Generalized Gradients of G-Structures and Kählerian Twistor Spinors

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Abstract

We consider problems related to two main research directions: on the one hand, generalized gradients and, on the other hand, a special class of spinors on Kähler spin manifolds.
We introduce generalized gradients in the general context of G-structures. They are natural first order differential operators acting on sections of vector bundles associated to irreducible G-representations. We study their geometric and analytic properties, in particular we show their conformal invariance and give a new proof of Branson's classification of minimal elliptic operators that are naturally constructed from generalized gradients.
On Kähler spin manifolds, Kählerian twistor spinors are a natural analogue of twistor spinors on Riemannian spin manifolds. They are defined as sections in the kernel of a first order differential operator adapted to the Kähler structure, called Kählerian twistor (Penrose) operator. We study the properties of Kählerian twistor spinors and give a complete description of compact simply-connected Kähler spin manifolds of constant scalar curvature carrying such spinors. We show that the existence of Kählerian twistor spinors is related to the lower bound of the spectrum of the Dirac operator.

Kurzzusammenfassung

Wir betrachten Fragestellungen bezüglich zweier Hauptforschungsrichtungen: einerseits verallgemeinerte Gradienten und andererseits eine spezielle Klasse von Spinoren auf Kählerschen Mannigfaltigkeiten.
Wir führen verallgemeinerte Gradienten auf G-Strukturen ein. Diese sind natürliche Differentialoperatoren erster Ordnung, die auf Schnitten von zu irreduziblen G-Darstellungen assozierten Vektorbündeln wirken. Wir untersuchen deren geometrischen und analytischen Eigenschaften, insbesondere zeigen wir deren konforme Invarianz und geben einen neuen Beweis für Bransons Klassifikation von minimalen elliptischen Differentialoperatoren, die natürlich aus verallgemeinerten Gradienten konstruiert werden.
Auf Kählerschen Spin-Mannigfaltigkeiten sind Kählersche Twistorspinoren ein natürliches Analogon von Twistorspinoren auf Riemannschen Spin-Mannigfaltigkeiten. Diese sind definiert als Schnitte im Kern eines an die Kählersche Struktur angepassten Differentialoperators erster Ordnung, des sogenanntes Kählerschen Twistor (Penrose)-Operators. Wir untersuchen die Eigenschaften der Kählerschen Twistorspinoren und geben eine vollständige Beschreibung der kompakten einfach-zusammenhängenden Kählerschen Spin-Mannigfaltigkeiten mit konstanter Skalarkrümmung, die solche Spinoren zulassen. Wir zeigen, dass die Existenz von Kählerschen Twistorspinoren in Verbindung mit der unteren Abschätzung des Spektrums des Dirac-Operators steht.

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Contents

Introduction

As the title indicates, in this thesis we are concerned with problems related to two main research directions: on the one hand, generalized gradients and, on the other hand, a special class of spinors on Kähler spin manifolds.

In the first part of the thesis we introduce generalized gradients in the general context of G-structures and study their geometric and analytic properties, in particular their conformal invariance and ellipticity.

In the second part we consider Kählerian twistor spinors, investigate their relationship to eigenvalue estimates of the Dirac operator and classify the compact simply-connected Kähler spin manifolds of constant scalar curvature carrying such spinors.

The two parts of the thesis are strongly related to each other, since Kählerian twistor spinors naturally arise as sections in the kernel of an $\mathrm{U}(n)$-generalized gradient, namely the so-called Kählerian twistor operator. Furthermore, one of the ingredients used in the classification of Kähler manifolds admitting such spinors are the Weitzenböck formulas, which are a universal technical tool involving generalized gradients. However, while the first part may be situated at the intersection of conformal and Riemannian geometry, representation theory and analysis on manifolds, the second one is mainly based on tools in spectral geometry and foremost in Kähler spin geometry.

In the sequel we first give a brief overview on the general context in which our problems are naturally placed, then we state the main results of our work and finally present the structure of the thesis.

A. General Setting

The notion of generalized gradients, also called Stein-Weiss operators, was first introduced by E. Stein and G. Weiss, [64], on an oriented Riemannian manifold, as a generalization of the Cauchy-Riemann equations. They are first order differential operators acting on sections of vector bundles associated to irreducible representations of the special orthogonal group (or of the spin group if the manifold is spin), which are given by the following universal construction: one projects onto an irreducible subbundle the covariant derivative induced on the associated vector bundle by the Levi-Civita connection (or, more generally, by any metric connection).

Some of the most important first order differential operators which naturally appear in geometry are, up to normalization, particular cases of generalized gradients. For example, on a Riemannian manifold, the exterior differential acting on differential forms, its formal adjoint, the codifferential, and the conformal Killing operator on 1-forms are generalized gradients. On a spin manifold classical examples of generalized gradients are the Dirac operator, the twistor (or Penrose) operator and the Rarita-Schwinger operator.

An essential property of generalized gradients is their invariance at conformal changes of the metric. This property was noticed for the first time by N. Hitchin in 1974, [29], in the case of the Dirac and the twistor operator in spin geometry and it turned out to have important consequences in physics. Two years later, H. Fegan, [19], showed that, up to the composition with a bundle map, the only conformally invariant first order differential operators between vector bundles associated to the bundle of oriented frames are the generalized gradients. Further results in this direction were obtained by Y. Homma, [31], [32], [34], for the conformal invariance of generalized gradients associated to $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$-structures. For these subgroups Homma's proof of conformal invariance is given by explicit computations based on the relationship between the enveloping algebra of the Lie algebra of the structure group and the algebraic structure of the principal symbols of generalized gradients.

On an oriented Riemannian manifold, generalized gradients naturally give rise, by composition with their formal adjoints, to second order differential operators acting on sections of associated vector bundles. Particularly important are the extreme cases of linear combinations of such second order operators: if the linear combination provides a zero-order operator, then it is a curvature term and one obtains a so-called Weitzenböck formula; if the linear combination is a second order differential operator, then it is interesting to determine when it is elliptic.

The importance of Weitzenböck formulas comes from the fact that they relate the local differential geometry to global topological properties by the so-called Bochner method, which has many applications in various problems.

For instance, Weitzenböck formulas are used to prove the vanishing of the Betti numbers under suitable curvature assumptions, the non-existence of positive scalar curvature metrics on spin manifolds with non-vanishing $\widehat{A}$-genus or eigenvalue estimates for Laplace and Dirac-type operators. The general pattern of the Bochner technique is the following: given a solution of a system of partial differential equations of geometric origin on a compact manifold, if it is assumed that a strict inequality is imposed on an appropriate geometric quantity, then the solution must vanish identically. This technique and its applications have been explained for instance by J.-P. Bourguignon in the survey [10] or, with more details, by H. Wu in [68].

There have been given two different approaches to a systematic study of all possible Weitzenböck formulas. In [63], U. Semmelmann and G. Weingart provide a unified treatment of the construction of Weitzenböck formulas for the irreducible non-symmetric holonomy groups, by giving on the one hand a recursion procedure for the construction of a basis of the space of Weitzenböck formulas and, on the other hand, by characterizing Weitzenböck formulas as eigenvectors of an explicitly known matrix. Another approach was found by Y. Homma, who described all Weitzenböck formulas in [33], [32], [31] and [34], separately for Riemannian, Kähler, hyper-Kähler, respectively quaternionic-Kähler manifolds. His method is based on the algebraic structure of the principal symbols, which is determined from their relationship to the universal enveloping algebra of the corresponding Lie algebra.

The other extreme case is when linear combinations of generalized gradients composed with their formal adjoints yield elliptic second order differential operators. The classical example here is the Laplacian acting on differential forms, which is obtained by assembling two generalized gradients, namely the exterior differential and the codifferential. All minimal elliptic operators provided by this construction were classified by Th. Branson, [13], who showed that it is enough to take surprisingly few generalized gradients in order to obtain an elliptic operator. It thus turned out that Laplace-type operators represent the generic case. Namely, apart from a few known exceptions, each minimal elliptic operator is given by a pair of generalized gradients. The arguments used by Th. Branson are based on tools and techniques of harmonic analysis and explicit computations of the spectra of generalized gradients on the sphere. Partial results regarding the ellipticity of natural first order operators were previously obtained by J. Kalina, A. Pierzchalski and P. Walczak, [37], who showed that the only generalized gradient which is strongly elliptic is given by the projection onto the Cartan summand. Furthermore, the projection onto its complement is also elliptic, by a result of E. Stein and G. Weiss, [64].

Closely related to such elliptic operators is the existence of refined Kato inequalities. This relationship was first remarked by J.-P. Bourguignon, [10]. He pointed out that in all geometric settings where refined Kato inequalities occurred, the sections under consideration are tensor fields of a certain type which are solutions of a natural linear first order injectively elliptic system.

Kato inequalities are estimates in Riemannian geometry, which have proven to be a powerful technique for linking vector-valued and scalar-valued problems in the analysis on manifolds. The classical Kato inequality may be stated as follows. For any section φ of a Riemannian or Hermitian vector bundle E endowed with a metric connection ∇ over a Riemannian manifold (M, g), at any point where φ does not vanish, the following inequality holds:

$$|d|\varphi|| \leq |\nabla\varphi|.$$

In many geometric situations the classical Kato inequality is not sufficient to obtain the desired results and there have been considered refined Kato inequalities of the form: $|d|\varphi|| \leq k|\nabla\varphi|$, for a constant $k < 1$. For example, such estimates occur in Yau's proof of the Calabi conjecture or in Bernstein's problem for minimal hypersurfaces in $\mathbb{R}^n$. The explicit computation of optimal Kato constants for all natural first order differential operators was given by D. Calderbank, P. Gauduchon and M. Herzlich, [18]. More precisely, the formula depends only on the conformal weights, which are real numbers computed from representation theoretical data of the Lie algebra of the structure group.

A central problem in spectral geometry is to find estimates for the eigenvalues of the classical elliptic differential operators on compact manifolds. Among the most important operators are the Laplace and the Yamabe operator on Riemannian manifolds and the Dirac and the twistor operator on spin manifolds.

In the sequel we turn our attention to the Dirac operator and its eigenvalue estimates. The Dirac operator is defined as the composition of the covariant derivative induced on the spinor bundle by the Levi-Civita connection and the Clifford multiplication. It was first introduced by Dirac on the Minkowski space using Pauli matrices and in its current form by M.F. Atiyah and I.M. Singer, [3]. It is an elliptic self-adjoint first order differential operator who acts on sections of Clifford bundles (in particular on the spinor bundle, if the manifold is spin). The Dirac operator has found many applications. For instance, it plays a fundamental role in the index theorems of Atiyah-Singer, [3], in the classification of manifolds of positive scalar curvature, in non-commutative geometry and Seiberg-Witten theory.

From the general theory of elliptic operators on compact manifolds it follows that the spectrum of the Dirac operator is discrete and the multiplicities of its eigenvalues are finite. When the scalar curvature, S, of the spin manifold is positive, it is possible to obtain a lower bound for the eigenvalues of the Dirac operator D, or, more precisely, for its square, using the so-called Lichnerowicz-Schrödinger formula, which is a typical example of a Weitzenböck formula: $D^2 = \nabla^*\nabla + \frac{S}{4}$, where $\nabla^*\nabla$ is the rough Laplacian acting on the spinor bundle. Integrating on the compact manifold yields the inequality: $\lambda^2 \geq \frac{1}{4}\inf_M S$, for each eigenvalue λ of the Dirac operator. However, this inequality is not optimal, since the equality case cannot hold unless S vanishes everywhere.

The first optimal lower bound for the eigenvalues of the Dirac operator was obtained in 1980 by Th. Friedrich, [20], using the twistor operator introduced by R. Penrose. He proved that on a compact spin manifold (M^n, g) of positive scalar curvature S, the first eigenvalue λ of D satisfies the inequality:

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S,$$

whose limiting case is characterized by the existence of real Killing spinors or, equivalently, by constant scalar curvature and the existence of twistor spinors.

Twistor spinors are defined as sections in the kernel of the twistor operator, which is an example of a generalized gradient and is given by the projection of the covariant derivative onto the Cartan summand of the tensor product $\mathrm{T}^*M \otimes \Sigma M$ (where T^*M is the cotangent bundle and ΣM is the spinor bundle). More precisely, a twistor spinor $\varphi \in \Gamma(\Sigma M)$ is a solution of the equation:

$$\nabla_X \varphi = -\frac{1}{n} X \cdot D\varphi,$$

where the dot denotes the Clifford multiplication. An important special class of twistor spinors is formed by Killing spinors, which are defined by the following linear differential equation: $\nabla_X \varphi = \alpha X \cdot \varphi$, for some constant α. They are closely related to the spectrum of the Dirac operator, as remarked above. Killing spinors also play an important role in physics. They were first introduced in general relativity as a tool to construct first integrals of the free geodesic motion and occurred more recently as supersymmetries in 10 and 11-dimensional supergravity theories.

The general geometric description of simply-connected manifolds carrying Killing spinors was obtained in 1993 by Ch. Bär, [5], using the correspondence between Killing spinors on M and parallel spinors on the Riemannian cone over M. He showed that each simply-connected manifold admitting Killing spinors belongs to one of the following families: spheres (in all dimensions), Sasaki-Einstein manifolds (in odd dimensions), 3-Sasakian manifolds (in dimensions of the form $4k+3$), manifolds admitting a nearly parallel G_2-structure (in dimension 7), nearly Kähler manifolds (in dimension 6).

There is an important class of manifolds, namely, the Kähler ones, that does not occur in this list. Indeed, it has already been shown in 1984 by O. Hijazi, [27], that Kähler spin manifolds do not carry any nontrivial Killing spinors. Moreover, in 1992 K.-D. Kirchberg, [42], proved that if the scalar curvature is nonzero, then there do not exist any nontrivial twistor spinors. In particular, this shows that Friedrich's inequality is not sharp on Kähler manifolds. In 1986, K.-D. Kirchberg, [40], improved this inequality and found the optimal one on compact Kähler manifolds. Making use of the Kähler form, he showed that every eigenvalue λ of the Dirac operator on a compact Kähler manifold (M^{2m}, g, J) of positive scalar curvature S satisfies the following inequalities:

$$\lambda^2 \geq \begin{cases} \frac{m+1}{4m} \inf_M S, & \text{if } m \text{ is odd,} \\ \frac{m}{4(m-1)} \inf_M S, & \text{if } m \text{ is even.} \end{cases} \tag{0.1}$$

The limiting manifolds of these inequalities of odd complex dimension are characterized by the existence of Kählerian Killing spinors, i.e. a pair of spinors (φ, ψ) satisfying the equations:

$$\begin{cases} \nabla_X \varphi = \alpha(X + iJX) \cdot \psi, \\ \nabla_X \psi = \alpha(X - iJX) \cdot \varphi, \end{cases} \tag{0.2}$$

for all vector fields X, where α is a real constant. Similarly, in even complex dimension, the equality case in (0.1) is characterized by the existence of spinors φ satisfying the following equations:

$$\begin{cases} \nabla_X \varphi = -\frac{1}{n}(X - iJX) \cdot D\varphi, \\ \nabla_X (D\varphi) = -\frac{1}{4}(Ric(X) - iJRic(X)) \cdot \varphi. \end{cases} \tag{0.3}$$

The limiting Kähler manifolds for these inequalities were geometrically described by A. Moroianu, [51], [54], in 1994 for odd complex dimension, respectively in 1999 for even complex dimension. He proved that in dimension $8\ell+2$ the only limiting Kähler manifold is $\mathbb{C}P^{4\ell+1}$ and in dimension $8\ell+6$ the limiting manifolds are the twistor spaces over quaternionic Kähler manifolds of positive scalar curvature. The proof of A. Moroianu is based on the notion of projectable spinors, which he introduced in [52]. He considered the S^1-principal bundle UM associated to a maximal square root of the canonical bundle of M with the metric induced by the connection of the canonical bundle. He showed that UM naturally admits a spin structure and each Kählerian Killing spinor on M induces a projectable Killing spinor on UM.

The situation is different in even complex dimension, where the absence of Kählerian Killing spinors does not allow the use of projectable spinors. It can be directly checked that the Riemannian product of a flat 2-dimensional torus and a manifold admitting Kählerian Killing spinors is a limiting manifold of even complex dimension. In 1990 it was conjectured by A. Lichnerowicz, [48], that all limiting Kähler manifolds are obtained in this way. A. Moroianu first showed in [53] that the Ricci tensor of a limiting Kähler manifold M^n of even complex dimension has two eigenvalues: 0 and $\frac{1}{n-2}S$ with multiplicities 2, respectively $n-2$. The second step in the proof of the Lichnerowicz conjecture was established by A. Moroianu in [54] in the framework of Spin^c geometry. An alternative proof was provided later, in 2001, based on a result of V. Apostolov, T. Drăghici and A. Moroianu, [2], regarding the splitting of a Kähler manifold whose Ricci tensor has two non-negative eigenvalues.

B. Main Original Results

Our main results in the first part of the thesis are the conformal invariance of G-generalized gradients and a new proof of Branson's classification of minimal first order elliptic operators naturally arising from generalized gradients.

We study generalized gradients in the more general context of G-structures, when there is a reduction of the structure group of the tangent bundle of a Riemannian manifold (M^n, g) to a closed subgroup G of $\mathrm{SO}(n)$. In particular, we make no assumptions on the holonomy group. Essentially, we consider G to be one of the groups that arise on interesting geometric structures, *i.e.* G is $\mathrm{SO}(n)$, $\mathrm{U}(\frac{n}{2})$, $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})$, $\mathrm{Sp}(1)\cdot\mathrm{Sp}(\frac{n}{4})$, G_2 or $\mathrm{Spin}(7)$ (where for the last two groups the dimension of the manifold is $n = 7$, respectively $n = 8$).

We show that the construction of classical generalized gradients for the structure groups $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$, introduced by E. Stein and G. Weiss, [64], can be carried over to G-structures. In order to define G-generalized gradients the only condition that has to be fulfilled is to have well-defined projections from the tensor product of the cotangent bundle and an irreducible associated G-bundle, $T^*M \otimes VM$ (which is the target bundle of any covariant derivative) onto its irreducible subbundles. This condition is satisfied if and only if the decomposition of such a tensor product is multiplicity-free, which is true due to a technical result in representation theory (see Theorem 1.7). This result is not new, it follows from more general results in representation theory, known as generalized Clebsch-Gordan theorems, which were proven, for instance, by J.R. Stembridge in [65]. The statement of Theorem 1.7 appears also in the paper [63] of U. Semmelmann and G. Weingart. Our contribution is its direct detailed proof. Essentially, Theorem 1.7 states that for any of the groups G considered above, if τ the restriction of the standard representation of $\mathrm{SO}(n)$ to G and λ an irreducible G-representation, then the decomposition of the tensor product $\tau \otimes \lambda = \mathbb{R}^n \otimes_{\mathbb{R}} \lambda \cong \mathbb{C}^n \otimes_{\mathbb{C}} \lambda$ is multiplicity-free and described as follows:

$$\tau \otimes \lambda = \underset{\varepsilon \subset \lambda}{\oplus} (\lambda + \varepsilon), \tag{0.4}$$

where by $\varepsilon \subset \lambda$ we denote the relevant weights of λ. Moreover, these relevant weights are completely described by a certain selection rule (for details see Theorem 1.7). The decomposition (0.4) carries over to vector bundles as follows: $T^*M \otimes_{\mathbb{R}} V_\lambda M = (T^*M)^{\mathbb{C}} \otimes_{\mathbb{C}} V_\lambda M = \underset{\varepsilon \subset \lambda}{\oplus} V_{\lambda+\varepsilon} M$.

The G-generalized gradients acting on sections of an irreducible associated vector bundle $V_\lambda M$, are then defined by the composition $P_\varepsilon^{\nabla^\lambda} = \Pi_\varepsilon \circ \nabla^\lambda$, where ∇^λ is the connection induced on $V_\lambda M$ by a G-connection ∇ of the G-structure and Π_ε is the projection onto the subbundle $V_{\lambda+\varepsilon} M$.

The main property of G-generalized gradients that we proved is their conformal invariance. More precisely, for each G-generalized gradient defined by the minimal G-connection and denoted by $P_\varepsilon^{G,\lambda}$, we show that there exists a weight, called conformal weight, relative to which it is conformally invariant. These conformal weights turn out to be exactly the eigenvalues of the symmetric operator $B_{\mathfrak{g}}^\lambda$, defined by:

$$B_{\mathfrak{g}}^\lambda : (\mathbb{R}^n)^* \otimes V_\lambda \to (\mathbb{R}^n)^* \otimes V_\lambda, \quad B_{\mathfrak{g}}^\lambda(\alpha \otimes v) = \sum_{i=1}^n e_i^* \otimes d\lambda(\mathrm{pr}_{\mathfrak{g}}(e_i \wedge \alpha))v.$$

The eigenspaces of B coincide, except for a special case, with the irreducible components given by the decomposition (0.4). Its corresponding eigenvalues, denoted by w_ε, are explicitly computed in terms of the representation theory of the Lie algebra $\mathfrak{g}$ of G, according to Fegan's Lemma, [19].

We summarize the results concerning the conformal invariance of G-generalized gradients obtained in § 1.2 in the following:

Theorem 0.1. *Any G-generalized gradient $P_\varepsilon^{G,\lambda}$ is conformally invariant relative to the weight w_ε and this is the only weight with respect to which*

$P_\varepsilon^{G,\lambda}$ *is conformally invariant. With respect to two conformally related G-structures* GM *and* $\bar{G}M$*, for* $\bar{g} = e^{2u}g$ *and* $\bar{G}M \hookrightarrow \mathrm{SO}_{\bar{g}}M$*, the conformal invariance relating the corresponding generalized gradients is expressed in the following form:*

$$\bar{P}_\varepsilon^{G,\lambda} \circ \phi_{w_\varepsilon}^{G,\bar{G}} = \phi_{w_\varepsilon - 1}^{G,\bar{G}} \circ P_\varepsilon^{G,\lambda},$$

where for any weight w*,* $\phi_w^{G,\bar{G}}$ *denotes the following isomorphism between the associated vector bundles* $V_\lambda^G M := GM \times_G V$ *and* $V_\lambda^{\bar{G}} M := \bar{G}M \times_G V$*:*

$$\phi_w^{G,\bar{G}} : V_\lambda^G M \to V_\lambda^{\bar{G}} M, \quad [(e_1, \dots, e_n), v] \mapsto [(e^{-u}e_1, \dots, e^{-u}e_n), e^{wu}v].$$

The conformal invariance of G-generalized gradients generalizes the results obtained by Y. Homma, [31]– [34], for Kählerian, hyper-Kählerian and quaternionic-Kählerian generalized gradients. At the same time it provides a uniform and direct proof in all these cases, avoiding the specific computations from the original proofs for each of the subgroups $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$. Furthermore, Theorem 0.1 motivates the terminology for the operator B called in literature the conformal weight operator (see [63]).

The proof of Theorem 0.1 is provided in the framework of conformal geometry and uses Weyl structures. An important role is played by the properties of the conformal weight operator B, whose eigenvalues are exactly the corresponding conformal weights of the generalized gradients.

The last result in the first part of the thesis is a new proof of Branson's classification of natural first order minimal elliptic operators. The original proof in [13] uses tools of harmonic analysis, which as powerful as they are, seem to be specific for the structure groups $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$. We propose a different approach, which is mainly based on the representation theory of the Lie algebra $\mathfrak{so}(n)$ and on the relationship between ellipticity and Kato constants, as explained in § 2.2.

The tool used in our proof is on the one hand, the explicit computation of the optimal Kato constants in terms of representation theoretical data provided by D. Calderbank, P. Gauduchon and M. Herzlich, [18], and, on the other hand, the branching rules for the special orthogonal group. The main idea is that the argument in [18] may be in a certain way reversed: while in [18] the task is to establish for each natural elliptic operator an explicit formula for its optimal Kato constant (assuming the Branson's list of minimal elliptic operators), our goal is to analyze to which extend the computations of the Kato constants rely on this assumption on the ellipticity and how Branson's list could be recovered. We mention that there is an exceptional case, corresponding to the zero-weight (see Remark 2.29), that is not recovered by our proof. The arguments suggest that they should carry over to other subgroups G of $\mathrm{SO}(n)$, in order to provide the classification of natural elliptic operators constructed from G-generalized gradients. However, it is still work in progress and in this direction we only have partial results, particularly for the exceptional group G_2.

In the second part of the thesis we get into the realm of Kähler spin geometry and study a special class of spinors, the so-called Kählerian twistor spinors. Our motivation is two-fold, on the one hand we are looking for an analogue of the notion of twistor spinors on Kähler manifolds and on the other, we want to describe the limiting manifolds of Kirchberg's refined inequality for the smallest eigenvalue of the square of the Dirac operator restricted to an irreducible subbundle of the spinor bundle. It turns out that these two problems are related to each other and the main result is the geometric description of these limiting manifolds or, equivalently, of compact simply-connected Kähler spin manifolds of constant scalar curvature carrying nontrivial Kählerian twistor spinors. Along the way we prove other intermediary results which are of interest in their own, for instance the description of Kählerian twistor spinors on Kähler-Einstein manifolds and on Kähler products.

As mentioned above, K.-D. Kirchberg proved in [42] that a Kähler manifold does not admit any nontrivial twistor spinors, unless its scalar curvature is zero. It is thus natural to ask for an analogue class of spinors on Kähler manifolds, defined by a twistorial equation adapted to the Kähler structure. These spinors are called Kählerian twistor spinors and are defined in the following way. On a Kähler spin manifold (M^{2m}, g, J), the spinor bundle ΣM splits into $\mathrm{U}(m)$-irreducible subbundles: $\Sigma M = \oplus_{r=0}^{m} \Sigma_r M$, where $\Sigma_r M$ is the eigenbundle of the Clifford multiplication with the Kähler form for the eigenvalue $i(2r-m)$. For each $0 \leq r \leq m$, there is defined a Kählerian twistor operator by the projection of the covariant derivative onto the Cartan summand of the tensor product $\mathrm{T}^*M \otimes \Sigma_r M$. The sections in the kernel of this first order differential operator are the Kählerian twistor spinors. Explicitly, they satisfy the equations:

$$\begin{cases} \nabla_{X^+}\varphi = -\frac{1}{2(m-r+1)} X^+ \cdot D^- \varphi, \\ \nabla_{X^-}\varphi = -\frac{1}{2(r+1)} X^- \cdot D^+ \varphi, \end{cases} \tag{0.5}$$

where $X^{\pm}$ denote the projections of X onto $T^{1,0}M$, respectively $T^{0,1}M$, the $(\pm i)$-eigenbundles of J, and $D^{\pm}$ are defined by $D^{\pm} = \sum_{i=1}^{2m} e_i^{\pm} \cdot \nabla_{e_i^{\mp}}$. A slightly different notion of "Kählerian twistor spinors" has been introduced by K.-D. Kirchberg, [41], and O. Hijazi, [28]. These form a special class of Kählerian twistor spinors (defined more naturally as sections in the kernel of the Kählerian twistor operator), which we call special Kählerian twistor spinors and are characterized by a further condition, namely to be in the kernel of D^- or D^+.

As in the Riemannian case, Kählerian twistor spinors are closely related to the spectrum of the Dirac operator. We note that the spinors characterizing the limiting manifolds of Kirchberg's inequalities, *i.e.* those satisfying the equations (0.2), respectively (0.3), are in particular Kählerian twistor spinors in $\Sigma_{\frac{m\pm 1}{2}}M$, respectively $\Sigma_{\frac{m}{2}\pm 1}M$. It is thus natural to study Kählerian twistor spinors also as a generalization of these two important special cases.

Considering the splitting of the spinor bundle into irreducible subbundles, there exists a refinement of Kirchberg's inequality (0.1). Namely, the first eigenvalue, λ^2, of the square of the Dirac operator restricted to $\Sigma_r M$ on a compact Kähler manifold (M^{2m}, g, J) of positive scalar curvature S satisfies the following inequality for $0 \leq r \leq \frac{m}{2}$ (for $\frac{m}{2} < r \leq m$ there is a similar one):

$$\lambda^2 \geq \frac{r+1}{2(2r+1)} \inf_M S.$$

The limiting manifolds are characterized by constant scalar curvature and the existence of nontrivial Kählerian twistor spinors in $\Sigma_r M$. We obtained their geometric description as follows (see Theorem 5.15).

Theorem 0.2. *Let (M^{2m}, g, J) be a compact simply-connected spin Kähler manifold of constant scalar curvature admitting nontrivial Kählerian twistor spinors in $\Sigma_r M$ for an r with $0 < r < m$. Then M is the product of a Ricci-flat manifold M_1 and an irreducible Kähler-Einstein manifold M_2, which must be one of the limiting manifolds of Kirchberg's inequality* (0.1) *in odd complex dimensions. More precisely, there exist left (right) Kählerian twistor spinors in at most one such $\Sigma_r M$ with $r < \frac{m}{2}$ ($r > \frac{m}{2}$) and they are of the form:*

$$\psi = \xi_0 \otimes \varphi_r \quad (\psi = \xi_{2r-m-1} \otimes \varphi_{m-r+1}),$$

where $\xi_0 \in \Gamma(\Sigma_0 M_1)$ ($\xi_{2r-m-1} \in \Gamma(\Sigma_{2r-m-1} M_1)$) is a parallel spinor and $\varphi_r \in \Gamma(\Sigma_r M_2)$ ($\varphi_{m-r+1} \in \Gamma(\Sigma_{m-r+1} M_2)$) is a left (right) Kählerian twistor spinor. In particular, the complex dimension of the Kähler-Einstein manifold M_2 is $2r+1$ (resp. $2(m-r)+1$).

In particular, for $r = \frac{m}{2} \pm 1$, the complex dimension of the Ricci-flat factor is 1 and we reobtain the limiting manifolds of Kirchberg's inequalities for even complex dimension. Thus, this result may be considered as a generalization of A. Moroianu's description of limiting Kähler manifolds in even complex dimension. However, we use his classification in the odd-dimensional case. In particular, Theorem 0.2 answers a question raised by K.-D. Kirchberg in [41] and, in a certain sense, completes the picture in the Kähler case.

The proof of Theorem 0.2 is done in more steps, which we briefly present here. Firstly we construct a connection, called Kählerian twistor connection, such that Kählerian twistor spinors are in one-to-one correspondence to parallel sections of this connection. The explicit computation of the curvature of the Kählerian twistor connection allows us to derive some useful formulas, that represent the starting point for the main arguments. We then show that on a compact Kähler spin manifold of constant scalar curvature all Kählerian twistor spinors are special Kählerian twistor spinors. The key point in the proof is that the existence of such a nontrivial spinor imposes strong restrictions on the Ricci tensor, namely it only has two constant non-negative eigenvalues. This has been proven by A. Moroianu, [53], in the special case of limiting manifolds of Kirchberg's inequality for even complex dimension and we notice that his method works for any bundle $\Sigma_r M$. By a result of V. Apostolov, T. Drăghici and A. Moroianu, [2], it follows that the Ricci tensor must be parallel. Thus, assuming the manifold to be simply-connected, it must split, by de Rham's decomposition theorem, in a product of irreducible Kähler-Einstein manifolds. Analyzing Kählerian twistor spinors on a product (see Theorem 5.12), it turns out that one of the factors is Ricci-flat and the other is Kähler-Einstein admitting itself special Kählerian twistor spinors. The problem is thus reduced to the study of Kähler-Einstein manifolds, where we show that the only nontrivial non-extremal Kählerian twistor spinors are the Kählerian Killing spinors (see Proposition 5.8).

Furthermore, we consider a larger class of manifolds, the weakly Bochner flat manifolds, and show that the existence of Kählerian twistor spinors already implies constancy of the scalar curvature. It remains an open question for further research whether this implication holds for other interesting classes of manifolds, for instance for the extremal Kähler ones or even for any Kähler manifold.

C. Outline of the Thesis

In the sequel we briefly present the organization of the thesis.

In **Chapter 1** we first describe the classical generalized gradients of the special orthogonal group SO(n) on a Riemannian manifold and of Spin(n) on a spin manifold. Then we introduce the generalized gradients associated to a G-structure for a closed subgroup G of SO(n). For this, we prove a technical representation-theoretical result concerning the multiplicity-free decomposition of the tensor product of an irreducible G-representation with the restriction to G of the standard SO(n)-representation. We use Weyl structures to show that the generalized gradients of a G-structure defined by the minimal G-connection are conformally invariant. In the last section, § 1.3, we explain how generalized gradients naturally give rise to second order differential operators, whose linear combinations yield two interesting extreme cases: Weitzenböck formulas and second order elliptic differential operators. For the study of Weitzenböck formulas we give a brief overview on the two systematic approaches provided by U. Semmelmann and G. Weingart, [63], and by Y. Homma, [33]. The second order elliptic differential operators are separately discussed in Chapter 2.

In **Chapter 2** we first present the general setting and state in Theorems 2.12 and 2.13 Branson's classification of minimal first order elliptic operators that naturally arise from generalized gradients. Then we turn our attention to Kato inequalities and their relationship to ellipticity. In § 2.2 we present the main steps for the explicit computation of optimal Kato constants in terms of representation theoretical data, which was obtained by D. Calderbank, P. Gauduchon and M. Herzlich, [18]. In § 2.3 we give a new proof of Branson's classification result. Our approach uses on the one hand the observation that certain arguments used by D. Calderbank, P. Gauduchon and M. Herzlich in the computation of the Kato constant may be reversed and, on the other hand, the branching rules for the special orthogonal group.

Chapter 3 is mainly a short introduction in spin geometry on Kähler manifolds, where we introduce the notations and some results needed in Chapters 4 and 5.

In **Chapter 4** we first define the main objects, the Kählerian twistor spinors, and study some particular cases. In § 4.3 we construct a connection called Kählerian twistor connection, such that Kählerian twistor spinors are in one-to-one correspondence to parallel sections of this connection. Furthermore, in § 4.4, we explicitly compute the curvature of the Kählerian twistor connection, which allows us to derive some useful formulas, that represent the starting point for the main arguments.

Chapter 5. First we show in § 5.1 that, on a compact Kähler spin manifold of constant scalar curvature, all Kählerian twistor spinors are special Kählerian twistor spinors. Then, in § 5.2, we prove the key fact that the existence of such a nontrivial spinor imposes strong restrictions on the Ricci tensor, namely it only has two constant eigenvalues. We conclude by a result of V. Apostolov, T. Drăghici and A. Moroianu, [2], that the Ricci tensor must be parallel. Thus, assuming the manifold simply-connected, it must be, by de Rham's decomposition theorem, a product of irreducible Kähler-Einstein manifolds. In § 5.4, analyzing Kählerian twistor spinors on a product, it turns out that one of the factors is Ricci-flat and the other is Kähler-Einstein admitting itself special Kählerian twistor spinors (see Theorem 5.12). Turning our attention in § 5.3 to Kähler-Einstein manifolds, we show that the only nontrivial non-extremal Kählerian twistor spinors are the Kählerian Killing spinors. The main result giving the geometric description of compact simply-connected Kähler manifolds of constant scalar curvature admitting Kählerian twistor spinors is then provided in § 5.5 by Theorem 5.15. In the last section, § 5.6, we show that on the larger class of weakly Bochner flat manifolds the existence of Kählerian twistor spinors already implies the constancy of the scalar curvature.

In **Appendix A.** we collect some formulas of spin geometry, that we use in the computations, particularly in §§ 4.3 and 4.4.

For the reader's convenience we have compiled both an index of notational conventions and an index of terms.

All manifolds and geometric objects on them are supposed to be differentiable of class $\mathcal{C}^\infty$. Throughout this work we use the summation convention: unless otherwise indicated, summation is implied whenever repeated indices occur in an expression.

Part I

Generalized Gradients of G-Structures

Chapter 1

Natural Differential Operators of G-structures

The concept of G-structure provides a unified manner to treat most of the interesting geometric structures. In the sequel we study natural differential operators associated to G-structures. Firstly we consider generalized gradients of a G-structure and prove their conformal invariance. Then we explain how generalized gradients naturally give rise to second order differential operators, whose linear combinations yield two interesting extreme cases: Weitzenböck formulas and second order elliptic differential operators, which we separately discuss in more detail in § 1.3 and in Chapter 2, respectively.

1.1 Generalized Gradients

Some of the most important first order differential operators which naturally appear in geometry are, up to normalization, particular cases of generalized gradients. For example, on a Riemannian manifold, the exterior differential acting on differential forms, its formal adjoint, the codifferential, the conformal Killing operator on 1-forms are generalized gradients. On a spin manifold, classical examples of generalized gradients are the Dirac operator, the twistor (or Penrose) operator and the Rarita-Schwinger operator.

The notion of generalized gradients, also called *Stein-Weiss operators*, was first introduced by E. Stein and G. Weiss, [64], on an oriented Riemannian manifold, *i.e.* for the structure group $\mathrm{SO}(n)$, as a generalization of the

Cauchy-Riemann equations. They give a general construction of first order differential operators that we recall in the sequel. We then show how this construction can be carried over to G-structures.

1.1.1 SO(n) and Spin(n)-generalized gradients

Let us first state the general context and fix the notation. We begin by briefly recalling the representation theoretical background needed to define the generalized gradients. The description of the representations of $\mathrm{SO}(n)$, or its Lie algebra $\mathfrak{so}(n)$, differs slightly according to the parity of n. We write $n = 2m$ if n is even and $n = 2m + 1$ if n is odd, where m is then the rank of $\mathfrak{so}(n)$. Let $\{e_1, \dots, e_n\}$ be a fixed oriented orthonormal basis of $\mathbb{R}^n$, so that $\{e_i \wedge e_j\}_{i<j}$ is a basis of the Lie algebra $\mathfrak{so}(n) \cong \Lambda^2\mathbb{R}^n$. We also fix a $\mathfrak{h}$ of $\mathfrak{so}(n)$ by the basis $\{e_1 \wedge e_2, \dots, e_{2m-1} \wedge e_{2m}\}$ and denote the dual basis of $\mathfrak{h}^*$ by $\{\varepsilon_1, \dots, \varepsilon_m\}$. The is then normalized such that this basis is orthonormal. Roots and weights are given by their coordinates with respect to the orthonormal basis $\{\varepsilon_i\}_{i=\overline{1,m}}$. Thus, irreducible $\mathfrak{so}(n)$-representations are parametrized by *dominant weights*, *i.e.* the weights λ whose coordinates are either all integers or all half-integers, $(\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$, which satisfy one of the inequality:

$$\begin{aligned} \lambda_1 \geq \lambda_2 \geq \cdots \lambda_{m-1} \geq |\lambda_m|, &\quad \text{if } n = 2m, \text{ or} \\ \lambda_1 \geq \lambda_2 \geq \cdots \lambda_{m-1} \geq \lambda_m \geq 0, &\quad \text{if } n = 2m+1. \end{aligned} \tag{1.1}$$

Through this parametrization a dominant weight λ is the highest weight of the corresponding representation. The representations of $\mathfrak{so}(n)$ are in one-to-one correspondence with the representations of the corresponding simply-connected Lie group, *i.e.* $\mathrm{Spin}(n)$, the universal covering of $\mathrm{SO}(n)$. The representations which factor through $\mathrm{SO}(n)$ are exactly those with $\lambda \in \mathbb{Z}^m$. With a slight abuse of notation we use the same symbol for an irreducible representation and its highest weight. Thus, an irreducible representation of $\mathfrak{so}(n)$ will be identified with its highest weight $\lambda \in \mathfrak{h}^*$ or, alternatively, will be denoted by V_λ. For example, in this notation, the standard representation denoted by τ is given by the weight $(1, 0, \dots, 0)$; the weight $(1, \dots, 1, 0, \dots, 0)$ (with p ones) corresponds to the p-form representation $\Lambda^p\mathbb{R}^n$, whereas the dominant weights $\lambda = (1, \dots, 1, \pm 1)$, for $n = 2m$, correspond to selfdual, respectively antiselfdual m-forms; the representation of totally symmetric traceless tensors $S^p_0\mathbb{R}^n$ has highest weight $(p, 0, \dots, 0)$.

In order to have this one-to-one correspondence between finite dimensional irreducible representations and highest weights, it is necessary to consider only complex representations. Since we want the standard representation τ to model the tangent bundle of a manifold, it is natural to consider the real standard representation $\mathbb{R}^n$. On the other hand, it is useful in many arguments to have this parametrization given by the highest weight and thus, to consider τ to be the complex representation $\mathbb{C}^n$. In a certain sense, we may use both representations alternatively, because the only place where this problem comes up is in the decomposition of a tensor product of the form $\tau \otimes \lambda$, where λ is any complex irreducible representation and in which case the following isomorphism holds: $\mathbb{R}^n \otimes_{\mathbb{R}} V_\lambda \cong \mathbb{C}^n \otimes_{\mathbb{C}} V_\lambda$. Thus, from now on, τ will denote either the real or the complex standard representation, as needed in the context.

The following so-called *classical selection rule* describes the decomposition of the tensor product $\tau \otimes \lambda$ into irreducible $\mathfrak{so}(n)$-representations, where τ is the standard representation and λ is any irreducible representation.

Lemma 1.1. *An irreducible representation of weight μ occurs in the decomposition of $\tau \otimes \lambda$ if and only if the following two conditions are fulfilled:*

(i) $\mu = \lambda \pm \varepsilon_j$, *for some* j, *or* $n = 2m+1$, $\lambda_m > 0$ *and* $\mu = \lambda$,

(ii) μ *is a dominant weight,* i.e. *satisfies the inequality* (1.1).

A proof of this selection rule follows from Theorem 1.7, where we consider more generally G-representations, for subgroups G of $\mathrm{SO}(n)$. We adopt the same terminology as in [18] and [63] and call the weights satisfying (i) *virtual weights* associated to λ and *effective weights* if they also satisfy (ii); the weights ε of the standard representation τ, $\varepsilon \in \{0, \pm\varepsilon_1, \dots, \pm\varepsilon_m\}$, with the property that $\mu_\varepsilon := \lambda + \varepsilon$ occurs in the decomposition of $\tau \otimes \lambda$, are also called *relevant weights*. We write $\varepsilon \subset \lambda$ for a relevant weight for a given irreducible representation λ, so that the decomposition of the tensor product may be expressed as follows:

$$\tau \otimes \lambda = \bigoplus_{\varepsilon \subset \lambda} \mu_\varepsilon. \tag{1.2}$$

The essential property of the decomposition (1.2) is that it is multiplicity-free, *i.e.* the isotypical components are actually irreducible (this is a special case of Theorem 1.7, to which we refer for the proof). It thus follows that the projections onto each irreducible summand μ_ε in the splitting are well-defined; we denote them by Π_ε.

Let (M, g) be a Riemannian manifold, $\mathrm{SO}_g M$ the principal $\mathrm{SO}(n)$-bundle of oriented orthonormal frames and ∇ any metric connection, considered either as a connection 1-form on $\mathrm{SO}_g M$ or as a covariant derivative on the tangent bundle $\mathrm{T}M$.

The tool for transferring representation theory on manifolds is the construction of associated vector bundles. We recall that, in general, to any G-principal bundle GM on M and any representation V of the group G, $\rho : G \to \mathrm{Aut}(V)$, there is an *associated vector bundle* VM on M defined by $VM := GM \times_G V := (GM \times V)/\rho$, where the quotient is given with respect to the following right action of the group G:

$$(p, v) \cdot g = (p \cdot g, \rho(g^{-1})v), \quad \text{for all } p \in GM, g \in G, v \in V.$$

In our case we consider vector bundles associated to $\mathrm{SO}_g M$ and irreducible $\mathrm{SO}(n)$-representations of highest weight λ and denote them by $V_\lambda M$. For instance, the tangent bundle is associated to the standard representation $\tau : \mathrm{SO}(n) \hookrightarrow \mathrm{GL}(\mathbb{R}^n)$ and the bundle of p-forms is associated to the irreducible representation of highest weight $\lambda = (\underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0)$. We identify the cotangent bundle T^*M and the tangent bundle $\mathrm{T}M$ using the metric g, since they are associated to equivalent $\mathrm{SO}(n)$-representations. The decomposition (1.2) carries over to the associated vector bundles:

$$\mathrm{T}^*M \otimes V_\lambda M \cong \mathrm{T}M \otimes V_\lambda M \cong \bigoplus_{\varepsilon \subset \lambda} V_{\mu_\varepsilon} M \tag{1.3}$$

and the corresponding projections are also denoted by Π_ε.

A metric connection ∇ on $\mathrm{SO}_g M$ induces a connection on any associated bundle $V_\lambda M$ denoted by $\nabla^\lambda : \Gamma(V_\lambda M) \to \Gamma(\mathrm{T}^*M \otimes V_\lambda M)$. The generalized gradients are then built-up by projecting the induced covariant derivative onto the irreducible subbundles $V_{\mu_\varepsilon} M$ given by the splitting (1.3).

Definition 1.2. Let (M, g) be a Riemannian manifold, ∇ a metric connection and $V_\lambda M$ the vector bundle associated to the irreducible $\mathrm{SO}(n)$-representation of highest weight λ. For each relevant weight ε of λ, *i.e.* for each irreducible component in the decomposition of $\mathrm{T}^*M \otimes V_\lambda M$, there is a *generalized gradient* $P_\varepsilon^{\nabla^\lambda}$ defined by the composition:

$$\Gamma(V_\lambda M) \xrightarrow{\nabla^\lambda} \Gamma(\mathrm{T}M \otimes V_\lambda M) \xrightarrow{\Pi_\varepsilon} \Gamma(V_{\lambda+\varepsilon} M). \tag{1.4}$$

The classical case is when ∇ is the Levi-Civita connection. However, any metric connection may be used to define the generalized gradients. Those defined by the Levi-Civita connection play an important role since they are conformal invariant, as we show in § 1.2.

Example 1.3. We consider the bundle of p-forms, $\Lambda^p M$, on a Riemannian manifold (M^n, g) and assume for simplicity that $n = 2m+1$ and $p \leq m-1$. The highest weight of the representation is $\lambda_p = (1, \ldots, 1, 0, \ldots, 0)$ and, from the selection rule in Lemma 1.1, it follows that there are three relevant weights for λ_p, namely $-\varepsilon_p$, ε_{p+1} and ε_1. The tensor product then decomposes as follows:

$$\mathrm{T}M \otimes \Lambda^p M \cong \Lambda^{p-1} M \oplus \Lambda^{p+1} M \oplus \Lambda^{p,1} M,$$

where the last irreducible component is the Cartan summand corresponding to the highest weight equal to the sum of the highest weights of the factors of the tensor product, *i.e.* to $\lambda_p + \varepsilon_1$. The generalized gradients in this case are, up to a constant factor, the following: the codifferential, δ, the exterior derivative, d, respectively the so-called *twistor operator*, T.

