin(7)-structure on $I \times T^1M^4$, for some interval $I \subset \mathbb{R}$.

□

$SU(2)$ and $SU(3)$-Structures on TM^3

In this section we will describe a construction which yields certain $SU(2)$-structures on T^1M and $SU(3)$-structures on $TM \setminus \{0\}$, where (M,g) is a 3-dimensional Riemannian manifold with tangent bundle $\pi : TM \to M$. Like in the previous section, the Einstein condition for (M,g) is encoded in certain torsion components of the structures. Since M is 3-dimensional, the Einstein condition is of course much more restrictive than in the 4-dimensional case. The tensors K, $\widehat{g}$, θ, I, ω, r, μ^v, μ^h, N^v and N^v are defined like in Definition 6.2 from the previous section.

Definition 6.10. Let ε_M be the induced volume form on (M,g). We define the following forms on $TM \setminus \{0\}$:

$$\begin{aligned}
\beta_2(X,Y) &:= \varepsilon_M(KN^v, KX, KY),\\
\beta_3(X,Y) &:= \varepsilon_M(\pi_* N^h, \pi_* X, \pi_* Y),\\
\omega_2 &:= \beta_2 - \beta_3,\\
\omega_3(X,Y) &:= \beta_2(\theta X, Y) - \beta_2(\theta Y, X),\\
\rho &:= \mu^v \wedge \omega_2 - \mu^h \wedge \omega_3,\\
\widehat{\rho} &:= \mu^v \wedge \omega_3 + \mu^h \wedge \omega_2.
\end{aligned}$$

Moreover, we define forms on $i : T^1M \hookrightarrow TM$ by

$$\begin{aligned}
\omega_1 &:= i^*\omega.\\
\alpha &:= \mu^h.
\end{aligned}$$

Definition 6.11. For a given basis field $\{e_1, e_2, e_3\}$ on $U \subset M$, we wish to associate a basis field on the open subset

$$\tilde{U} := \{u \in \pi^{-1}(U) \mid \lambda_1^2 + \lambda_2^2 \neq 0, \text{ where } u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3\} \subset TM \setminus \{0\}.$$

For $u \in \tilde{U}$ with $u = \sum \lambda_i e_i$, let

$$E_1(u) := \frac{1}{r(u)}\big(\lambda_1 v_u(e_1) + \lambda_2 v_u(e_2) + \lambda_3 v_u(e_3)\big),$$
$$E_3(u) := \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2}}\big(-\lambda_2 v_u(e_1) + \lambda_1 v_u(e_2)\big),$$
$$E_5(u) := \frac{1}{r(u)\sqrt{\lambda_1^2+\lambda_2^2}}\big(-\lambda_1\lambda_3 v_u(e_1) - \lambda_2\lambda_3 v_u(e_2) + (\lambda_1^2+\lambda_2^2) v_u(e_3)\big).$$

Then E_1, E_3 and E_5 are orthonormal and we obtain an orthonormal basis field via

$$E_2 := IE_1, \quad E_4 := IE_3 \quad \text{and} \quad E_6 := IE_5.$$

Note also that $E_1(u) = \frac{1}{r(u)} v_u(u) = N^v(u)$ and $E_2(u) = IN^v(u) = N^h(u)$ holds. In particular, we may choose $e_1, e_2, e_3 \in T_pM$ such that $e_1 = \frac{u}{\|u\|}$ holds, for a fixed vector $u \neq 0$. This yields

$$E_{2i}(u) = h_u(e_i) \quad \text{and} \quad E_{2i-1}(u) = v_u(e_i).$$

By choosing local normal coordinates around p, we may extend e_1, e_2, e_3 via parallel translation to a local basis field. Then

$$(\nabla^g_{e_i} e_j)_p = 0 \quad \text{and hence} \quad [e_i, e_j]_p = 0$$

hold at p. We refer to the corresponding basis field $(E_1, .., E_6)$ as an adapted frame at u.