If M has, in addition, a spin structure, then the classes of bundles and generalized gradients are enriched. More precisely, as mentioned above, the irreducible $\mathrm{Spin}(n)$-representations, and thus the irreducible associated $\mathrm{Spin}(n)$-bundles, correspond exactly to the irreducible $\mathfrak{so}(n)$-representations, so that they are parametrized by all the dominant weights $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$ satisfying the inequalities (1.1). The irreducible representations of $\mathrm{Spin}(n)$ with $\lambda \in (\frac{1}{2} + \mathbb{Z})^m$ are exactly those that do not factor through $\mathrm{SO}(n)$.

Let us recall here the definition of a spin structure, which will also be needed in Part II.

Definition 1.4. Let (M, g) be an oriented Riemannian manifold and $\mathrm{SO}_g M$ its bundle of orthonormal frames. The manifold M is called *spin* if there exists a 2-fold covering $\mathrm{Spin}_g M$ of $\mathrm{SO}_g M$ with projection $\theta : \mathrm{Spin}_g M \to \mathrm{SO}_g M$, satisfying the following conditions:

1. $\mathrm{Spin}_g M$ is a principal bundle over M with structure group $\mathrm{Spin}(n)$,
2. If we denote by ϕ the canonical projection of $\mathrm{Spin}(n)$ onto $\mathrm{SO}(n)$, then for every $u \in \mathrm{Spin}_g M$ and $a \in \mathrm{Spin}(n)$ we have $\theta(ua) = \theta(u)\phi(a)$, *i.e.* the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Spin}(n) & \xrightarrow{a\mapsto ua} & \mathrm{Spin}_g M & & \\ & & & \searrow & \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \theta} & & M \\ & & & \nearrow & \\ \mathrm{SO}(n) & \xrightarrow[A\mapsto\theta(u)A]{} & \mathrm{SO}_g M & & \end{array}$$

The bundle $\mathrm{Spin}_g M$ is called a *spin structure*.

An oriented Riemannian manifold M is spin if and only if its second Stiefel-Whitney class, $w_2(M)$, vanishes.

The n-dimensional Clifford algebra $\mathrm{C}l(n)$ has, up to equivalence, exactly one irreducible complex representation Σ_n for n even and two irreducible complex representations $\Sigma_n^{\pm}$ for n odd. In the last case, these two irreducible representations become equivalent when restricted to $\mathrm{Spin}(n)$ and this restriction is also denoted by Σ_n. For n even, the restriction to $\mathrm{Spin}(n)$ splits with respect to the action of the volume element: $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ and the elements of Σ_n^+ and Σ_n^- are usually called *positive*, respectively *negative half-spinors*. In the parametrization given by the dominant weights we have: Σ_{2m+1} has highest weight $(\frac{1}{2}, \dots, \frac{1}{2})$ and $\Sigma_{2m}^{\pm}$ has highest weight $(\frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2})$. For arbitrary n, Σ_n is called the *complex spin representation* and it defines a complex vector bundle associated to the spin structure, called the *complex spinor bundle* and denoted by ΣM. The sections of the spinor bundle ΣM are called *spinor fields* or, shortly, *spinors*.

On a Riemannian spin manifold (M, g), the $\mathrm{Spin}(n)$-generalized gradients $P_\varepsilon^{\nabla^\lambda}$ are similarly defined by the composition:

$$\Gamma(V_\lambda M) \xrightarrow{\nabla^\lambda} \Gamma(\mathrm{T}M \otimes V_\lambda M) \xrightarrow{\Pi_\varepsilon} \Gamma(V_{\lambda+\varepsilon} M). \tag{1.5}$$

The only difference to (1.4) is that now λ is an irreducible $\mathrm{Spin}(n)$-representation, ∇ a connection on the principal bundle $\mathrm{Spin}_g M$ and ∇^λ the induced connection on the associated bundle $V_\lambda M$. A special case is when ∇ is the connection canonically induced on $\mathrm{Spin}_g M$ by the Levi-Civita connection and is then denoted by ∇^g. This connection is used to define the generalized gradients in the following two examples.

Example 1.5. The spinor representation $\rho_n : \mathrm{Spin}(n) \to \mathrm{Aut}(\Sigma_n)$, with n odd, is irreducible and the tensor product bundle splits into two irreducible subbundles as follows:

$$\mathrm{T}M \otimes \Sigma M \cong \Sigma M \oplus \ker(c),$$

where $c : \mathrm{T}M \times \Sigma M \to \Sigma M$ denotes the *Clifford multiplication* of a vector field with a spinor. Thus, in this case, there are two generalized gradients: the *Dirac operator* D, which is locally explicitly given by the formula:

$$D\varphi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\varphi, \quad \text{for all } \varphi \in \Gamma(\Sigma M),$$

where $\{e_i\}_{i=\overline{1,n}}$ is a local orthonormal basis and the middle dot is a simplified notation for the Clifford multiplication, and the *twistor (Penrose) operator* T:

$$T_X\varphi = \nabla_X\varphi + \frac{1}{n}X \cdot D\varphi.$$

For n even, the spinor representation is not irreducible, so that the spinor bundle splits into two subbundles: $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. The decomposition of the tensor product is then given by:

$$\mathrm{T}^*M \otimes \Sigma^{\pm} M = \Sigma^{\mp} M \oplus \ker(c).$$

Again the projections onto the first summand correspond to the Dirac operator and onto $\ker(c)$ to the twistor operator.

Example 1.6 (Rarita-Schwinger Operator)**.** Let $n \geq 3$ be odd and consider the so-called *twistor bundle*, *i.e.* the target bundle of the twistor operator acting on spinors, denoted by $\Sigma_{3/2}M$. This is the vector bundle associated to the irreducible $\mathrm{Spin}(n)$-representation with highest weight $(\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$. If $n \geq 5$, it follows from the selection rule in Lemma 1.1 that there are four relevant weights: 0, $-\varepsilon_1$, $+\varepsilon_1$, $+\varepsilon_2$ and the corresponding four gradient targets are: $\Sigma_{3/2}M$ itself, the spinor bundle ΣM, the associated vector bundles to the irreducible representations of highest weights $(\frac{5}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, respectively. If $n = 3$ the last of these targets is missing.

The corresponding generalized gradient for the representation $\lambda = (\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and the relevant weight $\varepsilon = 0$ is denoted by $D_{3/2} := P_0^{(3/2,1/2,\ldots,1/2)}$. This operator is well-known especially in the physics literature and is called the *Rarita-Schwinger operator.*

If n is even, $n = 2m$, then the two bundles defined by the Cartan summand in $\mathrm{T}^*M \otimes \Sigma^\pm M$ have highest weights $(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2})$ and the corresponding Rarita-Schwinger operators are the generalized gradients denoted by:

$$D^\pm_{3/2} = P^{(3/2,1/2,\dots,1/1,\pm 1/2)}_{\mp\varepsilon_m}, \quad D^\pm_{3/2} : \Gamma(\Sigma^\pm_{3/2}M) \to \Gamma(\Sigma^\mp_{3/2}M).$$

Another possible realization of the Rarita-Schwinger operator is as the projection of the twisted Dirac operator D_T on $\Sigma_{3/2}M$ (see *e.g.* [66] for a detailed description):

$$\Gamma(\Sigma_{3/2}M) \hookrightarrow \Gamma(\mathrm{T}M \otimes \Sigma M) \xrightarrow{\nabla} \Gamma(\mathrm{T}^*M \otimes \mathrm{T}M \otimes \Sigma M) \xrightarrow{c\otimes 1} \Gamma(\mathrm{T}M \otimes \Sigma M) \to \Gamma(\Sigma_{3/2}M).$$

1.1.2 Generalized gradients of G-structures

Essentially the same construction as above may be used to define generalized gradients associated to a G-structure. On a differentiable manifold M of dimension n we denote by GL_nM the bundle of linear frames over M. If G is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, then a *$G$-structure* on M is a differentiable subbundle of GL_nM with structure group G. The existence of a G-structure on a manifold is a topological condition. For example, an oriented Riemannian metric is equivalent to an $\mathrm{SO}(n)$-structure, an almost Hermitian structure to a $\mathrm{U}(\frac{n}{2})$-structure, a conformal class of Riemannian metrics to a $\mathrm{CO}(n)$-structure. In order to construct geometric first order differential operators of a G-structure, the natural starting point is, as in the $\mathrm{SO}(n)$-case, a connection on the principal G-bundle. It will be considered alternatively either as a connection 1-form with values in $\mathfrak{g}$, the Lie algebra of G, or as a G-equivariant horizontal distribution H. In general, a connection ∇ on a principal G-bundle GM induces a connection on any vector bundle VM associated to a representation $\rho : G \to \mathrm{Aut}(V)$. One possible way to describe the induced connection ∇^ρ is to give its horizontal distribution, which is the projection onto GM of the G-equivariant horizontal distribution H:

$$H_{[p,v]} := d\mathrm{pr}_{(p,v)}(H_p \oplus 0), \quad \mathrm{pr} : GM \times V \to (GM \times V)/_\rho.$$

As we are mainly interested in Riemannian geometry, we will consider in the sequel G to be one of the subgroups of $\mathrm{SO}(n)$ which arise in important geometric situations and are mostly encountered in literature. These groups are exactly the ones in Berger's list of holonomy groups. Thus, in the sequel, we assume that

$$G \in \left\{ \mathrm{SO}(n), \mathrm{U}\left(\frac{n}{2}\right), \mathrm{SU}\left(\frac{n}{2}\right), \mathrm{Sp}\left(\frac{n}{4}\right), \mathrm{Sp}(1)\cdot\mathrm{Sp}\left(\frac{n}{4}\right), G_2, \mathrm{Spin}(7) \right\}, \quad (1.6)$$

where in the last two cases the dimension of the manifold is assumed to be 7, respectively 8. We notice that all the groups in (1.6) are compact. Moreover, their Lie algebras are simple, except for $\mathfrak{u}(\frac{n}{2}) = i\mathbb{R} \oplus \mathfrak{su}(\frac{n}{2})$, which has a 1-dimensional center and $\mathfrak{sp}(1) \oplus \mathfrak{sp}(\frac{n}{4})$, which is semisimple.

Every finite-dimensional representation of a compact Lie group G is equivalent to a unitary one, so that it can be decomposed as a direct sum of irreducible representations. Thus, without loss of generality, we consider in the sequel complex finite-dimensional irreducible representations of G, which are parametrized by the dominant weights. In Table 1.1 we wrote down the suitable positivity conditions that define the dominant weights for each group in the list (1.6). The coordinates of the weights are given with respect to a chosen basis of the dual of a Cartan subalgebra of the Lie algebra $\mathfrak{g}$.

The general setting is now the following: there exists a G-structure on M and a G-connection ∇, *i.e.* a connection whose holonomy group is a subset of G, $Hol(\nabla) \subseteq G$, and for any finite-dimensional complex irreducible G-representation of highest weight λ, $V_\lambda M$ denotes, as usual, the associated vector bundle.

We consider the restriction of the standard representation of the special orthogonal group, $\tau : \mathrm{SO}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{R})$, to the subgroup G and denote it further by τ. This restriction is the defining representation of the subgroup G and the associated vector bundle is just the tangent bundle $\mathrm{T}M$. The tangent and the cotangent bundle, $\mathrm{T}M$ and T^*M, will be identified, since they are associated to equivalent G-representations, as $G \subset \mathrm{SO}(n)$. The real G-representation τ is irreducible, because the groups in (1.6) are holonomy groups, which are known to act transitively on the unit sphere in $\mathbb{R}^n$. The restriction to G of the standard complex representation, which we still denote by τ, remains an irreducible representation, except for the unitary and special unitary group. For these groups the complexified tangent bundle splits into two irreducible subbundles:

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \quad (1.7)$$

which represent the $(\pm i)$-eigenspaces of the corresponding almost complex structure of the manifold.

The main ingredient needed to define the notion of G-generalized gradient is the following representation theoretical result (see also [63] for the statement of Theorem 1.7, which we prove here with all the details).

Theorem 1.7. *Let G be one of the groups in* (1.6)*, τ the restriction of the standard representation to G and λ an irreducible G-representation. The decomposition of the tensor product $\tau \otimes \lambda = \mathbb{R}^n \otimes_{\mathbb{R}} \lambda \cong \mathbb{C}^n \otimes_{\mathbb{C}} \lambda$ is described as follows:*

$$\tau \otimes \lambda = \bigoplus_{\varepsilon \subset \lambda} (\lambda + \varepsilon), \tag{1.8}$$

where by $\varepsilon \subset \lambda$ we denote the relevant weights of λ. Moreover, the relevant weights are described by the following selection rule: a weight ε of τ is relevant for λ if and only if $\lambda + \varepsilon$ is a dominant weight, with the exception of the weight $\varepsilon = 0$, which occurs only for the groups G_2 and $\mathrm{SO}(n)$ with n odd. For $\varepsilon = 0$ a stronger condition must be fulfilled, namely: $\lambda - \lambda_\tau$, respectively $\lambda - \lambda_\Sigma$ are dominant weights, where λ_τ and λ_Σ are the highest weight of the standard representation of $\mathfrak{g}_2$, respectively of the spinor representation of $\mathfrak{so}(n)$. The decomposition (1.8) *carries over to vector bundles and we get:*

$$\mathrm{T}^*M \otimes_{\mathbb{R}} V_\lambda M = (\mathrm{T}^*M)^{\mathbb{C}} \otimes_{\mathbb{C}} V_\lambda M = \bigoplus_{\varepsilon \subset \lambda} V_{\lambda+\varepsilon} M. \tag{1.9}$$

Proof: The decomposition (1.8) is a special case of an important result in representation theory, sometimes called the general Clebsch-Gordan theorem, which provides formulas for the multiplicities of the irreducible components of the tensor product of two irreducible representations of a semisimple Lie algebra. More precisely, if $\mathfrak{g}$ is a complex semisimple Lie algebra, an ordering of the roots is chosen and W^+ denotes the corresponding set of dominant weights of $\mathfrak{g}$, then the decomposition of the tensor product of any pair of irreducible representations, μ and λ with $\mu, \lambda \in W^+$, has the form:

$$\mu \otimes \lambda = \bigoplus_{\nu \in W^+} c(\nu; \mu, \lambda)\nu, \tag{1.10}$$

where the coefficients $c(\nu; \lambda, \nu)$ are the tensor product multiplicities giving the number of times the irreducible representation ν occurs as a summand in the decomposition. The tensor product is called *multiplicity-free* if $c(\nu; \mu, \lambda) \leq 1$, for all ν. Multiplicity-free tensor products are important because their decompositions are canonical and therefore the projections onto the irreducible components are well-defined.

There are different methods to compute the multiplicities $c(\nu; \mu, \lambda)$, most of them being based on Weyl's character formula or, equivalently, on Kostant's formula for the multiplicity of a weight. For instance, the more particular question on which tensor products of simple Lie algebras are multiplicity-free has been completely answered. The pairs of fundamental weights ω_1, ω_2 such that the tensor product $m_1\omega_1 \otimes m_2\omega_2$ is multiplicity-free for all $m_1, m_2 \geq 0$ are classified by P. Littelmann, [50], and, more generally, the multiplicity-free tensor products of simple Lie algebras have been completely classified by J. R. Stembridge, [65], and independently, for the exceptional Lie algebras, by R. King and B. Wybourne, [39]. From their classification it follows in particular that the decomposition of $\tau \otimes \lambda$ is multiplicity-free. However, this tensor product is a very special case, because τ is the standard representation. We provide here a direct proof, also based on Weyl's character formula, of the fact that the splitting (1.8) is multiplicity-free, since this is the main property that enables us to define G-generalized gradients. Moreover, the argument yields the stated characterization of the relevant weights.

First we need to make a short digression to the representation theory of semisimple Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r and $\mathfrak{h}$ a Cartan subalgebra. Choose an ordering of the roots and denote by $\omega_1, \dots, \omega_r$ the fundamental weights of $\mathfrak{g}$, which span the *Weyl chamber* denoted by WC:

$$WC = \{\alpha_1\omega_1 + \cdots + \alpha_r\omega_r |\, \alpha_1, \dots, \alpha_r \geq 0\}. \tag{1.11}$$

Then, the dominant weights are, by definition, the weights in this Weyl chamber. The so-called *Weyl vector*, δ, is defined to be half the sum of the positive roots, or equivalently, the sum of the fundamental weights, so that it lies in the interior of the Weyl chamber. The *strict Weyl chamber* SWC is the translation of WC with δ:

$$SWC = WC + \delta = \{\alpha_1\omega_1 + \cdots + \alpha_r\omega_r |\, \alpha_1, \dots, \alpha_r \geq 1\}. \tag{1.12}$$

The group $\mathfrak{W}$ generated by reflections through the hyperplanes orthogonal to the roots is called the *Weyl group* of the Lie algebra $\mathfrak{g}$. The length $|w|$ of a Weyl group element w is the length of the shortest word representing that element in terms of the standard generators, which are the reflections given by the simple roots.

Let Λ be the weight lattice of $\mathfrak{g}$, which is freely generated by the set of fundamental weights, and let $\mathbb{Z}\Lambda$ be the integral group ring on the abelian group Λ. We write $e(\nu)$ for the basis element of $\mathbb{Z}\Lambda$ corresponding to the weight ν, so that elements of $\mathbb{Z}\Lambda$ are expressions of the form $\sum n_\nu e(\nu)$, with all but a finite number of integers n_ν being zero. The *character* of a representation V

is defined as:

$$\mathrm{ch}(V) = \sum_{\nu} \dim(V_\nu) e(\nu) \in (\mathbb{Z}\Lambda)^{\mathfrak{W}}, \tag{1.13}$$

where V_ν is the weight space of V for the weight ν, $\dim(V_\nu)$ its multiplicity and $(\mathbb{Z}\Lambda)^{\mathfrak{W}}$ the ring of invariant elements under the action of the Weyl group. It can be directly checked that the character map, $\mathrm{ch} : R(\mathfrak{g}) \to \mathbb{Z}\Lambda$, is a ring homomorphism between the representation ring of the semisimple Lie algebra $\mathfrak{g}$ and $\mathbb{Z}\Lambda$.

For any weight ν, let A_ν be the following element in $\mathbb{Z}\Lambda$:

$$A_\nu = \sum_{w \in \mathfrak{W}} (-1)^{|w|} e(w(\nu)).$$

Let $\lambda, \mu \in WC$ be dominant weights and V_λ, V_μ the irreducible representations of highest weights λ, respectively μ. Weyl's character formula may be expressed in the following three equivalent ways:

$$\mathrm{ch}(V_\lambda) = \frac{A_{\lambda+\delta}}{A_\delta}, \tag{1.14a}$$

$$\mathrm{ev}_{\mu+\delta}(A_\delta \cdot \mathrm{ch}(V_\lambda)) = \delta_{\mu\lambda}, \tag{1.14b}$$

$$\mathrm{ev}_{\mu+\delta}(A_\delta \cdot \mathrm{ch}(W)) = \dim(\mathrm{Hom}_{\mathfrak{g}}(V_\mu, W)), \tag{1.14c}$$

where W is any finite-dimensional representation of $\mathfrak{g}$, $\delta_{\mu\lambda}$ is Kronecker's delta and $\mathrm{ev}_\mu f$ denotes the evaluation of an element f of $\mathbb{Z}\Lambda$, considered as a linear map with finite support on Λ, $f : \Lambda \to \mathbb{Z}$, at the point $\mu \in \Lambda$.

Since $\mathrm{ch} : R(\mathfrak{g}) \to \mathbb{Z}\Lambda$ is a ring homomorphism, we have in particular:

$$\mathrm{ch}(V_\mu \otimes V_\lambda) = \mathrm{ch}(V_\mu) \cdot \mathrm{ch}(V_\lambda),$$

which together with the Weyl character formula yields the following expression for the multiplicities $c(\nu; \lambda, \mu)$ of the tensor product:

$$\begin{aligned} c(\nu; \mu, \lambda) &= \dim(\mathrm{Hom}_{\mathfrak{g}}(V_\nu, V_\mu \otimes V_\lambda)) \overset{(1.14c)}{=} \mathrm{ev}_{\nu+\delta}(A_\delta \cdot \mathrm{ch}(V_\mu) \cdot \mathrm{ch}(V_\lambda)) \\ &\overset{(1.14a)}{=} \sum_{w \in \mathfrak{W}} (-1)^{|w|} \mathrm{ev}_{\nu+\delta}[e(w(\lambda + \delta)) \cdot \mathrm{ch}(V_\mu)]. \end{aligned} \tag{1.15}$$

Now let G be one of the groups in (1.6). Then, except for $\mathfrak{u}(n)$, the Lie algebras of the others groups are semisimple and we may thus use the previously mentioned notions to analyze the tensor product decomposition (1.8).

One can verify that the same arguments work also for $\mathfrak{u}(n)$, since the only difference between $\mathfrak{u}(n)$ and the simple Lie algebra $\mathfrak{su}(n)$ is its 1-dimensional center.

As usual, τ is the restriction of the standard representation of $\mathrm{SO}(n)$ to G, or the corresponding Lie algebra representation of $\mathfrak{g}$, and the weights of τ will be denoted here by $\varepsilon_1, \dots, \varepsilon_n$. It can easily be checked that each weight of τ has multiplicity 1. As we show in the sequel, this fact essentially implies that the decomposition (1.8) is multiplicity-free.

The character of τ is then given by $\mathrm{ch}(\tau) = \sum_{i=1}^n e(\varepsilon_i)$ and for $\mu = \tau$, the formula (1.15) yields:

$$\begin{aligned} c(\nu; \tau, \lambda) &= \sum_{w \in \mathfrak{W}} (-1)^{|w|} \mathrm{ev}_{\nu+\delta}[e(w(\lambda+\delta)) \cdot \sum_{i=1}^n e(\varepsilon_i)] \\ &= \sum_{w \in \mathfrak{W}} \sum_{i=1}^n (-1)^{|w|} \mathrm{ev}_{\nu+\delta} e(w(\lambda+\delta)+\varepsilon_i) \\ &= \sum_{w \in \mathfrak{W}} \sum_{i=1}^n (-1)^{|w|} \delta_{\nu+\delta, w(\lambda+\varepsilon_i+\delta)}, \end{aligned} \tag{1.16}$$

where for the last equality we used the property of the Weyl group to leave invariant the set of weights of any representation, in particular those of τ.

Let $\varepsilon \in \{\varepsilon_1, \dots, \varepsilon_n\}$ be one of the weights of τ. From (1.16) it follows that in order to compute the multiplicities $c(\nu; \tau, \lambda)$ we have to compare $\nu + \delta$ with $w(\lambda + \varepsilon + \delta)$, for all elements w of the Weyl group.

An important property of the weights of τ, that can be explicitly checked for each of the groups G in (1.6), is that when expressing them in terms of the fundamental weights $\omega_1, \dots, \omega_r$, the coefficients are either 0 or ± 1, except for two weights with coefficient -2, respectively 2, which only occur for G_2 and $\mathrm{SO}(n)$ with n odd. These cases are treated separately and explain the supplementary condition of the selection rule for the zero weight for these groups.

The above mentioned property and the description (1.12) of the strict Weyl chamber show that $\lambda + \varepsilon + \delta \in WC$. Since the Weyl group $\mathfrak{W}$ acts simply transitively on the set of Weyl chambers, it follows that, for any element $w \neq \mathrm{id}$ of $\mathfrak{W}$, $w(\lambda + \varepsilon + \delta)$ belongs to a different Weyl chamber, so that it cannot be equal to $\nu + \delta$, which lies in the strict Weyl chamber SWC. Thus, the sum over the elements of $\mathfrak{W}$ in (1.16) reduces to one term corresponding to $w = \mathrm{id}$ and we get:

$$c(\nu; \tau, \lambda) = \sum_{i=1}^{n} \delta_{\nu+\delta, \lambda+\varepsilon_i+\delta} = \sum_{i=1}^{n} \delta_{\nu, \lambda+\varepsilon_i}.$$

Hence the only multiplicities different from zero in the tensor product $\tau \otimes \lambda$ are $c(\lambda + \varepsilon; \tau, \lambda) = 1$, for some weight ε of τ, such that $\lambda + \varepsilon$ is dominant. This proves the decomposition (1.8) and the selection rule, except for the zero weight.

If G is G_2 or $\mathrm{SO}(n)$ with n odd, the problem is when $\lambda + \varepsilon + \delta \notin WC$, which may only occur if ε is exactly the weight ε_0 of τ whose expression in terms of the fundamental weights has a coefficient equal to -2. Thus, the condition $\lambda + \varepsilon_0 + \delta \notin WC$ is equivalent to $\lambda - \lambda_\tau$, respectively $\lambda - \lambda_\Sigma$ not being dominant for the group G_2, respectively $\mathrm{SO}(n)$. Then there is exactly one element w_0 in W, given by the reflection to one of the walls of the Weyl chamber WC, that maps $\lambda + \varepsilon_0 + \delta$ in WC. It can be directly checked that $w_0(\lambda + \varepsilon_0 + \delta) = \delta$, so that the only coefficient that changes compared to the above computation is $c(0; \tau, \lambda)$. More precisely, it decreases by 1 and thus becomes equal to zero, completing the proof for these special cases. □

From Theorem 1.7, the tensor product $\mathrm{T}^*M \otimes V_\lambda M$ is multiplicity-free, so that its decomposition is unique and the projections on the irreducible subbundles are well-defined (otherwise one may just define the projections onto the isotypical components of the tensor product). This allows us to define the G-generalized gradients as follows:

Definition 1.8. Let (M, g) be a Riemannian manifold carrying a G-structure and λ be a G-irreducible representation. Then the decomposition of the tensor product of the cotangent bundle with the associated vector bundle:

$$\mathrm{T}^*M \otimes V_\lambda M = \bigoplus_{\varepsilon \subset \lambda} V_{\lambda+\varepsilon} M$$

is completely described by Theorem 1.7 and the *G-generalized gradients* $P_\varepsilon^{\nabla^\lambda}$ acting on sections of $V_\lambda M$ are defined by the composition:

$$\Gamma(V_\lambda M) \xrightarrow{\nabla^\lambda} \Gamma(\mathrm{T}M^* \otimes V_\lambda M) \xrightarrow{\Pi_\varepsilon} \Gamma(V_{\lambda+\varepsilon} M), \qquad (1.17)$$

where ∇^λ is the connection induced on $V_\lambda M$ by a G-connection ∇ and Π_ε is the projection onto the subbundle $V_{\lambda+\varepsilon} M$.

In the sequel we sometimes drop the group G, which may be easily deduced from the context, and just call these operators generalized gradients. The Definition 1.8 may be given more generally, not just for the geometrically interesting groups in (1.6), but for any group $G \subset \mathrm{GL}(n)$, which satisfies the technical condition given by (1.8), *i.e.* such that the tensor product of any G-irreducible representation with the restriction of the standard $\mathrm{GL}(n)$-representation is multiplicity-free.

Notice that $N(\lambda) := \sharp\{\varepsilon \,|\, \varepsilon \text{ is relevant for } \lambda\} \leq \dim(\tau)$, so that there are at most n generalized gradients for each dominant weight λ and this is the generic case.

Example 1.9. If G is $\mathrm{U}(\frac{n}{2})$ or $\mathrm{SU}(\frac{n}{2})$, then the decomposition (1.7) implies that the covariant derivative splits as follows: $\nabla^\lambda = (\nabla^\lambda)^{1,0} + (\nabla^\lambda)^{0,1}$ with respect to the almost complex structure. Consequently, the set of $\mathrm{U}(\frac{n}{2})$ or $\mathrm{SU}(\frac{n}{2})$-generalized gradients acting on sections of an irreducible vector bundle $V_\lambda M$ splits into two subsets, namely the sets of gradients factorizing over the complementary projections:

$$\begin{array}{ccccc} & & \Gamma(T^{0,1}M\otimes_{\mathbb{C}} V_\lambda M) & \xrightarrow{\Pi_\varepsilon} & \Gamma(V_{\lambda+\varepsilon}M)\\ & \overset{\mathrm{pr}^{1,0}}{\nearrow} & & & \\ \Gamma(V_\lambda M)\xrightarrow{\nabla^\lambda}\Gamma(T^{\mathbb{C}}M\otimes_{\mathbb{C}}V_\lambda M) & & & & \\ & \underset{\mathrm{pr}^{0,1}}{\searrow} & & & \\ & & \Gamma(T^{1,0}M\otimes_{\mathbb{C}}V_\lambda M) & \xrightarrow{\Pi_{\varepsilon'}} & \Gamma(V_{\lambda+\varepsilon'}M)\end{array}$$

and which are called *holomorphic*, respectively *anti-holomorphic* generalized gradients. This apparently skewed notation is due to the isomorphisms $T^{1,0} \cong (T^{0,1})^*$ and $T^{0,1} \cong (T^{1,0})^*$. We notice that the weights of $T^{1,0}$ are equal to those of $T^{0,1}$ with opposite sign.

On a Kähler spin manifold of complex dimension m, examples of $\mathrm{U}(m)$-generalized gradients are the projections of the Dirac operator, D^+ and D^-, defined by (3.6), and the twistor operator of type r, T_r, given by (4.4), which act on each of the irreducible subbundles $\Sigma_r M$ of the spinor bundle, as explained in detail in § 3 and § 4.1.

Remark 1.10. The exceptional case of the zero weight in the selection rule in Theorem 1.7 provides interesting generalized gradients, which have the same source and target bundle, so that in particular they have spectra. In this case the projection Π_0 composed with the covariant derivative carries sections of $V_\lambda M$ to sections of a copy of $V_\lambda M$ which is a subbundle in $\mathrm{T}^*M \otimes V_\lambda M$. In order to use the same realization of $V_\lambda M$ as both source and target bundle for a realization Π_0^{self} of Π_0, one needs a choice of normalization. First, one normalizes the Hermitian inner product on $\mathrm{T}^*M \otimes V_\lambda M$ such that $|\xi \otimes v|^2 = |\xi|^2|v|^2$ and then normalizes Π^{self} such that $(\Pi_0^{\mathrm{self}})^2 = \Pi_0^*\Pi_0$. This determines Π_0^{self} up to multiplication by ± 1. Examples of such generalized gradients are the following: the Dirac operator, the Rarita-Schwinger operator, $*d$ acting on $\frac{n-1}{2}$-forms in odd dimension $n \geq 3$. Explicit computations of the spectra of these operators have been done on certain manifolds, for instance by Th. Branson, [14], on spheres. In general, there exist only estimates of the eigenvalues of these operators. An illustration of this fact is given in § 3.2 for the Dirac operator, where we describe the known estimates for its eigenvalues and the corresponding limiting manifolds.

1.2 The Conformal Invariance of Generalized Gradients

In this section we show that the generalized gradients associated to a G-structure are conformally invariant. First we describe the known classical case of generalized gradients of the special orthogonal group $\mathrm{SO}(n)$ on a Riemannian manifold and of $\mathrm{Spin}(n)$, the two-fold covering of $\mathrm{SO}(n)$, on a spin manifold. Then we prove the conformal invariance in the general case of a G-structure, when assuming a reduction of the structure group of the tangent bundle to a closed subgroup G of $\mathrm{SO}(n)$. In § 1.1 we defined generalized gradients for any metric connection, respectively G-connection. However, in order to ensure their conformal invariance, all generalized gradients considered throughout this section are constructed using the Levi-Civita connection, respectively the minimal G-connection.

The property of conformal invariance was noticed for the first time in spin geometry by N. Hitchin in 1974, [29], for the Dirac and Penrose operator and it turned out to have important consequences in physics. Two years later, H. Fegan, [19], showed that, up to the composition with a bundle map, the only conformally invariant first order differential operators between vector bundles associated to the bundle of oriented frames are the generalized gradients. Further results were obtained by Y. Homma, [31], [32], [34], for the conformal invariance of generalized gradients associated to U(n), Sp(n) and Sp(1)·Sp(n)-structures. For these subgroups Homma's proof of the conformal invariance is given by explicit computations, which are based on the relation between the enveloping algebra of the Lie algebra $\mathfrak{g}$ and the algebraic structure of the principal symbols of generalized gradients.

In the sequel we use the framework of conformal geometry to show the conformal invariance of generalized gradients associated to the minimal connection of any G-structure. This approach leads us on the one hand to a general result and on the other hand gives a uniform and direct proof of the known cases, avoiding the specific computations for each subgroup.

1.2.1 SO(n) and Spin(n)-generalized gradients

As noticed by P. Gauduchon, [24], the suitable approach to the study of conformal invariance is to make use of Weyl connections. An important role is played by the conformal weight operator, whose eigenvalues are exactly the corresponding conformal weights of the generalized gradients (which is also the motivation for its name).

Let (M, c) be an oriented n-dimensional manifold with a conformal structure c, *i.e.* an equivalence class of Riemannian metrics, where two metrics are equivalent $\bar{g} \sim g$ if there exists a function $u : M \to \mathbb{R}$ such that $\bar{g} = e^{2u} g$. In the language of G-structures this is equivalent to a reduction of the structure group of the tangent bundle to the conformal group

$$\mathrm{CO}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) |\, A^t A = aI_n, a > 0\},$$

which is also described as follows:

$$\mathrm{CO}(n) = \mathbb{R}^*_+ \times \mathrm{SO}(n) = \{aA |\, a \in \mathbb{R}^*_+, A \in \mathrm{SO}(n)\}.$$

Let $\mathrm{CO}_n M$ denote the principal bundle of oriented c-orthonormal frames on (M, c) and $\mathrm{SO}_g M$ the bundle of oriented orthonormal frames for a fixed metric g in the conformal class.

Each irreducible representation of CO(n), $\tilde{\lambda} : \mathrm{CO}(n) \to \mathrm{Aut}(V)$, is identified with a couple (λ, w), where λ is the restriction of $\tilde{\lambda}$ to SO(n), which is still an irreducible representation, and w, the conformal weight of $\tilde{\lambda}$, is determined by the restriction of $\tilde{\lambda}$ to $\mathbb{R}^*_+$, which has the following form:

$$\tilde{\lambda}(a) = a^w \cdot I, \quad a \in \mathbb{R}^*_+,$$

where I is the identity of V and the conformal weight w is a real or complex number, depending on whether V is a real or complex representation. As in the previous section, we use the description of an irreducible representation by its highest weight $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$, where $m = [\frac{n}{2}]$ is the rank of $\mathfrak{so}(n)$.

Let $V_{\tilde{\lambda}}M$ denote the vector bundle on M associated to $\tilde{\lambda}$ and to the principal $\mathrm{CO}(n)$-fiber bundle CO_nM and call it of conformal weight w (and the same for the sections of $V_{\tilde{\lambda}}M$). Notice that any vector bundle on M determined by GL_nM and a linear representation of $\mathrm{GL}(n, \mathbb{R})$ has a *natural conformal weight*, which is the conformal weight of the restriction of this representation to $\mathrm{CO}(n)$. For example, the natural conformal weight[1] of $\mathrm{T}M$ is 1 and of T^*M is -1.

Let us first recall the definition of a Weyl structure.

Definition 1.11. A *Weyl structure* on (M, c) is a linear connection D on $\mathrm{T}M$, which is *conformal*, *i.e.* induced by a $\mathrm{CO}(n)$-equivariant connection on CO_nM and *symmetric*, *i.e.* has no torsion.

A Weyl structure D is called *closed* if it is locally the Levi-Civita connection of a local metric in the conformal class c and it is called *exact* if it is globally the Levi-Civita connection of a metric in c. In both cases, locally or globally, the metric is uniquely defined up to a constant by the Weyl structure.

The Weyl structures on the conformal manifold (M, c) form an affine space modeled on the space of real 1-forms on M. More precisely, two Weyl structures are related by:

$$D_2 = D_1 + \tilde{\theta}, \tag{1.18}$$

where θ is a real 1-form on M and $\tilde{\theta}$ is the 1-form with values in the adjoint bundle (the one associated to the adjoint representation of $\mathrm{CO}(n)$ on its Lie algebra $\mathfrak{co}(n)$), identified with θ by:

$$\tilde{\theta}(X) = \theta(X) \cdot I + \theta \wedge X,$$

[1] We adopt the convention in [24], in contrast to [19] and [30], where the sign is opposite, since the principal bundle considered there is the one of co-frames.

where $\theta \wedge X$ is the skew-symmetric endomorphism defined as:

$$(\theta \wedge X)(Y) = \theta(Y)X - g(X,Y)\theta^\sharp,$$

for any metric g in the conformal class and $\theta^\sharp$ the dual of θ with respect to the metric g.

Each conformal connection on $\mathrm{T}M$, in particular any Weyl structure D, induces for each linear representation $\tilde{\lambda} : \mathrm{CO}(n) \to \mathrm{Aut}(V)$ a covariant derivative $D^{\tilde{\lambda}}$ on the associated vector bundle $V_{\tilde{\lambda}}M$. If D_1 and D_2 are related by (1.18), then, for $\tilde{\lambda} = (\lambda, w)$, the induced covariant derivatives $D_1^{\tilde{\lambda}}$ and $D_2^{\tilde{\lambda}}$ satisfy:

$$D_2^{\tilde{\lambda}} = D_1^{\tilde{\lambda}} + d\tilde{\lambda}(\tilde{\theta}) = D_1^{\tilde{\lambda}} + \sum_{i=1}^{n} e_i^* \otimes d\lambda(\theta \wedge e_i) + w\,\theta \otimes I, \tag{1.19}$$

where $\{e_i\}_{i=\overline{1,n}}$ is a c-orthonormal frame at the point considered and $\{e_i^*\}_{i=\overline{1,n}}$ is the (algebraic) dual frame.

By τ we denote, as usual, the standard representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$, identified with its dual $(\mathbb{R}^n)^*$, and also the representation of conformal weight -1 of $\mathrm{CO}(n)$ on $(\mathbb{R}^n)^*$. The associated vector bundles to τ and CO_nM, respectively SO_gM, are canonically identified to T^*M.

Consider an irreducible representation of $\mathrm{CO}(n)$, $\tilde{\lambda} : \mathrm{CO}(n) \to \mathrm{Aut}(V)$, of conformal weight w and λ the highest weight of its restriction to $\mathrm{SO}(n)$. The tensor product $\tau \otimes \tilde{\lambda}$ has then conformal weight $w-1$ and it has the same multiplicity-free decomposition into irreducible $\mathrm{CO}(n)$ or $\mathrm{SO}(n)$-subrepresentation, as described by Lemma 1.1:

$$\tau \otimes \lambda = \bigoplus_{\varepsilon \subset \lambda} \mu_\varepsilon, \tag{1.20}$$

where $\mu_\varepsilon = \lambda + \varepsilon$ and ε belongs to the set of weights of τ which are relevant with respect to λ. We introduce the following notation: $\mu_0 = \lambda$, $\mu_{i,\pm} = \lambda \pm \varepsilon_i$, for $i = 1, \ldots, m$, where $m = [\frac{n}{2}]$.

For each relevant weight ε of λ, we introduced in Definition 1.2 a generalized gradient acting on sections of the associated vector bundle $V_\lambda M$ (here we specify the metric g in the notation, because we shall compare operators associated to different metrics):

$$P_\varepsilon^{g,\lambda} = \Pi_\varepsilon \circ \nabla^{g,\lambda}, \tag{1.21}$$

where $\nabla^{g,\lambda}$ is the connection induced on $V_\lambda M$ by the Levi-Civita connection ∇^g of (M, g).

In the same way, for each conformal weight w, each relevant weight ε of λ determines on any conformal manifold (M, c) a family of generalized gradients, parametrized by the Weyl structures, acting on the associated vector bundle $V_{\tilde\lambda}M$, where $\tilde\lambda = (\lambda, w)$, as follows:

$$P_\varepsilon^{D,\tilde\lambda} = \Pi_\varepsilon \circ D^{\tilde\lambda}, \tag{1.22}$$

where $D^{\tilde\lambda}$ is the connection induced on $V_{\tilde\lambda}M$ by a Weyl connection D of (M, c) and Π_ε the projection of $\mathrm{T}^*M \otimes V_{(\lambda,w)}M$ onto its subbundle $V_{(\lambda+\varepsilon,w-1)}M$.

Definition 1.12. The operator $P_\varepsilon^{g,\lambda}$ is called *conformally invariant relative to the conformal weight* w if the operators $P_\varepsilon^{D,\tilde\lambda}$ defined by (1.22) do not depend on the Weyl structure D.

It turns out that these conformal weights, with respect to which the generalized gradients are conformally invariant, are exactly the eigenvalues of the so-called conformal weight operator. Let us first recall its definition (here we use the identification of the adjoint representation $\mathfrak{so}(n)$ with $\Lambda^2\mathbb{R}^n$ through $\langle A, x\wedge y\rangle = \langle Ax, y\rangle$, where the scalar product $\langle\cdot,\cdot\rangle$ on $\Lambda^2\mathbb{R}^n$ is the one induced via Gram's determinant):

Definition 1.13. The *conformal weight operator* of an $\mathrm{SO}(n)$-representation λ, $\lambda : \mathrm{SO}(n) \to \mathrm{Aut}(V)$, is the symmetric endomorphism defined as follows:

$$B^\lambda : (\mathbb{R}^n)^* \otimes V \to (\mathbb{R}^n)^* \otimes V, \quad B^\lambda(\alpha \otimes v) = \sum_{i=1}^n e_i^* \otimes d\lambda(e_i \wedge \alpha)v, \tag{1.23}$$

where $\{e_i\}_{\overline{1,n}}$ is an orthonormal basis of $\mathbb{R}^n$ and $\{e_i^*\}_{\overline{1,n}}$ its dual basis. The operator B^λ is $(\tau \otimes \tilde\lambda)$-equivariant for any conformal weight w, $\tilde\lambda = (\lambda, w)$. We also denote by B^λ the induced endomorphism on the associated bundle $\mathrm{T}^*M \otimes V_{\tilde\lambda}M$.

Using the conformal weight operator, the difference between the connections induced on $V_{\tilde\lambda}M$ by two Weyl structures D_1 and D_2 related by (1.18) is given as follows:

$$(D_2^{\tilde\lambda} - D_1^{\tilde\lambda})\xi = w\theta \otimes \xi - B^\lambda(\theta \otimes \xi), \quad \text{for all } \xi \in \Gamma(V_{\tilde\lambda}M). \tag{1.24}$$

Since the algebraic endomorphism B^λ is CO(n)-equivariant and the decomposition (1.20) is multiplicity-free, it follows from Schur's Lemma that B^λ acts on each irreducible component of the decomposition (1.20) by multiplication with a scalar.

We show in the sequel that the conformal weight operator B^λ may be expressed in terms of the Casimir operators and then explicitly compute its eigenvalues. Let us recall that the *(normalized) Casimir operator* C^λ of any SO(n)-representation λ is defined by:

$$C^\lambda = -\sum_{i<j} d\lambda(e_i \wedge e_j) \circ d\lambda(e_i \wedge e_j), \tag{1.25}$$

where as above, $\{e_i\}_{\overline{1,n}}$ is an orthonormal basis of $\mathbb{R}^n$.

If the representation λ is irreducible, it follows from Schur's Lemma that C^λ is a scalar equal to $c(\lambda) \cdot I$, where $c(\lambda)$, called the *Casimir number*, is a real positive number, except for the trivial representation, when it is zero. The Casimir numbers are normalized such that $c(\tau) = n - 1$. Freudenthal's formula gives the following expression for the Casimir numbers of an irreducible representation of $\mathfrak{so}(n)$ of highest weight λ:

$$c(\lambda) = \langle \lambda + \delta, \lambda + \delta \rangle - \langle \delta, \delta \rangle = \langle \lambda, \lambda + 2\delta \rangle, \tag{1.26}$$

where δ is the Weyl vector of $\mathfrak{so}(n)$, defined as half the sum of the positive roots, so that its components are $\delta_i = \frac{n-2i}{2}$, for $i = 1, \dots, m$.

Lemma 1.14 (Fegan's Lemma, [19]). *The conformal weight operator B^λ is equal to*

$$B^\lambda = \frac{1}{2}(C^{\tau\otimes\lambda} - I|_{\mathbb{R}^*} \otimes C^\lambda - C^\tau \otimes I|_V). \tag{1.27}$$

It is then straightforward the following:

Corollary 1.15. *The conformal weight operator B^λ acts on each irreducible summand $\lambda + \varepsilon$ in the decomposition $\tau \otimes \lambda = \bigoplus_{\varepsilon\subset\lambda} (\lambda + \varepsilon)$ by multiplication with the scalar $w_\varepsilon(\lambda)$ given by:*

$$w_\varepsilon(\lambda) = \frac{1}{2}(c(\lambda + \varepsilon) - c(\lambda) - (n - 1)). \tag{1.28}$$

From Freudenthal's formula (1.26) *we then get for* $i = 1, \dots, m$*:*

$$\begin{aligned} w_0(\lambda) &= \frac{1-n}{2}, \\ w_{i,+}(\lambda) := w_{\varepsilon_i}(\lambda) &= 1 + \lambda_i - i, \\ w_{i,-}(\lambda) := w_{-\varepsilon_i}(\lambda) &= 1 - n - (\lambda_i - i). \end{aligned} \tag{1.29}$$

To simplify notation, we shall sometimes omit the highest weight λ in the denomination of the conformal weights.

Remark 1.16. The formulas (1.29) together with the conditions on the weights to be relevant show that the conformal weights are ordered as follows:

$$w_{1,+} > \cdots > w_{m,+} > w_0 \geq w_{m,-} > \cdots > w_{1,-},$$

if n is odd, $n = 2m+1$, and $w_0(\lambda) = w_{m,-}(\lambda)$ if and only if $\lambda_m = 0$. But, from the selection rule given by Lemma 1.1, it follows that this equality case cannot occur for relevant weights, since if $\lambda_m = 0$, then neither w_0 nor $w_{m,-}$ is a relevant weight. For n even, $n = 2m$, the conformal weights are ordered as follows:

$$w_{1,+} > \cdots > w_{m-1,+} > \{w_{m,+}, w_{m,-}\} > w_{m-1,-} > \cdots > w_{1,-},$$

where $w_{m,+}(\lambda) - w_{m,-}(\lambda) = 2\lambda_m$, so that $w_{m,+}(\lambda) \neq w_{m,-}(\lambda)$ unless $\lambda_m = 0$. Hence, the conformal weights are almost always distinct. Thus, except for the cases of irreducible representations λ with $\lambda_m = 0$, it turns out that the decomposition (1.20) corresponds exactly to the eigenspaces of the conformal weight operator B^λ, which may be expanded as: $B^\lambda = \sum_{\varepsilon \subset \lambda} w_\varepsilon(\lambda)\Pi_\varepsilon$.