The following Lemma can be easily verified:

Lemma 6.12. In terms of the dual basis $(E^1, .., E^6)$ of $(E_1, .., E_6)$, the forms from Definition 6.2 and 6.10 are locally given by

(1) $\omega = E^{12} + E^{34} + E^{56}$,
(2) $\beta_2 = E^{35}$,
(3) $\beta_3 = E^{46}$,
(4) $\alpha = E^2$,
(5) $\omega_1 = E^{34} + E^{56}$,
(6) $\omega_2 = E^{35} - E^{46}$,
(7) $\omega_3 = E^{36} + E^{45}$,
(8) $\rho = E^{135} - E^{146} - E^{245} - E^{236}$,
(9) $\widehat{\rho} = E^{145} + E^{136} + E^{235} - E^{246}$.

□

Definition 6.13. From the previous Lemma we see immediately that the restriction of $(\alpha, \omega_1, \omega_2, \omega_3)$ to T^1M defines a $SU(2)$-structure on T^1M. Moreover, (ω, ρ) defines a $SU(3)$-structure on $TM \setminus \{0\}$.

To study the type of these structures, we have to compute the analogue of Lemma 6.5:

Lemma 6.14. Fix $u \in TM \setminus \{0\}$ and let $E_1, .., E_6$ be an adapted frame at u with dual frame $E^1, .., E^6$. Then the following formulas hold at u:

$$\begin{aligned} dE^2 &= \frac{1}{r}(E^{34} + E^{56}), \\ dE^4 &= \frac{1}{r}E^{23}, \\ dE^6 &= \frac{1}{r}E^{25}. \end{aligned}$$

The exterior derivative of the vertical forms involves the curvature R of (M, g). Let

$$\Omega_{2k-1} := R_{121k}E^{24} + R_{131k}E^{26} + R_{231k}E^{46},$$

then we have at u

$$\begin{aligned} dE^1 &= 0, \\ dE^3 &= r\Omega_3 + \frac{1}{r}E^{13}, \\ dE^5 &= r\Omega_5 + \frac{1}{r}E^{15}. \end{aligned}$$

Proof: The proof is completely analogue to the proof of Lemma 6.5. We use $dE^k(E_i, E_j) = -E^k[E_i, E_j]$ and

$$h_u(e_i) \cdot r = h_u(e_i) \cdot \lambda_j = 0,$$

$$v_u(e_i) \cdot r = \delta_{1i} \quad \text{and} \quad v_u(e_i) \cdot \lambda_j = \delta_{ij},$$

which imply

$$\begin{aligned} v_u(e_i) \cdot \frac{\lambda_a}{r} &= \frac{1}{r}(\delta_{ia} - \delta_{1i}\delta_{1a}), \\ v_u(e_i) \cdot \frac{\lambda_a}{\sqrt{\lambda_1^2 + \lambda_2^2}} &= \frac{1}{r}(\delta_{ia} - \delta_{1i}\delta_{1a}), \\ v_u(e_i) \cdot \frac{\lambda_a\lambda_b}{r\sqrt{\lambda_1^2 + \lambda_2^2}} &= \frac{1}{r}(\delta_{ia}\delta_{1b} + \delta_{ib}\delta_{1a} - 2\delta_{1a}\delta_{1b}\delta_{1i}). \end{aligned}$$

Now we obtain

$$\begin{array}{lll} [E_1, E_2] = 0, & [E_1, E_4] = 0, & [E_1, E_6] = 0, \\ [E_2, E_3] = -\frac{1}{r}E_4, & [E_2, E_5] = -\frac{1}{r}E_6, & [E_3, E_4] = -\frac{1}{r}E_2, \\ [E_3, E_6] = 0, & [E_4, E_5] = 0, & [E_5, E_6] = -\frac{1}{r}E_2, \end{array}$$

which yields the desired formula for dE^{2k}. To compute dE^{2k-1}, observe that

$$dE^{2k-1}(E_{2i-1}, E_{2j}) = 0$$

and by Lemma 6.1

$$\begin{aligned}
dE^{2k-1}(E_{2i}(u), E_{2j}(u)) &= dE^{2k-1}(h_u(e_i), h_u(e_j)) \\
&= h_u(e_i) \cdot E^{2k-1}(h(e_j)) - h_u(e_j) \cdot E^{2k-1}(h(e_i)) \\
&\quad - E^{2k-1}[h(e_i), h(e_j)]_u \\
&= -E^{2k-1}[h(e_i), h(e_j)]_u \\
&= E^{2k-1}(v_u(R(e_i, e_j)u) \\
&= r(u) R_{ij1k}.
\end{aligned}$$