Lemma 1.17. *The operator* $P_\varepsilon^{g,\lambda}$ *is conformally invariant relative to the conformal weight* w_ε *and this is the only conformal weight with respect to which this operator is conformally invariant.*

Proof: Composing (1.24) with the projection Π_ε we get

$$(P_\varepsilon^{D_2,\tilde\lambda} - P_\varepsilon^{D_1,\tilde\lambda})(\xi) = (w - w_\varepsilon)\Pi_\varepsilon(\theta \otimes \xi), \quad \text{for all } \xi \in \Gamma(V_{\tilde\lambda}M). \tag{1.30}$$

Thus, it follows that $P_\varepsilon^{g,\lambda}$ is conformally invariant if and only if $w = w_\varepsilon$, since otherwise we would get the contradiction that $\Pi_\varepsilon \equiv 0$. $\square$

The relation between the formulation of the conformal invariance with respect to the Weyl connections and the one with respect to the Levi-Civita connections of conformally related metrics follows from Proposition 1.27, which holds in a more general setting for G-generalized gradients. In the case of the $\mathrm{SO}(n)$-structure, the minimal connection is just the Levi-Civita connection, so that we obtain the equivalence of the following two statements:

(1) $P_\varepsilon^{g,\lambda}$ is conformally invariant relative to the conformal weight w_ε.

(2) If $\bar{g}$ is a metric conformally related to g, $\bar{g} = e^{2u}g$, then the corresponding operators induced by the Levi-Civita connections are related by:

$$P_\varepsilon^{\bar{g},\lambda} \circ \phi_{w_\varepsilon}^{g,\bar{g}} = \phi_{w_\varepsilon - 1}^{g,\bar{g}} \circ P_\varepsilon^{g,\lambda}, \tag{1.31}$$

where, for any conformal weight w, $\phi_w^{g,\bar{g}}$ is the isomorphism between the induced vector bundles defined by:

$$\phi_w^{g,\bar{g}} : V_\lambda^g M \to V_\lambda^{\bar{g}} M, \quad [(e_1, \dots, e_n), v] \mapsto [(e^{-u}e_1, \dots, e^{-u}e_n), e^{wu}v].$$

Remark 1.18. The relation (1.31) expressing the conformal invariance of the generalized gradients may be rewritten in the following form, which is usually encountered in literature:

$$P_\varepsilon^{\bar{g},\lambda} = e^{(w_\varepsilon - 1)u} \phi^{g,\bar{g}} \circ P_\varepsilon^{g,\lambda} \circ e^{-w_\varepsilon u} (\phi^{g,\bar{g}})^{-1}, \tag{1.32}$$

using the following identification of the associated vector bundles that does not take into account any conformal weight, but has the advantage of being an isometry:

$$\phi^{g,\bar{g}} : V_\lambda^g M \to V_\lambda^{\bar{g}} M, \quad [s, v] \mapsto [\Phi^{g,\bar{g}}(s), v],$$

where $\Phi^{g,\bar{g}}$ is the isomorphism of the $\mathrm{SO}(n)$-principal bundles of the two conformally related metrics $\bar{g} = e^{2u}g$:

$$\Phi^{g,\bar{g}} : \mathrm{SO}_g M \to \mathrm{SO}_{\bar{g}} M, \quad \{e_1, \dots, e_n\} \mapsto \{e^{-u}e_1, \dots, e^{-u}e_n\}. \tag{1.33}$$

Remark 1.19. If V is not just a representation of $\mathrm{SO}(n)$, but is the restriction of a representation of the general linear group $\mathrm{GL}(n, \mathbb{R})$, then, as noticed above, V has a natural conformal weight given by the restriction of the representation to $\mathrm{CO}(n) \subset \mathrm{GL}(n, \mathbb{R})$. In this case one may canonically identify the associated bundles to this representation and to the principal bundles $\mathrm{SO}_g M$, $\mathrm{SO}_{\bar{g}} M$ and $\mathrm{CO}_n M$. When both representations of highest weight λ, respectively $\lambda+\varepsilon$, come from representations of $\mathrm{GL}(n,\mathbb{R})$, the corresponding generalized gradients associated to the conformally related metrics g and $\bar{g}$ are related as follows, after having identified the associated vector bundles to the ones associated to $\mathrm{CO}_n M$:

$$P_\varepsilon^{\bar{g},\lambda} = e^{(w_\varepsilon - \omega_{\lambda+\varepsilon} - 1)u} P_\varepsilon^{g,\lambda} e^{-(w_\varepsilon - \omega_\lambda)u}, \tag{1.34}$$

where w_ε is the conformal weight and ω_λ, $\omega_{\lambda+\varepsilon}$ are the natural conformal weights of the representations λ, respectively $\lambda+\varepsilon$.

Example 1.20. We consider the bundle of p-forms, $\Lambda^p M$, on a Riemannian manifold (M^n, g), as in Example 1.20. The highest weight of the representation is $\lambda_p = (1,\dots,1,0,\dots,0)$ and as it is the restriction of a $\mathrm{GL}(n,\mathbb{R})$-representation, it has a natural conformal weight which is equal to $-p$. As we had already seen, there are three relevant weights for λ_p: $-\varepsilon_p$, ε_{p+1} and ε_1 and the tensor product decomposes as follows:

$$\mathrm{T}M \otimes \Lambda^p M \cong \Lambda^{p-1}M \oplus \Lambda^{p+1}M \oplus \Lambda^{p,1}M,$$

where the last irreducible component is the Cartan summand corresponding to the highest weight $\lambda_p + \varepsilon_1$. The eigenvalues of the conformal weight operator are then given by (1.29): $w_{p,-}(\lambda_p) = -n+p$, $w_{p+1,+}(\lambda_p) = -p$ and $w_{1,+}(\lambda_p) = 1$. The relation (1.34) implies that the conformal invariance of these generalized gradients, acting on the vector bundles associated to $\mathrm{CO}_n M$, may be expressed as follows: $d^{\bar{g}} = d^g$, $\delta^{\bar{g}} = e^{(-n+2p-2)u}\delta^g e^{(n-2p)u}$. The first equality just expresses the obvious fact that the exterior derivative d is independent of the metric. The conformal invariance of the twistor operator is given by substituting $w_\varepsilon = w_{1,+}(\lambda_p) = 1$ into (1.32):

$$T^{\bar{g}} = \phi^{g,\bar{g}} \circ T^g \circ e^u (\phi^{g,\bar{g}})^{-1}.$$

On a spin manifold, the generalized gradients associated to the group $\mathrm{Spin}(n)$ are also conformally invariant. The situation is very similar to the one for the group $\mathrm{SO}(n)$, since their Lie algebras are canonically identified. In the sequel we write down explicitly this property of conformal invariance on spin manifolds and illustrate it for the most well-known operators: the Dirac operator, the twistor (Penrose) operator and the Rarita-Schwinger operator.

The approach is the same as for the special orthogonal group, using in this case the Weyl structures on a spin conformal manifold. We first recall its definition. The *conformal spin group* is the group $\mathrm{CSpin}(n) = \mathrm{Spin}(n) \times \mathbb{R}_+^*$, which is the universal cover of the conformal group $\mathrm{CO}(n)$. Its Lie algebra is $\mathfrak{cspin}_n \cong \mathfrak{so}_n \oplus \mathbb{R} \cong \mathfrak{co}(n)$.

Definition 1.21. A *spin structure* on a conformal manifold (M, c) is given by a principal $\mathrm{CSpin}(n)$-bundle $\mathrm{CSpin}_n M$ together with a projection θ, such that the following diagram commutes for every $\tilde{u} \in \mathrm{CSpin}_n M$:

$$\begin{array}{ccccc} \mathrm{CSpin}(n) & \xrightarrow{a \mapsto \tilde{u}a} & \mathrm{CSpin}_n M & & \\ \Big\downarrow \phi & & \Big\downarrow \theta & \searrow & \\ & & & & M, \\ & & & \nearrow & \\ \mathrm{CO}(n) & \xrightarrow[A \mapsto \theta(\tilde{u})A]{} & \mathrm{CO}_n M & & \end{array}$$

where ϕ is the canonical projection of $\mathrm{CSpin}(n)$ onto $\mathrm{CO}(n)$.

Similarly to the Riemannian case, if M has a spin structure, then any Weyl structure D induces a connection on $\mathrm{CSpin}_n M$, and therefore a covariant derivative on each associated vector bundle to a representation of $\mathrm{CSpin}(n)$. The description of the representations of $\mathrm{CSpin}(n)$ is analogous to the one for $\mathrm{CO}(n)$: each irreducible representation $\tilde{\lambda}$ of $\mathrm{CSpin}(n)$ is given by a couple (λ, w), where λ is the restriction of $\tilde{\lambda}$ to $\mathrm{Spin}(n)$, which is still an irreducible representation, and w, the conformal weight of $\tilde{\lambda}$, is determined by the restriction of $\tilde{\lambda}$ to $\mathbb{R}_+^*$: $\tilde{\lambda}(a) = a^w \cdot I$.

Definition 1.12 of conformal invariance relative to a conformal weight may thus be carried over to generalized gradients of $\mathrm{Spin}(n)$. Since the formula (1.23) defining the conformal weight operator only involves the representation of the Lie algebra $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$, it follows that B^λ is well-defined for any irreducible representation λ of $\mathrm{Spin}(n)$. Consequently, its eigenvalues, called conformal weights, are also given by (1.27) or more explicitly by (1.29) and Remark 1.16 still holds in this case.

The argument in the proof of Lemma 1.17 also works for $\mathrm{Spin}(n)$-generalized gradients. The equivalence to the formulation with respect to the induced Levi-Civita connections of conformally related metrics follows similarly from Proposition 1.27. In order to give the precise statement we first need to recall the isomorphism between the spinor bundles of conformally related metrics.

Let (M^n, g) be a Riemannian spin manifold and consider the conformal change of the metric given by $\bar{g} = e^{2u}g$. As above, $\Phi^{g,\bar{g}}$ denotes the isomorphism of the two SO(n)-principal fiber bundles given by (1.33). Then there is a spin structure induced on $(M^n, \bar{g})$, which is defined up to isomorphism by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}_g M & \xrightarrow{\tilde{\Phi}^{g,\bar{g}}} & \mathrm{Spin}_{\bar{g}} M \\ \downarrow{\scriptstyle \theta_g} & & \downarrow{\scriptstyle \theta_{\bar{g}}} \\ \mathrm{SO}_g M & \xrightarrow[\Phi^{g,\bar{g}}]{} & \mathrm{SO}_{\bar{g}} M \end{array}$$

It then follows, by an analogue argument to the one in the proof of Proposition 1.27, that the following two statements are equivalent:

(1) $P_\varepsilon^{g,\lambda}$ is conformally invariant relative to the conformal weight w_ε.

(2) If $\bar{g}$ is a metric conformally related to g, $\bar{g} = e^{2u}g$, then the corresponding operators induced by the Levi-Civita connections are related by:

$$P_\varepsilon^{\bar{g},\lambda} \circ \tilde{\phi}_{w_\varepsilon}^{g,\bar{g}} = \tilde{\phi}_{w_\varepsilon - 1}^{g,\bar{g}} \circ P_\varepsilon^{g,\lambda}, \qquad (1.35)$$

where, for any conformal weight w, $\tilde{\phi}_w^{g,\bar{g}}$ is the isomorphism between the associated vector bundles defined by:

$$\tilde{\phi}_w^{g,\bar{g}} : V_\lambda^g M \to V_\lambda^{\bar{g}} M, \quad [s, v] \mapsto [\tilde{\Phi}^{g,\bar{g}}(s), e^{wu}v].$$

Example 1.22. The spinor representation $\rho_n : \mathrm{Spin}(n) \to \mathrm{Aut}(\Sigma_n)$, with n odd, is irreducible and in this case there are two generalized gradients (see Example 1.5): the Dirac operator D and the twistor operator T. The highest weight is $\rho_n = (\frac{1}{2}, \ldots, \frac{1}{2})$ and the conformal weights are given by (1.29): $w_0(\rho_n) = \frac{1-n}{2}$, $w_{1,+}(\rho_n) = \frac{1}{2}$. It then follows that D is conformally invariant relative to the conformal weight $\frac{1-n}{2}$ and T is conformally invariant relative to the weight $\frac{1}{2}$. If g and $\bar{g}$ are two conformally related metrics, $\bar{g} = e^{2u}g$, by (1.35) we have:

$$D^{\bar{g}} \circ \tilde{\phi}_{\frac{1-n}{2}}^{g,\bar{g}} = \tilde{\phi}_{-\frac{1+n}{2}}^{g,\bar{g}} \circ D^g, \qquad (1.36)$$

$$T^{\bar{g}} \circ \tilde{\phi}_{\frac{1}{2}}^{g,\bar{g}} = \tilde{\phi}_{-\frac{1}{2}}^{g,\bar{g}} \circ T^g. \qquad (1.37)$$

In the case when n is even, $n = 2m$, the spinor bundle splits in two irreducible subbundles $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ with highest weight $\rho_n^+ = (\frac{1}{2}, \dots, \frac{1}{2})$, respectively $\rho_n^- = (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$. In each case there are again two generalized gradients, the Dirac and the twistor operator (again see Example 1.5):

$$D : \Gamma(\Sigma^\pm M) \to \Gamma(\Sigma^\mp M),$$

$$T : \Gamma(\Sigma^\pm M) \to \Gamma(\ker(c)),$$

and their conformal weights, given by (1.29), are: $w_{m,-}(\rho_n^+) = w_{m,+}(\rho_n^-) = \frac{1-n}{2}$, respectively $w_{1,+}(\rho_n^+) = w_{1,+}(\rho_n^-) = \frac{1}{2}$.

Remark 1.23. In a simplified notation we consider:

$$\bar{\cdot} : \Sigma M = \mathrm{Spin}_g M \times_{\rho_n} \Sigma_n \to \Sigma \overline{M} = \mathrm{Spin}_{\bar{g}} M \times_{\rho_n} \Sigma_n$$
$$[s, \varphi] \mapsto [\tilde{\Phi}^{g,\bar{g}}(s), \varphi],$$

which is an isometry with respect to the Hermitian product on the spinor bundles. We may then rewrite the conformal invariance of D and T in the following more familiar expression (where $\overline{D}$ and $\overline{T}$ denote the operators associated to the metric $\bar{g}$):

$$\overline{D}(e^{-\frac{n-1}{2}u}\bar{\varphi}) = e^{-\frac{n+1}{2}u}\overline{D\varphi}, \tag{1.38}$$

$$\overline{T}(e^{\frac{u}{2}}\bar{\varphi}) = e^{-\frac{u}{2}}\overline{T\varphi}. \tag{1.39}$$

We mention that the conformal invariance of T is usually written in the following form:

$$\overline{T}_X(e^{\frac{u}{2}}\bar{\varphi}) = e^{\frac{u}{2}}\overline{T_X\varphi}, \tag{1.40}$$

where $T_X : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$, so that on the right side is the same conformal weight $\frac{1}{2}$.

The conformal invariance of these operators was first established by N. Hitchin, [29]. The original proof is given by an explicit computation, using the following relation between the Levi-Civita connections induced on the spinor bundles of two metrics g and $\bar{g} = e^{2u}g$ in the same conformal class:

$$\overline{\nabla}_X\overline{\varphi} = \overline{\nabla_X\varphi} - \frac{1}{2}\overline{X \cdot du \cdot \varphi} - \frac{1}{2}X(u)\overline{\varphi}, \tag{1.41}$$

for every $\varphi \in \Gamma(\Sigma M)$ and $X \in \Gamma(\mathrm{T}M)$, where $\overline{X}$ is given by $\overline{X} = e^{-u}X$.

Example 1.24. Another important generalized gradient is the so-called Rarita-Schwinger operator (see Example (1.6)), $D_{3/2} : \Sigma_{3/2}M \to \Sigma_{3/2}M$, for n odd and $D^{\pm}_{3/2} : \Sigma^{\pm}_{3/2}M \to \Sigma^{\mp}_{3/2}M$, for n even. It has the same conformal weight as the Dirac operator, namely $\frac{1-n}{2}$: for $n = 2m+1$, we get by (1.29) $w_0(\lambda) = \frac{1-n}{2}$, for any highest weight λ, and for $n = 2m$ we have $w_{m,-}((\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})) = w_{m,+}((\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})) = \frac{1-n}{2}$. Thus, the Rarita-Schwinger operator fulfills the same relation by conformal changes of the metric as the Dirac operator (with the notations of the previous remark, while here the sections φ and $\overline{\varphi}$ are in the twistor bundles associated to the principal bundle $\mathrm{Spin}_g M$, respectively $\mathrm{Spin}_{\bar{g}} M$):

$$\overline{D_{3/2}}(e^{-\frac{n-1}{2}u}\bar{\varphi}) = e^{-\frac{n+1}{2}u}\overline{D_{3/2}\varphi}. \tag{1.42}$$

1.2.2 Generalized gradients of G-structures

Suppose now that the Riemannian manifold (M, g) admits a reduction of the orthonormal frame bundle $\mathrm{SO}_g M$ to a subbundle GM with structure group G, where G is a closed subgroup of the special orthogonal group $\mathrm{SO}(n)$. First we need to establish which special connection plays for a G-structure the role of the Levi-Civita connection and should be used to define the generalized gradients, such that they are conformally invariant. This is the so-called *minimal connection* of a G-structure, which is, in a certain sense, the G-connection that is as close as possible to the Levi-Civita connection.

Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be the Lie algebra of G and decompose the Lie algebra $\mathfrak{so}(n)$ of all skew-symmetric matrices as the direct sum of $\mathfrak{g}$ and its orthogonal complement: $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$. Denote by $\mathrm{pr}_{\mathfrak{g}}$ and $\mathrm{pr}_{\mathfrak{g}^{\perp}}$ the projections onto $\mathfrak{g}$ and $\mathfrak{g}^{\perp}$, respectively.

The Levi-Civita connection seen as a connection form is a 1-form ω^{LC} on $\mathrm{SO}_g M$ with values in the Lie algebra $\mathfrak{so}(n)$. Restricting ω^{LC} to the subbundle GM and decomposing it with respect to the above splitting, we get:

$$\omega^{LC} = \mathrm{pr}_{\mathfrak{g}}(\omega^{LC}) + \mathrm{pr}_{\mathfrak{g}^{\perp}}(\omega^{LC}) =: \omega^G + T, \tag{1.43}$$

where ω^G is a connection form on GM and T is a 1-form on M with values in the associated vector bundle $GM \times_G \mathfrak{g}^{\perp}$. The connection corresponding to the connection 1-form ω^G is called the *minimal connection* of the G-structure and is denoted by ∇^G. T is called the *intrinsic torsion* of the G-structure and is a measure for how much the G-structure fails to be integrable. More precisely, a G-structure is integrable if and only if its intrinsic torsion vanishes, which means that the Levi-Civita connection restricts to GM and its holonomy group is a subgroup of G. Otherwise stated, the intrinsic torsion is the obstruction for the Levi-Civita connection to be a G-connection.

In the sequel we consider the generalized gradients of a G-structure defined by its minimal connection. If g and $\bar{g}$ are two conformally related metrics, there is a corresponding conformal change of the G-structure, denoted by $\bar{G}M$, which is the image of GM under the principal bundle isomorphism between SO_gM and $\mathrm{SO}_{\bar{g}}M$ given by (1.33). We thus obtain the following commutative diagram, where all the arrows are the natural inclusions:

$$\begin{array}{ccccc} GM & \hookrightarrow & (\mathbb{R}^*_+ \times G)M & \hookleftarrow & \bar{G}M \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{SO}_gM & \hookrightarrow & \mathrm{CO}_nM & \hookleftarrow & \mathrm{SO}_{\bar{g}}M \end{array}$$

The right and left squares of the diagram are still commutative when considering the canonical connections, *i.e.* the extension D^G of the minimal connection D^G to $(\mathbb{R}^*_+ \times G)M$ coincides with the projection onto $\mathbb{R} \oplus \mathfrak{g} \subset \mathfrak{co}(n)$ of the extension D^g of the Levi-Civita connection ∇^g to CO_nM.

We now consider the same construction of the differential operators as above, using the minimal connection of the G-structure and its extension to the $(\mathbb{R}^*_+ \times G)$-principal bundle. Here G is as in § 1.1 one of the subgroups in the list (1.6). Hence, G acts irreducibly on τ and the decomposition (1.20) is now done into G-irreducible components. We consider the G-generalized gradients introduced in Definition 1.8:

$$P^{G,\lambda}_{\varepsilon} = \Pi_{\varepsilon} \circ \nabla^{G,\lambda}, \tag{1.44}$$

where λ is the highest weight of an irreducible G-representation, $V_\lambda M$ is the associated vector bundle, $\nabla^{G,\lambda}$ is the connection on $V_\lambda M$ induced by the minimal connection ∇^G and Π_ε is the projection of the bundle $\mathrm{T}^*M \otimes V_\lambda M$ onto the subbundle $V_{\lambda+\varepsilon}M$.

For any conformal weight w, there is an irreducible representation $\tilde{\lambda} = (\lambda, w)$ of $\mathbb{R}^*_+ \times G$ and on the associated bundle $V_{\tilde{\lambda}}M$ the operators are similarly defined by:

$$P^{D,\tilde{\lambda}}_{\varepsilon} = \Pi_{\varepsilon} \circ D^{G,\tilde{\lambda}}, \tag{1.45}$$

for any connection D^G given by the projection onto $(\mathbb{R}_+^* \times G)M$ of a Weyl connection D on $\mathrm{CO}_n M$. We have the following analogous definition:

Definition 1.25. The operator $P_\varepsilon^{G,\lambda}$ is called *conformal invariant relative to the conformal weight* w if the operators defined by (1.45) do not depend on the Weyl structure whose projection onto the principal subbundle $(\mathbb{R}_+^* \times G)M$ defines the generalized gradients $P_\varepsilon^{D,\tilde{\lambda}}$.

The *conformal weight operator* may be defined for any G-representation V, $\lambda : G \to \mathrm{Aut}(V)$, as follows (see [63]):

$$B_{\mathfrak{g}}^\lambda : (\mathbb{R}^n)^* \otimes V \to (\mathbb{R}^n)^* \otimes V, \quad B_{\mathfrak{g}}^\lambda(\alpha \otimes v) = \sum_{i=1}^n e_i^* \otimes d\lambda(\mathrm{pr}_{\mathfrak{g}}(e_i \wedge \alpha))v, \tag{1.46}$$

where $\{e_i\}_{\overline{1,n}}$ is an orthonormal basis of $\mathbb{R}^n$ and $\{e_i^*\}_{\overline{1,n}}$ the dual basis.

The eigenvalues of $B_{\mathfrak{g}}^\lambda$ are then computed according to Fegan's Lemma:

$$B_{\mathfrak{g}}^\lambda = \frac{1}{2}(C^{\tau \otimes \lambda} - I|_{\mathbb{R}^*} \otimes C^\lambda - C^\tau \otimes I|_V), \tag{1.47}$$

where the Casimir operator of an irreducible G-representation λ is given by:

$$C^\lambda = -\sum_\alpha d\lambda(X_\alpha) \circ d\lambda(X_\alpha),$$

for an orthonormal basis $\{X_\alpha\}_\alpha$ of $\mathfrak{g}$ with respect to the invariant scalar product induced on $\mathfrak{g} \subset \mathfrak{so}(n) \cong \Lambda^2\mathbb{R}^n$, $\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr}(XY)$. Usually it is convenient to compute the Casimir operators with respect to a chosen scalar product and then to renormalize them. The Casimir numbers may thus be computed by Freudenthal's formula (1.26): $c(\lambda) = \langle \lambda, \lambda + 2\delta \rangle$, where δ is the Weyl vector of $\mathfrak{g}$ and are then renormalized as follows:

$$c^{\Lambda^2}(\lambda) = 2\frac{\dim(\mathfrak{g})}{n}\frac{c(\lambda)}{c(\tau)}.$$

Thus, the conformal weight operator $B_{\mathfrak{g}}^\lambda$ acts on each G-irreducible component in the decomposition $\tau \otimes \lambda = \bigoplus_{\varepsilon \subset \lambda} (\lambda + \varepsilon)$ by multiplication with the scalar w_ε given by:

$$\begin{aligned} w_\varepsilon = w_\varepsilon(\lambda) &= \frac{1}{2}(c^{\Lambda^2}(\lambda+\varepsilon) - c^{\Lambda^2}(\lambda) - c^{\Lambda^2}(\tau)) \\ &= \frac{\dim(\mathfrak{g})}{n}\frac{|\varepsilon|^2 + 2\langle\lambda+\delta,\varepsilon\rangle - \langle\tau+2\delta,\tau\rangle}{\langle\tau+2\delta,\tau\rangle}. \end{aligned} \tag{1.48}$$

In particular, it follows that the eigenspaces of the conformal weight operator B^λ are compatible with the irreducible decomposition of the tensor product $\tau\otimes\lambda$. Notice that if two conformal weights of $B^\lambda_{\mathfrak{g}}$ are equal, then the eigenspace to this eigenvalue is equal to the sum of the corresponding irreducible components.

The analogue of Lemma 1.17 holds in general for G-structures:

Lemma 1.26. *The G-generalized gradient $P^{G,\lambda}_\varepsilon$ is conformally invariant relative to the conformal weight w_ε and this is the only conformal weight with respect to which this operator is conformally invariant.*

Proof: Let D_1 and D_2 be any two Weyl connections. Then there is a real 1-form θ on M such that (1.18) holds. As above D^G_i denotes the projection of D_i onto the principal $\mathbb{R}^*_+\times G$-subbundle. The connections induced by D^G_1 and D^G_2 on the associated vector bundle $V_{\tilde\lambda}M$ are then related as follows:

$$D^{G,\tilde\lambda}_2 = D^{G,\tilde\lambda}_1 + d\tilde\lambda(\mathrm{pr}_{\mathfrak{g}}(\tilde\theta)) = D^{G,\tilde\lambda}_1 + \sum_{i=1}^n e^*_i\otimes d\lambda(\mathrm{pr}_{\mathfrak{g}}(\theta\wedge e_i)) + w\,\theta\otimes I, \tag{1.49}$$

where $\{e_i\}_{i=1,\dots,n}$ is a conformal frame and $\{e^*_i\}$ the dual frame. Thus, with respect to the conformal weight operator, we get:

$$(D^{G,\tilde\lambda}_2 - D^{G,\tilde\lambda}_1)(\xi) = w\,\theta\otimes\xi - B^\lambda_{\mathfrak{g}}(\theta\otimes\xi), \quad \text{for all } \xi\in\Gamma(V_{\tilde\lambda}M). \tag{1.50}$$

Projecting now the equation (1.50) onto the component $\lambda+\varepsilon$ of the decomposition of the tensor product $\tau\otimes\lambda$, we get:

$$(P^{D_2,\tilde\lambda}_\varepsilon - P^{D_1,\tilde\lambda}_\varepsilon)(\xi) = (w-w_\varepsilon)\Pi_\varepsilon(\theta\otimes\xi), \quad \text{for all } \xi\in\Gamma(V_{\tilde\lambda}M). \tag{1.51}$$

Hence, the generalized gradient $P^{G,\lambda}_\varepsilon$ is conformally invariant relative to the conformal weight w if and only if $w = w_\varepsilon$. □

The next result expresses the conformal invariance directly in terms of the minimal connections of two conformally related G-structures.

Proposition 1.27. *The following statements are equivalent:*

(1) $P_\varepsilon^{G,\lambda}$ *is conformally invariant relative to the conformal weight* w_ε*.*

(2) *If* GM *and* $\bar{G}M$ *are conformally related* G*-structures, for* $\bar{g} = e^{2u}g$ *and* $\bar{G}M \hookrightarrow \mathrm{SO}_{\bar{g}}M$*, then the corresponding generalized gradients are related by:*

$$\bar{P}_\varepsilon^{G,\lambda} \circ \phi_{w_\varepsilon}^{G,\bar{G}} = \phi_{w_\varepsilon - 1}^{G,\bar{G}} \circ P_\varepsilon^{G,\lambda}, \tag{1.52}$$

where for any conformal weight w*,* $\phi_w^{G,\bar{G}}$ *is the isomorphism between the associated vector bundles* $V_\lambda^G M := GM \times_G V$ *and* $V_\lambda^{\bar{G}} M := \bar{G}M \times_G V$*, defined by:*

$$\phi_w^{G,\bar{G}} : V_\lambda^G M \to V_\lambda^{\bar{G}} M, \quad [(e_1, \dots, e_n), v] \mapsto [(e^{-u}e_1, \dots, e^{-u}e_n), e^{wu}v].$$

Proof: We consider the following diagram:

$$\begin{array}{ccccc}
\Gamma(V_\lambda^G M) & \xleftarrow[\sim]{\phi_{w_\varepsilon}^G} & \Gamma(V_{(\lambda, w_\varepsilon)} M) & \xrightarrow[\sim]{\phi_{w_\varepsilon}^{\bar{G}}} & \Gamma(V_\lambda^{\bar{G}} M) \\
\downarrow{\scriptstyle \nabla^{G,\lambda}} & & {\scriptstyle D^{G,(\lambda,w_\varepsilon)}}\downarrow\downarrow{\scriptstyle D^{\bar{G},(\lambda,w_\varepsilon)}} & & \downarrow{\scriptstyle \nabla^{\bar{G},\lambda}} \\
\Gamma(\mathrm{T}^*M \otimes V_\lambda^G M) & & \Gamma(\mathrm{T}^*M \otimes V_{(\lambda,w_\varepsilon)} M) & & \Gamma(\mathrm{T}^*M \otimes V_\lambda^{\bar{G}} M) \\
\downarrow{\scriptstyle \Pi_\varepsilon} & & \downarrow{\scriptstyle \Pi_\varepsilon} & & \downarrow{\scriptstyle \Pi_\varepsilon} \\
\Gamma(V_{\mu_\varepsilon}^G M) & \xleftarrow[\sim]{\phi_{w_\varepsilon - 1}^G} & \Gamma(V_{(\mu_\varepsilon, w_\varepsilon - 1)} M) & \xrightarrow[\sim]{\phi_{w_\varepsilon - 1}^{\bar{G}}} & \Gamma(V_{\mu_\varepsilon}^{\bar{G}} M)
\end{array}$$

where D^G is the minimal connection of the G-structure extended to the $(\mathbb{R}_+^* \times G)$-principal bundle (which may be also seen as the projection onto $\mathfrak{g} \oplus \mathbb{R}$ of the Weyl connection given by the extension of the Levi-Civita connection of the metric g) and $D^{G,(\lambda,w_\varepsilon)}$ is the induced connection on the vector bundle $V_{\tilde{\lambda}} M$ associated to the irreducible representation $\tilde{\lambda} = (\lambda, w_\varepsilon)$. The isomorphisms in the diagram are defined by:

$$\phi_w^G : V_{(\lambda,w)} M \to V_\lambda^G M, \quad \phi_w^G([(f_1, \dots, f_n), v]) = [(e_1, \dots, e_n), a^w v],$$

where $f_i = ae_i$, $i = 1, \dots, n$ and $\{e_i\}_{\overline{1,n}}$ is an orthonormal basis with respect to the metric g. With this notation, we have: $\phi_w^{G,\bar{G}} = \phi_w^{\bar{G}} \circ (\phi_w^G)^{-1}$. The left and right "squares" of the diagram commute by the definition of the induced connection on an associated vector bundle.

Suppose now that (1) holds. Then the whole diagram commutes, since then for the conformal weight w_ε the compositions "in the middle" give the same operator (as $P_\varepsilon^{D,\tilde\lambda} = \Pi_\varepsilon \circ D^{G,\tilde\lambda}$ does not depend on the Weyl structure). The composition on its "boundary" gives (2).

If (1.52) holds for a conformal weight w and for any two conformally related G-structures, then the above diagram is commutative. Thus, the operator $P_\varepsilon^{D,\tilde\lambda}$ is the same for all the Weyl structures given by the minimal connections of conformally related G-structures. Then, (1.51) implies $(w - w_\varepsilon)\Pi_\varepsilon(\theta \otimes \xi) = 0$, for any exact 1-form θ on M and any $\xi \in \Gamma(V_{\tilde\lambda} M)$. At some fixed point on M, it follows that $(w - w_\varepsilon)\Pi_\varepsilon(\alpha \otimes v) = 0$, for all $\alpha \otimes v \in (\mathbb{R}^n)^* \otimes V$, which shows that $w = w_\varepsilon$. Substituting in (1.51), it follows that $P_\varepsilon^{g,\lambda}$ is conformally invariant relative to the conformal weight w_ε. □

In conclusion, the explicit formula (1.48) allows us to compute the eigenvalues of the conformal weight operator $B_{\mathfrak{g}}^\lambda$ of a G-structure for any irreducible representation λ, where G is a subgroup of the special othogonal group, and thus, by Lemma 1.26 and Proposition 1.27, to determine the conformal weights of the G-generalized gradients. For completeness we give in Table 1.1 the explicit values of the conformal weights of G-generalized gradients for all subgroups G in (1.6). As an example we consider the unitary group, which is a special case, since the complexified tangent bundle is not irreducible.

Example 1.28. If $G = \mathrm{U}(n) \subset \mathrm{SO}(2n)$, then $\mathrm{T}M^{\mathbb{C}}$ splits into two irreducible subspaces. Hence the $\mathrm{U}(n)$-generalized gradients are of two kinds: holomorphic and anti-holomorphic, as explained in Example 1.9. In [32] they were called *Kählerian gradients*, since if the metric is Kähler, the $\mathrm{U}(n)$-gradients are given by the Levi-Civita which coincides in this case with the minimal connection of the integrable $\mathrm{U}(n)$-structure. Formula (1.48) implies that the conformal weights are given by: $w_{i,-} = -\lambda_i + i - n$, $w_{i,+} = \lambda_i - i + 1$, for $i = 1, \ldots, n$.

The conformal invariance of G-generalized gradients has the following straightforward, but important consequences:

Corollary 1.29. *Let (M, g) be a Riemannian manifold which admits a G-structure, for some group $G \subset \mathrm{SO}(n)$. Then the dimension of the kernel of any G-generalized gradient,* $\dim(\ker(P_\varepsilon^{G,\lambda}))$, *is the same for all metrics conformally related to g.*

Corollary 1.30. *If $P_\varepsilon^{G,\lambda}$ is a G-generalized gradient with conformal weight $w_\varepsilon(\lambda)$, then its formal adjoint is, up to a constant, equal to $P_{-\varepsilon}^{G,\lambda+\varepsilon}$ and is conformally invariant with respect to the conformal weight $w_{-\varepsilon}(\lambda + \varepsilon)$.*

Table 1.1: Conformal weights of G-generalized gradients

dim(M)	Group	Geometry	Highest weight	Conformal Weights
$2m$	Spin($2m$) SO($2m$)	(spin) oriented Riemannian	$\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$ $\lambda_1 \geq \dots \geq \lambda_{m-1} \geq \lvert\lambda_m\rvert$	$w_{i,-} = 1 - 2m - \lambda_i + i,\ i = \overline{1,m}$ $w_{i,+} = 1 + \lambda_i - i,\ i = \overline{1,m}$
$2m+1$	Spin($2m+1$) SO($2m+1$)	(spin) oriented Riemannian	$\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$ $\lambda_1 \geq \dots \geq \lambda_m \geq 0$	$w_0 = -m$ $w_{i,-} = -2m - \lambda_i + i,\ i = \overline{1,m}$ $w_{i,+} = 1 + \lambda_i - i,\ i = \overline{1,m}$
$2m$	SU(m) U(m)	(special) almost Hermitian	$\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$ $\lambda_1 \geq \dots \geq \lambda_m$	$w_{i,-} = -\lambda_i + i - m,\ i = \overline{1,m}$ $w_{i,+} = \lambda_i - i + 1,\ i = \overline{1,m}$
$4m$	Sp(m)	almost hyper-Hermitian	$\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$ $\lambda_1 \geq \dots \geq \lambda_m \geq 0$	$w_{i,-} = -\lambda_i + i - 2m - 1,\ i = \overline{1,m}$ $w_{i,+} = \lambda_i - i + 1,\ i = \overline{1,m}$
$4m$	Sp(1)·Sp(m)	almost quaternion-Hermitian	$\beta \in \mathbb{Z},\ \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$ $\beta \geq 0,\ \lambda_1 \geq \dots \geq \lambda_m \geq 0$	$w_{\pm,i,-} = 1/2(-\lambda_i \pm \beta/m + i - 2m - 1 - 1/m \pm 1/m),$ $w_{\pm,i,+} = 1/2(\lambda_i \pm \beta/m - i + 1 - 1/m \pm 1/m),\ i = \overline{1,m}$
7	G_2	G_2-structure	$\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ $\lambda_1 \geq \lambda_2 \geq 0$	$w_{1,\pm} = -(5/3 \mp 5/3) \pm 1/3(2\lambda_1 + \lambda_2)$ $w_{2,\pm} = -(5/3 \mp 4/3) \pm 1/3(\lambda_1 + 2\lambda_2)$ $w_{3,\pm} = -(5/3 \mp 1/3) \pm 1/3(\lambda_1 - \lambda_2)$
8	Spin(7)	Spin(7)-structure	$\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$	$w_{1,\pm} = -(9/4 \mp 9/4) \pm 1/2(\lambda_1 + \lambda_2 + \lambda_3)$ $w_{2,\pm} = -(9/4 \mp 7/4) \pm 1/2(\lambda_1 + \lambda_2 - \lambda_3)$ $w_{3,\pm} = -(9/4 \mp 3/4) \pm 1/2(\lambda_1 - \lambda_2 + \lambda_3)$ $w_{4,\pm} = -(9/4 \mp 1/4) \pm 1/2(\lambda_1 - \lambda_2 - \lambda_3)$

Remark 1.31. We notice that in general it is not straightforward to construct higher order conformally invariant differential operators composing first order ones. The composition of two generalized gradients does not need to be conformally invariant, unless the corresponding conformal weights are related by $w_{\varepsilon_2}(\lambda+\varepsilon_1) = w_{\varepsilon_1}(\lambda) - 1$, in which case the composition $P^{G,\lambda+\varepsilon_1}_{\varepsilon_2} \circ P^{G,\lambda}_{\varepsilon_1}$ is a second order conformally invariant differential operator. An interesting particular case is the one when we compose a generalized gradient with its formal adjoint. For instance, the above condition is not fulfilled for the Laplace operator Δ acting on p-forms: it implies that $d\delta$ acting on p-forms is conformally invariant if and only if $p = \frac{n}{2}+1$ and similarly δd is conformally invariant if and only if $p = \frac{n}{2} - 1$, showing that $\Delta = d\delta + \delta d$ is not conformally invariant. Instead, the Laplace operator might be modified by the scalar curvature in order to make it conformally invariant. More precisely, the following formula: $Y_g = 4\frac{n-1}{n-2}\Delta_g + scal_g$ defines the so-called *conformal Laplacian* or *Yamabe operator* on an n-dimensional Riemannian manifold (M, g), for $n \geq 3$.

As applications of this property of conformal invariance we mention for instance the conformal relation between Killing and twistor spinors (see § 4 for the definition of these special classes of spinors), which together with the classification of manifolds admitting Killing spinors established by Ch. Bär, [5], yields a description of the manifolds carrying twistor spinors (which satisfy a more general equation than the Killing spinors). Another example is the conformal invariance of the Yamabe operator which plays a crucial role in the solution of the Yamabe problem of finding a metric of constant scalar curvature in a given conformal class on the manifold M.

1.3 Weitzenböck Formulas

A natural and universal way to construct second order differential operators acting on sections of associated vector bundles on a Riemannian manifold is to consider the composition of generalized gradients with their formal metric adjoints.

As in the previous sections, we consider a Riemannian manifold (M, g) admitting a G-structure with a fixed G-connection, $V_\lambda M$ is the vector bundle on M associated to the irreducible G-representation of highest weight λ and

P^λ_ε is a G-generalized gradient: $P^\lambda_\varepsilon : \Gamma(V_\lambda M) \to \Gamma(V_{\lambda+\varepsilon} M)$. Any subset $I \subset \{\varepsilon | \, \varepsilon$ relevant weight for $\lambda\}$ defines the following operator:

$$P^\lambda_I := \sum_{\varepsilon \in I} a_\varepsilon \, (P^\lambda_\varepsilon)^* \circ P^\lambda_\varepsilon, \tag{1.53}$$

as a linear combination with constant coefficients a_ε.

There are two interesting extreme cases for linear combinations of generalized gradients, *i.e.* for operators defined by (1.53):

1. The first case is when P^λ_I is a zero-order differential operator, which then yields a so-called Weitzenböck formula, provided that the connection used to define the generalized gradients is the Levi-Civita connection. This assumption is needed for the symmetries of the curvature of the Levi-Civita connection, which ensure that the zero-order differential operator is equal to a certain curvature term.
2. The other extreme case is when P^λ_I is an elliptic second order differential operator. This is a condition depending only on the principal symbol of the operator, so that it is independent of the chosen defining connection.

We analyze these two extreme cases in more detail, the first case in the sequel and the second one in Chapter 2.

In differential geometry, Weitzenböck formulas play an important role in relating the local geometry to global topological properties by the so-called Bochner method. Weitzenböck formulas occur in various problems and have many applications, for instance in proving vanishing results and eigenvalue estimates for geometric differential operators. In this thesis we will use a Weitzenböck formula adapted to the Kähler structure (see (5.1)) as one of the tools for providing the complete description of manifolds carrying a special class of spinors, the so-called Kählerian twistor spinors.

It turned out that the natural setting for a unified treatment of all Weitzenböck formulas that occurred in different geometric contexts is provided by generalized gradients. Thus, a Weitzenböck formula is mainly defined on a Riemannian manifold as a zero-order linear combination of differential operators obtained by composing generalized gradients with their adjoints, as in (1.53). For the sake of completeness, we briefly present in this section the two different approaches that have been given to a systematic study of all possible Weitzenböck formulas, by U. Semmelmann and G. Weingart, [63], and by Y. Homma, [33]. Furthermore, we show how a characterization of Weitzenböck formulas for the Levi-Civita connection given in [63] can be carried over to the more general case of the minimal connection of a G-structure with totally skew-symmetric torsion, where we give a sufficient condition to obtain Weitzenböck formulas (Lemma 1.33).

In [63], U. Semmelmann and G. Weingart provided a unified construction of Weitzenböck formulas for the irreducible non-symmetric holonomy groups, by giving on the one hand a recursion procedure for the construction of a basis of the space of Weitzenböck formulas and, on the other hand, by characterizing reduced Weitzenböck formulas as eigenvectors of an explicitly known matrix. Another approach was given by Y. Homma, who described all Weitzenböck formulas in [33], [32], [31] and [34], separately for Riemannian, Kähler, hyper-Kähler, respectively quaternionic-Kähler manifolds. His method is based on the algebraic structure of the principal symbols, which is determined from their relationship to the universal enveloping algebra of the corresponding Lie algebra.

As in § 1.1, we consider G-generalized gradients acting on sections of a vector bundle $V_\lambda M$ associated to an irreducible G-representation of highest weight λ. By Theorem 1.7, the decomposition of the tensor product $T \otimes V_\lambda$ (here we denote by $T := \tau$ the restriction to G of the standard $\mathrm{SO}(n)$-representation) is explicitly described as:

$$T \otimes V_\lambda = \bigoplus_{\varepsilon \subset \lambda} V_{\lambda+\varepsilon}, \tag{1.54}$$

where $\varepsilon \subset \lambda$ denotes the set of relevant weights of λ.

In [63] is studied the space of all Weitzenböck formulas on $V_\lambda M$, denoted by $\mathcal{W}(V_\lambda)$, defined in a more general sense as the space of all linear combinations of the form (1.53). Namely, if we simplify notation and drop λ, we have:

$$\mathcal{W}(V_\lambda) := \left\{ \sum_\varepsilon a_\varepsilon \, P_\varepsilon^* P_\varepsilon \right\}. \tag{1.55}$$

The reduced Weitzenböck formulas are those linear combinations which are bundle endomorphisms depending on the Riemannian curvature:

$$\sum_\varepsilon a_\varepsilon \, P_\varepsilon^* P_\varepsilon \; = \text{curvature action.} \tag{1.56}$$

The key point in [63] is the observation that the space of Weitzenböck formulas, $\mathcal{W}(V_\lambda)$, can be identified with the vector space $\mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda)$ of $\mathfrak{g}$-invariant endomorphisms of $T\otimes V_\lambda$, and thus is an algebra. Since the decomposition (1.54) is multiplicity-free, the projections onto the irreducible summands are well-defined and we denoted them by Π_ε. The algebra $\mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda)$ is then commutative and is spanned by the pairwise orthogonal idempotents Π_ε.

The space of Weitzenböck formulas has the following different realizations (see [63, Definition 3.1]):

$$\mathcal{W}(V_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(T\otimes T\otimes V_\lambda, V_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}(V_\lambda)) = \mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda), \tag{1.57}$$

where the first identification is given as follows. Each $F\in \mathrm{Hom}_{\mathfrak{g}}(T\otimes T\otimes V_\lambda, V_\lambda)$ defines by composition with ∇^2 a second order differential operator acting on sections of $V_\lambda M$:

$$\Gamma(V_\lambda M) \xrightarrow{\nabla^2} \Gamma(TM\otimes TM\otimes V_\lambda M) \xrightarrow{F} \Gamma(V_\lambda M).$$

As an element of $\mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda)$, F is expanded in the basis given by the projections Π_ε: $F=\sum_\varepsilon f_\varepsilon\,\Pi_\varepsilon$. A straightforward computation shows that the composition of a generalized gradient with its formal adjoint can be expressed as follows: $P_\varepsilon^* P_\varepsilon = -\Pi_\varepsilon(\nabla^2)$. Thus, we obtain:

$$F(\nabla^2) = -\sum_\varepsilon f_\varepsilon\, P_\varepsilon^* P_\varepsilon\,.$$

This gives the first identification in (1.57), showing that the coefficients of the Weitzenböck formula corresponding to F are equal to the coordinates of F in the basis $\{\Pi_\varepsilon\}_\varepsilon$, with opposite sign. For example, the rough Laplacian $\nabla^*\nabla = \sum_\varepsilon P_\varepsilon^* P_\varepsilon$ corresponds to the linear map $a\otimes b\otimes\varphi\mapsto -\langle a,b\rangle\varphi$.