Using

$$\begin{aligned}
dE^{2k-1}(E_{2i-1}(u), E_{2j-1}(u)) &= dE^{2k-1}(v_u(e_i), v_u(e_j)) \\
&= v_u(e_i) \cdot E^{2k-1}(v(e_j)) - v_u(e_j) \cdot E^{2k-1}(v(e_i)) \\
&\quad - E^{2k-1}[v(e_i), v(e_j)]_u \\
&= v_u(e_i) \cdot E^{2k-1}(v(e_j)) - v_u(e_j) \cdot E^{2k-1}(v(e_i)),
\end{aligned}$$

we compute

E_{2i-1}	E_{2j-1}	dE^1	dE^3	dE^5
E_1	E_3	0	$\frac{1}{r(u)}$	0
E_1	E_5	0	0	$\frac{1}{r(u)}$
E_3	E_5	0	0	0

and obtain the above formulas for the vertical forms.

□

We can now compute the exterior derivatives of the forms from Definition 6.10.

Proposition 6.15. Let $E_1, .., E_6$ be an adapted frame at $u \in TM \setminus \{0\}$ and let

$$\Omega_{2k-1} := R_{121k} E^{24} + R_{131k} E^{26} + R_{231k} E^{46},$$

where R is the curvature tensor of g. If ric denotes the Ricci tensor of R, then the following formulas hold at u:

$$
\begin{aligned}
d\mu^v &= 0,\\
d\alpha &= \frac{1}{r}\omega_1,\\
d\omega &= 0,\\
d\omega_1 &= \frac{1}{r}\mu^v \wedge \omega_1,\\
d\omega_2 &= r(\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5) - \frac{1}{r}\alpha \wedge \omega_3 + \frac{2}{r}\mu^v \wedge \beta_2,\\
d\omega_3 &= -r\mathrm{ric}_{11}\alpha \wedge \beta_3 + \frac{2}{r}\alpha \wedge \beta_2 + \frac{1}{r}\mu^v \wedge \omega_3,\\
d(\alpha \wedge \omega_2) &= -r\alpha \wedge (\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5) + \frac{2}{r}\mu^v \wedge \alpha \wedge \beta_2,\\
d(\alpha \wedge \omega_3) &= \frac{1}{r}\mu^v \wedge \alpha \wedge \omega_3,\\
d\rho &= r\mu^v \wedge (E^3 \wedge \Omega_5 - E^5 \wedge \Omega_3),\\
d\widehat{\rho} &= r\mathrm{ric}_{11}\mu^v \wedge \alpha \wedge \beta_3 - r\alpha \wedge (\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5).
\end{aligned}
$$

PROOF: The first three equations follow immediately from Lemma 6.14, since

$$d\omega = -\mu^v \wedge dE^2 + d(E^{34}+E^{56}) = -\mu^v \wedge dE^2 + d(r dE^2) = -\mu^v \wedge dE^2 + \mu^v \wedge dE^2 = 0.$$

For the other equations we compute $d\omega_1 = dr \wedge d\alpha = \frac{1}{r}\mu^v \wedge \omega_1$,

$$
\begin{aligned}
d\omega_2 &= d(E^{35} - E^{46})\\
&= r\Omega_3 \wedge E^5 + \frac{1}{r}E^{135} - E^3 \wedge (r\Omega_5 + \frac{1}{r}E^{15}) - \frac{1}{r}E^{236} + \frac{1}{r}E^{425}\\
&= r(\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5) - \frac{1}{r}\alpha \wedge \omega_3 + \frac{2}{r}\mu^v \wedge \beta_2
\end{aligned}
$$

and

$$
\begin{aligned}
d\omega_3 &= d(E^{36} + E^{45})\\
&= (r\Omega_3 + \frac{1}{r}E^{13}) \wedge E^6 - \frac{1}{r}E^{325} + \frac{1}{r}E^{235} - E^4 \wedge (r\Omega_5 + \frac{1}{r}E^{15})\\
&= r(\Omega_3 \wedge E^6 - E^4 \wedge \Omega_5) + \frac{2}{r}\alpha \wedge \beta_2 + \frac{1}{r}\mu^v \wedge \omega_3\\
&= -r\mathrm{ric}_{11}\alpha \wedge \beta_3 + \frac{2}{r}\alpha \wedge \beta_2 + \frac{1}{r}\mu^v \wedge \omega_3.
\end{aligned}
$$

Since $\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_3 = 0$, we get $d(\alpha \wedge \omega_2) = -\alpha \wedge d\omega_2$ and $d(\alpha \wedge \omega_3) = -\alpha \wedge d\omega_3$ and hence the seventh and eighth equation. Eventually,

$$\rho = \mu^v \wedge \omega_2 - \mu^h \wedge \omega_3 \quad \text{and} \quad \widehat{\rho} = \mu^v \wedge \omega_3 + \mu^h \wedge \omega_2$$

yield

$$
\begin{aligned}
d\rho &= -\mu^v \wedge d\omega_2 - d(\alpha \wedge \omega_3)\\
&= -r\mu^v \wedge (\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5) + \frac{1}{r}\mu^v \wedge \alpha \wedge \omega_3 - \frac{1}{r}\mu^v \wedge \alpha \wedge \omega_3\\
&= r\mu^v \wedge (E^3 \wedge \Omega_5 - E^5 \wedge \Omega_3)
\end{aligned}
$$

and

$$
\begin{aligned}
d\widehat{\rho} &= -\mu^v \wedge d\omega_3 + d(\alpha \wedge \omega_2) \\
&= r\mathrm{ric}_{11}\mu^v \wedge \alpha \wedge \beta_3 - \frac{2}{r}\mu^v \wedge \alpha \wedge \beta_2 - r\alpha \wedge (\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5) + \frac{2}{r}\mu^v \wedge \alpha \wedge \beta_2 \\
&= r\mathrm{ric}_{11}\mu^v \wedge \alpha \wedge \beta_3 - r\alpha \wedge (\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5).
\end{aligned}
$$

□

We can now study the $SU(3)$-structure (ω, ρ) on $TM \setminus \{0\}$ from Definition 6.13. Since always $d\omega = 0$ holds, the intrinsic torsion satisfies by Lemma 3.23 and Proposition 3.25

$$\mathcal{T} \in I_0\mathfrak{su}(3) \oplus \mathfrak{su}(3).$$

From the proof of Theorem 3.28 we see that

$$
\begin{aligned}
\mathcal{T} \in \mathfrak{su}(3) \text{ and } \eta = 0 \quad &\Leftrightarrow \quad d\rho = 0, \\
\mathcal{T} \in I_0\mathfrak{su}(3) \text{ and } \eta = 0 \quad &\Leftrightarrow \quad d\widehat{\rho} = 0.
\end{aligned}
$$

By Proposition 6.15 we have

$$
\begin{aligned}
d\rho &= r\mu^v \wedge (E^3 \wedge \Omega_5 - E^5 \wedge \Omega_3) \\
&= r(R_{1231}E^{1234} + R_{1331}E^{1236} - R_{3213}E^{1346} \\
&\quad + R_{1221}E^{1245} - R_{1321}E^{1256} + R_{2312}E^{1456})
\end{aligned}
$$

and hence $d\rho = 0$ is equivalent to $\mathrm{ric} = 0$. Similarly,

$$
\begin{aligned}
d\widehat{\rho} &= r\mathrm{ric}_{11}\mu^v \wedge \alpha \wedge \beta_3 - r\alpha \wedge (\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5) \\
&= r(\mathrm{ric}_{11}E^{1246} + R_{2312}E^{2456} - R_{3213}E^{2346}) \\
&= 0
\end{aligned}
$$