The algebra $\mathcal{W}(V_\lambda)$ has a canonical involution, a twist $\sigma:\mathcal{W}(V_\lambda)\to\mathcal{W}(V_\lambda)$, such that a Weitzenböck formula reduces to a pure curvature expression if and only if it is an eigenvector of σ with eigenvalue -1. More precisely, the twist σ is defined in the realization of the space of Weitzenböck formulas as $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T\otimes V_\lambda, V_\lambda)$ by the precomposition with the twist:

$$\sigma: T\otimes T\otimes V_\lambda \to T\otimes T\otimes V_\lambda, \quad a\otimes b\otimes v\mapsto b\otimes a\otimes v.$$

The decomposition of the space $\mathcal{W}(V_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}(V_\lambda))$ into the

(± 1)-eigenspaces of σ is the following:

$$\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}(V_\lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}(V_\lambda)) \oplus \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Sym}^2 T, \mathrm{End}(V_\lambda)). \tag{1.58}$$

The fact that a Weitzenböck formula $F \in \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}(V_\lambda))$ induces a zero-order operator which is a pure curvature term is proved by the following computation:

$$F(\nabla^2 \varphi) = \frac{1}{2} \sum_{i,j} F(e_i \otimes e_j \otimes (\nabla^2_{e_i,e_j} - \nabla^2_{e_j,e_i})\varphi) = \frac{1}{2} F_{e_i \otimes e_j} R_{e_i,e_j} \varphi, \tag{1.59}$$

where $\{e_i\}_{i=\overline{1,n}}$ is an orthonormal basis of T (and also a local orthonormal basis of the tangent bundle, parallel at the point where the computations are made) and φ is a section of $V_\lambda M$. Conversely, if F induces a zero-order operator, then in particular the principal symbol of $F(\nabla^2)$ vanishes: $0 = \sigma_{F(\nabla^2)}(\xi)(\varphi) = F_{\xi,\xi}\varphi$ (for any cotangent vector ξ and $\varphi \in \Gamma(V_\lambda M)$), showing that F is skew-symmetric in the first two arguments: $F \in \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}(V_\lambda))$.

The classical examples of reduced Weitzenböck formulas like the original Weitzenböck formula in Riemannian geometry:

$$\Delta = \nabla^* \nabla + q(R)$$

or the Schrödinger-Lichnerowicz formula in spin geometry (see also (3.9)):

$$D^2 = \nabla^* \nabla + \frac{1}{4} S$$

reduce in this setting to the fact that $\Delta - \nabla^*\nabla$ and $D^2 - \nabla^*\nabla$ are eigenvectors of σ of eigenvalue -1 and thus pure curvature expressions.

Remark 1.32. We notice that, whereas the different realizations of the space of Weitzenböck formulas in (1.57) are valid for any G-generalized gradients, in order to establish the equivalence between the skew-symmetric morphisms $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}(V_\lambda))$ and the Weitzenböck formulas reducing to pure curvature terms, it is important to consider those generalized gradients defined by the Levi-Civita connection. It is the vanishing of the torsion of the connection ∇ that implies the last equality in (1.59). In particular, this means that the Levi-Civita connection restricts to the G-structure, so that the minimal G-connection coincides with the Levi-Civita connection and the holonomy group must be contained in G.

In the more general setting of generalized gradients of G-structures, we have already noticed in § 1.2.2 that in order to obtain "nice" properties (for instance the conformal invariance of G-generalized gradients) one has to consider the minimal connection. The following result assures that also in this case, assuming some extra conditions, the skew-symmetric morphisms provide pure curvature terms.

Lemma 1.33. *Let ∇^G be the minimal connection of a G-structure and suppose that it has totally skew-symmetric torsion T. We consider now the space of Weitzenböck formulas given by* (1.55)*, where the generalized gradients are defined by the connection ∇^G. Then, any Weitzenböck formula $F \in \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}(V_\lambda))$ with the property that $F|_{\mathfrak{g}^\perp} \equiv 0$ induces a pure curvature term as follows:*

$$F((\nabla^G)^2\varphi) = \frac{1}{2}\sum_{i,j} F_{e_i\otimes e_j}(R^G_{e_i,e_j}\varphi),$$

where R^G denotes the curvature tensor of the minimal connection ∇^G.

Proof: If the torsion T^G of the minimal connection of a G-structure is totally skew-symmetric, then it may be identified with the intrinsic torsion T, defined by (1.43) as a 1-form on M with values in the associated vector bundle $GM \times_G \mathfrak{g}^\perp$. More precisely, when both are regarded as 3-forms on M, they are related as follows: $T^G = -2T$, so that T^G may also be seen as a 1-form with values in $GM \times_G \mathfrak{g}^\perp$.

We start as in (1.59) and, assuming that the frame $\{e_i\}_{i=\overline{1,n}}$ is parallel at the point where the computations are made, we have:

$$\begin{aligned} F((\nabla^G)^2\varphi) &= \sum_{i,j} F(e_i\otimes e_j\otimes(\nabla^G_{e_i}\nabla^G_{e_j}\varphi)) \\ &= \frac{1}{2}\sum_{i,j} F(e_i\otimes e_j\otimes(\nabla^G_{e_i}\nabla^G_{e_j}\varphi - \nabla^G_{e_j}\nabla^G_{e_i}\varphi)) \\ &= \frac{1}{2}\sum_{i,j} F_{e_i\otimes e_j}(R^G_{e_i,e_j}\varphi - \nabla^G_{T^G(e_i,e_j)}\varphi). \end{aligned} \tag{1.60}$$

It is thus enough to show that the last term above vanishes:

$$\begin{aligned}\sum_{i,j} F_{e_i,e_j}(\nabla^G_{T^G(e_i,e_j)}\varphi) &= \sum_{i,j,k} F(e_i,e_j)(T^G(e_i,e_j,e_k)\nabla^G_{e_k}\varphi)\\ &= 2\sum_{i<j}\sum_k F(e_i\wedge e_j)(\langle T^G(e_k), e_i\wedge e_j\rangle \nabla^G_{e_k}\varphi) \\ &= 2\sum_k F(T^G(e_k))(\nabla^G_{e_k}\varphi) = 0,\end{aligned} \tag{1.61}$$

where for the second equality we used that T^G is totally skew-symmetric and the last equality follows from the assumption that $F|_{\mathfrak{g}^\perp} \equiv 0$. □

An important example of a Weitzenböck formula $F \in \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}(V_\lambda))$ satisfying the assumptions of Lemma 1.33, and thus providing a pure curvature term, is given by the conformal weight operator $B^\lambda_{\mathfrak{g}}$, defined by (1.46). Its importance is explained in the discussion below. We mention that in the sequel we consider only Weitzenböck formulas defined by the Levi-Civita connection, as in [63].

We now describe the recursion procedure given in [63] for constructing a basis of the space $\mathcal{W}(V_\lambda)$ of all Weitzenböck formulas, such that the basis vectors are eigenvectors of σ with alternating eigenvalues ± 1. This recursion procedure makes essential use of a fundamental reduced Weitzenböck formula, namely the one corresponding to the conformal weight operator $B := B^\lambda_{\mathfrak{g}}$, defined by (1.46) (where again we simplify notation and drop the indices λ and $\mathfrak{g}$):

$$B : (\mathbb{R}^n)^* \otimes V \to (\mathbb{R}^n)^* \otimes V, \quad B(\alpha\otimes v) = \sum_{i=1}^n e_i^* \otimes d\lambda(\mathrm{pr}_{\mathfrak{g}}(e_i\wedge\alpha))v,$$

where $\{e_i\}_{\overline{1,n}}$ is an orthonormal basis of $\mathbb{R}^n$ and $\{e_i^*\}_{\overline{1,n}}$ the dual basis.

We recall that the eigenspaces of the conformal weight operator B are, except for a special case, exactly the irreducible summands in the decomposition (1.54) (see § 1.2.2). Its eigenvalues, the conformal weights, which we denoted by w_ε, are explicitly computed and listed in Table 1.1.

From the definition of the conformal weight operator it is obvious that the endomorphism B belongs to the (-1)-eigenspace of the twist σ and thus induces a pure curvature term, $B(\nabla^2)$, on the vector bundle $V_\lambda M$. This curvature term can be explicitly described using an orthonormal basis $\{X_\alpha\}$ of $\mathfrak{g}$ with respect to the scalar product induced on $\mathfrak{g} \subset \Lambda^2 T$. The curvature operator $R : \Lambda^2\mathrm{T}M \to \mathfrak{g}M$ associated to the Riemannian curvature tensor R of M gives rise to the following global section of the universal enveloping algebra bundle associated to the holonomy reduction:

$$q(R) := \sum_\alpha X_\alpha R(X_\alpha) \in \Gamma(\mathcal{U}(\mathfrak{g})M).$$

A direct computation yields then the so-called *universal Weitzenböck formula* (see [63, Proposition 3.6]):

$$B(\nabla^2) = -\sum_{\varepsilon\subset\lambda} w_\varepsilon\, P_\varepsilon^* P_\varepsilon \;= q(R).$$

In many cases the conformal weight operator B generates all Weitzenböck formulas. More precisely, the following algebra isomorphism holds if all conformal weights are pairwise different (see [63, Proposition 3.7]):

$$\mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda) \cong \mathbb{C}(B)/\langle \min(B)\rangle,$$

where $\min(B)$ denotes the minimal polynomial of the endomorphism B.

The conformal weight operator B is the first step of the recursion procedure. The output of the recursion procedure is a sequence of B-polynomials $p_i(B)$, whose first two terms are id and B and, more generally, $p_{2i}(B)$ is in the $(+1)$ and p_{2i+1} in the (-1)-eigenspace of σ. The coefficient of $P_\varepsilon^* P_\varepsilon$ in the Weitzenböck formula corresponding to $p_i(B)$ is equal to $p_i(w_\varepsilon)$.

In order to describe precisely the recursion procedure, one has to consider a further endomorphism of the space of Weitzenböck formulas, whose eigenspaces correspond to a finer decomposition of this space than (1.58). This is the so-called *classifying endomorphism* K of $\mathcal{W}(V_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(T\otimes T\otimes V_\lambda, V_\lambda)$, defined by the precomposition with the $\mathfrak{g}$-invariant endomorphism:

$$T\otimes T\otimes V_\lambda \to T\otimes T\otimes V_\lambda, \quad a\otimes b\otimes v \mapsto -\sum_\alpha X_\alpha a\otimes X_\alpha b\otimes v,$$

where $\{X_\alpha\}_\alpha$ is an orthonormal basis of $\mathfrak{g}$. The essential property of K is that it is compatible with the decomposition of the tensor product into irreducible summands: $T\otimes T = \oplus_\alpha W_\alpha$, in the following sense. The classifying endomorphism K is diagonalizable on $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}(V_\lambda))$ with eigenspaces $\mathrm{Hom}_{\mathfrak{g}}(W_\alpha, \mathrm{End}(V_\lambda))$ and the corresponding eigenvalues are explicitly given by: $\kappa_{W_\alpha} = \frac{1}{2}\mathrm{Cas}^{\Lambda^2}_{W_\alpha} - \mathrm{Cas}^{\Lambda^2}_{T}$.

The main result providing the recursion formula is now basically a direct consequence of Fegan's Lemma and Schur's Lemma:

Theorem 1.34. *(Recursion Formula, [63]) The actions of K, B and σ on $\mathcal{W}(V_\lambda)$ by precomposition are related as follows:*

$$K + B + \sigma B \sigma = Cas_T^{\Lambda^2} = -\frac{2}{n}\dim(\mathfrak{g}). \tag{1.62}$$

This result yields a recursion formula for K-eigenvectors and allows the construction of a complete eigenbasis for $\mathcal{W}(V_\lambda)$ of σ.

Corollary 1.35. *(Basic Recursion Procedure, [63]) Let $F \in \mathcal{W}(V_\lambda)$ be an eigenvector of the twist σ and of the classifying endomorphism K: $\sigma(F) = \pm F$, $K(F) = \kappa F$. Then, the new Weitzenböck formula:*

$$F_{new} = \left(B - \frac{\mathrm{Cas}_T^{\Lambda^2} - \kappa}{2} \right) \circ F$$

is again a σ-eigenvector with $\sigma(F_{new}) = \mp 1$. In particular, it holds:

$$\mathrm{id}_{new} = B, \quad B_{new} = B^2 - \frac{1}{4}\mathrm{Cas}_{\mathfrak{g}}^{\Lambda^2} B.$$

Corollary 1.36. *(Orthogonal Recursion Procedure, [63]) Let $p_0(B), \ldots, p_k(B)$ be a sequence of polynomials obtained by the Gram-Schmidt orthonormalization procedure to the powers $\mathrm{id}, B, \ldots, B^k$ of the conformal weight operator B. If all these polynomials are σ-eigenvectors and $p_k(B)$ is moreover a K-eigenvector, then the orthogonal projection $p_{k+1}(B)$ of B^{k+1} onto the orthogonal complement of the span of $\mathrm{id}, B, \ldots, B^k$ is again a σ-eigenvector.*

These recursion procedures are applied in [63] for G equal to one of the groups $\mathrm{SO}(n)$, G_2, Spin_7 and, in an adapted version, for Kähler geometry, $G = \mathrm{U}(n)$.

Apart from these recursion procedures, it turns out that the formula (1.62) may be considered directly as a matrix equation for the unknown matrix of the twist σ. This provided in [63] an explicit computation for the coefficients $\sigma_{\varepsilon,\varepsilon'}$ of the matrix of σ with respect to the basis given by the projections Π_ε: $\sigma(\Pi_\varepsilon) = \sum_{\varepsilon'} \sigma_{\varepsilon,\varepsilon'} \Pi_{\varepsilon'}$, for manifolds of holonomy $\mathrm{SO}(n)$, $\mathrm{U}(n)$, G_2 or $\mathrm{Spin}(7)$. As noticed above, the twist σ classifies all reduced Weitzenböck formulas

$F \in \mathcal{W}(V_\lambda)$. Consequently, the matrix expressions for the twist σ in the basis $\{\Pi_\varepsilon\}_\varepsilon$ of $\mathcal{W}(V_\lambda)$ makes it possible to check this condition effectively for a given Weitzenböck formula.

A different approach to the description of all Weitzenböck formulas was provided by Y. Homma, [33]. His strategy is to study the algebraic structure of the symbols of the operators and to show how they are controlled by the enveloping algebra of the Lie algebra $\mathfrak{g}$. He relates Weitzenböck formulas that reduce to pure curvature expressions to higher Casimir elements of the enveloping algebra. This gives a universal and direct construction of the coefficients a_ε and of the curvature actions in (1.56).

In the rest of this section we briefly present the main ideas of the approach in [33], without getting into the technical details. In [33] are described the reduced Weitzenböck formulas on Riemannian manifolds for the generalized gradients defined by the Levi-Civita connection. The method developed in [33] was then carried over by Y. Homma also to Kähler, hyper-Kähler and quaternionic Kähler manifolds in [32], [31], respectively [34], using the same main scheme, but specific computations for each group involved ($\mathrm{U}(n)$, $\mathrm{Sp}(n)$, respectively $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$).

The key observation in [33] is that the formulas in the universal enveloping algebra of the complexification of $\mathfrak{so}(n)$, $\mathfrak{so}(n,\mathbb{C})$, with certain symmetries involving the so-called *higher Casimir elements*, yield reduced Weitzenböck formulas on Riemannian manifolds. The intermediary step is constituted by the so-called *Clifford homomorphisms*, which, on the one hand, are the principal symbols of generalized gradients and, on the other hand, are related to the higher Casimir elements. These relations may be visualized in the following scheme, that we explain in the sequel:

higher Casimir elements	$\longleftrightarrow$	Clifford homomorphisms	$\longleftrightarrow$	reduced Weitzenböck formulas

Casimir elements are defined as elements in the center $\mathcal{Z}$ of the enveloping algebra $\mathcal{U}(\mathfrak{so}(n,\mathbb{C}))$. The center $\mathcal{Z}$ is characterized as the invariant subalgebra in $\mathcal{U}(\mathfrak{so}(n,\mathbb{C}))$ under the adjoint action of $\mathrm{SO}(n)$. An algebraic basis of $\mathcal{Z}$ is constructed as follows. As in § 1.1, we denote by $\{e_i\}_{i=\overline{1,n}}$ a fixed oriented orthonormal basis of $\mathbb{R}^n$ and denote by $e_{ij} := e_i \wedge e_j$, such that $\{e_{ij} | 1 \leq i < j \leq n\}$ constitute an orthonormal basis of $\mathfrak{so}(n) \cong \Lambda^2\mathbb{R}^n$. For each positive integer q, we consider the following element in $\mathcal{U}(\mathfrak{so}(n,\mathbb{C}))$:

$$e_{ij}^q = \begin{cases} \sum\limits_{1\le i_1,i_2,\dots,i_{q-1}\le n} e_{ii_1}e_{i_1i_2}\cdots e_{i_{q-1}j}, & q\ge 1,\\ \delta_{ij}, & q=0. \end{cases} \tag{1.63}$$

A straightforward computation shows that the traces $c_q := \sum_i e_{ii}^q$, for all $q \ge 0$, called *higher Casimir elements*, belong to the center $\mathcal{Z}$. Notice that the usual Casimir element defined by (1.25) corresponds to c_2.

In the case of $n = 2m+1$, the higher Casimir elements $\{c_q\}_q$ generate the center $\mathcal{Z}$ algebraically. In the case $n = 2m$, another Casimir element is needed in order to generate $\mathcal{Z}$, namely the so-called *Pfaffian element*, pf, defined by:

$$\mathrm{pf} := \frac{1}{(2i)^m m!}\sum_{\sigma\in\mathfrak{S}_{2m}} \mathrm{sgn}(\sigma)e_{\sigma(1)\sigma(2)}e_{\sigma(3)\sigma(4)}\cdots e_{\sigma(2m-1)\sigma(2m)},$$

where $\mathfrak{S}_{2m}$ is the permutation group of $\{1,\dots,2m\}$. By Schur's Lemma it follows that every Casimir element acts as a constant on each irreducible $\mathfrak{so}(n)$-representation. The eigenvalue of each c_q and of the Pfaffian element on an irreducible $\mathfrak{so}(n)$-representation of highest weight λ is denoted by $c_q(\lambda)$, respectively $\mathrm{pf}(\lambda)$. These eigenvalues are explicitly computed only in terms of the conformal weights of $\mathfrak{so}(n)$ ([33, Proposition 4.14]).

It turned out that many computations simplify if one considers the *translated Casimir elements* defined by: $\tilde{c}_q := \sum_i \tilde{e}_{ii}^q$, where $\tilde{e}_{ij}^q$ are given by the same formula (1.63), but with translated elements $\tilde{e}_{ij} = e_{ij} + \frac{n-1}{2}\delta_{ij}$ (notice that the same translation with $\frac{n-1}{2}$ is also considered in § 2.2 for the conformal weight operator, see (2.9)).

The following result in [33] gives in a certain sense the algebraic counterpart of the reduced Weitzenböck formulas:

Theorem 1.37 ([33]). *Any translated element $\tilde{e}_{ij}^q$ is a linear combination of $\{\tilde{e}_{ji}^p\}_{p=0,\dots,q}$, whose coefficients are translated Casimir elements:*

$$\tilde{e}_{ij}^q = (-1)^q\tilde{e}_{ji}^q - \frac{1-(-1)^q}{2}\tilde{e}_{ji}^{q-1} + \sum_{p=0}^{q-1}(-1)^p\tilde{c}_{q-1-p}\tilde{e}_{ji}^p. \tag{1.64}$$

By analogy to the Dirac operator, whose principal symbol is given by the Clifford multiplication, the principal symbols of generalized gradients are called in [33] *Clifford homomorphisms.* These Clifford homomorphisms satisfy the identities corresponding to (1.56) at the symbol level. More precisely, at the level of vector spaces, they are defined as follows: for any relevant weight ε of λ, each vector $a \in T$ defines a linear map

$$p_\varepsilon(a) : V_\lambda \to V_{\lambda+\varepsilon}, \quad p_\varepsilon(a)v := \Pi_\varepsilon(a \otimes v).$$

The adjoint operator of $p_\varepsilon(a)$ with respect to the inner products on V_λ and $V_{\lambda+\varepsilon}$ is denoted by $(p_\varepsilon(a))^*$. The linear maps $p_\varepsilon(a)$ and $(p_\varepsilon(a))^*$ are called the *Clifford homomorphisms associated to λ and ε.* The orthogonal projection $\Pi_\varepsilon : T \otimes V_\lambda \to V_{\lambda+\varepsilon}$ is then realized as follows:

$$\Pi_\varepsilon(a \otimes v) = \sum_{i=1}^{n} (p_\varepsilon(e_i))^* p_\varepsilon(a) e_i \otimes v.$$

Clifford homomorphisms are related to the higher Casimir elements of the enveloping algebra by the following result:

Proposition 1.38. *([33]) The Clifford homomorphisms $\{p_\varepsilon\}_\varepsilon$ satisfy the relations:*

$$\sum_{\varepsilon \subset \lambda} (w_\varepsilon)^q \sum_{i=1}^{n} (p_\varepsilon(e_i))^* p_\varepsilon(e_i) = c_q(\lambda)\mathrm{id}_{V_\lambda}, \ \textit{for each } q \geq 0, \tag{1.65}$$

where $\varepsilon \subset \lambda$ denotes the set of relevant weights of λ and $\{w_\varepsilon\}_\varepsilon$ are the conformal weights of $\mathfrak{so}(n)$. For $n = 2m$, the Pfaffian element satisfies a similar relation:

$$\sum_{\varepsilon \subset \lambda} \mathrm{pf}(\lambda + \varepsilon) \sum_{i=1}^{n} (p_\varepsilon(e_i))^* p_\varepsilon(e_i) = 2m\mathrm{pf}(\lambda)\mathrm{id}_{V_\lambda}. \tag{1.66}$$

The Clifford homomorphisms are compatible with the action of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$, so that they extend to vector bundle homomorphisms. For each vector field $X = \sum_i X^i e_i$, the bundle homomorphism $p_\varepsilon(X) \in \mathrm{Hom}(V_\lambda M, V_{\lambda+\varepsilon} M)$ is defined by:

$$V_\lambda M \ni [e, v] \mapsto \sum_i X^i [e, p_\varepsilon(e_i)v] \in V_{\lambda+\varepsilon} M.$$

Any generalized gradient P_ε may be then expressed as follows:

$$P_\varepsilon(\varphi) = \Pi_\varepsilon\left(\sum_{i=1}^{n} e_i^* \otimes \nabla_{e_i}\varphi\right) = \sum_{i=1}^{n} p_\varepsilon(e_i)\nabla_{e_i}\varphi,$$

for any section $\varphi \in \Gamma(V_\lambda M)$. The second order differential operator $P_\varepsilon^* P_\varepsilon$ is realized as:

$$P_\varepsilon^* P_\varepsilon = -\sum_{i,j} (p_\varepsilon(e_i))^* p_\varepsilon(e_j)\nabla^2_{e_i,e_j}, \tag{1.67}$$

where $\nabla^2_{X,Y} := \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$, for all vector fields X, Y.

In order to give the reduced Weitzenböck formulas, we need to consider also the curvature actions in (1.56). The (translated) curvature endomorphisms on the associated vector bundle $V_\lambda M$ are defined by:

$$\tilde{R}^q_\lambda := \sum_{i,j} \lambda(\tilde{e}^q_{ij}) R_\lambda(e_i, e_j), \text{ for each } q \geq 1,$$

where λ is canonically extended to a representation of the enveloping algebra and R_λ denotes the curvature tensor of the connection induced on $V_\lambda M$ by the Levi-Civita connection: $R_\lambda(X,Y) := \nabla^2_{X,Y} - \nabla^2_{Y,X}$, for any vector fields X, Y. When n is even, there is also a curvature endomorphism related to the Pfaffian element:

$$R^{\mathrm{pf}}_\lambda := \sum_{i,j} \lambda(\mathrm{pf}_{ij}) R_\lambda(e_i, e_j),$$

where $\{\mathrm{pf}_{ij}\}_{i,j=\overline{1,2m}}$ are the elements of the enveloping algebra defined by:

$$\mathrm{pf}_{ij} := \begin{cases} \mathrm{pf}, & i = j, \\ (-1)^{i+j} \frac{2m}{(2i)^m m!} \sum\limits_{\sigma \in \mathfrak{S}^{ij}_{2m}} \mathrm{sgn}(\sigma) e_{\sigma(1)\sigma(2)} \cdots e_{\sigma(2m-1)\sigma(2m)}, & i < j, \\ -\mathrm{pf}_{ji}, & i > j, \end{cases}$$

where $\mathfrak{S}^{ij}_{2m}$ is the permutation group of $\{1, \ldots, 2m\} \setminus \{i, j\}$.

The main result (see [33, Theorem 7.1]) now essentially follows from the formula (1.64) in the enveloping algebra, using the relationship between higher Casimir elements and Clifford homomorphisms given by (1.65) and (1.66) and also the expression (1.67) for the operators occurring in reduced Weitzenböck formulas.

Theorem 1.39. *([33]) Let $\{P_\varepsilon\}_{\varepsilon\subset\lambda}$ be the generalized gradients acting on sections of $V_\lambda M$. Then all (independent) reduced Weitzenböck formulas are given as follows:*

$$\sum_{\varepsilon\subset\lambda}\left\{\sum_{p=0}^{2q-1}\tilde{c}_{2q-1-p}(\lambda)(-\widetilde{w}_\varepsilon)^p\right\}P_\varepsilon^* P_\varepsilon = \tilde{R}_\lambda^{2q}, \text{ for } q \geq 1,$$

and, when n is even, there is also the following reduced Weitzenböck formula:

$$\sum_{\varepsilon\subset\lambda} 2(\mathrm{pf}(\lambda)-\mathrm{pf}(\lambda+\varepsilon))P_\varepsilon^* P_\varepsilon = R_\lambda^{\mathrm{pf}},$$

where $\{\widetilde{w}_\varepsilon\}_\varepsilon$ are the translated conformal weights of $\mathfrak{so}(n)$, which are explicitly given by (2.10).

Chapter 2

Elliptic Operators and Kato Inequalities

On a Riemannian manifold, generalized gradients naturally give rise to second order differential operators, by composing each generalized gradient with its formal adjoint. In the previous chapter, § 1.3, we considered linear combinations of these operators that sum up to a zero-order differential operator and provide Weitzenböck formulas. In this chapter we analyze the other interesting extreme case, namely when this construction yields second order elliptic differential operators. The main result here is a new proof of Branson's classification, [13], of such elliptic operators for generalized gradients between vector bundles with structure group $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. The method we use for the proof is completely different from the one in [13], which seems to be specific for these two structure groups. The approach we give is mainly based on the representation theory of the Lie algebra $\mathfrak{so}(n)$ and on the relationship between ellipticity and Kato constants, which we explain in § 2.2. The arguments suggest that they should carry over to get similar classification results for generalized gradients of G-structures.

Firstly we present the general setting and state Branson's classification result. Then we turn our attention to Kato inequalities and their relationship to ellipticity. An essential tool for our proof is the explicit computation of the optimal Kato constants in terms of representation theoretical data, which was done by D. Calderbank, P. Gauduchon and M. Herzlich, [18]. The starting point in [18] is the list of elliptic second order differential operators provided by Branson's classification. For each such operator is given an explicit formula for its optimal Kato constant. Our main observation is that this argument may be reversed and we recover the ellipticity using this direct computation, as we show in § 2.3.

2.1 Ellipticity of Generalized Gradients

The purpose of this section is to present Branson's classfication result of second order elliptic operators that naturally arise from generalized gradients. We begin by briefly explaining the notions of ellipticity needed in the sequel and analyze the main properties of these operators given as linear combinations of generalized gradients composed with their formal adjoints. Then, in Theorems 2.12 and 2.13 we state the classification result of Th. Branson, [13], of all operators of this type which are elliptic. In particular, this classification shows that ellipticity is attained by assembling surprisingly few generalized gradients. From the construction and the arguments used it follows that the classification is valid for all Riemannian manifolds or (if a spin structure enters) Riemannian spin manifolds.

Let us first recall the definition of the principal symbol of a differential operator, which is a simple invariant way to refer to its highest order part. If E and F are smooth vector bundles over the manifold M and $P:\Gamma(E)\to\Gamma(F)$ is a linear differential operator of order k, then at every point $x \in M$ and for every $\xi \in T^*_x M$ one can associate an algebraic object, the *principal symbol* $\sigma_\xi(P;x)$, or simply $\sigma_\xi(P)$. If, in local coordinates, $Pu = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha u$, where a_α are $\dim(F) \times \dim(E)$ matrices, then $\sigma_\xi(P;x)$ is the matrix

$$\sigma_\xi(P;x) = i^k \sum_{|\alpha|\leq k} a_\alpha(x)\xi^\alpha,$$

with the notation $\xi^\alpha = \prod_j \xi^{\alpha_j}$ and $i = \sqrt{-1}$. It is usually convenient to delete the factor i^k when M is a real manifold, as it is in our context. To define the principal symbol invariantly, let E_x and F_x be the fibers of E and F at $x \in M$, let $u \in \Gamma(E)$ with $u(x) = z$ and $\varphi \in C^\infty(M)$ such that $\varphi(x) = 0$, $d\varphi(x) = \xi$, then $\sigma_\xi(P;x) : E_x \to F_x$ is the following endomorphism

$$\sigma_\xi(P;x)z = \frac{i^k}{k!} P(\varphi^k u)|_x. \tag{2.1}$$

Example 2.1. On a vector bundle E over a manifold M, any connection $\nabla : \Gamma(E) \to \Gamma(\mathrm{T^*M} \otimes E)$ satisfies $\nabla(\varphi u) = d\varphi \otimes u + \varphi\nabla u$, for any sections

$\varphi \in C^\infty(M)$, $u \in \Gamma(E)$. Consequently, the principal symbol of ∇ is given by: $\sigma_\xi(\nabla)u = i\xi \otimes u$. Alternatively, in the convention without the coefficient i^k, the principal symbol is just the identity: $\sigma = \mathrm{id} : \Gamma(\mathrm{T}^*\mathrm{M}\otimes E) \to \Gamma(\mathrm{T}^*\mathrm{M}\otimes E)$. It follows that the principal symbol of any G-generalized gradient, which is defined by (1.17) as a projection of a G-connection onto an irreducible subbundle: $P_\varepsilon := P_\varepsilon^{\nabla^\lambda} := \Pi_\varepsilon \circ \nabla^\lambda$, is given exactly by the projection Π_ε which defines it, $\sigma_{P_\varepsilon} = \Pi_\varepsilon : \Gamma(\mathrm{T}^*\mathrm{M} \otimes V_\lambda M) \to \Gamma(V_{\lambda+\varepsilon}M)$.

Definition 2.2. A linear differential operator $P : \Gamma(E) \to \Gamma(F)$ is *elliptic at a point* $x \in M$ if the symbol $\sigma_\xi(P; x)$ is an isomorphism for every real section $\xi \in \mathrm{T}^*_x\mathrm{M} \setminus \{0\}$. P is *elliptic* if it is elliptic at all points $x \in M$.

Example 2.3. Classical examples of elliptic operators are $\bar{\partial}$, the Cauchy-Riemann operator acting on complex-valued functions (or more general on forms of type $(0, q)$ on a complex manifold), which is a first order operator and the Laplacian Δ acting on p-forms, which is of second order. We shall come back to these examples later on, as they are special cases of elliptic operators obtained from generalized gradients.

Ellipticity is an algebraic property of a differential operator which implies analytic conclusions. The theory of linear elliptic operators is very important and highly-developed, but in our context we consider a special case of elliptic operators and, as we shall see, without loss of generality, the problem may be reduced to analyzing first order differential operators. Obviously, from Definition 2.2, a necessary condition for the existence of an elliptic operator between two vector bundles is that they have the same rank. So that in order to talk about the ellipticity of generalized gradients, which act between irreducible vector bundles of (usually) different ranks, we need to consider the following notion of ellipticity.

Definition 2.4. A linear differential operator $P : \Gamma(E) \to \Gamma(F)$ is *underdetermined elliptic at a point* $x \in M$ if its symbol $\sigma_\xi(P; x)$ is surjective for every real section $\xi \in \mathrm{T}^*_x\mathrm{M} \setminus \{0\}$. P is *overdetermined* (or *injectively*) *elliptic at a point* $x \in M$ if $\sigma_\xi(P; x)$ is injective for every real section $\xi \in \mathrm{T}^*_x\mathrm{M} \setminus \{0\}$. P is called *(injectively) strongly elliptic* if $\sigma_\xi(P; x)$ is injective for every complex cotangent vector $\xi \in (\mathrm{T}^*_x\mathrm{M})^{\mathbb{C}} \setminus \{0\}$.

Remark 2.5. Since the principal symbol of a G-generalized gradient P_ε is given by the projection Π_ε defining it, the above notions of ellipticity may be easily rephrased in terms of this projection as follows: P_ε is underdetermined (respectively overdetermined) elliptic if an only if $\Pi_\varepsilon \circ (\xi \otimes \cdot) : V_\lambda \to V_{\lambda+\varepsilon}$ is surjective (respectively injective), for each nonzero section $\xi \in \Gamma(\mathrm{T}^*_x\mathrm{M})$. Thus, the generalized gradient P_ε is (strongly) injectively elliptic if and only if Π_ε is non-vanishing on each decomposable element, *i.e.*

$$\Pi_\varepsilon(\xi \otimes v) = 0 \Rightarrow \xi \otimes v = 0,$$

where $\xi \in \Gamma(\mathrm{T}^*\mathrm{M})$ (respectively $\xi \in \Gamma((\mathrm{T}^*\mathrm{M})^{\mathbb{C}})$) and $v \in \Gamma(V_\lambda M)$.

Remark 2.6. The property of an operator to be strongly elliptic obviously implies that it is elliptic. The converse is not true and a counterexample is provided by the Dirac operator D on a spin manifold, whose principal symbol is given by the Clifford multiplication, $\sigma_\xi(D)(\varphi) = \xi \cdot \varphi$.

The general setting considered in the sequel is the following: (M, g) is a Riemannian (spin) manifold, λ is a dominant weight of $\mathfrak{so}(n)$ and $V_\lambda M$ is the associated vector bundle to the irreducible representation of highest weight λ. For any subset I of the set of relevant weights of λ, which is completely determined by the selection rule in Lemma 1.1, denote by P_I the following differential operator:

$$D_I := \sum_{\varepsilon \in I} P^*_\varepsilon P_\varepsilon \,, \tag{2.2}$$

where $P_\varepsilon := \Pi_\varepsilon \circ \nabla$ is the generalized gradient. This is a simplified notation, where the highest weight λ is omitted, but may be easily deduced from the context. It is then natural to ask

Question 2.7. Given λ, for which subsets I is the operator D_I elliptic?

The complete answer to this question was given by Th. Branson, [13]. In this section we restate his result in our notation and in § 2.3 we give a new proof of it.

First notice that Question 2.7 regarding second order differential operators may be reduced to first order ones. If we denote by P_I the following first order operator:

$$P_I := \sum_{\varepsilon \in I} P_\varepsilon, \tag{2.3}$$

then $D_I = P^*_I P_I$ and the following equivalence holds:

Lemma 2.8. *The operator D_I is elliptic if and only if P_I is injectively elliptic.*

This equivalence is a simple consequence of the behavior of principal symbols, namely that: $\sigma_{P_I^* P_I} = (\sigma_{P_I})^* \circ \sigma_{P_I}$, where $(\sigma_{P_I})^*$ is the Hermitian adjoint of σ_{P_I}. In the sequel we shall shortly say that P_I is elliptic instead of injectively elliptic.

It follows that D_I is elliptic if and only if the projection

$$\Pi_I := \sum_{\varepsilon \in I} \Pi_\varepsilon : T \otimes V_\lambda \to \bigoplus_{\varepsilon \in I} V_{\lambda+\varepsilon} \tag{2.4}$$

is injective when restricted to the set of decomposable elements in $T \otimes V_\lambda$. Thus, the ellipticity of the operators D_I is reduced to a question about the representation theory of $\mathfrak{so}(n)$, without reference to any particular manifold. This remark is the starting point in the original proof given by Th. Branson for the classification of elliptic Stein-Weiss operators. This remark also shows that, in contrast to the situation in § 1.3, where the connection ∇ defining the generalized gradients is the Levi-Civita connection, here ∇ can be any metric connection.

The fact that each projection in (2.4) is onto a different direct summand has the following straightforward, but important consequences:

(1) If instead of the operators P_I given by (2.4), we consider, more generally, operators of the form $\sum_{\varepsilon \in I} a_\varepsilon P_\varepsilon$ with nonzero coefficients, then such an operator is elliptic if and only if P_I is. Thus, the ellipticity only depends on the subset I and not on the coefficients, unlike in the case of Weitzenböck formulas, see § 1.3, where these coefficients play a very important role.

(2) If $I_1 \subset I_2$ and P_{I_1} is elliptic, then also P_{I_2} is elliptic. Hence the interesting operators are the *minimal elliptic* operators P_I, *i.e.* such that there is no proper subset of I which still defines an elliptic operator. It is this set of minimal elliptic operators that was determined by Th. Branson. In a certain sense, the bigger the set I is, the greater is the probability for P_I to be elliptic. For instance, if I is the whole set of weights of the standard representation, then the operator is the *rough Laplacian* $\nabla^*\nabla$, which is, of course, elliptic.

The following results concerning Question 2.7 have been shown prior to Branson's classification. Recall that the *Cartan summand* of two irreducible representations λ and μ is the subrepresentation of $\lambda \otimes \mu$ of highest weight $\lambda + \mu$.

Proposition 2.9 (J. Kalina, A. Pierzchalski and P. Walczak, [37]). *For any irreducible representation λ, the projection onto the Cartan summand of*

$\tau \otimes \lambda$ defines a strongly elliptic first order differential operator, also called top gradient*, and this is the only generalized gradient with this property.*

In fact, in [37], J. Kalina, A. Pierzchalski and P. Walczak proved a more general version of Proposition 2.9, for the irreducible decomposition of the tensor product of two irreducible representations of any compact semisimple Lie group: among all operators defined by projections from a tensor product to its irreducible subbundles, only the one given by the projection onto the Cartan summand is strongly elliptic.

The proof of Proposition 2.9 in [37] uses an approximation of the highest weight vector with converging sequences in order to show that the top gradient is strongly elliptic, while the other implication is more or less straightforward. Namely, if $v \in V_\lambda$ and $w \in V_\mu$ are respectively the highest weight vectors, then $v \otimes w$ is the highest weight vector in $V_\lambda \otimes V_\mu$ and thus belongs to the Cartan summand. Then $v \otimes w$ is a nontrivial element in the kernel of any other projection different from the one onto the Cartan summand. Moreover, this shows that the operator defined by any other set of projections not containing the Cartan projection is not strongly elliptic.

For example, from Proposition 2.9, it follows that the twistor operator T acting on p-forms (see Example 1.3) is strongly elliptic.

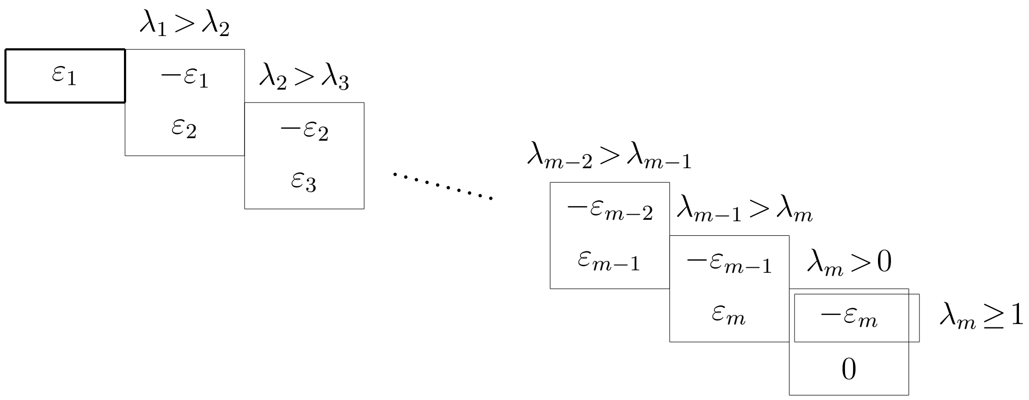
Proposition 2.10 (E. Stein and G. Weiss, [64])**.** *For any irreducible representation λ, the projection onto the complement of the Cartan summand,* i.e. *when the set I is equal to the whole set of relevant weights except for the highest weight of τ, defines an elliptic operator P_I.*

Example 2.11. In Example 1.3 the complement of the Cartan projection defines the operator $P = d + \delta$, which, by Proposition 2.10, is (injectively) elliptic and, by the above construction, just gives rise to the *Laplacian* acting on p-forms: $\Delta = d\delta + \delta d = (d + \delta)^*(d + \delta)$.

Branson's classification essentially says that the Laplacian is not a special case, but the generalized gradients usually break up into pairs or singletons which are elliptic. Before stating it, we give a graphical interpretation of the classical selection rule in Lemma 1.1 for the special orthogonal group, which can be found in [63] and simplifies the task of finding the relevant weights. In our context the graphical interpretation is helpful to better visualize the classification of elliptic operators, which turns out to be strongly related to the selection rule.

First we consider the case of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$ with n odd, $n = 2m+1$. We use the same notation as above: $\{\pm\varepsilon_1, \dots, \pm\varepsilon_m, 0\}$ are the weights of the defining complex representation τ of $\mathrm{SO}(n)$ and V_λ is the irreducible representation of highest weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{m-1} \geq \lambda_m \geq 0) \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$, where λ_i are the coordinates of λ with respect to the orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_m\}$. The decision criterion given in Lemma 1.1 for the decomposition of the tensor product $\tau \otimes \lambda$ can be read in the following diagram featuring the weights of τ and labeled boxes. A weight ε is relevant for an irreducible representation λ if and only if the coordinates $\lambda_1, \dots, \lambda_m$ of λ satisfy all the inequalities labeling the boxes containing ε.

Diagram 2.1: Selection Rule for $\mathrm{SO}(2m + 1)$ or $\mathrm{Spin}(2m + 1)$

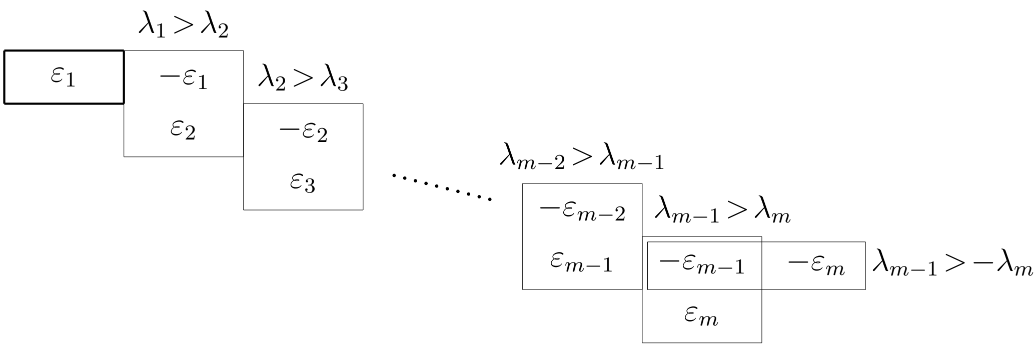


Theorem 2.12 (Th. Branson, [13]). *Let (M, g) be a Riemannian (spin) manifold of odd real dimension $n = 2m + 1$ and $V_\lambda M$ the associated vector bundle to an irreducible* $\mathrm{SO}(n)$*- (or* $\mathrm{Spin}(n)$*)-representation of highest weight λ. For any subset I of the set of relevant weights of λ, the corresponding operator $P_I = \sum_{\varepsilon \in I} \Pi_\varepsilon \circ \nabla$ is a minimal elliptic operator if and only if I is one of the following sets:*

1. *$\{\varepsilon_1\}$ (strongly elliptic),*
2. *$\{0\}$, if λ is properly half-integral,*
3. *$\{-\varepsilon_i, \varepsilon_{i+1}\}$, for $i = 1, \dots, m - 1$,*
4. *$\{-\varepsilon_m, 0\}$, if λ is integral.*

For the case of SO(n) or Spin(n) with n even, $n = 2m$, there is a similar graphical interpretation of the relevant weights of an irreducible representation $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m|) \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$, where again λ_i are the coordinates of λ with respect to the orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_m\}$.

Diagram 2.2: Selection Rule for SO($2m$) or Spin($2m$)



Theorem 2.13 (Th. Branson, [13]). *Considering the same assumptions as in Theorem 2.12, but now on a Riemannian (spin) manifold of even real dimension $n = 2m$, the operator $P_I = \sum_{\varepsilon \in I} \Pi_\varepsilon \circ \nabla$ is minimal elliptic if and only if I is one of the following sets:*

1. $\{\varepsilon_1\}$ *(strongly elliptic),*
2. $\{-\varepsilon_m\}$, *if* $\lambda_m > 0$,
3. $\{\varepsilon_m\}$, *if* $\lambda_m < 0$,
4. $\{-\varepsilon_i, \varepsilon_{i+1}\}$, *for* $i = 1, \ldots, m-2$,
5. $\{-\varepsilon_{m-1}, \varepsilon_m\}$, *if* $\lambda_m \geq 0$,
6. $\{-\varepsilon_{m-1}, -\varepsilon_m\}$, *if* $\lambda_m \leq 0$.

Theorems 2.12 and 2.13 give a complete answer to the Question 2.7, by restating the results for the second order differential operators $D_I = P_I^* P_I$. Note that in the list of minimal elliptic operators no operator $P_\varepsilon^* P_\varepsilon$ appears twice, except for $P^*_{-\varepsilon_{m-1}}P_{-\varepsilon_{m-1}}$ in the case when n is even and $\lambda_m = 0 \neq \lambda_{m-1}$. The list is also exhaustive, *i.e.* each $P^*_\varepsilon P_\varepsilon$ occurs, except for n odd and λ properly half-integral, when $P^*_{-\varepsilon_m}P_{-\varepsilon_m}$ does not occur in the list. Thus, apart from these exceptions, the subsets I defining the minimal elliptic operators form a partition of the set of weights of the standard representation τ.

Remark 2.14. A priori it is not clear that the ellipticity of the differential operator $D_I : V_\lambda M \to V_\lambda M$ defined by a certain subset I is independent of the given highest weight λ (of course here are considered only the highest weights for which all the elements in I are relevant weights). This follows from Theorems 2.12 and 2.13 and no other direct way of proving it is known.

Our aim is to give a new proof of Theorems 2.12 and 2.13 in § 2.3, which we hope is better suited as a starting point for an analogous classification of elliptic operators defined by G-generalized gradients for the subgroups considered in (1.6). For this reason we only mention here for the proof of these theorems, that the arguments used by Th. Branson involve tools and techniques of harmonic analysis, explicit computations of the spectra of generalized gradients on the sphere and a strong irreducibility property of principal series representations of the group $\mathrm{Spin}_0(n+1,1)$. For details we refer the reader to Th. Branson's paper [13].

Remark 2.15. The $\mathrm{GL}(n)$- and $\mathrm{O}(n)$-generalized gradients which are elliptic have been studied by J. Kalina, B. Ørsted, A. Pierzchalski, P. Walczak and G. Zhang, [36], in an elementary way using the language of Young diagrams. However, they find only the top gradient and miss the other elliptic generalized gradients in the list of Branson, because they restrict to a certain special class, the so-called "up gradients", as pointed out in [13].