is equivalent to $\mathrm{ric} = 0$. Hence we showed

Corollary 6.16. The $SU(3)$-structure on $TM \setminus \{0\}$ from Definition 6.13 is always of type $\mathcal{T} \in I_0\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ and

$$\mathcal{T} \in \mathfrak{su}(3),\ \eta = 0 \quad \Leftrightarrow \quad \mathcal{T} \in I_0\mathfrak{su}(3),\ \eta = 0 \quad \Leftrightarrow \quad \mathcal{T} = 0,\ \eta = 0 \quad \Leftrightarrow \quad R = 0,$$

where R is the curvature tensor of (M, g).

□

Now consider the $SU(2)$-structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on T^1M from Definition 6.13. By Proposition 6.15

$$d\omega_1 = 0 \quad \text{and} \quad d(\alpha \wedge \omega_3) = 0$$

always hold and

$$\begin{aligned} d\omega_2 &= \Omega_3 \wedge E^5 - E^3 \wedge \Omega_5 - \alpha \wedge \omega_3 \\ &= -R_{1221}E^{245} + R_{1321}E^{256} - R_{2312}E^{456} \\ &\quad - R_{1231}E^{234} - R_{1331}E^{236} + R_{3213}E^{346} \\ &\quad - E^{236} - E^{245}. \end{aligned}$$

Hence the equation $d\omega_2 = \lambda(\alpha \wedge \omega_3) = \lambda(E^{236} + E^{245})$, for some constant λ, is equivalent to $\mathrm{ric} = -2(1+\lambda)g$. Moreover,

$$\begin{aligned} d(\alpha \wedge \omega_2) &= -\alpha \wedge (\Omega_3 \wedge E^5 - E^3 \wedge \Omega_5) \\ &= R_{2312}E^{2456} - R_{3213}E^{2346} = 0 \end{aligned}$$

is equivalent to $\mathrm{ric}_{12} = \mathrm{ric}_{13} = 0$, i.e. (M, g) being Einstein. Note also that

$$d\omega_3 = -\mathrm{ric}_{11}\alpha \wedge \beta_3 + 2\alpha \wedge \beta_2 \neq 0,$$

i.e. the structure is never parallel.

Corollary 6.17. The $SU(2)$-structure on T^1M from Definition 6.13 always satisfies $d\omega_1 = 0$ and $d(\alpha \wedge \omega_3) = 0$. Moreover,

$$\begin{aligned} d(\alpha \wedge \omega_2) = 0 \quad &\Leftrightarrow \quad (M, g) \text{ is Einstein.} \\ d\omega_2 = \lambda(\alpha \wedge \omega_3) \quad &\Leftrightarrow \quad (M, g) \text{ is Einstein with } \mathrm{ric} = -2(1+\lambda)g. \end{aligned}$$

The structure is never parallel.

□

G_2-Structures on S^1-Bundles over M^6

Let (M^6, g, ω, I) be a Kähler manifold with canonical S^1-bundle

$$\pi : K \subset \Lambda^{(3,0)}T^*M^6 \to M^6.$$

In this section we define a G_2-structure on the total space K. Our approach is motivated by Example 1, p.84 from [**5**]. In addition to the result from [**5**], we study the possible torsion types and show that the G_2-structure is always hypo and hence a candidate to solve the corresponding embedding problem.

The connection 1-form $\mathcal{Z}^g$ on K, induced by the Levi-Civita connection, is a $\mathfrak{u}_1 = i\mathbb{R}$ valued 1-form and satisfies by Proposition 5.4

$$d\mathcal{Z}^g = i\pi^*\varrho,$$

where ϱ is the Ricci form of g. We denote by ξ the vertical lift of $i \in \mathfrak{u}_1$, i.e.

$$\xi_\Phi = R_{\Phi *} i,$$

for all $\Phi \in K$. Then we obtain a metric $\bar{g}$ on K by

$$\bar{g}(\xi, \xi) := 1, \quad \bar{g}(\xi, h^g X) := 0 \quad \text{and} \quad \bar{g}(h^g X, h^g Y) := g(X, Y),$$

where $X, Y \in TM^6$ and h^g denotes the horizontal lift to K w.r.t. the connection $\mathcal{Z}^g$. Since $\mathcal{Z}^g(\xi) = i$, the dual $\alpha := \xi \lrcorner \bar{g}$ of ξ is given by

$$\alpha = -i\mathcal{Z}^g.$$

The Kähler form pulls back to a 2-form

$$\bar{\omega} := \pi^* \omega$$

on K and we set

$$\bar{\sigma} := \frac{1}{2} \bar{\omega}^2.$$

There are tautological 3-forms on K, defined by

$$\begin{aligned} \rho(U, V, W) &:= \mathrm{Re}(\Phi)(\pi_* U, \pi_* V, \pi_* W), \\ \widehat{\rho}(U, V, W) &:= \mathrm{Im}(\Phi)(\pi_* U, \pi_* V, \pi_* W), \end{aligned}$$

for $U, V, W \in T_\Phi K$. Lifting a Cayley frame for the Kähler structure on M^6 to K and extending the lift by the vector field ξ, yields a Caley frame for the G_2-structure

Definition 6.18.

$$\begin{aligned} \varphi &:= \rho + \alpha \wedge \bar{\omega}, \\ \psi &:= \bar{\sigma} - \alpha \wedge \widehat{\rho}. \end{aligned}$$

Note that the metric of this structure is just $g(\varphi) = \bar{g}$. To compute the torsion type of this structure we need to compute the exterior derivatives of the tensors ρ and $\widehat{\rho}$. Using

$$\begin{aligned} d\rho(U_0, .., U_3) = & \sum_{k=0}^{3} (-1)^k U_k \cdot (\rho(U_1, .., \hat{U}_k, .., U_3)) \\ & + \sum_{0 \le k, l \le 3} (-1)^{k+l} \rho([U_k, U_l], U_1, .., \hat{U}_k, .., \hat{U}_l, .., U_3), \end{aligned}$$

together with Lemma 1.11 and $\xi \lrcorner \rho = 0$, we get

Lemma 6.19.

$$\begin{aligned} d\rho &= i\mathcal{Z}^g \wedge \widehat{\rho} = -\alpha \wedge \widehat{\rho}, \\ d\widehat{\rho} &= -i\mathcal{Z}^g \wedge \rho = \alpha \wedge \rho. \end{aligned}$$

□

The Kähler condition yields

$$d\bar{\omega} = d\bar{\sigma} = 0 \quad \text{and} \quad d\alpha = \pi^* \varrho.$$

Hence we compute

$$\begin{aligned} d\psi &= -\pi^* \varrho \wedge \widehat{\rho} + \alpha \wedge \alpha \wedge \rho \\ &= -\pi^* \varrho \wedge \widehat{\rho}. \end{aligned}$$

Since $\pi^* \varrho$ is a $(1,1)$ form w.r.t. the pullback of I to the horizontal distribution on K, and $\rho + i\widehat{\rho}$ is a $(3,0)$ form, we see that on the six-dimensional horizontal distribution the $(4,1)$ form $\pi^* \varrho \wedge (\rho + i\widehat{\rho})$ vanishes. Since the Ricci form $\pi^* \varrho$ is a real form, we obtain

$$\pi^* \varrho \wedge \rho = \pi^* \varrho \wedge \widehat{\rho} = 0$$

and, in particular, $d\psi = 0$. Now

$$d\varphi = -\alpha \wedge \widehat{\rho} + \pi^* \varrho \wedge \bar{\omega}$$

shows that $d\varphi \neq 0$ and we compute

$$\begin{aligned} & d\varphi = \lambda\psi \\ \Leftrightarrow \; & -\alpha \wedge \widehat{\rho} + \pi^*(\varrho \wedge \omega) = \lambda\bar{\sigma} - \lambda\alpha \wedge \widehat{\rho} \\ \Leftrightarrow \; & (\lambda - 1)\alpha \wedge \widehat{\rho} + \pi^*(\varrho \wedge \omega - \frac{1}{2}\lambda\omega^2) = 0 \\ \Leftrightarrow \; & \lambda = 1 \quad \text{and} \quad \varrho \wedge \omega = \frac{1}{2}\omega^2. \end{aligned}$$