2.2 Optimal Kato Constants

Kato inequalities are estimates in Riemannian geometry, which have proved to be a powerful technique for linking vector-valued and scalar-valued problems in analysis on manifolds. The classical Kato inequality may be stated as follows. For any section φ of a Riemannian or Hermitian vector bundle E endowed with a metric connection ∇ over a Riemannian manifold (M, g), at any point where φ does not vanish, the following inequality holds:

$$|d|\varphi|| \leq |\nabla\varphi|. \tag{2.5}$$

This estimate is a direct consequence of the Schwarz inequality applied to the identity $d(|\varphi|^2) = 2\langle\nabla\varphi, \varphi\rangle$, which is given by the fact that the connection is metric. Thus, equality is attained at a point $x \in M$ if and only if $\nabla\varphi$ is a multiple of φ at x, *i.e.* if there exists a 1-form α such that

$$\nabla\varphi = \alpha \otimes \varphi. \tag{2.6}$$

Whenever (2.6) has no solutions in the corresponding geometric setting, there exist refined Kato inequalities, which are of the form

$$|d|\varphi|| \leq k|\nabla\varphi|, \tag{2.7}$$

with a constant $k < 1$. For example, such estimates occur in Yau's proof of the Calabi conjecture or in the Bernstein problem for minimal hypersurfaces in $\mathbb{R}^n$. It turns out that the knowledge of the best constant k plays a key role in such proofs. For a survey of these techniques see the introductory part in [18] and the references therein.

The principle underlying the existence of refined Kato inequalities was first remarked by J.-P. Bourguignon, [10]. He pointed out that in all geometric settings where refined Kato inequalities occured, the sections under consideration are solutions of a natural linear first order injectively elliptic system and that in such a situation the equality case in (2.5) cannot be achieved, except at points where $\nabla\varphi = 0$. To see this, suppose that equality is attained at a point by a solution φ of such an elliptic system. At that point, $\nabla\varphi = \alpha \otimes \varphi$, for some 1-form α. A natural first order operator is one of the form P_I, that we considered in § 2.1, *i.e.* is given by a projection Π_I of the connection ∇ onto a natural subbundle of $\mathrm{T}^*\mathrm{M} \otimes E$. Hence $0 = \Pi_I(\nabla\varphi) = \Pi_I(\alpha \otimes \varphi)$ and, by the ellipticity of P_I, it follows that $\alpha \otimes \varphi = 0$, so $\nabla\varphi = 0$. It thus turns out that there is a strong relationship between the ellipticity of the operators P_I and the existence of refined Kato inequalities for sections in their kernel.

D. Calderbank, P. Gauduchon and M. Herzlich, [18], proved that there exists indeed a refined Kato inequality for sections in the kernel of any natural first order injectively elliptic operator P_I, which acts on sections of a vector bundle associated to an irreducible $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$-representation. They computed the optimal Kato constant k_I, which depends only on the choice of the elliptic operator, in terms of representation-theoretical data. More precisely, the formulas for the optimal Kato constants involve only the conformal weights of the generalized gradients, which are explicitly known (see *e.g.* Table 1.1).

In this section we present the main steps in the computation of the general expression for the optimal Kato constants, see [18].

The general setting is the same as in § 2.1: on a Riemannian (spin) manifold (M, g), $V_\lambda M$ is the vector bundle associated to the irreducible $\mathrm{SO}(n)$-respectively $\mathrm{Spin}(n)$-representation of highest weight $\lambda = (\lambda_1, \dots, \lambda_m)$; for any subset I of the set of relevant weights of λ, P_I is the operator defined by (2.3) acting on sections of $V_\lambda M$. The aim is to show that for each operator P_I which is injectively elliptic, there exists an optimal constant $k_I < 1$ such that the refined Kato inequality holds:

$$|d|\varphi|| \leq k_I |\nabla \varphi|, \qquad \text{for all } \varphi \in \ker(P_I), \tag{2.8}$$

and to give a formula for k_I, in terms of the conformal weights.

In Definition 1.13 we introduced the conformal weight operator B (we shall omit from now on the highest weight λ), whose eigenvalues, the conformal weights, are given by (1.29). The key property used in the sequel is that the conformal weights are strictly ordered (see Remark 1.16), with the exception of the case when n is even, $n = 2m$, $\lambda_m = 0$ and $w_{m,+} = w_{m,-}$, which is due to the fact that the two corresponding $\mathrm{SO}(n)$-irreducible representations are exchanged by a change of orientation, while their sum is an irreducible $\mathrm{O}(n)$-representation. In this exceptional case we consider in the sequel these two representations as one summand, so that the conformal weights of distinct projections are always different from each other.

It turns out that the computations are simplified if one considers the *translated conformal weight operator*:

$$\widetilde{B} : (\mathbb{R}^n)^* \otimes V_\lambda \to (\mathbb{R}^n)^* \otimes V_\lambda, \quad \widetilde{B} := B + \frac{n-1}{2}\mathrm{Id}, \tag{2.9}$$

whose eigenvalues are the *translated conformal weights*, for $i = 1, \dots, m$:

$$\begin{aligned} \widetilde{w}_0(\lambda) &= 0, \\ \widetilde{w}_{i,+}(\lambda) &= \lambda_i - i + \frac{n+1}{2}, \\ \widetilde{w}_{i,-}(\lambda) &= -\lambda_i + i - \frac{n-1}{2}, \end{aligned} \tag{2.10}$$

which obviously have the same strict ordering as the conformal weights. These translated conformal weights have the advantage that the virtual weights whose relevance depends on the same condition on λ, *i.e* that are in the same box in the Diagrams 2.1 and 2.2, sum up to zero if that condition is not fulfilled. For instance, if $\lambda_i = \lambda_{i+1}$, then the weights $-\varepsilon_i$ and ε_{i+1} are not relevant for λ and their corresponding conformal weights satisfy: $\widetilde{w}_{i,-}(\lambda) + \widetilde{w}_{i+1,+}(\lambda) = 0$. This cancellation property for non-relevant weights is useful for the forthcoming computations.

The strict ordering of the translated conformal weights allows us to rename them (and the corresponding summands in the decomposition of the tensor product $(\mathbb{R}^n)^* \otimes V_\lambda$) and to index them in a decreasing ordering as follows:

$$(\mathbb{R}^n)^* \otimes V_\lambda = \bigoplus_{i=1}^{N} V_i, \tag{2.11}$$

with

$$\widetilde{w}_1(\lambda) > \widetilde{w}_2(\lambda) > \cdots > \widetilde{w}_N(\lambda),$$

where N is the number of summands in the decomposition, *i.e.* the number of relevant weights for λ. This reordering of the indices of the conformal weights is then carried over to the corresponding weights of the standard representation and thus, the subsets I defining the operators P_I are subsets of $\{1, \ldots, N\}$.

Remark 2.16. Notice that, in the above notation, the weights which are in the same box in Diagram 2.1 and 2.2 are pairs of type $\{i, N+2-i\}$ and the list of minimal elliptic operators of the form P_I established by Th. Branson (see Theorems 2.12 and 2.13) is the following:

1. $P_{\{1\}}$;
2. $P_{\{\ell+1\}}$ if $N = 2\ell$ and $\lambda_m \neq 0$;
3. $P_{\{\ell\}}$ if $N = 2\ell - 1$ and λ is properly half-integral;
4. $P_{\{i,N+2-i\}}$ for $i = 2, \ldots, \ell - 1$;
5. $P_{\{\ell,\ell+2\}}$ if $N = 2\ell$;
6. $P_{\{\ell,\ell+1\}}$ if $N = 2\ell - 1$ and λ is integral.

This shows that the list of the minimal elliptic operators depends only on the ordering of the conformal weights.

The following result reduces the search for Kato inequalities to an algebraic problem. By $\widehat{I}$ we denote the complement of I in $\{1,\dots,N\}$.

Lemma 2.17. *Let I be a subset of $\{1,\dots,N\}$ and $P_I := \sum_{i\in I}\Pi_i \circ \nabla$ the corresponding operator. For any section φ in the kernel of P_I and at any point where φ does not vanish, the following inequality holds:*

$$|d|\varphi|| \leq k_I |\nabla\varphi|, \tag{2.12}$$

where the constant k_I, called Kato constant, *is defined by*

$$k_I := \sup_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha\otimes v)| = \sqrt{1 - \inf_{|\alpha|=|v|=1}|\Pi_I(\alpha\otimes v)|^2}, \tag{2.13}$$

where $\alpha \in (\mathbb{R}^n)^*$ and $v \in V_\lambda$. Furthermore, equality holds at a point if and only if $\nabla\varphi = \Pi_{\widehat{I}}(\alpha\otimes\varphi)$ for a 1-form α at that point, such that:

$$|\Pi_{\widehat{I}}(\alpha\otimes\varphi)| = k_I|\alpha\otimes\varphi|.$$

The proof of Lemma 2.17 is based, as for the classical Kato inequality, on purely algebraic refined Schwarz inequalities of the form:

$$\frac{|\langle\Phi, v\rangle|}{|v|} \leq k|\Phi|, \tag{2.14}$$

where $\Phi \in (\mathbb{R}^n)^*\otimes V_\lambda$ and $v \in V_\lambda$. The inequality (2.12) is obtained by lifting (2.14) to the associated vector bundles and putting $v = \varphi$ and $\Phi = \nabla\varphi$ for a section $\varphi \in \Gamma(V_\lambda M)$.

The first step in the minimization process for the computation of the Kato constant k_I, given by (2.13), is to use the classical Lagrange interpolation, in order to express each projection Π_j, for some $j \in \{1,\dots,N\}$, as follows:

$$\Pi_j = \prod_{k\neq j}\frac{\widetilde{B} - \widetilde{w}_k\mathrm{Id}}{\widetilde{w}_j - \widetilde{w}_k} = \frac{\sum_{k=0}^{N-1}\widetilde{w}_j^{N-1-k}A_k}{\prod_{k\neq j}(\widetilde{w}_j - \widetilde{w}_k)},$$

where $A_k := \sum_{\ell=0}^{k}(-1)^\ell\sigma_\ell(\widetilde{w})\widetilde{B}^{k-\ell}$ and $\sigma_i(\widetilde{w})$ is the i-th elementary symmetric function in the translated conformal weights $\widetilde{w}_1,\dots,\widetilde{w}_N$. By products (or powers) of endomorphisms we mean their composition. It further follows that the norms of the projections are given by:

$$|\Pi_j(\alpha\otimes v)|^2 = \langle \Pi_j(\alpha\otimes v), \alpha\otimes v\rangle = \frac{\displaystyle\sum_{k=0}^{N-1} \widetilde{w}_j^{N-1-k}\langle A_k(\alpha\otimes v), \alpha\otimes v\rangle}{\displaystyle\prod_{k\neq j}(\widetilde{w}_j - \widetilde{w}_k)}. \quad (2.15)$$

Formula (2.15) expressing the N quantities $|\Pi_j(\alpha\otimes v)|^2$ in terms of the other N quantities $\langle A_k(\alpha\otimes v), \alpha\otimes v\rangle$ is the key formula of the method in [18]. The main technical step is based on the one hand, on a precise expression for the traces of the operators $\widetilde{B}^j$ (which follows from the relationship between translated conformal weights and higher order Casimir operators) and, on the other hand, is based on a result due to Diemer and G. Weingart, who proved that each family of polynomials in $\widetilde{B}$ satisfying some special recurrence formula involving their traces has nice symmetry properties. We refer to [18] for the details and only state here the output of this technical analysis:

Lemma 2.18. *If N is odd, then $\langle A_{2j+1}(\alpha\otimes v), \alpha\otimes v\rangle = 0$, for each j.*
If N is even, then $\langle A_{2j+1}(\alpha\otimes v), \alpha\otimes v\rangle + \frac{1}{2}\langle A_{2j}(\alpha\otimes v), \alpha\otimes v\rangle = 0$, for each j.

Lemma 2.18 shows that in the expression (2.15) of $|\Pi_j(\alpha\otimes v)|^2$ half of the N quantities $\langle A_k(\alpha\otimes v), \alpha\otimes v\rangle$ vanish. Thus, each $|\Pi_j(\alpha\otimes v)|^2$ is given as an affine function in the remaining variables, that we denote as follows:

$$Q_k := (-1)^{k-1}\langle A_{2k-2}(\alpha\otimes v), \alpha\otimes v\rangle, \quad k = 2, \dots, \ell, \text{ for } N = 2\ell-1 \text{ or } N = 2\ell.$$

Notice that the first variable would be just constant, $Q_1 = 1$, since $A_0 = \mathrm{Id}$.

The equalities (2.15), for $j = 1, \dots, N$, then become for $N = 2\ell - 1$:

$$|\Pi_j(\alpha\otimes v)|^2 = \frac{\widetilde{w}_j^{2(\ell-1)} - \displaystyle\sum_{k=2}^{\ell}(-1)^k \widetilde{w}_j^{2(\ell-k)} Q_k}{\displaystyle\prod_{k\neq j}(\widetilde{w}_j - \widetilde{w}_k)} =: \pi_j(Q_2, \dots, Q_\ell), \quad (2.16)$$

and for $N = 2\ell$:

$$|\Pi_j(\alpha \otimes v)|^2 = \left(\widetilde{w}_j - \frac{1}{2}\right) \frac{\widetilde{w}_j^{2(\ell-1)} - \sum_{k=2}^{\ell} (-1)^k \widetilde{w}_j^{2(\ell-k)} Q_k}{\prod_{k \neq j} (\widetilde{w}_j - \widetilde{w}_k)} =: \pi_j(Q_2, \dots, Q_\ell). \tag{2.17}$$

Hence, the problem of estimating $\inf\limits_{|\alpha|=|v|=1} |\Pi_I(\alpha \otimes v)|^2$ (for a subset I corresponding to an elliptic operator) is reduced to minimizing this affine function over the admissible region in the $(\ell-1)$-dimensional affine space. The admissible region consists of the points Q of coordinates $\{Q_k\}_{k=\overline{2,\ell}}$, such that there exist unitary vectors $\alpha \in (\mathbb{R}^n)^*$ and $v \in V_\lambda$ with the property that for each $k = 2, \dots, \ell$ the following relation holds: $Q_k = (-1)^{k-1} \langle A_{2k-2}(\alpha \otimes v), \alpha \otimes v \rangle$. Thus, the search for Kato constants mainly reduces to linear programming.

The admissible region is contained in a convex in the Q-space, since $|\Pi_j(\alpha \otimes v)|^2 = \pi_j(Q)$ and each norm is non-negative and smaller than 1, if Q is an admissible point. More precisely, from (2.16) it follows that the point $Q = (Q_2, \dots, Q_\ell)$ is in the convex region $\mathcal{P}$ in $\mathbb{R}^{\ell-1}$ defined by the following system of linear inequalities for N odd, $N = 2\ell - 1$:

$$\sum_{k=2}^{\ell} (-1)^{j+k} \widetilde{w}_j^{2(\ell-k)} Q_k \geq (-1)^j \widetilde{w}_j^{2(l-1)}, \quad j = 1, \dots, 2\ell - 1, \tag{2.18}$$

with equality if and only if $|\Pi_j(\alpha \otimes v)|^2 = \pi_j(Q) = 0$. For N even, $N = 2\ell$, the system of linear inequalities is similarly obtained from (2.17), taking into account that the sign of the denominator in (2.17) is $(-1)^{j-1}$:

$$\begin{cases} \sum_{k=2}^{\ell} (-1)^{j+k} \widetilde{w}_j^{2(\ell-k)} Q_k \geq (-1)^j \widetilde{w}_j^{2(l-1)}, & 1 \leq j \leq \ell, \\ \sum_{k=2}^{\ell} (-1)^{j+k} \widetilde{w}_j^{2(\ell-k)} Q_k \leq (-1)^j \widetilde{w}_j^{2(l-1)}, & \ell + 1 \leq j \leq 2\ell, \end{cases} \tag{2.19}$$

and equality is attained if and only if $|\Pi_j(\alpha \otimes v)|^2 = \pi_j(Q) = 0$.

The convex region $\mathcal{P}$ defined by the system (2.18), respectively (2.19), turns out to have the following important properties, as explained in the sequel: it is compact, hence polyhedral, and one may identify its vertices as corresponding to the maximal non-elliptic operators. Since the norms are affine in the Q_k's, it then suffices to minimize over the set of vertices.

For a subset $J \subset \{1, \ldots, N\}$ with $\ell - 1$ elements, the intersection of the corresponding hyperplanes is the point denoted by Q^J:

$$\{Q^J\} := \bigcap_{j \in J} \{\pi_j(Q_2, \ldots Q_\ell) = 0\}, \tag{2.20}$$

whose coordinates are given by the elementary symmetric functions in the squares of the translated conformal weights: $Q_k^J = \sigma_{k-1}\big((\widetilde{w}_j^2)_{j \in J}\big)$. At the point Q^J, the affine functions π_j, defined by (2.16) for $N = 2\ell - 1$, take the values

$$\pi_j(Q^J) = \frac{\prod_{k \in J}(\widetilde{w}_j^2 - \widetilde{w}_k^2)}{\prod_{k \neq j}(\widetilde{w}_j - \widetilde{w}_k)} = \frac{\prod_{k \in J, k \neq j}(\widetilde{w}_j + \widetilde{w}_k)}{\prod_{k \in \widehat{J}, k \neq j}(\widetilde{w}_j - \widetilde{w}_k)} \varepsilon_j(J), \tag{2.21}$$

where $\varepsilon_j(J) = 0$ if $j \in J$ and 1 otherwise. Similarly, for $N = 2\ell$, the affine functions π_j defined by (2.17) take the values:

$$\pi_j(Q^J) = \left(\widetilde{w}_j - \frac{1}{2}\right) \frac{\prod_{k \in J}(\widetilde{w}_j^2 - \widetilde{w}_k^2)}{\prod_{k \neq j}(\widetilde{w}_j - \widetilde{w}_k)} = \left(\widetilde{w}_j - \frac{1}{2}\right) \frac{\prod_{k \in J, k \neq j}(\widetilde{w}_j + \widetilde{w}_k)}{\prod_{k \in \widehat{J}, k \neq j}(\widetilde{w}_j - \widetilde{w}_k)} \varepsilon_j(J). \tag{2.22}$$

The expression for the coordinates of Q^J follows from the fact that the affine function π_j is obtained by evaluating a polynomial independent of j on $\widetilde{w}_j^2$ and that the coefficients of a polynomial are given by the elementary symmetric functions of the roots.

In the case $N = 2\ell - 1$, the compactness of the convex region $\mathcal{P}$ is obtained by considering the subsets $J = \{2, \ldots, \ell\}$ and $J = \{\ell + 1, \ldots, 2\ell - 1\}$. The inverse of the Vandermonde system of inequalities (2.18) for $J = \{2, \ldots, \ell\}$ has non-negative entries, while for $J = \{\ell + 1, \ldots, 2\ell - 1\}$ it has non-positive entries and one gets:

Proposition 2.19. *If $N = 2\ell - 1$, then for $k = 2, \ldots, \ell$ the following inequalities hold:*

$$\sigma_{k-1}(\widetilde{w}_2^2, \ldots, \widetilde{w}_\ell^2) \leq Q_k \leq \sigma_{k-1}(\widetilde{w}_{\ell+1}^2, \ldots, \widetilde{w}_{2\ell-1}^2).$$

The lower bounds are all attained if and only if $\Pi_{\{2,\ldots,\ell\}}(\alpha \otimes v) = 0$, while the upper bounds are all attained if and only if $\Pi_{\{\ell+1,\ldots,2\ell-1\}}(\alpha \otimes v) = 0$. Both bounds are sharp if n is even or if n is odd and λ is integral, since *the operators* $P_{\{2,\dots,\ell\}}$ *and* $P_{\{\ell+1,\dots,2\ell-1\}}$ *are not injectively elliptic. In the case when* n *is odd and* λ *is properly half-integral, the above lower bound is not sharp, since* $P_{\{2,\dots,\ell\}}$ *is elliptic.*

In the case $N = 2\ell$, the compactness of the convex region is analogously obtained by taking the subsets $J = \{2, \dots, \ell\}$ and $J = \{\ell + 2, \dots, 2\ell\}$. Again the inverse of the Vandermonde systems have all entries of the same sign and one gets:

Proposition 2.20. *If* $N = 2\ell$*, then for* $k = 2, \dots, \ell$ *the following inequalities hold:*

$$\sigma_{k-1}(\widetilde{w}_2^2, \dots, \widetilde{w}_\ell^2) \le Q_k \le \sigma_{k-1}(\widetilde{w}_{\ell+2}^2, \dots, \widetilde{w}_{2\ell}^2).$$

The lower bounds are all attained if and only if $\Pi_{\{2,\dots,\ell\}}(\alpha \otimes v) = 0$*, while the upper bounds are all attained if and only if* $\Pi_{\{\ell+2,\dots,2\ell\}}(\alpha \otimes v) = 0$*. These bounds are always sharp by the non-ellipticity of* $P_{\{2,\dots,\ell\}}$ *and* $P_{\{\ell+2,\dots,2\ell\}}$*.*

As there exists a set of minimal elliptic operators, there also exists a set of *maximal non-elliptic* operators, *i.e.* the set of operators P_I which are non-elliptic and I has maximal cardinality. Theorems 2.12 and 2.13 provide us also the set of maximal non-elliptic operators, which are explicitly described as follows.

Let $\mathcal{NE}$ denote the set of subsets of $\{1, \dots, N\}$ whose elements are obtained by choosing exactly one index in each of the sets $\{j, N+2-j\}$ for $2 \le j \le \ell$, if $N = 2\ell-1$ or $N = 2\ell$, giving $2^{\ell-1}$ elements in $\mathcal{NE}$. The elements of $\mathcal{NE}$ are then precisely the subsets of $\{1, \dots, N\}$ corresponding to the maximal non-elliptic operators, unless n is odd, $N = 2\ell - 1$ and λ is properly half-integral, in which case the subsets containing ℓ (which corresponds to the zero weight) are elliptic. This is called the *exceptional case* and is the only one when the Kato constant provided by Theorem 2.22 might not be optimal.

The set $\mathcal{NE}$ can easily be described in the graphical interpretation given by Diagrams 2.1 and 2.2 (with the remark that now the indices are considered according to the convention given by the decreasing ordering of the conformal weights): each element of $\mathcal{NE}$ contains exactly one index from each box containing two weights. For instance, for $n = 2m + 1$, if $\lambda_m = \frac{1}{2}$, then $-\varepsilon_m$ is not relevant and the zero weight forms one box, so that it is not taken in any subset in $\mathcal{NE}$; if $\lambda_m \ge 1$, then $\{-\varepsilon_m, 0\}$ are in the same box and one of them is chosen for each subset in $\mathcal{NE}$. For $n = 2m$, if $\lambda_{m-1} > \lambda_m > 0$, then $\{-\varepsilon_{m-1}, \varepsilon_m\}$ form one box and $\{-\varepsilon_m\}$ is alone in a box; whereas if $\lambda_{m-1} > -\lambda_m > 0$, then $\pm\varepsilon_m$ are interchanged (since the ordering of the corresponding conformal weights changes: $w_{m,+} - w_{m,-} = 2\lambda_m$), namely $\{-\varepsilon_{m-1}, -\varepsilon_m\}$ are in one box and $\{\varepsilon_m\}$ forms itself a box. This is in accordance with the classification of minimal elliptic operators: $\{-\varepsilon_m\}$ and $\{-\varepsilon_{m-1}, \varepsilon_m\}$ are elliptic if $\lambda_m > 0$, and $\{\varepsilon_m\}$ and $\{-\varepsilon_{m-1}, -\varepsilon_m\}$ are elliptic if $\lambda_m < 0$. In the special case when the weights $\pm\varepsilon_m$ are relevant and their conformal weights are equal: $w_{m,-} = w_{m,+}$, *i.e.* when $\lambda_{m-1} > \lambda_m = 0$, then, in our convention, the corresponding representations are considered as one summand $V_{\lambda-\varepsilon_m} \oplus V_{\lambda+\varepsilon_m}$ and in this case the last box in the Diagram 2.2 is formed by $\{-\varepsilon_{m-1}, \varepsilon_m\}$, and the corresponding projection to ε_m is here actually the projection onto $V_{\lambda-\varepsilon_m} \oplus V_{\lambda+\varepsilon_m}$.

In the sequel we call $\mathcal{NE}$ the set of maximal non-elliptic operators, which is true apart from the exceptional case. Notice that each subset in $\mathcal{NE}$ has exactly $\ell - 1$ elements, where ℓ gives the parity of N, *i.e.* $N = 2\ell - 1$ or $N = 2\ell$. Now we can give explicitly the description of the vertices of the polyhedral region $\mathcal{P}$ in $\mathbb{R}^{\ell-1}$ (we also here recall its complete proof given in [18], because we need the arguments in § 2.3):

Proposition 2.21. *The vertices of the polyhedron $\mathcal{P}$ are exactly the points Q^J, defined by* (2.20), *with $J \in \mathcal{NE}$, the set of maximal non-elliptic operators. In the exceptional case, when n is odd and λ is properly half-integral, only one inclusion holds, namely that the vertices are contained in the set $\mathcal{NE}$.*

Proof: Let us denote by $\mathcal{V}$ the set of vertices of the polyhedron $\mathcal{P}$ in $\mathbb{R}^{\ell-1}$. Then we have to show that, if we are not in the exceptional case, the following equality holds: $\mathcal{V} = \{Q^J |\, J \in \mathcal{NE}\}$. Notice that the vertices of $\mathcal{P}$ are characterized as follows:

$$\mathcal{V} = \{Q^J \,|\, |J| = \ell - 1, \Pi_j(Q^J) = 0, \text{for all } j \in J;\, \Pi_j(Q^J) > 0, \text{for all } j \in \widehat{J}\}.$$

The two inclusions are shown as follows.

(1) $\{Q^J |\, J \in \mathcal{NE}\} \subset \mathcal{V}$: for $J \in \mathcal{NE}$, P_J is not elliptic, so that there exists a decomposable element $\alpha \otimes v \in (\mathbb{R}^n)^* \otimes V_\lambda$ of norm one with $\Pi_j(\alpha \otimes v) = 0$, for all $j \in J$, which implies that $Q^J \in \mathcal{V}$.

(2) $J \notin \mathcal{NE}$ implies $Q^J \notin \mathcal{V}$ (where J is a subset of $\{1, \ldots, N\}$ with $\ell - 1$ elements, for $N = 2\ell$ or $N = 2\ell - 1$): if $J \notin \mathcal{NE}$, then it follows that the corresponding operator P_J is elliptic (otherwise J would be contained in a maximal non-elliptic set, but they all have exactly $\ell - 1$ elements and are

contained in $\mathcal{NE}$). In order to show that Q^J is not a vertex of the polyhedron $\mathcal{P}$, it is enough to find for any set J defining an elliptic operator P_J, an index i such that $\pi_i(Q^J) < 0$.

For N odd, equation (2.21) implies that for each $i \notin J$, $\Pi_i(Q^J)$ is nonzero and its sign is:

$$\operatorname{sgn}(\pi_i(Q^J)) = (-1)^{i-1}\operatorname{sgn}(\prod_{j\in J}(\widetilde{w}_i^2 - \widetilde{w}_j^2)).$$

There are exactly $\ell - 1$ couples of the type $(s, N+2-s)$ and, since $J \notin \mathcal{NE}$ and has $\ell - 1$ elements, there exists at least one such couple not contained in J.
The ordering of the squares of the translated conformal weights, that can be directly checked by the formulas (2.10), is the following ($N = 2\ell - 1$):

$$\widetilde{w}_1^2 > \widetilde{w}_{N+1}^2 > \widetilde{w}_2^2 > \widetilde{w}_N^2 > \cdots > \widetilde{w}_i^2 > \widetilde{w}_{N+2-i}^2 > \cdots > \widetilde{w}_\ell^2 > \widetilde{w}_{N+2-\ell}^2.$$

It then follows that for a couple $(s, N+2-s)$, $\widetilde{w}_s^2$ and $\widetilde{w}_{N+2-s}^2$ are adjacent in this ordering, so that the following signs are the same:

$$\operatorname{sgn}(\prod_{j\in J}(\widetilde{w}_s^2 - \widetilde{w}_j^2)) = \operatorname{sgn}(\prod_{j\in J}(\widetilde{w}_{N+2-s}^2 - \widetilde{w}_j^2)).$$

Since N is odd, s and $N+2-s$ have different parity, showing that $\pi_s(Q^J)$ and $\pi_{N+2-s}(Q^J)$ have opposite signs.

For N even, the only difference is the way the sign chances when passing from $i = s$ to $i = N+2-s$: the parity of i remains the same, but the sign of the factor $\left(\widetilde{w}_j - \frac{1}{2}\right)$ in (2.22) changes.
We notice that the arguments in (2) are also valid in the exceptional case, since $\mathcal{NE}$ still contains all subsets defining maximal non-elliptic operators. Thus, the inclusion $\mathcal{V} \subset \{Q^J | \, J \in \mathcal{NE}\}$ holds in all cases. □

As remarked above, in order to compute the Kato constant given by (2.13) it suffices to minimize or maximize over the set of vertices of the polyhedron $\mathcal{P}$. The identification of these vertices provided by Proposition 2.21 and the fact that the explicit values of the norms $|\Pi_j(\alpha \otimes v)|^2$ at each vertex turn out to be easily computed prove the main result:

Theorem 2.22 (D. Calderbank, P. Gauduchon and M. Herzlich, [18])**.** *Let I be a subset of $\{1, \ldots, N\}$ corresponding to an injectively elliptic operator $P_I = \sum_{i\in I} \Pi_i \circ \nabla$ acting on sections of $V_\lambda M$. Then a refined Kato inequality* *holds: $|d|\varphi|| \leq k_I |\nabla\varphi|$, for any section $\varphi \in \ker(P_I)$, outside the zero set of φ. If N is odd, the Kato constant k_I is given by the following expressions:*

$$k_I^2 = \max_{J \in \mathcal{NE}} \left(\sum_{i \in \widehat{I} \cap \widehat{J}} \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right) = 1 - \min_{J \in \mathcal{NE}} \left(\sum_{i \in I \cap \widehat{J}} \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right). \tag{2.23}$$

If N is even, the Kato constant k_I is similarly given by:

$$\begin{aligned} k_I^2 &= \max_{J \in \mathcal{NE}} \left(\sum_{i \in \widehat{I} \cap \widehat{J}} \left(\widetilde{w}_i - \frac{1}{2} \right) \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right) \\ &= 1 - \min_{J \in \mathcal{NE}} \left(\sum_{i \in I \cap \widehat{J}} \left(\widetilde{w}_i - \frac{1}{2} \right) \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right). \end{aligned} \tag{2.24}$$

These Kato constants are optimal, unless in the exceptional case when n and N are odd, $N = 2\ell + 1$, λ is properly half-integral and the set J achieving the extremum contains $\ell + 1$.

Another completely different approach to the computation of optimal Kato constants was provided, independently, by Th. Branson, [15], whose proofs rely on powerful techniques of harmonic analysis. One may say that the method in [18] (whose main steps we gave in this section) is the local method, relying on algebraic considerations on the conformal weights and a linear programming problem. On the other hand, the method in [15] is a global one, using the spectral computation on the round sphere in [13] and a result relating the spectrum of an operator to information on its symbol. The advantage of the local method is that it provides an explicit description of the sections satisfying the equality case of the refined Kato inequality, while the advantage of the global method is that it is always sharp (also in the exceptional case).

2.3 A New Proof of Branson's Classification

The aim of this section is to give a new proof of Branson's classification of natural first order minimal elliptic operators, [13], stated here in our notation in Theorems 2.12 and 2.13. The tools we use are on the one hand the computation of the Kato constant provided by D. Calderbank, P. Gauduchon and M. Herzlich in [18], whose main steps are given in the previous section, § 2.2, and, on the other hand, the branching rules for the special orthogonal group. The main idea is that the argument in [18] may be, in a certain sense, reversed: while in [18] the task is to establish for each natural elliptic operator an explicit formula of its optimal Kato constant, assuming known the list of Branson of minimal elliptic operators, our goal is to analyze to which extend the computations of the Kato constants rely on this assumption of ellipticity and how Branson's list could be recovered. Our proof does not cover a special case, that is explained in Remark 2.29, but has the advantage of avoiding the powerful tools of harmonic analysis used in [13]. Being based mainly on representation theoretical arguments, this proof suggests that it should carry over to other subgroups G of $\mathrm{SO}(n)$, in order to provide the classification of natural elliptic operators constructed from G-generalized gradients, as pointed out in Remark 2.30.

The starting point is the following straightforward observation:

Lemma 2.23. *Let k_I be the optimal Kato constant for the operator P_I, which is given by $k_I = \sup\limits_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha \otimes v)|$ (see Lemma 2.17). Then it holds:*

$$k_I < 1 \iff P_I \text{ is an elliptic operator.}$$

Proof: If $|\alpha| = |v| = 1$, then $1 = |\alpha \otimes \varphi|^2 = |\Pi_I(\alpha \otimes \varphi)|^2 + |\Pi_{\widehat{I}}(\alpha \otimes \varphi)|^2$, so that k_I is always smaller or equal to 1. Then, by negation, the equivalence in the statement is the same as the following equivalence:

$$k_I = 1 \iff P_I \text{ is not elliptic,}$$

which is a consequence of the definitions: $k_I = 1$ if and only if there exist α and v of norm 1 such that $|\Pi_{\widehat{I}}(\alpha \otimes v)| = 1$, which is then the same as $|\Pi_I(\alpha \otimes \varphi)| = 0$, or, equivalently, $\alpha \otimes \varphi \in \ker(P_I)$, meaning that P_I is not elliptic. □

Lemma 2.23 implies that the ellipticity of a natural first order differential operator P_I follows from the computation of its optimal Kato constant k_I. Thus, as soon as we are able to compute explicitly k_I (without using the ellipticity assumption) or to show that k_I is strictly less than 1, it follows that the operator P_I is elliptic. In the sequel we show that k_I is strictly bounded from above by 1 for the operators in Branson's list, *i.e.*, in the notation given by the decreasing ordering of the translated conformal weights, for all operators enumerated in Remark 2.16, except the case 3., which corresponds to the zero weight.

We use the same notation as in § 2.2 and notice that for the construction of the convex region $\mathcal{P}$, as well as for establishing its compactness, the only ingredient needed is the ordering of the translated conformal weights, which is provided by the explicit formulas (2.10).

The key observation is that the only step in the proof of the main result in [18] (and stated here in Theorem 2.22) where the ellipticity of the operators was used, is in the identification of the vertices of the polyhedral region, namely in Proposition 2.21. If we now consider the same set $\mathcal{NE}$ introduced in § 2.2:

$$\mathcal{NE} = \{J \subset \{1,\ldots,N\} \,|\, |J \cap \{i, N+2-i\}| = 1, \text{for } 2 \leq i \leq \ell\},$$

where $N = 2\ell - 1$ or $N = 2\ell$, then one inclusion established in Proposition 2.21 still holds, without any ellipticity assumption on the operators. More precisely, we get:

Lemma 2.24. *The vertices of the polyhedron $\mathcal{P}$ are given by a subset of $\mathcal{NE}$.*

Proof: As in the proof of Proposition 2.21, we start with a subset J in $\{1,\ldots,N\}$ with $\ell - 1$ elements, such that $J \notin \mathcal{NE}$ and show that the corresponding point Q^J defined by (2.20) is not a vertex of $\mathcal{P}$. It is enough to find an element $i \in \{1,\ldots,N\}$ such that $\pi_i(Q^J) < 0$. The argument in the same as in the proof of Proposition 2.21: by the definition of the set $\mathcal{NE}$ we find a couple $(s, N+2-s)$, for some $2 \leq s \leq \ell$, that is outside J and from the ordering of the translated conformal weights we conclude that $\pi_s(Q^J)$ and $\pi_{N+2-s}(Q^J)$ are nonzero and have opposite signs. □

From the inclusion $\mathcal{V} \subset \mathcal{NE}$ given by Lemma 2.24, the formula (2.13) for the Kato constant k_I and the expressions (2.16) and (2.17) for the norms of the projections, we get the following upper bound:

$$k_I^2 = \max_{Q\in\mathcal{P}} \left(\sum_{j\in\widehat{I}} \pi_j(Q) \right) = \max_{Q\in\mathcal{V}} \left(\sum_{j\in\widehat{I}} \pi_j(Q) \right) \leq \max_{J\in\mathcal{NE}} \left(\sum_{j\in\widehat{I}} \pi_j(Q^J) \right) =: c_I. \tag{2.25}$$

Thus, if we show for a subset $I \subset \{1,\ldots,N\}$ that $c_I < 1$, then from (2.25) and Lemma 2.23 it follows that the corresponding operator P_I is elliptic.

We notice that the formulas for the optimal Kato constant in Theorem 2.22 actually compute the values of the upper bound c_I, if we do not assume the ellipticity of any operator involved. This straightforward, but important remark provides the main argument in the new proof of Branson's classification.

Applying now Theorem 2.22 in the special case when the set I has only one element or two elements of the form $\{i, N+2-i\}$, one recovers the list of minimal elliptic operators as follows.

Corollary 2.25. *The upper bound c_I is strictly smaller than 1 for any of the following subsets I:*

1. $I = \{1\}$;
2. $I = \{\ell+1\}$ *if* $N = 2\ell$ *and* $\lambda_m \neq 0$;
3. $I = \{i, N+2-i\}$ *for* $i = 2, \dots, \ell$.

From the above discussion it then follows that the corresponding operators P_I are elliptic.

Proof: By Theorem 2.22, the upper bound c_I is given by the following formula, if $N = 2\ell - 1$:

$$\begin{aligned} c_I &= \max_{J \in \mathcal{NE}} \left(\sum_{j \in \widehat{I}} \pi_j(Q^J) \right) = \max_{J \in \mathcal{NE}} \left(\sum_{i \in \widehat{I} \cap \widehat{J}} \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right) \\ &= 1 - \min_{J \in \mathcal{NE}} \left(\sum_{i \in I \cap \widehat{J}} \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right), \end{aligned} \tag{2.26}$$

and if $N = 2\ell$:

$$\begin{aligned} c_I &= \max_{J \in \mathcal{NE}} \left(\sum_{j \in \widehat{I}} \pi_j(Q^J) \right) = \max_{J \in \mathcal{NE}} \left(\sum_{i \in \widehat{I} \cap \widehat{J}} \left(\widetilde{w}_i - \frac{1}{2} \right) \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right) \\ &= 1 - \min_{J \in \mathcal{NE}} \left(\sum_{i \in I \cap \widehat{J}} \left(\widetilde{w}_i - \frac{1}{2} \right) \frac{\prod_{j \in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right). \end{aligned} \tag{2.27}$$

The last expressions in (2.26) and (2.27) are particularly simple if the set I has just a few elements, as it is in our case.

1. Substituting $I = \{1\}$ in (2.26) and (2.27), the sums reduce to one element, since $I \cap \widehat{J} = \{1\}$ for any $J \in \mathcal{NE}$, and we get:

$$c_{\{1\}} = 1 - \min_{J \in \mathcal{NE}} \left(\frac{\prod_{j \in J} (\widetilde{w}_1 + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{i\}} (\widetilde{w}_1 - \widetilde{w}_j)} \right), \quad \text{if } N = 2\ell - 1, \tag{2.28}$$

$$c_{\{1\}} = 1 - \min_{J \in \mathcal{NE}} \left(\left(\widetilde{w}_1 - \frac{1}{2} \right) \frac{\prod_{j \in J} (\widetilde{w}_1 + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{1\}} (\widetilde{w}_1 - \widetilde{w}_j)} \right), \quad \text{if } N = 2\ell, \tag{2.29}$$

which implies that $c_{\{1\}} < 1$, because $\widetilde{w}_1$ is the biggest translated conformal weight: $\widetilde{w}_1^2 > \widetilde{w}_j^2$, for any $2 \leq j \leq N$ and $\widetilde{w}_1 = \lambda_1 + \frac{n-1}{2} > \frac{1}{2}$ (we assume always $n \geq 2$ and $\lambda_1 \neq 0$, otherwise λ is just the trivial representation).

2. If the dimension n is odd, $n = 2m + 1$, the case $N = 2\ell$ can only occur if $\lambda_m = \frac{1}{2}$, as can be easily seen in the Diagram 2.1 which illustrates the selection rule (since in all the other cases the weights come in pairs). In this case, the index $\ell + 1$, given by the decreasing ordering of the translated conformal weights, stays for the weight 0. If $n = 2m$ and $N = 2\ell$, then from Diagram 2.2, it follows that the index $\ell + 1$ stays either for the weight $-\varepsilon_m$, if $\lambda_m > 0$, or for the weight ε_m, if $\lambda_m < 0$ (since again the indices are given by the decreasing ordering of the translated conformal weights and $\widetilde{w}_{m,+} - \widetilde{w}_{m,-} = 2\lambda_m$). Substituting $I = \{\ell + 1\}$ in (2.27) reduces again the sum to one element and yields the following expression:

$$c_{\{\ell+1\}} = 1 - \min_{J \in \mathcal{NE}} \left(\frac{\widetilde{w}_{\ell+1} - \frac{1}{2}}{\widetilde{w}_{\ell+1} - \widetilde{w}_1} \cdot \frac{\prod_{j \in J} (\widetilde{w}_{\ell+1} + \widetilde{w}_j)}{\prod_{j \in \widehat{J} \setminus \{1, \ell+1\}} (\widetilde{w}_{\ell+1} - \widetilde{w}_j)} \right). \tag{2.30}$$

From the explicit values of the translated conformal weights given by (2.10), namely: $\widetilde{w}_{m,-} = -\lambda_m + m - \frac{n-1}{2}$ and $\widetilde{w}_{m,+} = \lambda_m - m + \frac{n+1}{2}$, it follows that for $n = 2m + 1$, as well as for $n = 2m$, the term $\left(\widetilde{w}_{\ell+1} - \frac{1}{2}\right)$ is strictly negative, and thus $\frac{\widetilde{w}_{\ell+1} - \frac{1}{2}}{\widetilde{w}_{\ell+1} - \widetilde{w}_1}$ is strictly positive. From the way the sets $J \in \mathcal{NE}$ are defined, by choosing exactly one element from each pair $\{i, 2\ell + 2 - i\}$ for $2 \leq i \leq \ell$, it follows that in the product in (2.30), there occur only factors of one of the following two types: $\frac{\widetilde{w}_{\ell+1} + \widetilde{w}_i}{\widetilde{w}_{\ell+1} - \widetilde{w}_{2\ell+2-i}}$ or $\frac{\widetilde{w}_{\ell+1} + \widetilde{w}_{2\ell+2-i}}{\widetilde{w}_{\ell+1} - \widetilde{w}_i}$ for some $2 \leq i \leq \ell$. From the ordering of the translated conformal weights it turns out that each such factor is strictly positive, showing thus that $c_{\{\ell+1\}} < 1$.

3. The ordering of the translated conformal weights implies the following

inequalities, for any $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, \ell\}$ and $j \neq i, N+2-i$:

$$\begin{aligned}
\frac{\widetilde{w}_i + \widetilde{w}_j}{\widetilde{w}_i - \widetilde{w}_{N+2-j}} &> \frac{\widetilde{w}_i + \widetilde{w}_{N+2-j}}{\widetilde{w}_i - \widetilde{w}_j} > 0, \quad \textit{if } i < j \text{ or } N+2-j < i, \\
\frac{\widetilde{w}_i + \widetilde{w}_{N+2-j}}{\widetilde{w}_i - \widetilde{w}_j} &> \frac{\widetilde{w}_i + \widetilde{w}_j}{\widetilde{w}_i - \widetilde{w}_{N+2-j}} > 0, \quad \textit{if } j < i < N+2-j.
\end{aligned}$$

If $N = 2\ell - 1$, then substituting I in (2.26) with a set formed by a pair of type $I = \{i, N+2-i\}$, with $i \in \{2, \ldots, \ell\}$, and using the above relations yields the following expression for the upper bound of the Kato constant:

$$c_I = 1 - \min\left(\frac{\widetilde{w}_i + \widetilde{w}_{2\ell+1-i}}{\widetilde{w}_i - \widetilde{w}_1}, \frac{\widetilde{w}_i + \widetilde{w}_{2\ell+1-i}}{\widetilde{w}_{2\ell+1-i} - \widetilde{w}_1}\right).$$

Similarly, if $N = 2\ell$, then substituting $I = \{i, N+2-i\}$ in (2.27) yields:

$$c_I = 1 - \min\left(\frac{(\widetilde{w}_i + \widetilde{w}_{2\ell+2-i})(\widetilde{w}_i - \frac{1}{2})}{(\widetilde{w}_i - \widetilde{w}_{\ell+1})(\widetilde{w}_i - \widetilde{w}_1)}, \frac{(\widetilde{w}_i + \widetilde{w}_{2\ell+2-i})(\widetilde{w}_{2\ell+2-i} - \frac{1}{2})}{(\widetilde{w}_{2\ell+2-i} - \widetilde{w}_{\ell+1})(\widetilde{w}_{2\ell+2-i} - \widetilde{w}_1)}\right).$$

The same argument as in the case 2. shows that $c_I < 1$. □

Corollary 2.25 proves that all the operators that come up in Branson's classification, and that we listed in Remark 2.16 in our notation, are elliptic, except for one special case that we explain in Remark 2.29. But our aim is to determine *all* minimal elliptic operators, so that we still have to eliminate the other possibilities. Namely, on the one hand, we have to show that the generalized gradients corresponding to an element in one of the sets obtained in the case 3. of Corollary 2.25 are not elliptic, and on the other hand, that there are no other combinations which provide elliptic operators. Thus, we have to find the maximal non-elliptic operators, in order to conclude that the elliptic operators found in Corollary 2.25 are all the minimal elliptic operators. The main tool we use for this is the branching rule of the special orthogonal group and the following necessary condition for ellipticity (see also [18]):

Lemma 2.26. *Let* $P_I : \Gamma(V_\lambda) \to \Gamma(\bigoplus_{i \in I} V_i)$ *be the operator corresponding to a subset* $I \subset \{1, \ldots, N\}$*, in the notation introduced by* (2.11)*. If there exists an irreducible* $\mathrm{SO}(n-1)$*-subrepresentation of* V_λ *that does not occur as* $\mathrm{SO}(n-1)$*-subrepresentation of* V_i *for any* $i \in I$*, then* P_I *is not elliptic.*

Proof: By Definition 2.4, P_I is (injectively) elliptic if its principal symbol,

$\Pi_I : (\mathbb{R}^n)^* \otimes V_\lambda \to \bigoplus_{i \in I} V_i$, is injective when restricted to the set of decomposable elements, *i.e.* if for any vector $\alpha \in (\mathbb{R}^n)^*$, $\alpha \neq 0$, the linear map:

$$V_\lambda \to \bigoplus_{i \in I} V_i, \quad v \mapsto \Pi_I(\alpha \otimes v)$$

is injective. Since $\mathrm{SO}(n)$ acts transitively on the unit sphere in $(\mathbb{R}^n)^*$, one may, without loss of generality, take α to be a unit vector. Then, the above map is $\mathrm{SO}(n-1)$-equivariant, where $\mathrm{SO}(n-1)$ is the stabilizer group of α under the $\mathrm{SO}(n)$-action on the sphere. The existence of an injective and $\mathrm{SO}(n-1)$-equivariant map between V_λ and $\bigoplus_{i \in I} V_i$ shows that any $\mathrm{SO}(n-1)$-subrepresentation of V_λ occurs in V_i for some $i \in I$. □

In order to use Lemma 2.26 we have to apply the branching rule for the restriction of an $\mathrm{SO}(n)$-representation to $\mathrm{SO}(n-1)$, which we recall in the sequel (see *e.g.* Theorem 9.16, [45]). We consider, as usual, the parametrization of irreducible $\mathrm{SO}(n)$-representations by dominant weights, *i.e.* the weights satisfying the inequalities (1.1).