By Schur's Lemma, $\omega\wedge \; : \Lambda^2 T^* M^6 \to \Lambda^4 T^* M^6$ defines an isomorphism. Hence $d\varphi = \lambda\psi$ is equivalent to $\lambda = 1$ and $\varrho = \frac{1}{2}\omega$. In summary we have, cf. **[5]** Example 1, p.84:

Theorem 6.20. The G_2-structure φ on K from Definition 6.18 is always hypo, but never parallel. The structure is nearly parallel with $d\varphi = \psi$ if and only if the underlying Kähler structure is Einstein with $\mathrm{ric} = \frac{1}{2}g$.

□

$SU(3)$-Structures on $S^1 \times S^1$-Bundles over M^4

Let (M^4, g, ω, I) be a Kähler manifold with canonical S^1-bundle

$$\pi_K : K \subset \Lambda^{(2,0)} T^* M^4 \to M^4.$$

The connection 1-form $\mathcal{Z}^g$ on K, induced by the Levi-Civita connection, is a $\mathfrak{u}_1 = i\mathbb{R}$ valued 1-form and satisfies by Proposition 5.4

$$d\mathcal{Z}^g = i\pi_K^* \varrho,$$

where ϱ is the Ricci form of g. More generally, every closed 2-form ϱ_P on M^4 such that

$$[\frac{\varrho_P}{2\pi}] \in H^2(M^4; \mathbb{Z}),$$

corresponds, up to isomorphism, to a S^1-bundle $\pi_P : P \to M^4$, together with a connection 1-form $\mathcal{Z}_P$, such that

$$d\mathcal{Z}_P = i\pi_P^* \varrho_P$$

holds, cf. [**44**]. Given such a form ϱ_P, we define a $SU(3)$-structure on the total space of $\pi : K\triangle P \to M^4$, where

$$K\triangle P := \{(\Phi, \Psi) \mid \pi_K(\Phi) = \pi_P(\Psi)\}$$

is the fibre product of K and P. Like in the previous section we have tautological 2-forms on K, given by

$$\begin{aligned}\omega_2(U, V) &:= \mathrm{Re}(\Phi)(\pi_* U, \pi_* V),\\ \omega_3(U, V) &:= \mathrm{Im}(\Phi)(\pi_* U, \pi_* V),\end{aligned}$$

for $U, V \in T_\Phi K$, which satisfy

$$\begin{aligned}d\omega_2 &= i\mathcal{Z}^g \wedge \omega_3,\\ d\omega_3 &= -i\mathcal{Z}^g \wedge \omega_2.\end{aligned}$$

We denote by ξ, ξ_P the vertical lift of $(i, 0), (0, i) \in \mathfrak{u}_1 \times \mathfrak{u}_1$ to K, P, respectively. Then we obtain a metric $\bar{g}$ on $K\triangle P$ by

$$\begin{aligned}\bar{g}(\xi, \xi) &:= \bar{g}(\xi_P, \xi_P) := 1\\ \bar{g}(\xi, \xi_P) &:= 0,\\ \bar{g}(\xi, hX) &:= \bar{g}(\xi_P, hX) := 0,\\ \bar{g}(hX, hY) &:= g(X, Y),\end{aligned}$$

where $X, Y \in TM^4$ and h denotes the horizontal lift to $K\triangle P$ w.r.t. the connection $(\mathcal{Z}^g, \mathcal{Z}_P)$. Then the dual 1-forms for ξ and ξ_P satisfy