Proposition 2.27 (Branching Rule for $\mathrm{SO}(n)$)**.**

(a) *For the group* $\mathrm{SO}(2m+1)$*, the irreducible representation with highest weight* $\lambda = (\lambda_1, \dots, \lambda_m)$ *decomposes with multiplicity* 1 *under* $\mathrm{SO}(2m)$*, and the representations of* $\mathrm{SO}(2m)$ *that appear are exactly those with highest weights* $\gamma = (\gamma_1, \dots, \gamma_m)$ *such that*

$$\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \dots \geq \lambda_{m-1} \geq \gamma_{m-1} \geq \lambda_m \geq |\gamma_m|. \tag{2.31}$$

(b) *For the group* $\mathrm{SO}(2m)$*, the irreducible representation with highest weight* $\lambda = (\lambda_1, \dots, \lambda_m)$ *decomposes with multiplicity* 1 *under* $\mathrm{SO}(2m-1)$*, and the representations of* $\mathrm{SO}(2m-1)$ *that appear are exactly those with highest weights* $\gamma = (\gamma_1, \dots, \gamma_{m-1})$ *such that*

$$\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \dots \geq \lambda_{m-1} \geq \gamma_{m-1} \geq |\lambda_m|. \tag{2.32}$$

From Proposition 2.27 we obtain the maximal non-elliptic operators as follows:

Corollary 2.28. *The maximal non-elliptic operators* P_J *are given exactly by the sets* J *in* $\mathcal{NE}$*, apart from the special case when* n *is odd,* $N = 2\ell - 1$ *and* $\lambda_m \geq 1$*. In this case the sets* J *of* $\mathcal{NE}$ *that do not contain* ℓ *(which corresponds to the weight* 0*) are maximal non-elliptic.*

Proof: We recall that the coordinates of a dominant weight λ are given with respect to the basis $\{\varepsilon_i\}_{i=\overline{1,m}}$ introduced in § 1.1.1. Here it is more convenient to consider the elements of a set J as weights of the standard representation, instead of the notation with indices corresponding to the ordering of the translated conformal weights.

Let J be a subset in $\mathcal{NE}$, *i.e.* J has cardinality $\ell - 1$, where $N = 2\ell$ or $N = 2\ell - 1$. If $n = 2m$, then J is obtained by choosing exactly one weight from each pair of relevant weights of type $\{-\varepsilon_i, \varepsilon_{i+1}\}$, for $1 \leq i \leq m-2$ and one weight from $\{-\varepsilon_{m-1}, \varepsilon_m\}$, if $\lambda_m > 0$, or one weight from $\{-\varepsilon_{m-1}, -\varepsilon_m\}$, if $\lambda_m < 0$. If $n = 2m+1$, then we consider the sets $J \in \mathcal{NE}$ obtained by choosing exactly one weight from each pair of relevant weights of type $\{-\varepsilon_i, \varepsilon_{i+1}\}$, for $1 \leq i \leq m-1$ and the weight $-\varepsilon_m$, if it is relevant.

For each such set J, it is enough to find an $\mathrm{SO}(n-1)$-subrepresentation of V_λ that does not occur in $\bigoplus\limits_{\varepsilon \in J} V_{\lambda+\varepsilon}$. By Corollary 2.28 it will then follow that the corresponding operator P_J is not elliptic. When enlarging the set J to some set J' by adding any other relevant weight, there is at least one subset I of J' which is equal to one of those listed in Corollary 2.25, showing that J' is elliptic. This means that J is maximal non-elliptic.

For $n = 2m$ we choose the irreducible $\mathrm{SO}(2m-1)$-subrepresentation of λ with highest weight $\gamma = (\gamma_1, \ldots, \gamma_{m-1})$, where the coordinates are defined by the following rule, for each $1 \leq i \leq m-2$:

$$\gamma_i = \begin{cases} \lambda_i, & \text{if } \lambda_i = \lambda_{i+1} \text{ or } -\varepsilon_i \in J \\ \lambda_{i+1}, & \text{if } \varepsilon_{i+1} \in J, \end{cases} \tag{2.33}$$

and

$$\gamma_{m-1} = \begin{cases} \lambda_{m-1}, & \text{if } \lambda_{m-1} = \lambda_m = 0 \text{ or } -\varepsilon_{m-1} \in J \\ \lambda_m, & \text{if } \varepsilon_m \in J \text{ and } \lambda_m > 0 \\ -\lambda_m, & \text{if } -\varepsilon_m \in J \text{ and } \lambda_m < 0. \end{cases} \tag{2.34}$$

We recall that the condition $\lambda_i = \lambda_{i+1}$, for $1 \leq i \leq m-2$, is equivalent to the fact that the weights $\{-\varepsilon_i, \varepsilon_{i+1}\}$ are not relevant for λ and $\lambda_{m-1} = \lambda_m = 0$ is the only case when $-\varepsilon_{m-1}$ is not relevant (see *e.g.* Diagram 2.2). The coordinates of γ fulfill the inequalities (2.32) for the representation λ, showing that γ is an irreducible $\mathrm{SO}(2m-1)$-subrepresentation of λ. On the other hand, it can be directly checked that the inequalities (2.32) are not satisfied anymore for any of the $\mathrm{SO}(2m)$-representations of highest weight $\lambda + \varepsilon$ with

$\varepsilon \in J$, showing that γ does not occur as $\mathrm{SO}(2m-1)$-subrepresentation in $\oplus_{\varepsilon \in J} V_{\lambda+\varepsilon}$.

For $n = 2m+1$ we similarly choose an irreducible $\mathrm{SO}(2m)$-subrepresentation of λ with highest weight $\gamma = (\gamma_1, \dots, \gamma_m)$, whose coordinates are defined by the following rule, for each $1 \leq i \leq m-1$:

$$\gamma_i = \begin{cases} \lambda_i, & \text{if } \lambda_i = \lambda_{i+1} \text{ or } -\varepsilon_i \in J \\ \lambda_{i+1}, & \text{if } \varepsilon_{i+1} \in J, \end{cases} \tag{2.35}$$

and $\gamma_m = \lambda_m$. It follows also in this case that the inequalities (2.31) are fulfilled for λ, but fail for any $\lambda + \varepsilon$ with $\varepsilon \in J$. The branching rule then implies that γ is an irreducible $\mathrm{SO}(2m)$-subrepresentation of V_λ which does not occur as subrepresentation in $\oplus_{\varepsilon \in J} V_{\lambda+\varepsilon}$. □

Remark 2.29. From Corollary 2.25 and Corollary 2.28 we recover Branson's classification of minimal elliptic operators, up to an exceptional case. Namely, when n is odd, $N = 2\ell - 1$ and $\lambda_m > 0$, then the zero weight is relevant. If λ is moreover properly half-integral, then the corresponding operator $P_\ell : V_\lambda M \to V_\lambda M$ is elliptic (by Branson's result), while if λ is integral, P_ℓ is not elliptic. Unfortunately this special case cannot be recovered by the above arguments, since they only involve the translated conformal weights, which are associated to the Lie algebra $\mathfrak{so}(n)$, so that they do not distinguish between the groups $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$. The argument based on the branching rule for establishing the maximal non-elliptic operators does not work either for the zero weight, since in this case the source and target representations are isomorphic.

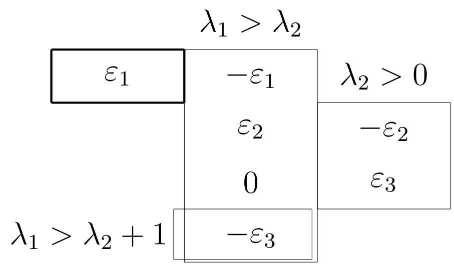
Remark 2.30. This new approach to the classification of minimal elliptic operators has the advantage that it is mainly based on representation theory and avoids the techniques of harmonic analysis which, as powerful as they are, seem to be particular for the special orthogonal group. Our hope is that this method can be carried over to the subgroups G in (1.6), in order to provide a similar classification of the minimal elliptic operators obtained from G-generalized gradients. We notice that the argument in Lemma 2.26 still works for any of these groups G (because they occur as holonomy groups, which are known to act transitively on the unit sphere) and combined with the branching rules of the groups involved yield a list of non-elliptic operators. This is work in progress and until now we only have partial results, particularly for the group G_2.

In the sequel we present our partial results regarding the ellipticity of G_2-generalized gradients. The group G_2 is the smallest of the exceptional Lie groups. It is a compact, simple and simply-connected group, defined as the automorphism group of the octonions algebra $\mathbb{O}$. Hence, we have the inclusion $G_2 \hookrightarrow \mathrm{SO}(7)$, where $\mathrm{SO}(7)$ is identified with the rotation group of the imaginary octonions. Alternatively, G_2 may be defined as the isotropy group of a 3-form of general type in $\Lambda^3(\mathbb{R}^7)$ or as the isotropy group of a real $\mathrm{Spin}(7)$-spinor. The Lie algebra $\mathfrak{g}_2$ is 14-dimensional and has rank 2.

We denote by T the complexification of the defining representation of G_2 (or, equivalently, of $\mathfrak{g}_2$) given by the restriction of the standard $\mathrm{SO}(7)$-representation. Thus, T is a 7-dimensional representation and it has too many weights to be orthonormal for any scalar product on the dual $\mathfrak{h}^*$ of a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2$. However, one can choose an ordering of the weights in $\mathfrak{h}^*$, so that the weights of T are totally ordered: $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > 0 > -\varepsilon_3 > -\varepsilon_2 > -\varepsilon_1$. In this notation, the fundamental weights of $\mathfrak{g}_2$ are given by: $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$; and the last weight of T is expressed as follows: $\varepsilon_3 = -\omega_2 + 2\omega_1$.

Any irreducible G_2-representation is parametrized by a dominant weight, λ, which is expressed as a positive integral combination of the fundamental weights: $\lambda = a\omega_1 + b\omega_2$, with $a, b \geq 0$, or, equivalently, with respect to the weights of T as: $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2$, with integers $\lambda_1 \geq \lambda_2 \geq 0$. The selection rule given by Theorem 1.7 for the decomposition of $T \otimes V_\lambda$ into irreducible G_2-representations may be visualized in following diagram, which is similar to the ones for the special orthogonal group (see Diagrams 2.1 and 2.2) and is found as well in [63]:

Diagram 2.3: Selection Rule for G_2



Let (M, g) be a Riemannian manifold carrying a G_2-structure and consider the G_2-generalized gradients (in a simplified notation: $P_\varepsilon := P_\varepsilon^{\nabla^\lambda} = \Pi_\varepsilon \circ \nabla^\lambda$) acting on sections of the associated vector bundle $V_\lambda M$, as defined by (1.17), where ∇ is any G_2-connection.

We now consider Question 2.7 for the group G_2. Namely, we ask for which subsets I of the set $\{\pm\varepsilon_i, 0\}_{i=1,3}$ of weights of the representation T is the operator $D_I := \sum_{\varepsilon\in I} P_\varepsilon^* P_\varepsilon$ elliptic. By Lemma 2.8, which holds for any group G, this question is equivalent to asking when the first order differential operator $P_I := \sum_{\varepsilon\in I} P_\varepsilon$ is (injectively) elliptic. The interesting cases are, as for $\mathrm{SO}(n)$, the minimal elliptic operators and the maximal non-elliptic operators.

We recall that the method used in our proof of Branson's classification for the case of $\mathrm{SO}(n)$ consists of two steps: first we proved that the Kato constant of the operators in Branson's list is strictly smaller then 1, which yields their ellipticity, and then we established the set of maximal non-elliptic operators, which implies that the operators in the list are *all* minimal elliptic operators. Our purpose is to use the same method for the group G_2. The results we have until now give a partial answer for the second step. Namely, we provide a list of non-elliptic operators, but it is still open if it contains all maximal non-elliptic operators. We obtain this list using the following necessary criterion for ellipticity, which is the analogue of Lemma 2.26:

Lemma 2.31. *Let $P_I : \Gamma(V_\lambda) \to \Gamma(\bigoplus_{\varepsilon\in I} V_\varepsilon)$ be the operator corresponding to a subset $I \subset \{\pm\varepsilon_i, 0\}_{i=1,3}$. If there exists an irreducible $\mathrm{SU}(3)$-subrepresentation of V_λ that does not occur as $\mathrm{SU}(3)$-subrepresentation of any V_ε, with $\varepsilon \in I$, then P_I is not elliptic.*

The proof of Lemma 2.31 is based on the same argument as for Lemma 2.26, which is straightforwardly carried over to the case of G_2. We only need to observe that the group G_2 acts transitively on the 6-dimensional sphere in $\mathbb{R}^7$ and the isotropy group of a point on the sphere is $\mathrm{SU}(3)$.

In order to apply Lemma 2.31, we need the branching rule for restricting a representation of G_2 to $\mathrm{SU}(3)$. This rule was established, for instance, by R. King and A. Qubanchi, [38], and can be formulated as follows:

Proposition 2.32. *(Branching Rule for G_2 to $\mathrm{SU}(3)$, [38]) The multiplicity of the $\mathrm{SU}(3)$-irreducible representation of highest weight $(\nu_1 \geq \nu_2 \geq 0)$ in the G_2-irreducible representation of highest weight $(\lambda_1 \geq \lambda_2 \geq 0)$ is given by:*

$$C^{(\lambda_1,\lambda_2)}_{(\nu_1,\nu_2)} = 1 + \min(\lambda_1-\lambda_2, \lambda_2, \nu_1-\nu_2, \nu_2, \lambda_1+\lambda_2-\nu_1, \lambda_1-\nu_1+\nu_2, \lambda_1-\nu_2, -\lambda_2+\nu_1), \tag{2.36}$$

where the minimum is taken only if all elements are positive; otherwise one takes -1, so that the coefficient is equal to 0.

Lemma 2.31 and Proposition 2.32 imply as follows that the differential operators P_I given by the subsets $I = \{\varepsilon_2, -\varepsilon_3\}$, respectively $I = \{-\varepsilon_1, -\varepsilon_2, -\varepsilon_3\}$, are not elliptic.

Let $I = \{\varepsilon_2, -\varepsilon_3\}$. It is enough to show that for any G_2-irreducible representation V_λ of highest weight ($\lambda_1 \geq \lambda_2 \geq 0$), there exists an SU(3)-irreducible representation of highest weight (ν_1, ν_2) which occurs in the decomposition of V_λ, but does not occur in the decompositions of $V_{\lambda+\varepsilon_2}$ and $V_{\lambda-\varepsilon_3}$. We check directly by (2.36) that the SU(3)-irreducible representation with highest weight $(\lambda_2, 0)$ fulfills these conditions, as follows:

$$\begin{aligned}
C^{(\lambda_1,\lambda_2)}_{(\lambda_2,0)} &= 1 + \min(\lambda_1 - \lambda_2, \lambda_2, \lambda_1, 0) \geq 1,\\
C^{(\lambda_1,\lambda_2+1)}_{(\lambda_2,0)} &= 1 + \min(\lambda_1 - \lambda_2 - 1, \lambda_2, 0, \lambda_1, -1) = 0,\\
C^{(\lambda_1-1,\lambda_2+1)}_{(\lambda_2,0)} &= 1 + \min(\lambda_1 - \lambda_2 - 2, \lambda_2, 0, \lambda_1 - 1, -1) = 0.
\end{aligned}$$

Let $I = \{-\varepsilon_1, -\varepsilon_2, -\varepsilon_3\}$. Similarly, we check that for the G_2-irreducible representation V_λ of highest weight ($\lambda_1 \geq \lambda_2 \geq 0$), there exists the SU(3)-irreducible representation of highest weight $(\lambda_1 + \lambda_2, \lambda_1)$ with multiplicity greater or equal to 1 in the decomposition of V_λ, but which does not occur in the decompositions of $V_{\lambda-\varepsilon_1}$, $V_{\lambda-\varepsilon_2}$ and $V_{\lambda-\varepsilon_3}$. Namely, it holds:

$$\begin{aligned}
C^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2,\lambda_1)} &= 1 + \min(\lambda_1 - \lambda_2, \lambda_2, \lambda_1, 0) \geq 1,\\
C^{(\lambda_1-1,\lambda_2)}_{(\lambda_1+\lambda_2,\lambda_1)} &= 1 + \min(\lambda_1 - \lambda_2 - 1, \lambda_2, \lambda_1, -1) = 0,\\
C^{(\lambda_1,\lambda_2-1)}_{(\lambda_1+\lambda_2,\lambda_1)} &= 1 + \min(\lambda_1 - \lambda_2, \lambda_2 - 1, \lambda_1, -1) = 0,\\
C^{(\lambda_1-1,\lambda_2+1)}_{(\lambda_1+\lambda_2,\lambda_1)} &= 1 + \min(\lambda_1 - \lambda_2 - 2, \lambda_2, \lambda_1 - 1, -1) = 0.
\end{aligned}$$

The following table summarizes the known results on the ellipticity of the operators of type P_I contructed from G_2-generalized gradients:

Singletons		Pairs		Triples	
$\{-\varepsilon_1\}$	non-elliptic	$\{-\varepsilon_1, -\varepsilon_2\}$	non-elliptic	$\{-\varepsilon_1, -\varepsilon_2, -\varepsilon_3\}$	non-elliptic
$\{\varepsilon_2\}$	non-elliptic	$\{-\varepsilon_1, -\varepsilon_3\}$	non-elliptic		
$\{-\varepsilon_2\}$	non-elliptic	$\{-\varepsilon_2, -\varepsilon_3\}$	non-elliptic		
$\{-\varepsilon_3\}$	non-elliptic	$\{\varepsilon_2, -\varepsilon_3\}$	non-elliptic		
$\{\varepsilon_1\}$	strongly elliptic				

We recall that the top gradient given by the weight ε_1 and corresponding to the projection onto the Cartan summand is, by Proposition 2.9, strongly elliptic. The other sets in the table, providing non-elliptic operators, follow from the two sets I considered above and from the remark that if P_I is non-elliptic, then all the operators corresponding to subsets of I are non-elliptic as well. We notice that there are two singletons for which the ellipticity question is still open, namely ε_3 and 0. For the weight 0, the source and target of the operator are both equal to V_λ, so that Lemma 2.31 cannot be applied. For the weight ε_3, it follows from a dimension argument (using Weyl's dimension formula) that if $\lambda_1 > 2\lambda_2 - 1$, then $\dim(V_\lambda) > \dim(V_{\lambda+\varepsilon_3})$, showing that in these cases the G_2-generalized gradient P_{ε_3} is not elliptic.

Part II

Kählerian Twistor Spinors

Chapter 3

Preliminaries: Spin Geometry on Kähler Manifolds

The purpose of this chapter is to fix the notations and to recall some results of spin geometry on Kähler manifolds that we shall need in the sequel.

3.1 The Decomposition of the Spinor Bundle

Let (M, g, J) be a Kähler manifold of real dimension $n = 2m$ with Riemannian metric g, complex structure J and Kähler form $\Omega = g(J\cdot, \cdot)$. The tangent and cotangent bundle are identified using the metric g. In the sequel $\{e_i\}_{i=\overline{1,n}}$ always denotes a local orthonormal frame and, where we do not write sums, we implicitly use the Einstein summation convention over repeated indices. The complexified tangent bundle splits into the $(\pm i)$-eigenbundles of the complex structure: $\mathrm{TM}^{\mathbb{C}} = \mathrm{TM}^{1,0} \oplus \mathrm{TM}^{0,1}$ and we denote the components of a vector field X with respect to this splitting as follows:

$$X^+ = \frac{1}{2}(X - iJX) \in \Gamma(\mathrm{TM}^{1,0}), \quad X^- = \frac{1}{2}(X + iJX) \in \Gamma(\mathrm{TM}^{0,1}).$$

We now assume on M the existence of a spin structure. On Kähler manifolds this is equivalent to the existence of a square root of the canonical bundle $K = \Lambda^{(m,0)}M$, *i.e.* a holomorphic line bundle L such that $K \cong L \otimes L$ (see [29]).

Let $\mathrm{Spin}_g M$ be the $\mathrm{Spin}(2m)$-principal bundle of the spin structure and denote by ΣM the associated spinor bundle: $\Sigma M = \mathrm{Spin}_g M \times_{\mathrm{Spin}(2m)} \Sigma$, where Σ is the 2^m-dimensional complex spin representation of $\mathrm{Spin}(2m)$. ΣM is a complex Hermitian vector bundle and its sections are called *spinor fields* (or shortly *spinors*).

The Clifford contraction $c : \mathrm{T^*M} \otimes \Sigma M \to \Sigma M$ is defined on each fiber by the Clifford multiplication on the spinor representation Σ. On decomposable elements we have $c(X \otimes \varphi) = X \cdot \varphi$. It is extended to a multiplication with k-forms. Each k-form α acts as an endomorphism of the spinor bundle, which is locally given by:

$$\alpha \cdot \varphi = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2m} \alpha(e_{i_1}, \ldots, e_{i_k}) e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \varphi.$$

Consider now the Clifford multiplication with the complex volume form (with the orientation given by the complex structure): $\omega^{\mathbb{C}} = i^m \prod_{i=1}^m e_i \cdot Je_i$. Since the dimension is even, $\omega^{\mathbb{C}}$ has the eigenvalues $+1$ and -1 and the eigenspaces are inequivalent complex irreducible representations of $\mathrm{Spin}(2m)$ denoted by:

$$\Sigma = \Sigma^+ \oplus \Sigma^-. \tag{3.1}$$

As an endomorphism of the spinor bundle, the Kähler form is given locally by:

$$\Omega = \frac{1}{2} \sum_{j=1}^{n} e_j \cdot Je_j. \tag{3.2}$$

By a straightforward computation, it follows:

Lemma 3.1. *Under the action of the Kähler form Ω, the spinor bundle splits into the orthogonal sum of holomorphic subbundles:*

$$\Sigma M = \overset{m}{\underset{r=0}{\oplus}} \Sigma_r M, \tag{3.3}$$

where each $\Sigma_r M$ is the eigenbundle of Ω corresponding to the eigenvalue $i\mu_r = i(2r - m)$ and $\mathrm{rank}_{\mathbb{C}}(\Sigma_r M) = \binom{m}{r}$.

This decomposition corresponds to the one for $(0, *)$-forms in $(0, r)$-forms on M, if we consider the so-called *Hitchin representation* (see [29]) of the spin bundle of any almost Hermitian manifold: $\Sigma M \cong L \otimes \Lambda^{0,*}$, where L is the square root of the canonical bundle K of M determined by the spinorial structure.

Comparing the decomposition (3.1) with the finer one (3.3) we have:

$$\Sigma^+ M = \bigoplus_{\substack{0\leq r\leq 2m \\ r \text{ even}}} \Sigma_r M, \quad \Sigma^- M = \bigoplus_{\substack{0\leq r\leq 2m \\ r \text{ odd}}} \Sigma_r M.$$

On the spinor bundle ΣM there is a canonical $\mathbb{C}$-anti-linear real (resp. quaternionic) structure, $\mathfrak{j} : \Sigma M \to \Sigma M$, such that $\mathfrak{j}^2 = (-1)^{\frac{m(m+1)}{2}}$. The following property of $\mathfrak{j}$ holds:

$$\mathfrak{j} : \Sigma_r M \to \Sigma_{m-r} M, \quad \mathfrak{j}(Z \cdot \varphi) = \bar{Z} \cdot \mathfrak{j}(\varphi), \quad \text{for all } Z \in \Gamma(\mathrm{TM}^{\mathbb{C}}).$$

3.2 The Dirac Operator and Estimates for Its Eigenvalues

The Levi-Civita connection ∇ on TM induces a covariant derivative on ΣM, which we also denote by ∇. Since the Kähler form is parallel, ∇ preserves the splitting (3.3).

The *Dirac operator* is defined as the composition

$$\Gamma(\Sigma M) \xrightarrow{\nabla} \Gamma(\mathrm{T}^*\mathrm{M} \otimes \Sigma M) \xrightarrow{c} \Gamma(\Sigma M), \quad D = c \circ \nabla. \tag{3.4}$$

Explicitly D is locally given by

$$D = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}. \tag{3.5}$$

Associated with the complex structure J there is another "square root of the Laplacian", locally defined by

$$D^c = \sum_{j=1}^{n} J e_j \cdot \nabla_{e_j}.$$

D^c is also an elliptic self-adjoint operator and it follows easily that

$$(D^c)^2 = D^2 \quad \text{and} \quad DD^c + D^c D = 0.$$

Define now the two operators

$$D^+ = \frac{1}{2}(D - iD^c) = \sum_{j=1}^{n} e_j^+ \cdot \nabla_{e_j^-}, \quad D^- = \frac{1}{2}(D + iD^c) = \sum_{j=1}^{n} e_j^- \cdot \nabla_{e_j^+}, \tag{3.6}$$

which satisfy the relations

$$D = D^+ + D^-, \quad (D^+)^2 = 0, \quad (D^-)^2 = 0, \quad D^+D^- + D^-D^+ = D^2. \tag{3.7}$$

When restricting the Dirac operator to $\Sigma_r M$, it acts as follows:

$$D = D^- + D^+ : \Gamma(\Sigma_r M) \to \Gamma(\Sigma_{r-1} M) \oplus \Gamma(\Sigma_{r+1} M),$$

because of the following result which can be checked by straightforward computation.

Lemma 3.2. *For any tangent vector field X and $r \in \{0, \ldots, m\}$ one has*

$$X^+ \cdot \Sigma_r M \subseteq \Sigma_{r+1} M \quad X^- \cdot \Sigma_r M \subseteq \Sigma_{r-1} M, \tag{3.8}$$

with the convention that $\Sigma_{-1} M = \Sigma_{m+1} M = M \times \{0\}$. Thus, if we denote by c_r the restriction of the Clifford contraction to $\mathrm{T^*M} \otimes \Sigma_r M$, then c_r splits as follows:

$$c_r = c_r^- \oplus c_r^+ : \mathrm{TM} \otimes \Sigma_r M \to \Sigma_{r-1} M \oplus \Sigma_{r+1} M.$$

One of the main tools for the study of the Dirac operator is the Schrödinger-Lichnerowicz formula:

$$D^2 = \nabla^* \nabla + \frac{1}{4} S, \tag{3.9}$$

where $\nabla^* \nabla$ is the Laplacian on the spinor bundle and S is the scalar curvature of M.

Let us recall here for later use the lower bounds for the spectrum of the Dirac operator on Riemannian and Kähler manifolds. The first such inequality was obtained by Th. Friedrich, [20], who showed that on an n-dimensional compact Riemannian spin manifold (M, g) each eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S. \tag{3.10}$$

Of course, this inequality gives new information only if the scalar curvature is positive, in which case we denote the smallest possible eigenvalue by $\lambda_0 = \sqrt{\frac{n}{4(n-1)} \inf_M S}$. In [20] it is shown that a limiting manifold for (3.10) is characterized by the existence of a special spinor. More precisely, we have:

Theorem 3.3. *Let (M, g) be a Riemannian manifold which admits an eigenspinor φ of the Dirac operator D with the smallest eigenvalue λ_0. Then the manifold is Einstein and φ is a Killing spinor for the Killing constant $-\frac{\lambda_0}{n}$,* i.e. *satisfies the equation*

$$\nabla_X \varphi = -\frac{\lambda_0}{n} X \cdot \varphi, \tag{3.11}$$

for all vector fields X on M. Conversely, if φ is a nontrivial spinor on M satisfying the equation (3.11) *for some real constant, then g is an Einstein metric and (M, g) is a limiting manifold for* (3.10), *φ being an eigenspinor of D of the smallest eigenvalue λ_0.*

The complete simply connected Riemannian manifolds carrying real Killing spinors have been described by Ch. Bär, [5]. The main tool in his proof is the cone construction. He shows that Killing spinors correspond to fixed points of the holonomy group of the cone and then uses the Berger-Simons classification of possible holonomy groups.

On Kähler manifolds the inequality (3.10) is always strict since a Kähler manifold does not admit Killing spinors. It was improved by K.-D. Kirchberg, who showed that each eigenvalue λ of the Dirac operator on a $2m$-dimensional compact spin Kähler manifold (M, g, J) satisfies

$$\lambda^2 \geq \frac{m+1}{4m} \inf_M S, \quad \text{if } m \text{ is odd}, \tag{3.12}$$

$$\lambda^2 \geq \frac{m}{4(m-1)} \inf_M S, \quad \text{if } m \text{ is even}. \tag{3.13}$$

Again we can only get new information about the eigenvalues from these inequalities if the scalar curvature is positive. In this case we denote the smallest possible eigenvalues by $\lambda_0^{odd} := \sqrt{\frac{m+1}{4m}\inf_M S}$ and $\lambda_0^{even} := \sqrt{\frac{m}{4(m-1)}\inf_M S}$.

The limiting manifolds of Kirchberg's inequalities are also characterized by the existence of spinors satisfying a certain differential equation. More precisely, K.-D. Kirchberg, [43], and O. Hijazi, [28], proved:

Theorem 3.4. *Let (M, g, J) be a compact Kähler spin manifold of complex dimension $m = 2l + 1$ which admits an eigenspinor φ of D corresponding to the smallest eigenvalue λ_0^{odd}. Then the metric g is Einstein and the spinor $\varphi = \varphi_l + \varphi_{l+1} \in \Gamma(\Sigma_l M \oplus \Sigma_{l+1} M)$ is a Kählerian Killing spinor for the Killing constant $-\frac{\lambda_0^{odd}}{m+1}$,* i.e. *its components satisfy the equations:*

$$\begin{aligned} \nabla_X \varphi_l &= -\frac{\lambda_0^{odd}}{m+1} X^- \cdot \varphi_{l+1}, \\ \nabla_X \varphi_{l+1} &= -\frac{\lambda_0^{odd}}{m+1} X^+ \cdot \varphi_l, \end{aligned} \tag{3.14}$$

for any vector field X. Conversely, if $\varphi = \varphi_l + \varphi_{l+1} \in \Gamma(\Sigma_l M \oplus \Sigma_{l+1} M)$ is a spinor on M satisfying the equations (3.14) *for some real constant, then g is an Einstein metric and (M, g, J) is a limiting manifold for* (3.12)*, φ being an eigenspinor of D corresponding to the smallest eigenvalue λ_0^{odd}.*

In the case of even complex dimension, the characterization of limiting manifolds in terms of special spinors has been given by K.-D.Kirchberg, [41], and in the following stronger version by P. Gauduchon, [25].

Theorem 3.5. *Let (M, g, J) be a compact Kähler spin manifold of complex dimension $m = 2l \geq 4$ which admits an eigenspinor φ of D to the smallest eigenvalue λ_0^{even}. Then the manifold has constant scalar curvature and $\varphi = \varphi_{l-1} + \mathfrak{j}(\varphi'_{l-1})$, where $\varphi_{l-1}, \varphi'_{l-1} \in \Gamma(\Sigma_{l-1} M)$ are spinors satisfying the following equation:*

$$\nabla_X \varphi_{l-1} = -\frac{1}{m} X^- \cdot D\varphi_{l-1}, \tag{3.15}$$

for any vector field X. Conversely, if $\varphi = \varphi_{l-1} + \mathfrak{j}(\varphi'_{l-1}) \in \Gamma(\Sigma_{l-1} M \oplus \Sigma_{l+1} M)$ is a spinor on M such that $\varphi_{l-1}, \varphi'_{l-1}$ satisfy the equation (3.15) *and if there exists a real nonzero constant λ such that $D^2 \varphi = \lambda^2 \varphi$, then the manifold has constant scalar curvature and is a limiting manifold for* (3.13)*, λ being equal to λ_0^{even}.*

These limiting manifolds have been classified by A. Moroianu in [51] for m odd and in [54] for m even. In the even complex dimension, the result was conjectured by A. Lichnerowicz, [48], who proved it under the assumption that the Ricci tensor is parallel.

Theorem 3.6. *The only limiting manifold for* (3.12) *in complex dimension* $4k+1$ *is the complex projective space* $\mathbb{C}P^{4k+1}$. *In complex dimension* $4k+3$ *the limiting manifolds are exactly the twistor spaces over quaternionic Kähler manifolds of positive scalar curvature.*

Theorem 3.7. *A Kähler manifold* M *of even complex dimension* $m \geq 4$ *is a limiting manifold for* (3.13) *if and only if its universal cover is isometric to a Riemannian product* $N \times \mathbb{R}^2$, *where* N *is a limiting manifold for the odd complex dimension* $m-1$ *and* M *is the suspension over a flat parallelogram of two commuting isometries of* N *preserving a Kählerian Killing spinor.*

Chapter 4

Kählerian Twistor Spinors

In this chapter we introduce the main objects, the Kählerian twistor spinors. We show that these spinors are in one-to-one correspondence to parallel sections of a connection called the Kählerian twistor connection, which we construct explicitly. Moreover, we compute the curvature of the Kählerian twistor connection and obtain formulas that are the starting point for the classification of Kähler spin manifolds carrying Kählerian twistor spinors.

4.1 Twistor Operators

Natural first order differential operators acting on sections of an associated vector bundle E over the manifold M are given by the composition of projections onto irreducible components of the tensor product $\mathrm{T}^*\mathrm{M} \otimes E$ with a covariant derivative on E, as we have seen in § 1.1. Then the principal symbol of the operator is the projection defining it. There is always a distinguished projection onto the so-called *Cartan summand*, whose highest weight is exactly the sum of the highest weights of the representations defining the bundles $\mathrm{T}^*\mathrm{M}$ and E.

Consider now the spinor bundle ΣM over a Riemannian spin manifold (M, g). Since the tensor product $\mathrm{TM} \otimes \Sigma M$ splits as $\mathrm{Spin}(n)$-representation as follows:

$$\mathrm{TM} \otimes \Sigma M \cong \Sigma M \oplus \ker(c),$$

we get two first order differential operators: the Dirac operator (see (3.4)), given by the projection (which is identified with the Clifford contraction) of the covariant derivative onto ΣM and the complementary operator given by the projection of the covariant derivative onto the Cartan summand $\ker(c)$.

In order to define the projections we need to consider an embedding of ΣM into the tensor product $\mathrm{TM}\otimes\Sigma M$, *i.e.* the right inverse of the contraction c, $\iota:\Sigma M\to \mathrm{TM}\otimes\Sigma M$, such that $c\circ\iota = id_{\Sigma M}$, which is given locally as follows:

$$\iota(\varphi) = -\frac{1}{n}\sum_{j=1}^{n} e_j\otimes e_j\cdot\varphi.$$

The *Riemannian twistor (Penrose) operator* is then defined by

$$T:\Sigma M\to\ker c,\quad T\varphi = \nabla\varphi + \frac{1}{n}e_j\otimes e_j\cdot D\varphi,$$

or, more explicitly, when applied to a vector field X:

$$T_X\varphi = \nabla_X\varphi + \frac{1}{n}X\cdot D\varphi. \tag{4.1}$$

Definition 4.1. Let (M,g) be a Riemannian manifold. A spinor $\varphi\in\Gamma(\Sigma M)$ is called *Riemannian twistor spinor* if it belongs to the kernel of the Riemannian twistor operator, *i.e.* if it satisfies the differential equation

$$\nabla_X\varphi = -\frac{1}{n}X\cdot D\varphi, \tag{4.2}$$

for all vector fields X.

Let us now consider the case of Kähler manifolds. It was proven in [42] by K.-D. Kirchberg that on a Kähler manifold with nonzero scalar curvature, the space of Riemannian twistor spinors is trivial (a different proof of this result is also given by O. Hijazi, [28], and for compact Kähler manifolds this vanishing result is due to A. Lichnerowicz, [49]). Thus, it is natural to consider a twistor operator adapted to the Kähler structure, by looking at the decomposition of each tensor product of the vector bundles $\mathrm{TM}\otimes\Sigma_r M$ (for $r=0,\dots,m$) into irreducible components under the action of the unitary group $\mathrm{U}(m)$. There are three irreducible summands:

$$\mathrm{TM}\otimes\Sigma_r M\cong\Sigma_{r-1}M\oplus\Sigma_{r+1}M\oplus\ker(c_r), \tag{4.3}$$

where c_r is the restriction of the Clifford contraction to $\Sigma_r M$ as in Lemma 3.2. Thus, there are three first order differential operators: the first two projections are given by c_r^-, respectively c_r^+, and the third one is the projection onto the Cartan summand, $\ker c_r$. As in the Riemannian case we need the two embeddings, which are locally given as follows:

$$\iota_r^- : \Sigma_{r-1}M \to \mathrm{TM} \otimes \Sigma_r M, \quad \iota_r^-(\varphi) = -\frac{1}{2(m-r+1)} \sum_{j=1}^n e_j \otimes e_j^+ \cdot \varphi,$$

$$\iota_r^+ : \Sigma_{r+1}M \to \mathrm{TM} \otimes \Sigma_r M, \quad \iota_r^+(\varphi) = -\frac{1}{2(r+1)} \sum_{j=1}^n e_j \otimes e_j^- \cdot \varphi.$$

The *Kählerian twistor (Penrose) operator* is defined as the projection of the covariant derivative onto the Cartan summand: $T_r : \Gamma(\Sigma_r M) \to \Gamma(\ker(c_r))$,

$$T_r\varphi = \nabla\varphi + \frac{1}{2(m-r+1)} e_j \otimes e_j^+ \cdot D^-\varphi + \frac{1}{2(r+1)} e_j \otimes e_j^- \cdot D^+\varphi,$$

or, more explicitly, when applied to a vector field X:

$$(T_r)_X\varphi = \nabla_X\varphi + \frac{1}{2(m-r+1)} X^+ \cdot D^-\varphi + \frac{1}{2(r+1)} X^- \cdot D^+\varphi. \tag{4.4}$$

The Kählerian twistor operator has already introduced *e.g.* by P. Gauduchon, [25]. Different approaches have been considered by O. Hijazi, [28], and by K.-D. Kirchberg, [41]. In Remark 4.8 we discuss the relationship between the various definitions of a twistor spinor adapted to the Kähler structure.

Definition 4.2. Let (M, g, J) be a Kähler manifold. A spinor $\varphi \in \Gamma(\Sigma_r M)$ is called *Kählerian twistor spinor* if it belongs to the kernel of the Kählerian twistor operator, *i.e.* if it satisfies the differential equations

$$\begin{cases} \nabla_{X^+}\varphi = -\frac{1}{2(m-r+1)} X^+ \cdot D^-\varphi, \\ \nabla_{X^-}\varphi = -\frac{1}{2(r+1)} X^- \cdot D^+\varphi, \end{cases} \tag{4.5}$$

for all vector fields X.

We shall denote by $\mathcal{KT}(r)$ the space of Kählerian twistor spinors in $\Sigma_r M$. It follows immediately from the defining equations (4.5) that the real (or quaternionic) structure of the spinor bundle, $\mathfrak{j}$, preserves these spaces:

$$\mathfrak{j} : \mathcal{KT}(r) \xrightarrow{\sim} \mathcal{KT}(m-r).$$

Thus, it is sufficient to study $\mathcal{KT}(r)$ for $0 \le r \le \frac{m}{2}$.

A special class of spinors in $\mathcal{KT}(r)$ are the special Kählerian twistor spinors defined as follows:

Definition 4.3. A Kählerian twistor spinor $\varphi \in \Gamma(\Sigma_r M)$ is *holomorphic*, respectively *anti-holomorphic Kählerian twistor spinor* if $D^+\varphi = 0$, respectively $D^-\varphi = 0$.

We denote the space of holomorphic and anti-holomorphic Kählerian twistor spinors in $\Sigma_r M$ by $\mathcal{HKT}(r)$, respectively $\mathcal{AKT}(r)$, and notice that they are interchanged by j:

$$\mathrm{j} : \mathcal{AKT}(r) \xrightarrow{\sim} \mathcal{HKT}(m-r). \tag{4.6}$$

Parallel spinors are of course the simplest examples of a twistor spinor of any kind, because, by definition, all components of the covariant derivative vanish. However, this condition is very restrictive, since parallel spinors only exist on Ricci-flat Kähler manifolds.

Directly from the decomposition (4.3) and using the embeddings ι_r^+ and ι_r^- we get the following Weitzenböck formula, relating the differential operators acting on sections of $\Sigma_r M$:

$$\nabla^*\nabla = \frac{1}{2(r+1)} D^- D^+ + \frac{1}{2(m-r+1)} D^+ D^- + T_r^* T_r. \tag{4.7}$$

This is just a special case of a general Weitzenböck formula (see § 1.3) which expresses the rough Laplacian $\nabla^*\nabla$ acting on an associated vector bundle E as the sum of all T^*T with T first order differential operator given by the projections of a covariant derivative onto the irreducible components of the tensor product $\mathrm{TM} \otimes E$.

Remark 4.4 (Relationship to the estimates of the eigenvalues of the Dirac operator). By Theorem 3.3, each eigenspinor corresponding to the smallest eigenvalue of the Dirac operator on a Riemannian manifold is a Killing spinor, thus in particular a twistor spinor. Moreover, the eigenspinors to the smallest eigenvalue are exactly the eigenspinors of D which are twistor spinors.

Similarly, on a Kähler manifold Theorems 3.4 and 3.5 imply that every eigenspinor of the Dirac operator corresponding to the smallest eigenvalue is a sum of two special Kählerian twistor spinors: if m is odd, then (3.14) implies that $\varphi \in \mathcal{AKT}(\frac{m-1}{2}) \oplus \mathcal{HKT}(\frac{m+1}{2})$ and if m is even, then (3.15) implies that

$\varphi \in \mathcal{AKT}(\frac{m}{2}-1) \oplus \mathcal{HKT}(\frac{m}{2}+1)$. Moreover, eigenspinors corresponding to the smallest eigenvalue are exactly the eigenspinors of D which are Kählerian twistor spinors. The geometric description of the limiting Kähler manifolds (Theorems 3.6 and 3.7) provides the first examples of manifolds admitting Kählerian twistor spinors. Thus, Kählerian twistor spinors may be seen as a generalization of these special spinors which naturally appear in the limiting case for the lower bound of the spectrum of the Dirac operator.

We now show that special Kählerian twistor spinors on a Kähler manifold of positive scalar curvature are exactly the eigenspinors with the smallest eigenvalue of the square of the Dirac operator restricted to an irreducible subbundle $\Sigma_r M$. This result has been proven by K.-D. Kirchberg, [41], and by O. Hijazi, [28]. Here we follow the argument given by P. Gauduchon, [25], and by U. Semmelmann, [60].

Lemma 4.5. *Let $\varphi \in \Gamma(\Sigma_r M)$. Then the following inequality holds*

$$|\nabla\varphi|^2 \geq \frac{1}{2(r+1)}|D^+\varphi|^2 + \frac{1}{2(m-r+1)}|D^-\varphi|^2, \tag{4.8}$$

with equality if and only if φ is a Kählerian twistor spinor ($T_r\varphi = 0$).

Proof: The statement of the lemma is a direct consequence of the following relation:

$$|\nabla\varphi|^2 = \frac{1}{2(r+1)}|D^+\varphi|^2 + \frac{1}{2(m-r+1)}|D^-\varphi|^2 + |T_r\varphi|^2,$$

which in turn is implied by the following equalities that are straightforward from the definition of the embeddings $\iota_r^{\pm}$:

$$|\iota_r^+(\varphi_{r+1})|^2 = \frac{1}{2(r+1)}|\varphi_{r+1}|^2, \quad |\iota_r^-(\varphi_{r-1})|^2 = \frac{1}{2(m-r+1)}|\varphi_{r-1}|^2,$$

$$\nabla\varphi = \iota_r^+(D^+\varphi) + \iota_r^-(D^-\varphi) + T_r\varphi.$$

□

The Lichnerowicz formula (3.9) yields the following:

Lemma 4.6. *Let φ be an eigenspinor of D^2, $D^2\varphi = \lambda\varphi$ that satisfies the inequality*

$$|\nabla\varphi|^2 \geq \frac{1}{k}|D\varphi|^2. \tag{4.9}$$

Then the following inequality holds:

$$\lambda \geq \frac{k}{k-1}\frac{1}{4}\inf_M S$$

and equality is attained if and only if S is constant and equality in (4.9) *holds at all points of the manifold.*

Proposition 4.7. *Let (M, g, J) be a Kähler manifold of positive scalar curvature. Then any eigenvalue λ of D^2 on $\Sigma_r M$ satisfies:*

$$\lambda \geq \frac{2(r+1)}{2r+1}\frac{1}{4}\inf_M S, \quad \text{if } r \leq \frac{m}{2} \tag{4.10}$$

and

$$\lambda \geq \frac{2(m-r+1)}{2m-2r+1}\frac{1}{4}\inf_M S, \quad \text{if } r > \frac{m}{2}. \tag{4.11}$$

Equality is attained if and only if the scalar curvature is constant and the corresponding eigenspinor is an anti-holomorphic (holomorphic) Kählerian twistor spinor if $r \leq \frac{m}{2}$ ($r > \frac{m}{2}$).

Proof: Let $\varphi \in \Gamma(\Sigma_r M)$ with $D^2\varphi = \lambda\varphi$. We distinguish two cases.

I. If $D^-\varphi = 0$, then $|D\varphi|^2 = |D^+\varphi|^2$ and (4.8) implies:

$$|\nabla\varphi|^2 \geq \frac{1}{2(r+1)}|D^+\varphi|^2 = \frac{1}{2(r+1)}|D\varphi|^2.$$

Applying Lemma 4.6, it follows that $\lambda \geq \frac{2(r+1)}{2r+1}\frac{1}{4}\inf_M S$.

II. If $D^-\varphi \neq 0$, then we apply the same argument for $\varphi^- := D^-\varphi \in \Gamma(\Sigma_{r-1}M)$ with $D^2\varphi^- = \lambda\varphi^-$. Then $D^-\varphi^- = 0$, so that $|D\varphi^-|^2 = |D^+\varphi^-|^2$ and from (4.8) it follows:

$$|\nabla\varphi^-|^2 \geq \frac{1}{2r}|D^+\varphi^-|^2 = \frac{1}{2r}|D\varphi^-|^2.$$

Applying again Lemma 4.6, it follows that $\lambda \geq \frac{2r}{2r-1}\frac{1}{4}\inf_M S > \frac{2(r+1)}{2r+1}\frac{1}{4}\inf_M S$.

The same argument applied to the cases when $D^+\varphi = 0$ and $D^+\varphi \neq 0$ shows that $\lambda \geq \frac{2(m-r+1)}{2m-2r+1}\frac{1}{4}\inf_M S$. If $r \leq \frac{m}{2}$, then $\frac{2(m-r+1)}{2m-2r+1}\frac{1}{4}\inf_M S \leq \frac{2(r+1)}{2r+1}\frac{1}{4}\inf_M S$ and thus follows (4.10). Similarly for $r > \frac{m}{2}$ it follows (4.11). The equality case follows from Lemmas 4.5 and 4.6. $\square$

Remark 4.8 (Relationship to other notions of Kählerian twistor spinors). The term "Kählerian twistor spinor" has already been used in the literature. A class of spinors with this name has been introduced by K.-D. Kirchberg, [41], and by O. Hijazi, [28]. We explain here the relationship between Definition 4.2 and these definitions.

In [41], K.-D. Kirchberg defined a Kählerian twistor spinor of type r (for $1 \leq r \leq m$) to be a spinor $\varphi \in \Gamma(\Sigma M)$ satisfying the equation

$$\nabla_X \varphi = -\frac{1}{4r}(X \cdot D\varphi + JX \cdot D^c\varphi). \tag{4.12}$$

He showed that a solution of (4.12) must lie in $\Gamma(\Sigma_{r-1}M \oplus \Sigma_{m-r+1}M)$. By rewriting this equation using the operators D^+ and D^-:

$$\begin{cases} \nabla_{X^+}\varphi = -\frac{1}{2r}X^+ \cdot D^-\varphi, \\ \nabla_{X^-}\varphi = -\frac{1}{2r}X^- \cdot D^+\varphi, \end{cases}$$

it follows that the spinors satisfying (4.12) are exactly the anti-holomorphic and holomorphic Kählerian twistor spinors in $\Sigma_{r-1}M$, respectively $\Sigma_{m-r+1}M$, see Definition 4.3. K.-D. Kirchberg, [42], further proved some vanishing results for these spinors, which we shall also obtain in § 5.3 (for Kähler-Einstein manifolds) and § 5.2 (for Kähler manifolds of constant scalar curvature) as a special case.

In [28], O. Hijazi considered as defining equation for a spinor $\varphi \in \Gamma(\Sigma M)$ the following slightly more general equation than (4.12):

$$\nabla_X \varphi = aX \cdot D\varphi + bJX \cdot D^c\varphi, \tag{4.13}$$

where a and b are any real numbers. For example, if $a = -\frac{1}{n}$ and $b = 0$, then a solution of (4.13) is a Riemannian twistor spinor (as (4.2) shows). Furthermore O. Hijazi proved that on a Kähler spin manifold with nonzero scalar curvature there exists a nontrivial solution of (4.13) if and only if $a = b = -\frac{1}{4(r+1)}$, for some integer r with $0 \leq r \leq m-2$, thus reducing equation (4.13) to (4.12). For $r = m-1$ it is proven ([28, Theorem 4.30] and [41, Proposition 11, Theorem 17]) that the solutions of the equation (4.13) are exactly the Riemannian twistor spinors and they are all trivial on a Kähler spin manifold of nonzero scalar curvature.

In order to better compare these definitions we notice that, using the operators D and D^c, the defining equation (4.5) for Kählerian twistor spinors can be rewritten as follows:

$$\begin{aligned}\nabla_X\varphi = &- \frac{m+2}{8(r+1)(m-r+1)}(X\cdot D\varphi + JX\cdot D^c\varphi)\\ &- \frac{m-2r}{8(r+1)(m-r+1)}i(JX\cdot D\varphi - X\cdot D^c\varphi).\end{aligned}$$

An important property of the special Kählerian twistor spinors noticed by K.-D. Kirchberg, [42], is that they are eigenspinors of the square of the Dirac operator, if we assume the scalar curvature to be constant. This is implied by the Lichnerowicz formula as follows.
Let $\varphi \in \mathcal{AKT}(r)$: $\nabla_X\varphi = -\frac{1}{2(r+1)}X^-\cdot D^+\varphi$. By differentiating once this defining equation and then contracting, we get

$$\nabla_{e_j}\nabla_{e_j}\varphi = -\frac{1}{2(r+1)}e_j^-\cdot\nabla_{e_j}D^+\varphi,$$

where $\{e_j\}_{j=\overline{1,n}}$ is a local orthonormal frame parallel at the point where the computations are made. Since $D^-\varphi = 0$, it follows

$$\nabla^*\nabla\varphi = \frac{1}{2(r+1)}D^-D^+\varphi = \frac{1}{2(r+1)}D^2\varphi,$$

which together with the Lichnerowicz formula (3.9) yields

$$D^2\varphi = \frac{r+1}{2(2r+1)}S\varphi. \tag{4.14}$$

Thus, if the scalar curvature S is constant, then φ is an eigenspinor of D^2. Similarly, if $\varphi \in \mathcal{HKT}(r)$, then we get $D^2\varphi = \frac{m-r+1}{2(2m-2r+1)}S\varphi$.

4.2 Particular Cases

We first look at extremal cases of Kählerian twistor spinors, *i.e.* of highest and lowest type and notice that they are always special Kählerian twistor spinors. Let $\varphi \in \Gamma(\Sigma_r M)$: if $r = 0$, then $D^-\varphi$ vanishes automatically and if $r = m$, then $D^+\varphi = 0$. Thus φ is an anti-holomorphic, respectively holomorphic Kählerian twistor spinor. Moreover, as shown by K.-D. Kirchberg, they are exactly the holomorphic, respectively anti-holomorphic sections in $\Sigma_0 M$, respectively $\Sigma_m M$ ([41, Theorem 12], with the remark that we use different conventions, namely S_0 in [41] corresponds to $\Sigma_m M$ in our notations):

$$\mathcal{KT}(0) = \mathcal{AKT}(0) = \bar{H}^0(M, \Sigma_0 M) = \jmath H^0(M, K^{\frac{1}{2}}),$$

$$\mathcal{KT}(m) = \mathcal{HKT}(m) = H^0(M, \Sigma_m M) = H^0(M, K^{\frac{1}{2}}).$$

Another special case is the middle of the dimension, when m is even and $r = \frac{m}{2}$. This is the only case when the coefficients of the defining equations (4.5) of a Kählerian twistor spinor are equal:

$$\nabla_X \varphi = -\frac{1}{m+2}(X^+ \cdot D^- \varphi + X^- \cdot D^+ \varphi). \tag{4.15}$$

We show that on a compact Kähler spin manifold of positive constant scalar curvature there does not exist any nontrivial solution of this twistorial equation, which means that there are no Kählerian twistor spinors in the middle dimension.

By differentiating and then contracting (4.15) we obtain

$$\nabla_{e_j} \nabla_{e_j} \varphi = -\frac{1}{m+2}(e_j^+ \cdot \nabla_{e_j^+} D^- \varphi + e_j^- \cdot \nabla_{e_j^-} D^+ \varphi),$$

where $\{e_j\}_{j=\overline{1,n}}$ is an orthonormal frame parallel at the point where the computation is made. Thus, it follows that

$$\nabla^* \nabla \varphi = \frac{1}{m+2}(D^- D^+ \varphi + D^- D^+ \varphi) = \frac{1}{m+2} D^2 \varphi,$$

which together with the Lichnerowicz formula (3.9) implies

$$D^2 \varphi = \frac{m+2}{4(m+1)} S \varphi,$$

showing that if the scalar curvature is constant, then φ is an eigenspinor with the eigenvalue $\lambda_0^{even} = \frac{m+2}{4(m+1)} S$, which is strictly smaller than $\frac{m}{4(m-1)} S$. This value is the lower bound given by Kirchberg's inequality (3.13) for m even. Thus φ must be zero.
If $S = 0$, then from the above relations we have $\nabla^* \nabla \varphi = D^2 \varphi = 0$, so that φ is a parallel spinor if the manifold M is compact.

4.3 The Kählerian Twistor Connection

The purpose of this section is to construct for each r (from now on we fix an r with $0 < r < m$ and $r \neq \frac{m}{2}$) a vector bundle endowed with a connection, called *Kählerian twistor connection*, such that Kählerian twistor spinors in $\Sigma_r M$ are in one-to-one correspondence to parallel sections of this connection. This allows us to conclude for instance, that the space of Kählerian twistor spinors is finite dimensional. The curvature of this connection provides useful formulas for computations with Kählerian twistor spinors, needed in § 5 to describe geometrically the Kähler manifolds admitting such spinors.

The idea of constructing a larger vector bundle with a suitable connection such that solutions of a certain equation correspond to parallel sections has often appeared in the literature. For example for Riemannian twistor spinors this construction was done by Th. Friedrich, [21], and for conformal Killing forms by U. Semmelmann, [61].

By the definition of a Kählerian twistor spinor, the covariant derivative of φ involves $\varphi^+ := D^+\varphi$ and $\varphi^- := D^-\varphi$. Hence, the first step will be the computation of the covariant derivatives of these sections, which yields an expression involving only zero-order terms and $D^2\varphi$. Then we compute the covariant derivative of $D^2\varphi$ and get an expression involving zero-order terms and the sections φ^+ and φ^-, showing that the system closes and thus defines a connection. More precisely, if we denote by $\hat{\varphi}$ the section $(\varphi, \varphi^+, \varphi^-, D^2\varphi)$ in $\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M \oplus \Sigma_r M$, then we have $\nabla_X \hat{\varphi} = B(X)\hat{\varphi}$, where $B(X)$ is a certain 4×4-matrix whose coefficients are endomorphisms of the spinor bundle, depending on the vector field X. The Kählerian twistor connection is then a connection in the bundle $\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M \oplus \Sigma_r M$, defined as $\hat{\nabla}_X = \nabla_X - B(X)$ and the Kählerian twistor spinors are given by the first component of parallel sections of $\hat{\nabla}$.

Let φ be a Kählerian twistor spinor in $\Sigma_r M$. First we derive some formulas relating the second order differential operators D^+D^-, D^-D^+ and D^2, when applied to a Kählerian twistor spinor. Since we asssumed that $r \neq m/2$, the system formed by the Weitzenböck formula (4.7) (using the fact that, by definition, $T_r\varphi = 0$) and the Lichnerowicz formula (3.9) can be inversed and we get the following relations:

$$D^+D^-\varphi = -\frac{(2r+1)(m-r+1)}{m-2r} D^2\varphi + \frac{(r+1)(m-r+1)}{2(m-2r)} S\varphi, \qquad (4.16)$$

$$D^-D^+\varphi = \frac{(2m-2r+1)(r+1)}{m-2r} D^2\varphi - \frac{(r+1)(m-r+1)}{2(m-2r)} S\varphi, \qquad (4.17)$$

$$D^+D^-\varphi = -\frac{(2r+1)(m-r+1)}{(2m-2r+1)(r+1)}D^-D^+\varphi + \frac{m-r+1}{2(2m-2r+1)}S\varphi, \quad (4.18)$$

$$D^-D^+\varphi = -\frac{(2m-2r+1)(r+1)}{(2r+1)(m-r+1)}D^+D^-\varphi + \frac{r+1}{2(2r+1)}S\varphi. \quad (4.19)$$

We now compute the covariant derivatives of $D^+\varphi$ and $D^-\varphi$ in the direction of a vector field X which is parallel at the point where the computations are done. The local orthonormal frame $\{e_j\}_{j=\overline{1,n}}$ is parallel at this point too.

$$\begin{aligned}
\nabla_{X^+}(D^+\varphi) &\overset{(A.6)}{=} D^+(\nabla_{X^+}\varphi) - \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi \\
&= -\frac{1}{2(m-r+1)}D^+(X^+\cdot D^-\varphi) - \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi \\
&\overset{(A.3)}{=} \frac{1}{2(m-r+1)}X^+\cdot D^+D^-\varphi - \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi \\
&\overset{(4.16)}{=} -\frac{2r+1}{2(m-2r)}X^+\cdot D^2\varphi + \frac{r+1}{4(m-2r)}SX^+\cdot\varphi - \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi.
\end{aligned}$$

$$\begin{aligned}
\nabla_{X^-}(D^+\varphi) &\overset{(A.7)}{=} D^+(\nabla_{X^-}\varphi) = -\frac{1}{2(r+1)}D^+(X^-\cdot D^+\varphi) \\
&\overset{(A.3)}{=} \frac{1}{2(r+1)}[X^-\cdot D^+(D^+\varphi) + 2\nabla_{X^-}(D^+\varphi)] = \frac{1}{r+1}\nabla_{X^-}(D^+\varphi),
\end{aligned}$$

so that $\nabla_{X^-}(D^+\varphi) = 0$.

These two equations give the second row of the connection in (4.25). Similarly, for the covariant derivative of $D^-\varphi$ we have

$$\begin{aligned}
&\nabla_{X^-}(D^-\varphi) = \frac{2m-2r+1}{2(m-2r)}X^-\cdot D^2\varphi - \frac{m-r+1}{4(m-2r)}SX^-\cdot\varphi - \frac{1}{2}\mathrm{Ric}(X^-)\cdot\varphi, \\
&\nabla_{X^+}(D^-\varphi) = 0,
\end{aligned}$$

which yield the third row of the connection in (4.25).

For the last component of the connection we compute the covariant derivative of $D^2\varphi$.

$$\nabla_X(D^2\varphi) \overset{(A.10)}{=} D^2(\nabla_X\varphi) - \frac{1}{2}D(\mathrm{Ric})(X)\cdot\varphi + \nabla_{\mathrm{Ric}(X)}\varphi - e_j\cdot e_i\cdot\nabla_{\nabla_{e_j}\nabla_{e_i}X}\varphi. \quad (4.20)$$

We now compute separately the terms appearing in (4.20). The first one is:

$$D^2(\nabla_X\varphi)=-\frac{1}{2(m-r+1)}D^2(X^+\cdot D^-\varphi)-\frac{1}{2(r+1)}D^2(X^-\cdot D^+\varphi), \quad (4.21)$$

$$\begin{aligned} D^2(X^+\cdot D^-\varphi) &\overset{(A.8)}{=} X^+\cdot D^2 D^-\varphi-\mathrm{Ric}(X^+)\cdot D^-\varphi+D^2(X^+)\cdot D^-\varphi \\ &\overset{(3.7)}{=} X^+\cdot D^-D^+D^-\varphi-\mathrm{Ric}(X^+)\cdot D^-\varphi+D^2(X^+)\cdot D^-\varphi \\ &\overset{(4.18)}{=} \frac{m-r+1}{2(2m-2r+1)}X^+\cdot D^-(S\varphi)-\mathrm{Ric}(X^+)\cdot D^-\varphi+D^2(X^+)\cdot D^-\varphi \\ &= \frac{m-r+1}{2(2m-2r+1)}SX^+\cdot D^-\varphi+\frac{m-r+1}{2(2m-2r+1)}X^+\cdot\partial(S)\cdot\varphi \\ &\quad -\mathrm{Ric}(X^+)\cdot D^-\varphi+D^2(X^+)\cdot D^-\varphi \\ &= \frac{m-r+1}{2(2m-2r+1)}SX^+\cdot D^-\varphi-\frac{m-r+1}{2(2m-2r+1)}\partial(S)\cdot X^+\cdot\varphi \\ &\quad -\frac{m-r+1}{2m-2r+1}X^+(S)\varphi-\mathrm{Ric}(X^+)\cdot D^-\varphi+D^2(X^+)\cdot D^-\varphi, \end{aligned}$$

$$\begin{aligned} D^2(X^-\cdot D^+\varphi) &= \frac{r+1}{2(2r+1)}SX^-\cdot D^+\varphi-\frac{r+1}{2(2r+1)}\bar\partial(S)\cdot X^-\cdot\varphi \\ &\quad -\frac{r+1}{2r+1}X^-(S)\varphi-\mathrm{Ric}(X^-)\cdot D^+\varphi+D^2(X^-)\cdot D^+\varphi. \end{aligned}$$

Replacing these terms in (4.21) it follows

$$\begin{aligned} D^2(\nabla_X\varphi) &= -\frac{1}{4(2m-2r+1)}SX^+\cdot D^-\varphi+\frac{1}{4(2m-2r+1)}\partial(S)\cdot X^+\cdot\varphi \\ &\quad +\frac{1}{2(2m-2r+1)}X^+(S)\varphi+\frac{1}{2(m-r+1)}\mathrm{Ric}(X^+)\cdot D^-\varphi \\ &\quad -\frac{1}{4(2r+1)}SX^-\cdot D^+\varphi+\frac{1}{4(2r+1)}\bar\partial(S)\cdot X^-\cdot\varphi \qquad (4.22) \\ &\quad +\frac{1}{2(2r+1)}X^-(S)\varphi+\frac{1}{2(r+1)}\mathrm{Ric}(X^-)\cdot D^+\varphi \\ &\quad -\frac{1}{2(m-r+1)}D^2(X^+)\cdot D^-\varphi-\frac{1}{2(r+1)}D^2(X^-)\cdot D^+\varphi. \end{aligned}$$

The third term in (4.20) is given by

$$\begin{aligned} &\nabla_{\mathrm{Ric}(X)}\varphi=\nabla_{\mathrm{Ric}(X^+)}\varphi+\nabla_{\mathrm{Ric}(X^-)}\varphi \\ &=-\frac{1}{2(m-r+1)}\mathrm{Ric}(X^+)\cdot D^-\varphi-\frac{1}{2(r+1)}\mathrm{Ric}(X^-)\cdot D^+\varphi \end{aligned} \qquad (4.23)$$

and the last term in (4.20) is

$$\begin{aligned} e_j \cdot e_i \cdot \nabla_{\nabla_{e_j}\nabla_{e_i}X}\varphi &= -\frac{1}{2(m-r+1)} e_j \cdot e_i \cdot \nabla_{e_j}\nabla_{e_i}X^+ \cdot D^-\varphi \\ &\quad -\frac{1}{2(r+1)} e_j \cdot e_i \cdot \nabla_{e_j}\nabla_{e_i}X^- \cdot D^+\varphi \qquad (4.24) \\ &= -\frac{1}{2(m-r+1)} D^2(X^+)\cdot D^-\varphi - \frac{1}{2(r+1)} D^2(X^-)\cdot D^+\varphi. \end{aligned}$$

Substituting the formulas (4.22), (4.23) and (4.24) in (4.20), we get the following equality, which yields the last row of the connection matrix (4.25):

$$\begin{aligned} \nabla_X(D^2\varphi) = &-\frac{1}{2} D(\mathrm{Ric})(X)\cdot\varphi + \frac{1}{2(2m-2r+1)} X^+(S)\varphi + \frac{1}{2(2r+1)} X^-(S)\varphi \\ &+ \frac{1}{4(2r+1)} \bar{\partial}(S)\cdot X^- \cdot \varphi + \frac{1}{4(2m-2r+1)} \partial(S)\cdot X^+ \cdot \varphi \\ &- \frac{1}{4(2r+1)} SX^- \cdot D^+\varphi - \frac{1}{4(2m-2r+1)} SX^+ \cdot D^-\varphi. \end{aligned}$$

Hence, we have shown that if $\varphi \in \Gamma(\Sigma_r M)$ is a Kählerian twistor spinor, then the four-tuple $(\varphi, \varphi^+ := D^+\varphi, \varphi^- := D^-\varphi, D^2\varphi)$ is parallel with respect to the following connection, denoted by $\hat{\nabla}$, which we call *Kählerian twistor connection*:

$$\begin{pmatrix} \nabla_X & \frac{1}{2(r+1)}X^-\cdot & \frac{1}{2(m-r+1)}X^+\cdot & 0 \\ -\frac{r+1}{4(m-2r)}SX^+\cdot + \frac{1}{2}\mathrm{Ric}(X^+)\cdot & \nabla_X & 0 & \frac{2r+1}{2(m-2r)}X^+\cdot \\ \frac{m-r+1}{4(m-2r)}SX^-\cdot + \frac{1}{2}\mathrm{Ric}(X^-)\cdot & 0 & \nabla_X & -\frac{2m-2r+1}{2(m-2r)}X^-\cdot \\ A(X) & \frac{1}{4(2r+1)}SX^-\cdot & \frac{1}{4(2m-2r+1)}SX^+\cdot & \nabla_X \end{pmatrix}, \qquad (4.25)$$

where we denote by $A(X)$ the following endomorphism of the spinor bundle:

$$\begin{aligned} A(X) :=& \frac{1}{2} D(\mathrm{Ric})(X) - \frac{1}{4(2m-2r+1)} (dS)^- \cdot X^+ - \frac{1}{4(2r+1)} (dS)^+ \cdot X^- \\ &- \frac{1}{2(2m-2r+1)} X^+(S) - \frac{1}{2(2r+1)} X^-(S). \end{aligned}$$

Moreover, it follows that any parallel section is of the form $(\varphi, \varphi^+, \varphi^-, D^2\varphi)$ with φ a Kählerian twistor spinor:

Proposition 4.9. *There is a one-to-one correspondence between Kählerian twistor spinors in $\Sigma_r M$ and the parallel sections of the vector bundle $\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M \oplus \Sigma_r M$ with respect to the connection $\hat{\nabla}$ given by* (4.25). *The explicit bijection is given by $\varphi \mapsto \hat{\varphi} = (\varphi, \varphi^+, \varphi^-, D^2\varphi)$.*

Proof: In the above discussion we have seen that the function $\varphi \mapsto \hat{\varphi}$ takes values in the space of parallel sections with respect to the connection (4.25) and it is obviously injective. Thus we only need to prove its surjectivity.

Let $(\varphi, \psi, \xi, \eta) \in \Gamma(\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M \oplus \Sigma_r M)$ be a parallel section with respect to (4.25). Since the first row of this connection is exactly the Kählerian twistor operator, it follows by contractions that the first three components are $(\varphi, \varphi^+, \varphi^-)$, where φ is a Kählerian twistor spinor. From

$$\nabla_{X^-}\varphi = -\frac{1}{2(r+1)} X^- \cdot \psi$$

we get by contraction that $\psi = D^-\varphi$. Similarly, from

$$\nabla_{X^-}\varphi = -\frac{1}{2(m-r+1)} X^+ \cdot \xi$$

we get $\xi = D^+\varphi$. Substituting now ψ and ξ in the first row yields that φ is a Kählerian twistor spinor. For the last component of the four-tuple we may compare for example the second row of the connection matrix applied to the parallel sections $(\varphi, \psi = \varphi^+, \xi = \varphi^-, \eta)$ and $(\varphi, \varphi^+, \varphi^-, D^2\varphi)$ obtaining:

$$X^+ \cdot \eta = X^+ \cdot D^2\varphi,$$

which contracted yields $\eta = D^2\varphi$. □

If the manifold (M, g, J) is Kähler-Einstein, then $\mathrm{Ric}(X) = \frac{S}{n} X$ and the Kählerian twistor connection $\hat{\nabla}$ simplifies as follows:

$$\hat{\nabla}_X = \begin{pmatrix} \nabla_X & \frac{1}{2(r+1)} X^- \cdot & \frac{1}{2(m-r+1)} X^+ \cdot & 0 \\ -\frac{r(m+2)}{4m(m-2r)} S X^+ \cdot & \nabla_X & 0 & \frac{2r+1}{2(m-2r)} X^+ \cdot \\ \frac{(m-r)(m+2)}{4m(m-2r)} S X^- \cdot & 0 & \nabla_X & -\frac{2m-2r+1}{2(m-2r)} X^- \cdot \\ 0 & \frac{1}{4(2r+1)} S X^- \cdot & \frac{1}{4(2m-2r+1)} S X^+ \cdot & \nabla_X \end{pmatrix}. \tag{4.26}$$

4.4 The Curvature of the Kählerian Twistor Connection

In this section we compute the curvature of the Kählerian twistor connection. The first component of this curvature allows us to reduce the Kählerian twistor connection $\hat{\nabla}$ to one acting on a bundle of smaller rank, namely on $\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M$, which is given by the matrix (4.28).

Let $\varphi \in \Gamma(\Sigma_r M)$ be a Kählerian twistor spinor. As $\hat{\varphi} = (\varphi, \varphi^+, \varphi^-, \eta := D^2\varphi)$ is a parallel section of $\hat{\nabla}$ (see Proposition 4.9), then by definition the curvature of this connection vanishes on this section: $\hat{R}_{X,Y}(\hat{\varphi}) = 0$, for any vector fields X and Y. Thus, computing the components of $\hat{R}$ we get certain identities which by further contractions yield the formulas in Proposition 4.12.

The first component of $\hat{R}$ is computed as follows.

$$
\begin{aligned}
\hat{R}_{X,Y}\begin{pmatrix}\varphi\\ \varphi^+\\ \varphi^-\\ \eta\end{pmatrix}_1 &= \nabla_X\left(\nabla_Y\varphi + \frac{1}{2(r+1)}Y^-\cdot\varphi^+ + \frac{1}{2(m-r+1)}Y^+\cdot\varphi^-\right)\\
&+ \frac{1}{2(r+1)}X^-\cdot\left(-\frac{r+1}{4(m-2r)}SY^+\cdot\varphi + \frac{1}{2}\mathrm{Ric}(Y^+)\cdot\varphi + \nabla_Y\varphi^+\right)\\
&+ \frac{1}{2(m-r+1)}X^+\cdot\left(\frac{m-r+1}{4(m-2r)}SY^-\cdot\varphi + \frac{1}{2}\mathrm{Ric}(Y^-)\cdot\varphi + \nabla_Y\varphi^-\right)\\
&+ \frac{1}{4(m-2r)}\left(\frac{2r+1}{r+1}X^-\cdot Y^+ - \frac{2m-2r+1}{m-r+1}X^+\cdot Y^-\right)\cdot\eta\\
&- \nabla_Y\left(\nabla_X\varphi + \frac{1}{2(r+1)}X^-\cdot\varphi^+ + \frac{1}{2(m-r+1)}X^+\cdot\varphi^-\right)\\
&- \frac{1}{2(r+1)}Y^-\cdot\left(-\frac{r+1}{4(m-2r)}SX^+\cdot\varphi + \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi + \nabla_X\varphi^+\right)\\
&- \frac{1}{2(m-r+1)}Y^+\cdot\left(\frac{m-r+1}{4(m-2r)}SX^-\cdot\varphi + \frac{1}{2}\mathrm{Ric}(X^-)\cdot\varphi + \nabla_X\varphi^-\right)\\
&- \frac{1}{4(m-2r)}\left(\frac{2r+1}{r+1}Y^-\cdot X^+ - \frac{2m-2r+1}{m-r+1}Y^+\cdot X^-\right)\cdot\eta\\
&- \nabla_{[X,Y]}\varphi - \frac{1}{2(r+1)}[X,Y]^-\cdot\varphi^+ - \frac{1}{2(m-r+1)}[X,Y]^+\cdot\varphi^-
\end{aligned}
$$

$$= R_{X,Y}\varphi - \frac{S}{4(m-2r)} i\langle X, JY\rangle\varphi + \frac{1}{4(r+1)}\Big[X^-\cdot\mathrm{Ric}(Y^+) - Y^-\cdot\mathrm{Ric}(X^+)\Big]\cdot\varphi$$
$$+ \frac{1}{4(m-r+1)}\Big[X^+\cdot\mathrm{Ric}(Y^-) - Y^+\cdot\mathrm{Ric}(X^-)\Big]\cdot\varphi$$
$$+ \frac{1}{4(m-2r)}\Big[\frac{2r-m}{(r+1)(m-r+1)}(X^+\cdot Y^- - Y^+\cdot X^-) + \frac{2(2r+1)}{r+1} i\langle X, JY\rangle\Big]\cdot\eta.$$

Contracting this equation using the formulas in Appendix A we obtain:

$$0 = -\frac{(m+2)S}{8(r+1)(m-2r)} Y^-\cdot\varphi + \frac{(m+2)S}{8(m-r+1)(m-2r)} Y^+\cdot\varphi - \frac{i}{4(r+1)} Y^-\cdot\rho\cdot\varphi$$
$$+ \frac{i}{4(m-r+1)} Y^+\cdot\rho\cdot\varphi + \frac{m+1}{2(r+1)(m-2r)} Y^-\cdot\eta - \frac{m+1}{2(m-r+1)(m-2r)} Y^+\cdot\eta,$$

or, equivalently:

$$\frac{(m+2)S}{4(m-2r)} Y^-\cdot\varphi + \frac{i}{2} Y^-\cdot\rho\cdot\varphi - \frac{m+1}{m-2r} Y^-\cdot\eta = 0,$$

$$\frac{(m+2)S}{4(m-2r)} Y^+\cdot\varphi + \frac{i}{2} Y^+\cdot\rho\cdot\varphi - \frac{m+1}{m-2r} Y^+\cdot\eta = 0,$$

which both yield by a further contraction:

$$\frac{(m+2)S}{4(m-2r)}\cdot\varphi + \frac{i}{2}\cdot\rho\cdot\varphi - \frac{m+1}{m-2r}\cdot\eta = 0,$$

so that

$$D^2\varphi = \eta = \frac{(m+2)S}{4(m+1)}\varphi + \frac{m-2r}{2(m+1)} i\rho\cdot\varphi. \tag{4.27}$$

This relation allows us to reduce the connection to one acting on sections of the bundle $\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M$. Substituting (4.27) in the second and third row of the connection (4.25), we get the following connection, denoted by $\tilde{\nabla}$, with respect to which the triple $(\varphi, \varphi^+, \varphi^-)$ is parallel if $\varphi \in \mathcal{KT}(r)$:

$$\begin{pmatrix} \nabla_X & \frac{1}{2(r+1)}X^-\cdot & \frac{1}{2(m-r+1)}X^+\cdot \\ -\frac{1}{8(m+1)}SX^+\cdot + \frac{1}{2}\mathrm{Ric}(X^+)\cdot + \frac{2r+1}{4(m+1)} iX^+\cdot\rho\cdot & \nabla_X & 0 \\ -\frac{1}{8(m+1)}SX^-\cdot + \frac{1}{2}\mathrm{Ric}(X^-)\cdot - \frac{2m-2r+1}{4(m+1)} iX^-\cdot\rho\cdot & 0 & \nabla_X \end{pmatrix}. \tag{4.28}$$

As in Proposition 4.9, it follows that there is a one-to-one correspondence between Kählerian twistor spinors on $\Sigma_r M$ and parallel sections of the bundle $\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M$ with respect to the connection $\tilde{\nabla}$ given by (4.28). An immediate consequence is that the space of Kählerian twistor spinors is finite dimensional and an upper bound is given as follows.

Corollary 4.10. *Let (M, g, J) be a connected spin Kähler manifold. The dimension of the space of Kählerian twistor spinors in $\Sigma_r M$ is bounded by the rank of the vector bundle $\Sigma_r M \oplus \Sigma_{r+1} M \oplus \Sigma_{r-1} M$:*

$$\dim_{\mathbb{C}}(\mathcal{KT}(r)) \leq \binom{m}{r} + \binom{m}{r+1} + \binom{m}{r-1}.$$

Remark 4.11. Twistor operators are one of the typical examples of generalized gradients (see § 1.1). Twistor operators are strongly elliptic by a result in [64], implying directly that on compact spin Kähler manifolds the space of Kählerian twistor spinors is finite dimensional. However, our Corollary 4.10 is a purely local result: the manifold M is not assumed to be compact.

The second component of the curvature of the connection $\hat{\nabla}$, $\hat{R}_{X,Y}\begin{pmatrix}\varphi\\ \varphi^+\\ \varphi^-\end{pmatrix}_2$, is computed as follows.

$$\begin{aligned}
&\left(-\frac{1}{8(m+1)}SX^+ + \frac{1}{2}\mathrm{Ric}(X^+) + \frac{2r+1}{4(m+1)}iX^+\cdot\rho\right)\cdot\left(\nabla_Y\varphi + \frac{1}{2(r+1)}Y^-\cdot\varphi^+\right)\\
&+\frac{1}{2(m-r+1)}\left(-\frac{1}{8(m+1)}SX^+ + \frac{1}{2}\mathrm{Ric}(X^+) + \frac{2r+1}{4(m+1)}iX^+\cdot\rho\right)\cdot Y^+\cdot\varphi^-\\
&+\nabla_X\left(-\frac{1}{8(m+1)}SY^+\cdot\varphi + \frac{1}{2}\mathrm{Ric}(Y^+)\cdot\varphi + \frac{2r+1}{4(m+1)}iY^+\cdot\rho\cdot\varphi + \nabla_Y\varphi^+\right)\\
&-\left(-\frac{1}{8(m+1)}SY^+ + \frac{1}{2}\mathrm{Ric}(Y^+) + \frac{2r+1}{4(m+1)}iY^+\cdot\rho\right)\cdot\left(\nabla_X\varphi + \frac{1}{2(r+1)}X^-\cdot\varphi^+\right)\\
&-\frac{1}{2(m-r+1)}\left(-\frac{1}{8(m+1)}SY^+ + \frac{1}{2}\mathrm{Ric}(Y^+) + \frac{2r+1}{4(m+1)}iY^+\cdot\rho\right)\cdot X^+\cdot\varphi^-\\
&-\nabla_Y\left(-\frac{1}{8(m+1)}SX^+\cdot\varphi + \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi + \frac{2r+1}{4(m+1)}iX^+\cdot\rho\cdot\varphi + \nabla_X\varphi^+\right)\\
=&R_{XY}\varphi^+ + \frac{1}{4(r+1)}\Big[-\frac{1}{4(m+1)}S(X^+\cdot Y^- - Y^+\cdot X^-)\\
&+(\mathrm{Ric}(X^+)\cdot Y^- - \mathrm{Ric}(Y^+)\cdot X^-) + \frac{2r+1}{2(m+1)}i(X^+\cdot\rho\cdot Y^- - Y^+\cdot\rho\cdot X^-)\Big]\cdot\varphi^+
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4(m-r+1)}\Bigg[-\frac{1}{4(m+1)}S(X^+\cdot Y^+ - Y^+\cdot X^+)\\
&+(\mathrm{Ric}(X^+)\cdot Y^+ - \mathrm{Ric}(Y^+)\cdot X^+) + \frac{2r+1}{2(m+1)}i(X^+\cdot\rho\cdot Y^+ - Y^+\cdot\rho\cdot X^+\Bigg]\cdot\varphi^-\\
&+\Bigg[-\frac{1}{8(m+1)}(X(S)Y^+ - Y(S)X^+) + \frac{1}{2}((\nabla_X\mathrm{Ric})(Y^+) - (\nabla_Y\mathrm{Ric})(X^+))\\
&+\frac{2r+1}{4(m+1)}i(Y^+\cdot\nabla_X\rho - X^+\cdot\nabla_Y\rho)\Bigg]\cdot\varphi.
\end{aligned}
$$

Contracting this equation using the formulas in Appendix A we get

$$
\begin{aligned}
0 =&\frac{1}{2}\mathrm{Ric}(Y)\cdot\varphi^+ + \frac{1}{4(r+1)}\Bigg[\frac{1}{2(m+1)}S((m-r)Y^- + (r+1)Y^+)\\
&-\left((\frac{S}{2}+i\rho)\cdot Y^- + 2(r+1)\mathrm{Ric}(Y^+)\right)\\
&-\frac{2r+1}{2(m+1)}i(2(m-r)\cdot\rho\cdot Y^- + 2rY^+\cdot\rho - iSY^+)\Bigg]\cdot\varphi^+\\
&+\frac{1}{4(m-r+1)}\Bigg[\frac{m-r}{m+1}SY^+ - (\frac{S}{2}+i\rho)\cdot Y^+ - 2(m-r)\mathrm{Ric}(Y^+)\\
&-\frac{2r+1}{2(m+1)}i(2(2m-2r-1)Y^+\cdot\rho + 4i(m-r-1)\mathrm{Ric}(Y^+) + iSY^+)\Bigg]\cdot\varphi^-\\
&+\Bigg[-\frac{1}{8(m+1)}(dS\cdot Y^+ + 2(m-r)Y(S))\\
&+\frac{1}{2}\left(-i\nabla_{Y^+}\rho - \frac{1}{2}Y^+(S) + i\nabla_Y\rho + \frac{1}{2}Y(S)\right)\\
&+\frac{2r+1}{4(m+1)}i\left(\frac{1}{2}Y^+\cdot JdS - 2\nabla_{Y^+}\rho + 2(m-r)\nabla_Y\rho\right)\Bigg]\cdot\varphi\\
=&-\frac{1}{4(r+1)(m+1)}\Bigg[\frac{S}{2}(rY^+ + (r+1)Y^-) + ir(2r+1)Y^+\cdot\rho\\
&+i(r+1)(2m-2r+1)Y^-\cdot\rho + 2(r+1)(m-2r)\mathrm{Ric}(Y^-)\Bigg]\cdot\varphi^+\\
&+\frac{1}{4(m-r+1)(m+1)}\Bigg[\frac{mS}{2}Y^+ + (4r^2+4r-4rm-3m)iY^+\cdot\rho\\
&+2(m-r-1)(2r-m)\mathrm{Ric}(Y^+)\Bigg]\cdot\varphi^-
\end{aligned}
$$

$$-\frac{1}{4(m+1)}\Big[(m-r-1)Y^+(S)-(r+1)Y^-(S)+rY^+\cdot(dS)^+-(r+1)Y^+\cdot(dS)^-$$
$$-2(2r+1)(m-r-1)i\nabla_{Y^+}\rho-2(r+1)(2m-2r+1)i\nabla_{Y^-}\rho\Big]\cdot\varphi$$

By projecting this equality onto $\Sigma_r M$ and $\Sigma_{r+2}M$, it is equivalent to the following two equations:

$$\begin{aligned}
&+\left[\frac{S}{2}Y^-+i(2m-2r+1)Y^-\cdot\rho+2(m-2r)\mathrm{Ric}(Y^-)\right]\cdot\varphi^+\\
&-\frac{1}{m-r+1}\left[\frac{mS}{2}Y^++(4r^2+4r-4rm-3m)iY^+\cdot\rho\right.\\
&\left.+2(m-r-1)(2r-m)\mathrm{Ric}(Y^+)\right]\cdot\varphi^- \qquad (4.29)\\
&-\Big[-(m-r-1)Y^+(S)+(r+1)Y^-(S)+(r+1)Y^+\cdot(dS)^-\\
&-2(2r+1)(m-r-1)i\nabla_{Y^+}\rho+2(r+1)(2m-2r+1)i\nabla_{Y^-}\rho\Big]\cdot\varphi=0,
\end{aligned}$$

$$\left[\frac{S}{2}Y^++i(2r+1)Y^+\cdot\rho\right]\cdot\varphi^++(r+1)Y^+\cdot(dS)^+\cdot\varphi=0. \qquad (4.30)$$

The contraction of (4.30) yields (if $r\neq m-1$):

$$\left[\frac{S}{2}+(2r+1)i\rho\right]\cdot\varphi^++(r+1)(dS)^+\cdot\varphi=0,$$

so that

$$i\rho\cdot\varphi^+=-\frac{1}{2(2r+1)}S\varphi^+-\frac{r+1}{2r+1}(dS)^+\cdot\varphi. \qquad (4.31)$$

The contraction of (4.29) yields:

$$\begin{aligned}
0=&(m-r+1)S\varphi^++2(2r+1)(m-r+1)i\rho\cdot\varphi^+\\
&+\frac{2(r-m)(r+1)S}{m-r+1}\varphi^-+\frac{4(r+1)(r-m)(2r-2m-1)}{m-r+1}i\rho\cdot\varphi^-\\
&+2(r+1)(m-r+1)(dS)^+\cdot\varphi-4(r+1)(m-r)(dS)^-\cdot\varphi,
\end{aligned}$$

which, by projections onto $\Sigma_{r+1}M$, respectively $\Sigma_{r-1}M$, is equivalent to the following two equations:

$$i\rho\cdot\varphi^+ = -\frac{1}{2(2r+1)}S\varphi^+ - \frac{r+1}{2r+1}(dS)^+\cdot\varphi, \tag{4.32}$$

which is again the equation (4.31) (here it follows to be true also for $r=m-1$) and

$$i\rho\cdot\varphi^- = \frac{1}{2[2(m-r)+1]}S\varphi^- + \frac{m-r+1}{2(m-r)+1}(dS)^-\cdot\varphi. \tag{4.33}$$

Inserting the relations (4.32) and (4.33) back in (4.29) we get

$$\begin{aligned}
&(m-2r)\left[\frac{S}{2r+1}Y^- - 2\mathrm{Ric}(Y^-)\right]\cdot\varphi^+ \qquad (4.34)\\
&+\frac{(m-2r)(m-r-1)}{m-r+1}\left[\frac{S}{2m-2r+1}Y^+ - 2\mathrm{Ric}(Y^+)\right]\cdot\varphi^-\\
&+\Big[-(m-r-1)Y^+(S)+(r+1)Y^-(S)-\frac{(2r+1)(m-r-1)}{2m-2r+1}Y^+\cdot(dS)^-\\
&+\frac{(r+1)(2m-2r+1)}{2r+1}Y^-\cdot(dS)^+\Big]\cdot\varphi\\
&+2i\Big[(2r+1)(m-r-1)\nabla_{Y^+}\rho+(r+1)(2m-2r+1)\nabla_{Y^-}\rho\Big]\cdot\varphi=0,
\end{aligned}$$

which is equivalent to the following equations:

$$\begin{aligned}
i\nabla_{Y^+}\rho\cdot\varphi = &-\frac{(m-2r)}{(2r+1)(m-r+1)}\left[\frac{1}{2(2m-2r+1)}SY^+ - \mathrm{Ric}(Y^+)\right]\cdot\varphi^-\\
&+\frac{1}{2(2r+1)}Y^+(S)\cdot\varphi+\frac{1}{2(2m-2r+1)}Y^+\cdot(dS)^-\cdot\varphi,
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
i\nabla_{Y^-}\rho\cdot\varphi = &-\frac{m-2r}{(r+1)(2m-2r+1)}\left[\frac{S}{2(2r+1)}Y^- - \mathrm{Ric}(Y^-)\right]\cdot\varphi^+\\
&-\frac{1}{2(2m-2r+1)}Y^-(S)\cdot\varphi-\frac{1}{2(2r+1)}Y^-\cdot(dS)^+\cdot\varphi.
\end{aligned} \tag{4.36}$$

Now we do a similar computation for the third component of the curvature of the Kählerian twistor connection.

$$
\begin{aligned}
&\hat{R}_{X,Y}\begin{pmatrix}\varphi\\ \varphi^+\\ \varphi^-\end{pmatrix}_3 = -\left(\frac{1}{8(m+1)}SX^- - \frac{1}{2}\mathrm{Ric}(X^-) + \frac{2m-2r+1}{4(m+1)}iX^-\cdot\rho\right)\cdot\Big(\nabla_Y\varphi\\
&+\frac{1}{2(r+1)}Y^-\cdot\varphi^+ + \frac{1}{2(m-r+1)}Y^+\cdot\varphi^-\Big) + \Big(\frac{1}{8(m+1)}SY^- - \frac{1}{2}\mathrm{Ric}(Y^-)\\
&+\frac{2m-2r+1}{4(m+1)}iY^-\cdot\rho\Big)\cdot\left(\nabla_X\varphi + \frac{1}{2(r+1)}X^-\cdot\varphi^+ + \frac{1}{2(m-r+1)}\cdot X^+\cdot\varphi^-\right)\\
&+\nabla_X\left(-\frac{1}{8(m+1)}SY^-\cdot\varphi + \frac{1}{2}\mathrm{Ric}(Y^-)\cdot\varphi - \frac{2m-2r+1}{4(m+1)}iY^-\cdot\rho\cdot\varphi + \nabla_Y\varphi^-\right)\\
&-\nabla_Y\left(-\frac{1}{8(m+1)}SX^-\cdot\varphi + \frac{1}{2}\mathrm{Ric}(X^-)\cdot\varphi - \frac{2m-2r+1}{4(m+1)}iX^-\cdot\rho\cdot\varphi + \nabla_X\varphi^-\right)\\
&= R_{XY}\varphi^- + \frac{1}{4(r+1)}\Big[-\frac{1}{4(m+1)}S(X^-\cdot Y^- - Y^-\cdot X^-)\\
&+(\mathrm{Ric}(X^-)\cdot Y^- - \mathrm{Ric}(Y^-)\cdot X^-) - \frac{2m-2r+1}{2(m+1)}i(X^-\cdot\rho\cdot Y^- - Y^-\cdot\rho\cdot X^-)\Big]\cdot\varphi^+\\
&+\frac{1}{4(m-r+1)}\Big[-\frac{1}{4(m+1)}S(X^-\cdot Y^+ - Y^-\cdot X^+)\\
&+(\mathrm{Ric}(X^-)\cdot Y^+ - \mathrm{Ric}(Y^-)\cdot X^+) - \frac{2m-2r+1}{2(m+1)}i(X^-\cdot\rho\cdot Y^+ - Y^-\cdot\rho\cdot X^+)\Big]\cdot\varphi^-\\
&+\Big[-\frac{1}{8(m+1)}(X(S)Y^- - Y(S)X^-) + \frac{1}{2}((\nabla_X\mathrm{Ric})(Y^-) - (\nabla_Y\mathrm{Ric})(X^-))\\
&-\frac{2m-2r+1}{4(m+1)}i(Y^-\cdot\nabla_X\rho - X^-\cdot\nabla_Y\rho)\Big]\cdot\varphi.
\end{aligned}
$$

Contracting this equation using the formulas in Appendix A we get

$$
\begin{aligned}
0 =\ &\frac{1}{2}\mathrm{Ric}(Y)\cdot\varphi^-\\
&+\frac{1}{4(r+1)}\Big(\frac{r}{(m+1)}SY^- - \frac{S}{2}Y^- + iY^-\cdot\rho - 2(r-1)\mathrm{Ric}(Y^-)\\
&-\frac{2m-2r+1}{2(m+1)}i(-2(2r-1)Y^-\cdot\rho + iSY^- + 4i(r-1)\mathrm{Ric}(Y^-))\Big)\cdot\varphi^+\\
&+\frac{1}{4(m-r+1)}\Big[\frac{1}{2(m+1)}S(rY^+ + (m-r+1)Y^-) - \frac{S}{2}Y^+ + iY^+\cdot\rho\\
&-2(\mathrm{Ric}(Y^+) + (m-r+1)\mathrm{Ric}(Y^-))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2m-2r+1}{2(m+1)}i(2rY^+\cdot\rho+4ir\mathrm{Ric}(Y^+)+2(m-r)Y^-\cdot\rho+iSY^-)\Big]\cdot\varphi^-\\
&+\Big[\frac{1}{8(m+1)}(Y^-\cdot dS+2Y^-(S)-2rY(S))+\frac{1}{2}(-i\nabla_{Y^+}\rho+\frac{1}{2}Y^+(S))\\
&-\frac{2m-2r+1}{4(m+1)}i(\frac{1}{2}Y^-\cdot JdS-2\nabla_{Y^-}\rho+2r\nabla_Y\rho)\Big]\cdot\varphi\\
&=\frac{1}{4(m+1)}\Big[\frac{1}{r+1}\Big(\frac{mS}{2}Y^-+(4rm-m-4r^2+4r)iY^-\cdot\rho\\
&+2(r-1)(m-2r)\mathrm{Ric}(Y^-)\Big)\cdot\varphi^+-\frac{1}{m-r+1}\Big(\frac{S}{2}((m-r+1)Y^++(m-r)Y^-)\\
&-i((2r+1)(m-r+1)Y^++(2m-2r+1)(m-r)Y^-))\cdot\rho\\
&+2(m-r+1)(2r-m)\mathrm{Ric}(Y^+)\Big)\cdot\varphi^-+\Big((m-r+1)Y^+(S)-(r-1)Y^-(S)\\
&+(m-r+1)Y^-\cdot(dS)^+-(m-r)Y^-\cdot(dS)^--2(2r+1)(m-r+1)i\nabla_{Y^+}\rho\\
&-2(r-1)(2m-2r+1)i\nabla_{Y^-}\rho\Big)\Big]\cdot\varphi
\end{aligned}
$$

By projecting this equality onto $\Sigma_{r-2}M$ and Σ_rM, it is equivalent to the following two equations:

$$
\Big[\frac{S}{2}Y^--i(2m-2r+1)Y^-\cdot\rho\Big]\cdot\varphi^-+(m-r+1)Y^-\cdot(dS)^-\cdot\varphi=0 \tag{4.37}
$$

and

$$
\begin{aligned}
&-\Big[\frac{S}{2}Y^+-i(2r+1)Y^+\cdot\rho+2(2r-m)\mathrm{Ric}(Y^+)\Big]\cdot\varphi^-\\
&+\frac{1}{r+1}\Big[\frac{mS}{2}Y^-+(4rm-m-4r^2+4r)iY^-\cdot\rho+2(r-1)(m-2r)\mathrm{Ric}(Y^-)\Big]\cdot\varphi^+\\
&+\Big[(m-r+1)Y^+(S)-(r-1)Y^-(S)+(m-r+1)Y^-\cdot(dS)^+\\
&-2(2r+1)(m-r+1)i\nabla_{Y^+}\rho-2(r-1)(2m-2r+1)i\nabla_{Y^-}\rho\Big]\cdot\varphi=0.
\end{aligned} \tag{4.38}
$$

The contractions of the equations (4.37) and (4.38) yield again the equations (4.32) and (4.33). Also, by inserting (4.32) and (4.33) back into (4.38), we get an equation which is equivalent to (4.35) and (4.36).