$$\alpha := \xi \lrcorner \bar{g} = -i\mathcal{Z}^g \quad \text{and} \quad \alpha_P := \xi_P \lrcorner \bar{g} = -i\mathcal{Z}_P,$$

where we consider $\mathcal{Z}^g$ and $\mathcal{Z}_P$ as forms on $K\triangle P$ via the pull back under the canonical projections $K\triangle P \to K, P$. Hence the pull back of the tautological forms ω_2 and ω_3 to $K\triangle P$ satisfy

$$\begin{aligned}d\omega_2 &= -\alpha \wedge \omega_3,\\ d\omega_3 &= \alpha \wedge \omega_2\end{aligned}$$

and for the exterior derivatives of α and α_P we obtain

$$d\alpha = \pi^*\varrho \quad \text{and} \quad d\alpha_P = \pi^*\varrho_P.$$

We can easily find a Cayley frame to prove that

Definition 6.21.

$$\begin{aligned}\bar{\omega} &:= \alpha_P \wedge \alpha + \omega_3,\\ \rho &:= \alpha_P \wedge \pi^*\omega - \alpha \wedge \omega_2,\\ \widehat{\rho} &:= \alpha_P \wedge \omega_2 + \alpha \wedge \pi^*\omega,\end{aligned}$$

defines a $SU(3)$-structure on $K\triangle P$. Since $\pi^*\varrho_{(P)}$ is a $(1,1)$ form w.r.t. the pullback of I to the horizontal distribution on $K\triangle P$, and $\omega_2 + i\omega_3$ is a $(2,0)$ form, we see that on the four dimensional horizontal distribution the $(3,1)$ form $\pi^*\varrho_{(P)} \wedge (\omega_2 + i\omega_3)$ vanishes. Since the Ricci form $\pi^*\varrho$ is a real form, we obtain

$$\pi^*\varrho_{(P)} \wedge \omega_2 = \pi^*\varrho_{(P)} \wedge \omega_3 = 0.$$

Then

$$d\bar{\omega} = \pi^*\varrho_P \wedge \alpha - \alpha_P \wedge \pi^*\varrho + \alpha \wedge \omega_2 \neq 0$$

shows that the structure is never parallel, but satisfies

$$d\bar{\omega} \wedge \bar{\omega} = 0,$$

since $\omega_2 \wedge \omega_3 = 0$. The Kähler condition $d\omega = 0$ yields

$$d\rho = \pi^*(\varrho_P \wedge \omega) - \pi^*\varrho \wedge \omega_2 = \pi^*(\varrho_P \wedge \omega).$$

Hence the structure is hypo if $\varrho_P \wedge \omega = 0$. In contrast to the G_2-case, the structure turns out to be never nearly Kähler. In summary we have

Theorem 6.22. The $SU(3)$-structure on $K\triangle P$ from Definition 6.21 always satisfies $d\bar{\omega} \wedge \bar{\omega} = 0$. Moreover, the structure is hypo if in addition $\varrho_P \wedge \omega = 0$ holds. The structure is never parallel or nearly parallel.

□

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Köln, den ____________________ ____________________________

Sebastian Stock

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Köln, den ____________________ ____________________________

Sebastian Stock

Lebenslauf

Angaben zur Person:

Name:	Stock
Vorname:	Sebastian Wolfgang
Geburtsort:	Köln
Geburtsdatum:	22.08.1982
Nationalität:	deutsch
Familienstand:	ledig

Universitäts- und Schulausbildung:

seit 07/2009:	Stipendiand des Graduiertenkollegs 1269 der Universität zu Köln.
01/2008-06/2009:	Ph.D. Student an der McMaster University, Hamilton, Kanada.
04/2003-07/2007:	Mathematikstudium an der Universität zu Köln.
2002:	Allgemeinen Hochschulreife, Gymnasium der Stadt Kerpen.

Berufserfahrungen:

01/2008-06/2009:	Teaching Assistant an der McMaster University.
04/2004-07/2007:	Übungsgruppenleiter an der Universität zu Köln.
07/2002-04/2003:	Grundwehrdienst.

Köln, den ____________________

____
Sebastian Stock