We gather now the formulas that we obtained for the actions on a special Kählerian twistor spinor and deduce some new ones. Is is enough to consider anti-holomorphic Kählerian twistor spinors, since similar formulas are then fulfilled by holomorphic Kählerian twistor spinors. In fact, they are obtained by conjugating the ones for anti-holomorphic Kählerian twistor spinors and replacing the constant $k = \frac{1}{2(2r+1)}$ with $k = \frac{1}{2(2m-2r+1)}$.

Proposition 4.12. *Let (M, g, J) be a spin Kähler manifold and $\varphi \in \Gamma(\Sigma_r M)$ be an anti-holomorphic Kählerian twistor spinor for some fixed r, $0 < r < m$:*

$$\begin{cases} \nabla_{X^-}\varphi = -\frac{1}{2(r+1)} X^- \cdot \varphi^+, \\ \nabla_{X^+}\varphi = 0, \end{cases}$$

so that in particular $\varphi^- = D^-\varphi = 0$. Then the following formulas hold, where we denote by $k := \frac{1}{2(2r+1)}$:

$$(dS)^- \cdot \varphi = 0, \tag{4.39}$$

$$D^2\varphi = k(r+1)S\varphi, \tag{4.40}$$

$$\nabla_X\varphi^+ = -\frac{1}{2}\mathrm{Ric}(X^+) \cdot \varphi, \tag{4.41}$$

$$\mathrm{Ric}(X^-) \cdot \varphi = kSX^- \cdot \varphi, \tag{4.42}$$

$$i\rho \cdot \varphi = kS\varphi, \tag{4.43}$$

$$i\nabla_{X^-}\rho \cdot \varphi = -kX^-(S)\varphi - kX^- \cdot (dS)^+ \cdot \varphi = -k(X^- \wedge (dS)^+) \cdot \varphi, \tag{4.44}$$

$$i\nabla_{X^+}\rho \cdot \varphi = kX^+(S)\varphi, \tag{4.45}$$

$$i\rho \cdot \varphi^+ = -kS\varphi^+ - 2k(r+1)(dS)^+ \cdot \varphi, \tag{4.46}$$

$$\mathrm{Ric}(X^-) \cdot \varphi^+ = kSX^- \cdot \varphi^+ - 2k(r+1)X^-(S)\varphi, \tag{4.47}$$

$$i\nabla_{X^+}\rho\varphi^+ = -\mathrm{Ric}^2(X^+)\varphi + kS\mathrm{Ric}(X^+)\cdot\varphi - 2k(r+1)\nabla_{X^+}(dS)^+\varphi - kX^+(S)\varphi^+, \tag{4.48}$$

$$i\nabla_{X^-}\rho\cdot\varphi^+ = -2k(r+1)\nabla_{X^-}(dS)^+\cdot\varphi - 3kX^-(S)\varphi^+ - kX^-\cdot(dS)^+\cdot\varphi^+. \tag{4.49}$$

Proof: Equation (4.39) follows directly from (4.33) and $\varphi^- = 0$. Equation (4.40) is the property of a special Kählerian twistor spinor to be eigenspinor of D^2, which we have shown in (4.14).
Substituting (4.40) into the second row of the connection $\hat{\nabla}$ given by (4.25) we get (4.41). Similarly, by substituting (4.40) into the third row of the

connection $\hat{\nabla}$ given by (4.25) we get the equation (4.42). Equations (4.27) and (4.40) yield (4.43).

Substituting (4.47) in (4.36) yields (4.44) and similarly, substituting (4.39) and $\varphi^- = 0$ in (4.35) yields (4.45). Differentiating (4.43) we get

$$\begin{aligned}
&i\nabla_{X^-}\rho\cdot\varphi = -i\rho\cdot\nabla_{X^-}\varphi + kX^-(S)\varphi + kS\nabla_{X^-}\varphi\\
&= \quad kX^-(S)\varphi - \frac{1}{2(r+1)}kSX^-\cdot\varphi^+ + \frac{i}{2(r+1)}[X^-\cdot\rho - 2i\mathrm{Ric}(X^-)]\cdot\varphi^+\\
&\overset{(4.46)}{=} kX^-(S)\varphi - \frac{1}{2(r+1)}kSX^-\cdot\varphi^+ - \frac{1}{2(r+1)}[kSX^-\cdot\varphi^+ - 2\mathrm{Ric}(X^-)\cdot\varphi^+\\
&\qquad + 2k(r+1)X^-\cdot(dS)^+\cdot\varphi]\\
&= \quad kX^-(S)\varphi - \frac{1}{r+1}[kSX^- - \mathrm{Ric}(X^-)]\cdot\varphi^+ - kX^-\cdot(dS)^+\cdot\varphi,
\end{aligned}$$

which compared with (4.36) yields

$$\begin{aligned}
&-kX^-(S)\varphi + \frac{1}{r+1}[kSX^- - \mathrm{Ric}(X^-)]\cdot\varphi^+ =\\
&= \frac{(m-2r)}{(2m-2r+1)(r+1)}[kSX^- - \mathrm{Ric}(X^-)]\cdot\varphi^+ + \frac{1}{2(2m-2r+1)}X^-(S)\varphi,
\end{aligned}$$

thus proving (4.47). Equation (4.46) is just (4.32). Differentiating (4.46) we get

$$\begin{aligned}
i\nabla_X\rho\cdot\varphi^+ \;&=\; -i\rho\cdot\nabla_X\varphi^+ - kX(S)\varphi^+ - kS\nabla_X\varphi^+ - 2k(r+1)\nabla_X(dS)^+\cdot\varphi\\
&\quad - 2k(r+1)(dS)^+\cdot\nabla_X\varphi\\
&\overset{(4.41)}{=} \frac{1}{2}i\rho\cdot\mathrm{Ric}(X^+)\cdot\varphi - kX(S)\varphi^+ + \frac{1}{2}kS\mathrm{Ric}(X^+)\cdot\varphi\\
&\quad - 2k(r+1)\nabla_X(dS)^+\cdot\varphi + k(dS)^+\cdot X^-\cdot\varphi^+,
\end{aligned}$$

which together with the commutator relation

$$[i\rho\cdot\mathrm{Ric}(X^+)\cdot\varphi = i\mathrm{Ric}(X^+)\cdot\rho\cdot\varphi - 2\mathrm{Ric}^2(X^+)\cdot\varphi$$

and (4.43) yields (4.48) and (4.49). □

Chapter 5

The Geometric Description of Manifolds Carrying Kählerian Twistor Spinors

In this chapter we prove our main result, the classification of simply-connected compact Kähler spin manifolds of constant scalar curvature carrying Kählerian twistor spinors (Theorem 5.15). Let (M, g, J) be such a manifold. Briefly, the main steps of the proof, which we shall describe in detail, are the following:

1. All Kählerian twistor spinors on M are special Kählerian twistor spinors.
2. The Ricci tensor has two constant eigenvalues.
3. The Ricci tensor is parallel.
4. $M = M_1 \times M_2$ with M_1 Kähler-Einstein admitting Kählerian twistor spinors and M_2 a Ricci-flat Kähler manifold.

In the last section we relax the condition on the constancy of the scalar curvature to the metric being weakly Bochner flat.

5.1 Special Kählerian Twistor Spinors

We show that if the scalar curvature is constant, then each Kählerian twistor spinor is holomorphic or anti-holomorphic.

Proposition 5.1. *Let* (M, g, J) *be a compact Kähler spin manifold of positive constant scalar curvature and* $\varphi \in \Gamma(\Sigma_r M)$ $(0 < r < m)$ *a Kählerian twistor spinor . Then* φ *is an anti-holomorphic Kählerian twistor spinor if* $r < \frac{m}{2}$ *or a holomorphic Kählerian twistor spinor if* $r > \frac{m}{2}$*.*

Proof: The relation (4.7) and the Lichnerowicz formula (3.9) imply the following Weitzenböck formula:

$$\frac{2r+1}{2(r+1)} D^- D^+ \varphi + \frac{2m-2r+1}{2(m-r+1)} D^+ D^- \varphi = \frac{1}{4} S\varphi \tag{5.1}$$

If $r < \frac{m}{2}$, then $\frac{2r+1}{2(r+1)} < \frac{2m-2r+1}{2(m-r+1)}$ and from (5.1) it follows by integration (where we denote by $|| \cdot ||$ the global norm: $||\varphi||^2 = \int_M \langle \varphi, \varphi \rangle \mathrm{vol}_M$):

$$\frac{2r+1}{2(r+1)} ||D\varphi||^2 \leq \frac{1}{4} S ||\varphi||^2, \tag{5.2}$$

with equality if and only if $D^- \varphi = 0$. Further it follows that

$$\frac{2(r+1)}{2r+1} \frac{1}{4} S \leq \lambda_{min} \leq \frac{||D\varphi||^2}{||\varphi||^2} \leq \frac{2(r+1)}{2r+1} \frac{1}{4} S,$$

where the first inequality is given by (4.10) and the second inequality is the property of the Rayleigh quotient to have as its minimum the smallest eigenvalue λ_{min} of the operator (here D^2 acting on the Hilbert space of square integrable sections of $\Sigma_r M$). It then follows that equality must hold in (5.2), so that φ must be an anti-holomorphic Kählerian twistor: $D^- \varphi = 0$ and in particular an eigenspinor of D^2 with the smallest eigenvalue of D^2 on $\Sigma_r M$ (Proposition 4.7).

The same argument shows that for $r > \frac{m}{2}$, any Kählerian twistor spinor in $\Sigma_r M$ must be a holomorphic Kählerian twistor spinor.

If m even and $r = \frac{m}{2}$, then we have seen in § 4.2 that the only nontrivial Kählerian twistor spinors are the parallel ones. □

Remark 5.2. We notice that the condition for the scalar curvature to be positive in Proposition 5.1 is not restrictive. If $S \leq 0$, then taking in (5.1) the scalar product with φ and integrating over M yields $D^+ \varphi = D^- \varphi = 0$. As φ is also in the kernel of the twistor operator, it follows that $\nabla \varphi = 0$, implying that the manifold is Ricci-flat (and thus $S = 0$), unless $\varphi \equiv 0$.

5.2 The Eigenvalues of the Ricci Tensor

We now show that the Ricci tensor of a Kähler manifold of constant scalar curvature admitting a special Kählerian twistor spinor has two constant eigenvalues and is parallel. This result has been proven by A. Moroianu in [53] and [54] for the special case of a limiting even dimensional spin Kähler manifold for Kirchberg's inequality (3.13). We note that the same method can be applied when the existence of a special Kählerian twistor spinor is assumed. As shown in Proposition 5.1 this is no restriction, since all Kählerian twistor spinors are special Kählerian twistor spinors if S is constant.

Theorem 5.3. *The Ricci tensor of a Kähler spin manifold of constant scalar curvature admitting a Kählerian twistor spinor has two constant eigenvalues. If φ is an anti-holomorphic Kählerian twistor spinor in $\Sigma_r M$, then the Ricci tensor has the eigenvalues $\frac{S}{2(2r+1)}$ and 0, with multiplicities $2(2r+1)$ and $2(m-2r-1)$ respectively. If $\varphi \in \Gamma(\Sigma_r M)$ is a holomorphic Kählerian twistor spinor, then the Ricci tensor has the eigenvalues $\frac{S}{2(2m-2r+1)}$ and 0, with multiplicities $2(2m-2r+1)$ and $2(2r-m-1)$ respectively.*

Since the multiplicity of an eigenvalue is a positive integer, we get directly the following

Corollary 5.4. *On a Kähler spin manifold of positive constant scalar curvature, anti-holomorphic Kählerian twistor spinors may exist only in $\Sigma_r M$ with $r \leq \frac{m-1}{2}$ and holomorphic Kählerian twistor spinors may exist only in $\Sigma_r M$ with $r \geq \frac{m+1}{2}$.*

Remark 5.5. In the extremal cases, if m is odd and $r = \frac{m\pm1}{2}$, then the existence of a holomorphic (respectively anti-holomorphic) Kählerian twistor spinor $\varphi \in \Gamma(\Sigma_r M)$ implies that the Ricci tensor only has one eigenvalue, thus proving that the manifold must be Kähler-Einstein. As we move away from the middle dimension the multiplicity of the eigenvalue 0 of the Ricci tensor grows and the manifold is actually the product of a Kähler-Einstein manifold and a Ricci-flat one, if M is supposed to be simply-connected (see Theorem 5.15) below.

The proof of Theorem 5.3 follows from Lemmas 5.6 and 5.7. It is enough to consider anti-holomorphic Kählerian twistor spinors, since the holomorphic ones are obtained by applying the canonical $\mathbb{C}$-anti-linear quaternionic (resp. real) structure $\mathfrak{j}$ to the anti-holomorphic ones.

Lemma 5.6. *If $\varphi \in \Gamma(\Sigma_r M)$ is an anti-holomorphic Kählerian twistor spinor, then the following formulas hold (with the notation $K = kS = \frac{S}{2(2r+1)}$):*

$$\nabla_X \varphi^+ = -\frac{1}{2}\mathrm{Ric}(X^+) \cdot \varphi, \tag{5.3}$$

$$\mathrm{Ric}(X^-) \cdot \varphi = KX^- \cdot \varphi, \qquad \mathrm{Ric}(X^-) \cdot \varphi^+ = KX^- \cdot \varphi^+, \tag{5.4}$$

$$i\rho \cdot \varphi = K\varphi, \qquad i\rho \cdot \varphi^+ = -K\varphi^+, \tag{5.5}$$

$$\nabla_X \rho \cdot \varphi = 0, \qquad i\nabla_X \rho \cdot \varphi^+ = -(\mathrm{Ric}^2(X^+) - K\mathrm{Ric}(X^+)) \cdot \varphi. \tag{5.6}$$

Proof: These relations follow directly from the general formulas (4.41) - (4.49) for anti-holomorphic Kählerian twistor spinors by using the fact that the scalar curvature S is constant. □

Let us consider the 2-forms

$$\rho_s := \frac{1}{2}\sum_{i=1}^{n} e_i \wedge J\mathrm{Ric}^s(e_i) = \frac{1}{2}\sum_{i=1}^{n} e_i \cdot J\mathrm{Ric}^s(e_i).$$

We need the following two properties of these 2-forms: the commutating rule

$$\rho_s \cdot X = X \cdot \rho_s + 2J\mathrm{Ric}^s(X) \tag{5.7}$$

and the equivalence

$$\delta\rho_s = 0 \Leftrightarrow \nabla_{e_i}\mathrm{Ric}^s(e_i) = 0, \tag{5.8}$$

which follows from the following formula:

$$\delta\rho_s(X) = -\nabla_{e_i}\rho_s(e_i, X) = -\nabla_{e_i}\langle J\mathrm{Ric}^s(e_i), X\rangle = \langle\nabla_{e_i}\mathrm{Ric}^s(e_i), JX\rangle,$$

where X is a vector field and $\{e_i\}_{i=\overline{1,n}}$ a local orthonormal frame parallel at the point where the computation is done.

We consider then as in [53] the following statements:

(a_s) $tr(\mathrm{Ric}^s) = 2(2r+1)K^s$;

(b_s) $i\rho_s \cdot \varphi = K^s\varphi$;

(c_s) $i\rho_s \cdot \varphi^+ = -K^s\varphi^+$;

(d_s) $\nabla_X \rho_s \cdot \varphi = 0$;

(e_s) $i\nabla_X\rho_s \cdot \varphi^+ = -(\mathrm{Ric}^{s+1}(X^+) - K^s\mathrm{Ric}(X^+)) \cdot \varphi$;

(f_s) $\delta\rho_s = 0$.

From Lemma 5.6 it follows that these statements are true for $s = 1$ (constant scalar curvature implies $\delta\rho = -\frac{1}{2}JdS = 0$). We prove by induction that they are true for all $s \in \mathbb{N}$.

Lemma 5.7. *The following implications hold:*

1. $(a_s) \Rightarrow (b_s), (c_s)$;
2. $(b_s), (c_s) \Rightarrow (d_s), (e_s)$;
3. $(a_s), (f_{s-1}) \Rightarrow (f_s)$;
4. $(d_s), (e_s), (f_s) \Rightarrow (a_{s+1})$.

Proof: Let $x \in M$ and $\{X_i, X_a\}$ be an orthonormal basis adapted with respect to the Ricci tensor:

$$\mathrm{Ric}(X_a) = \mu_a X_a, \quad a \in \{1, \ldots, p\}, \mu_a \neq K,$$

$$\mathrm{Ric}(X_i) = KX_i, \quad i \in \{p+1, \ldots, n\}.$$

From (5.4) it follows that

$$X_a^- \cdot \varphi = 0 \tag{5.9}$$

and

$$X_a^- \cdot \varphi^+ = 0, \tag{5.10}$$

for all $a \in \{1, \ldots, p\}$.

1. Suppose that (a_s) is true. We then have:

$$\sum_{a=1}^{p}(\mu_a^s - K^s) = tr(\mathrm{Ric}^s) - nK^s = 2(2r+1-m)K^s \tag{5.11}$$

and using (5.9) we get:

$$(\rho_s - K^s\Omega) \cdot \varphi = \frac{1}{2}\sum_{a=1}^{p}(\mu_a^s - K^s)X_a \cdot JX_a \cdot \varphi = \frac{i}{2}\sum_{a=1}^{p}(\mu_a^s - K^s)X_a \cdot X_a \cdot \varphi$$
$$= \frac{i}{2}\sum_{a=1}^{p}(\mu_a^s - K^s) \cdot \varphi = i(m-2r-1)K^s\varphi.$$

Thus we obtain

$$i\rho_s \cdot \varphi = iK^s\Omega \cdot \varphi - (m-2r-1)K^s\varphi = (m-2r-m+2r+1)K^s\varphi = K^s\varphi.$$

Similarly, from (5.10) and (5.11), it follows

$$(\rho_s - K^s\Omega) \cdot \varphi^+ = i(m-2r-1)K^s\varphi^+,$$

so that

$$i\rho_s \cdot \varphi^+ = iK^s\Omega \cdot \varphi^+ - (m-2r-1)K^s\varphi = -K^s\varphi.$$

2. Using the relation (5.7) we obtain

$$\begin{aligned} i\nabla_X\rho_s \cdot \varphi &= i\nabla_X(\rho_s \cdot \varphi) - i\rho_s \cdot \nabla_X\varphi = K^s\nabla_X\varphi - i\rho_s \cdot \nabla_X\varphi \\ &= -\frac{1}{2(r+1)}(K^s - i\rho_s) \cdot X^- \cdot \varphi^+ \\ &= -\frac{1}{2(r+1)}(K^sX^- - iX^- \cdot \rho_s - 2\mathrm{Ric}^s(X^-)) \cdot \varphi^+ \\ &\overset{(c_s)}{=} -\frac{1}{r+1}(K^sX^- - \mathrm{Ric}^s(X^-)) \cdot \varphi^+ \overset{(5.4)}{=} 0 \end{aligned} \tag{5.12}$$

and similarly

$$\begin{aligned} i\nabla_X\rho_s \cdot \varphi^+ &= i\nabla_X(\rho_s \cdot \varphi^+) - i\rho_s \cdot \nabla_X\varphi^+ = K^s\nabla_X\varphi^+ - i\rho_s \cdot \nabla_X\varphi^+ \\ &= \frac{1}{2}(K^s + i\rho_s) \cdot \mathrm{Ric}(X^+) \cdot \varphi \\ &= \frac{1}{2}(K^s \cdot \mathrm{Ric}(X^+) + i\mathrm{Ric}(X^+) \cdot \rho_s - 2\mathrm{Ric}^{s+1}(X^+)) \cdot \varphi \\ &\overset{(b_s)}{=} -(\mathrm{Ric}^{s+1}(X^+) - K^s\mathrm{Ric}(X^+)) \cdot \varphi. \end{aligned} \tag{5.13}$$

3. From (a_s) we get

$$(\nabla_X\mathrm{Ric})(e_i, \mathrm{Ric}^{s-1}e_i) = tr(\nabla_X\mathrm{Ric} \circ \mathrm{Ric}^{s-1}) = \frac{1}{s}\nabla_X(tr(\mathrm{Ric}^s)) = 0.$$

Thus, considering X a vector field and $\{e_i\}_{i=\overline{1,n}}$ a local orthonormal frame parallel, both parallel in x and using (5.8), we obtain

$$\begin{aligned} 0 &= d\rho(Je_i, \mathrm{Ric}^{s-1}e_i, X) \\ &= (\nabla_{Je_i}\rho)(\mathrm{Ric}^{s-1}e_i, X) + (\nabla_{\mathrm{Ric}^{s-1}e_i}\rho)(X, Je_i) + (\nabla_X\rho)(Je_i, \mathrm{Ric}^{s-1}e_i) \end{aligned}$$

$= (\nabla_{Je_i}\mathrm{Ric})(J\mathrm{Ric}^{s-1}e_i, X) + (\nabla_{\mathrm{Ric}^{s-1}e_i}\mathrm{Ric})(JX, Je_i) - (\nabla_X\mathrm{Ric})(e_i, \mathrm{Ric}^{s-1}e_i)$
$= 2(\nabla_{e_i}\mathrm{Ric})(\mathrm{Ric}^{s-1}e_i, X) = 2\langle\nabla_{e_i}Ric^s(e_i), X\rangle - 2\mathrm{Ric}(\nabla_{e_i}\mathrm{Ric}^{s-1}(e_i), X),$

proving that (a_s) and (f_{s-1}) imply (f_s).

4. By contracting (e_s) we get

$$iD\rho_s \cdot \varphi^+ = -(tr(\mathrm{Ric}^{s+1}) - K^s tr(\mathrm{Ric}))\varphi. \tag{5.14}$$

As $D\rho_s = \delta\rho_s + d\rho_s \overset{(f_s)}{=} d\rho_s$, it follows that $D\rho_s$ is symmetric (*i.e.* $\langle D\rho_s \cdot \psi, \xi\rangle = \langle\psi, D\rho_s \cdot \xi\rangle$ for any spinors ψ and ξ, where $\langle\cdot,\cdot\rangle$ is the Hermitian scalar product on ΣM). Taking then the scalar product with φ in (5.14) yields

$$i(tr(\mathrm{Ric}^{s+1}) - K^s tr(\mathrm{Ric}))\langle\varphi,\varphi\rangle = \langle D\rho_s \cdot \varphi^+, \varphi\rangle = \langle\varphi^+, D\rho_s \cdot \varphi\rangle = 0,$$

and as the support of φ is dense in M, we obtain (a_{s+1}). □

The formulas (a_s) show that the sum of the s^{th} powers of the eigenvalues of Ric equals $2(2r+1)K^s$ for all s, so by Newton's relations this proves Theorem 5.3.

5.3 Kähler-Einstein Manifolds

In this section we show that on a Kähler-Einstein manifold there may only exist non-extremal Kählerian twistor spinors if its complex dimension, m, is odd. They must lie in $\Sigma_{\frac{m-1}{2}}M$ or $\Sigma_{\frac{m+1}{2}}M$ and are automatically Kählerian Killing spinors. Thus such a manifold is a limiting manifold for Kirchberg's inequality (3.12). These manifolds have been geometrically described by A. Moroianu (see Theorem 3.6).

Let (M, g, J) be a Kähler-Einstein manifold. Then the scalar curvature S is constant and by Proposition 5.1 it follows that all Kählerian twistor spinors are special Kählerian twistor spinors: if $0 \le r \le \frac{m}{2}$, then they must be anti-holomorphic Kählerian twistor spinors and if $\frac{m}{2} \le r \le m$, then they must be holomorphic Kählerian twistor spinors. As above, it is sufficient to consider anti-holomorphic Kählerian twistor spinors $\varphi \in \Gamma(\Sigma_r M)$ for a fixed r with $0 < r < \frac{m}{2}$.

As $\rho = \frac{1}{2m} S\Omega$, it follows that $i\rho \cdot \varphi = \frac{m-2r}{2m} S\varphi$ and from (4.27) we get

$$D^2\varphi = \frac{m^2 + m - 2mr + 2r^2}{2m(m+1)} S\varphi. \tag{5.15}$$

On the other hand, by Proposition 4.7, any anti-holomorphic Kählerian twistor spinor in $\Sigma_r M$ (for $0 \leq r < \frac{m}{2}$) is an eigenspinor of D^2 with the smallest possible eigenvalue on $\Sigma_r M$:

$$D^2\varphi = \frac{(r+1)S}{2(2r+1)} \varphi. \tag{5.16}$$

Comparing the eigenvalues in (5.15) and (5.16), we get for $r \leq \frac{m-1}{2}$:

$$\frac{(r+1)S}{2(2r+1)} = \frac{(m^2 + m - 2mr + 2r^2)S}{2m(m+1)},$$

which, since $S \neq 0$, is equivalent to $0 = -r(m-2r)(m-2r-1)$.

As $r \neq 0$ and $r \neq \frac{m}{2}$, it follows that the only possible value for r is $\frac{m-1}{2}$. Thus, except for parallel spinors (if S=0) and extremal spinors (those in $\Sigma_0 M$), anti-holomorphic Kählerian twistor spinors on a Kähler-Einstein manifold can only exist in $\Sigma_{\frac{m-1}{2}} M$ and they are by definition exactly the Kählerian Killing spinors. A similar result is true for holomorphic Kählerian twistor spinors and it might be obtained by considering the isomorphism (4.6) given by the quaternionic (resp. real) structure $\mathfrak{j}$.

Concluding, we have proven the following

Proposition 5.8. *On a spin Kähler-Einstein manifold of scalar curvature the only nontrivial Kählerian twistor spinors are the extremal ones in* $\Sigma_0 M$ *and* $\Sigma_m M$ *and the Kählerian Killing spinors in* $\Sigma_{\frac{m-1}{2}} M$ *and* $\Sigma_{\frac{m+1}{2}} M$ *(if* m *is odd).*

Combining Proposition 5.8 with the characterization in Proposition 3.4 of the limiting manifolds of Kirchberg's inequality and their geometric description given by A. Moroianu (see Theorem 3.6) we obtain

Theorem 5.9. *A Kähler-Einstein spin manifold admitting nontrivial and non-extremal Kählerian twistor spinors is either* $\mathbb{C}P^{4k+1}$ *or a twistor space over a quaternionic Kähler manifold of positive scalar curvature if* $m = 4k+3$.

Example 5.10 (The complex projective space)**.** The dimension of the space of Kählerian Killing spinors on $\mathbb{C}P^{2m}$ with $m = 2k-1$ is $\binom{2k}{k}$(see [43]).

Corollary 5.11. *(see [52]) Let (M, g, J) be a Kähler-Einstein manifold admitting nontrivial non-extremal Kählerian twistor spinors, which is not the complex projective space, then the dimension of their space is 2. More precisely:*

$$\dim_{\mathbb{C}}(\mathcal{KT}(\frac{m-1}{2})) = \dim_{\mathbb{C}}(\mathcal{KT}(\frac{m+1}{2})) = 1.$$

5.4 Kählerian Twistor Spinors on Kähler Products

We now study Kählerian twistor spinors on a product of compact spin Kähler manifolds and show that they are defined by parallel spinors on one of the factors and special Kählerian twistor spinors on the other factor. For twistor forms a similar result was obtained by A. Moroianu and U. Semmelmann, [56]. They showed that twistor forms on a product of compact Riemannian manifolds are defined by Killing forms on the factors.

Let $M = M_1 \times M_2$ be the product of two compact spin Kähler manifolds of real dimensions $2m$ and $2n$ respectively. Then M is also a spin Kähler manifold and its induced spinor bundle is identified with the tensor product of the spinor bundles of the factors:

$$\Sigma M \cong \Sigma M_1 \otimes \Sigma M_2,$$

with the Clifford multiplication given by:

$$(X_1 + X_2) \cdot (\psi_1 \otimes \psi_2) = X_1 \cdot \psi_1 \otimes \psi_2 + \bar{\psi}_1 \otimes X_2 \cdot \psi_2,$$

where $\bar{\psi}$ is the conjugate of the spinor with respect to the decomposition given by (3.1), $\Sigma M_1 = \Sigma_+ M_1 \oplus \Sigma_- M_1$.

We consider the decompositions of the spinor bundles of M_1 and M_2 with respect to their Kähler forms Ω_1, Ω_2 (Lemma 3.1): $\Sigma M_1 = \oplus_{k=0}^{m} \Sigma_k M_1$, $\Sigma M_2 = \oplus_{l=0}^{n} \Sigma_l M_2$. Then the corresponding decomposition of ΣM into eigenbundles of $\Omega = \Omega_1 + \Omega_2$ is:

$$\Sigma M = \overset{m+n}{\underset{r=0}{\oplus}} \Sigma_r M, \tag{5.17}$$

with

$$\Sigma_r M \cong \bigoplus_{k=0}^{r} \Sigma_k M_1 \otimes \Sigma_{r-k} M_2, \tag{5.18}$$

since the Kähler form Ω acts on a section of $\Sigma_k M_1 \otimes \Sigma_{r-k} M_2$ as:

$$\begin{aligned}\Omega \cdot (\psi_1 \otimes \psi_2) =& (\Omega_1 + \Omega_2) \cdot (\psi_1 \otimes \psi_2) = \Omega_1 \cdot \psi_1 \otimes \psi_2 + \psi_1 \otimes \Omega_2 \cdot \psi_2 \\ =& i(2r - m - n)\psi_1 \otimes \psi_2.\end{aligned}$$

Let us define the differential operators:

$$D_1^+ = \sum_{i=1}^{2m} e_i^+ \cdot \nabla_{e_i^-}, \quad D_2^+ = \sum_{j=1}^{2n} f_j^+ \cdot \nabla_{f_j^-},$$

where $\{e_i\}_{i=\overline{1,2m}}$ and $\{f_j\}_{j=\overline{1,2n}}$ denote local orthonormal basis of the tangent distributions to M_1, respectively M_2. Their adjoints are

$$D_1^- = \sum_{i=1}^{2m} e_i^- \cdot \nabla_{e_i^+}, \quad D_2^- = \sum_{j=1}^{2n} f_j^- \cdot \nabla_{f_j^+}.$$

The following relations are straightforward:

$$D^+ = D_1^+ + D_2^+, \quad D^- = D_1^- + D_2^-,$$

$$\begin{aligned}(D_1^+)^2 = (D_2^+)^2 = (D_1^-)^2 = (D_2^-)^2 = 0,\\ D_1^+ D_2^+ + D_2^+ D_1^+ = D_1^- D_2^- + D_2^- D_1^- = 0,\\ D_1^+ D_2^- + D_2^- D_1^+ = D_1^- D_2^+ + D_2^+ D_1^- = 0.\end{aligned}$$

We may suppose without loss of generality that one of the factors M_1 or M_2 is not Ricci-flat. Otherwise, $M = M_1 \times M_2$ is Ricci-flat and by the Lichnerowicz formula Kählerian twistor spinors are parallel.

Theorem 5.12. *Let $M = M_1 \times M_2$ be the product of two compact spin Kähler manifolds of dimensions $2m$, respectively $2n$ and suppose that M_2 is not Ricci-flat. Let $\psi \in \Gamma(\Sigma_r M)$ be a nontrivial Kählerian twistor spinor. Then ψ has the following form*

$$\psi = \xi_0 \otimes \varphi_r + \xi_m \otimes \varphi_{r-m}, \tag{5.19}$$

where ξ_0, ξ_m are parallel spinors in $\Sigma_0 M_1$, $\Sigma_m M_1$, φ_r is an anti-holomorphic

Kählerian twistor spinor in $\Sigma_r M_2$ (if $r \leq n$, otherwise $\varphi_r \equiv 0$) and φ_{r-m} is a holomorphic Kählerian twistor spinor in $\Sigma_{r-m} M_2$ (if $m \leq r$, otherwise $\varphi_{r-m} \equiv 0$). In particular, M_1 is a Ricci-flat manifold and M_2 carries special Kählerian twistor spinors in $\Sigma_r M_2$ or $\Sigma_{r-m} M_2$.

Proof: Let ψ be a Kählerian twistor spinor in $\Sigma_r M$:

$$\begin{cases} \nabla_{X^+}\psi = -\frac{1}{2(m+n-r+1)} X^+ \cdot D^-\psi, \\ \nabla_{X^-}\psi = -\frac{1}{2(r+1)} X^- \cdot D^+\psi, \end{cases} \tag{5.20}$$

for any vector field X tangent to M. With respect to the decomposition (5.18), ψ is written as

$$\psi = \psi_0 + \cdots + \psi_r,$$

with $\psi_k \in \Gamma(\Sigma_k M_1 \otimes \Sigma_{r-k} M_2)$, for $k = 0, \ldots, r$.

Projecting onto the components given by (5.18), the twistorial equation (5.20) is equivalent to the following two systems of equations:

For $X \in \Gamma(\mathrm{TM}_1)$:

$$\begin{cases} \nabla_{X^+}\psi_k = -\frac{1}{2(m+n-r+1)} X^+ \cdot (D_1^-\psi_k + D_2^-\psi_{k-1}), \\ \nabla_{X^-}\psi_k = -\frac{1}{2(r+1)} X^- \cdot (D_1^+\psi_k + D_2^+\psi_{k+1}) \end{cases} \tag{5.21}$$

and for $X \in \Gamma(\mathrm{TM}_2)$:

$$\begin{cases} \nabla_{X^+}\psi_k = -\frac{1}{2(m+n-r+1)} X^+ \cdot (D_1^-\psi_{k+1} + D_2^-\psi_k), \\ \nabla_{X^-}\psi_k = -\frac{1}{2(r+1)} X^- \cdot (D_1^+\psi_{k-1} + D_2^+\psi_k). \end{cases} \tag{5.22}$$

If $\{e_i\}_{i=\overline{1,2m}}$ is an orthonormal basis of the $2m$-dimensional manifold M_1, then we have on $\Sigma_r M_1$:

$$e_i^+ \cdot e_i^- = -2r, \quad e_i^- \cdot e_i^+ = -2(m-r).$$

By contracting (5.21) using the relations above, it follows that

$$\begin{aligned} D_1^-\psi_k = e_i^- \cdot \nabla_{e_i^+}\psi_k &= -\frac{1}{2(m+n-r+1)} e_i^- \cdot e_i^+ \cdot (D_1^-\psi_k + D_2^-\psi_{k-1}) \\ &= \frac{m-k+1}{m+n-r+1}(D_1^-\psi_k + D_2^-\psi_{k-1}), \end{aligned}$$

$$
\begin{aligned}
D_1^+\psi_k = e_i^+ \cdot \nabla_{e_i^-}\psi_k &= -\frac{1}{2(r+1)} e_i^+ \cdot e_i^- \cdot (D_1^+\psi_k + D_2^+\psi_{k+1}) \\
&= \frac{k+1}{r+1}(D_1^+\psi_k + D_2^+\psi_{k+1}),
\end{aligned}
$$

so that we get

$$
(r-k)D_1^+\psi_k = (k+1)D_2^+\psi_{k+1} \tag{5.23}
$$

and

$$
(n+k-r)D_1^-\psi_k = (m-k+1)D_2^-\psi_{k-1}. \tag{5.24}
$$

We distinguish three cases for $0 \leq r \leq m+n$:

I. Suppose that r is strictly smaller than m and n. For $k < r$, (5.23) and (5.24) imply:

$$
\begin{aligned}
D_1^- D_1^+\psi_k = \frac{k+1}{r-k} D_1^- D_2^+\psi_{k+1} &= -\frac{k+1}{r-k} D_2^+ D_1^-\psi_{k+1} \\
&= -\frac{(k+1)(m-k)}{(r-k)(n+k-r+1)} D_2^+ D_2^-\psi_k,
\end{aligned} \tag{5.25}
$$

which integrated over M yields $D_1^+\psi_k = D_2^-\psi_k = 0$, for all $k < r$. Similarly, it follows that $D_1^-\psi_k = D_2^+\psi_k = 0$, for all $k > 0$. As $D_1^-\psi_0 = D_2^-\psi_r = 0$ holds automatically, then (5.21) and (5.22) show that ψ_k are parallel spinors on M (and thus are zero, since M is not Ricci-flat) for $1 \leq k \leq r-1$. The first component $\psi_0 \in \Gamma(\Sigma_0 M_1 \otimes \Sigma_r M_2)$ satisfies the equations:

$$
\nabla_X \psi_0 = 0, \quad \text{for all } X \in \Gamma(\mathrm{TM}_1),
$$

$$
\nabla_{X^+}\psi_0 = 0, \quad \nabla_{X^-}\psi_0 = -\frac{1}{r+1} X^- \cdot D_2^+\psi_0, \quad \text{for all } X \in \Gamma(\mathrm{TM}_2)
$$

and $\psi_r \in \Gamma(\Sigma_r M_1 \otimes \Sigma_0 M_2)$ satisfies the equations:

$$
\nabla_{X^+}\psi_r = 0, \quad \nabla_{X^-}\psi_r = -\frac{1}{r+1} X^- \cdot D_2^+\psi_r, \quad \text{for all } X \in \Gamma(\mathrm{TM}_1),
$$

$$
\nabla_X \psi_r = 0, \quad \text{for all } X \in \Gamma(\mathrm{TM}_2)
$$

Thus $\psi_0 = \xi_0 \otimes \varphi_r$ with $\xi_0 \in \Gamma(\Sigma_0 M_1)$ a parallel spinor on M_1 and $\varphi_r \in \Gamma(\Sigma_r M_2)$ an anti-holomorphic Kählerian twistor spinor on M_2 ($D^-\varphi_r = 0$). Similarly $\psi_r = \xi_r \otimes \varphi_0$, in particular with $\varphi_0 \in \Gamma(\Sigma_0 M_2)$ a parallel spinor on M_2, but as M_2 is not Ricci-flat, this term must vanish.

II. If r is strictly larger than m and n, then by applying the real (resp. quaternionic) structure $\mathfrak{j}$ to a Kählerian twistor spinor in $\Sigma_r M$ we get one in $\Sigma_{m+n-r}M$, thus reducing to the first case. It then follows that a Kählerian twistor spinor $\psi \in \Gamma\Sigma_r M$ is of the form $\psi = \xi_m \otimes \varphi_{r-m}$ with $\xi_m \in \Gamma(\Sigma_m M_1)$ a parallel spinor on M_1 and $\varphi_{r-m} \in \Gamma(\Sigma_{r-m} M_2)$ a holomorphic Kählerian twistor spinor on M_2.

III. Let r be a number between m and n and suppose that $m \leq r \leq n$. Since $\Sigma_k M_1$ exist only for $0 \leq k \leq m$, then automatically $\psi_{m+1} = \cdots = \psi_r = 0$. Integrating (5.25) over M we obtain as above $D_1^+\psi_k = D_2^-\psi_k = 0$, for all $k \leq m-1$ and $D_1^-\psi_k = D_2^+\psi_k = 0$, for all $k \geq 1$. From (5.21) and (5.22) it follows that $\psi_1, \ldots, \psi_{m-1}$ are parallel spinors in ΣM and thus must vanish. The first component ψ_0 has as above the form $\psi_0 = \xi_0 \otimes \varphi_r$ with $\xi_0 \in \Gamma(\Sigma_0 M_1)$ a parallel spinor on M_1 and $\varphi_r \in \Gamma(\Sigma_r M_2)$ an anti-holomorphic Kählerian twistor spinor on M_2. The last component is of the form $\psi_m = \xi_m \otimes \varphi_{r-m}$ with $\xi_m \in \Gamma(\Sigma_m M_1)$ a parallel spinor on M_1 and $\varphi_{r-m} \in \Gamma(\Sigma_{r-m} M_2)$ a holomorphic Kählerian twistor spinor on M_2.

The last possible case is when $n \leq r \leq m$. The same argument as above holds with M_1 and M_2 interchanged. As M_2 is assumed not to be Ricci-flat, then it carries no parallel spinors, showing that there are no nontrivial Kählerian twistor spinors in this case. □

Remark 5.13. From Theorem 5.12 it follows in particular that on a product of two compact spin Kähler manifolds any Kählerian twistor spinor is a special Kählerian twistor spinor. Moreover, since one of the factors must be a Ricci-flat manifold, it follows that the second factor, which in turn carries special Kählerian twistor spinors, is an irreducible Kähler manifold with holonomy $U(m)$ (from Berger's list, where we eliminate the case of symmetric manifolds, which are in particular Kähler-Einstein and thus studied in Theorem 5.9).

If $n < m$, where m is the complex dimension of the Ricci-flat factor M_1 and n the complex dimension of the other factor M_2, then Theorem 5.12 implies that there are no nontrivial Kählerian twistor spinors in $\Sigma_r M$ for $n < r < m$.

5.5 The Geometric Description

In this section we give the main result (Theorem 5.15), which is now a consequence of Theorems 5.3 and 5.12 and the following splitting result proven by V. Apostolov, T. Drăghici and A. Moroianu, [2]:

Theorem 5.14. *Let (M, g, J) be a compact Kähler manifold whose Ricci tensor has two distinct constant non-negative eigenvalues λ and μ. Then the universal cover of (M, g, J) is the product of two simply-connected Kähler-Einstein manifolds with Einstein constants λ and μ, respectively.*

Theorem 5.15. *Let (M^{2m}, g, J) be a compact simply-connected spin Kähler manifold of constant scalar curvature admitting nontrivial Kählerian twistor spinors in $\Sigma_r M$ for an r with $0 < r < m$. Then M is the product of a Ricci-flat manifold M_1 and an irreducible Kähler-Einstein manifold M_2, which must be one of the manifolds described in Theorem 5.9. More precisely, there exist anti-holomorphic (holomorphic) Kählerian twistor spinors in at most one such $\Sigma_r M$ with $r < \frac{m}{2}$ $(r > \frac{m}{2})$ and they are of the form:*

$$\psi = \xi_0 \otimes \varphi_r \quad (\psi = \xi_{2r-m-1} \otimes \varphi_{m-r+1}), \tag{5.26}$$

where $\xi_0 \in \Gamma(\Sigma_0 M_1)$ $(\xi_{2r-m-1} \in \Gamma(\Sigma_{2r-m-1} M_1))$ is a parallel spinor and $\varphi_r \in \Gamma(\Sigma_r M_2)$ $(\varphi_{m-r+1} \in \Gamma(\Sigma_{m-r+1} M_2))$ is an anti-holomorphic (holomorphic) Kählerian twistor spinor. In particular, the complex dimension of the Kähler-Einstein manifold M_2 is $2r+1$ (resp. $2(m-r)+1$).

Proof: Let (M, g, J) be a Kähler manifold as in the hypothesis of the theorem and $\varphi \in \Gamma(\Sigma_r M)$ a Kählerian twistor spinor. By Proposition 5.1, φ is a special Kählerian twistor spinor and, as usual, we may suppose that it is an anti-holomorphic Kählerian twistor spinor. Then, by Theorem 5.3 the Ricci tensor has two constant eigenvalues: $\frac{S}{2(2r+1)}$ with multiplicity $2(2r+1)$ and 0 with multiplicity $2(m-2r-1)$. From Theorem 5.14, as M is supposed to be simply-connected, it follows that M is the product of a Ricci-flat manifold M_1 and a Kähler-Einstein manifold M_2 of scalar curvature equal to $\frac{S}{2(2r+1)}$. By Theorem 5.12, ψ is of the form (5.26) with ξ_0 is a parallel spinor in $\Sigma_0 M_1$ and φ_r is an anti-holomorphic Kählerian twistor spinor in $\Sigma_r M_2$. We then conclude by applying Theorem 5.9. We notice that this result together with Corollary 5.11 also provides the dimension of the space of Kählerian twistor spinors. □

Remark 5.16. This result can be seen as a generalization of the geometric description of limiting even dimensional Kähler manifolds for Kirchberg's inequality (3.13) using the characterization in Theorem 3.5. Thus, if M is a limiting Kähler manifold of even complex dimension $m = 2l$, then it admits an anti-holomorphic Kählerian twistor spinor in $\Sigma_{l-1} M$ (or equivalently a holomorphic Kählerian twistor spinor in $\Sigma_{l+1} M$). By Theorem 5.15, M is then the product of a 2-dimensional Ricci-flat manifold M_1 and a $(4l-2)$-dimensional Kähler-Einstein manifold M_2, which is a limiting manifold for Kirchberg's inequality (3.12) in odd dimensions.

Remark 5.17. If in Theorem 5.15, (M, g) is not assumed to be simply-connected, then its universal Riemannian cover $(\widetilde{M}, \tilde{g})$ carries a unique spin structure. By a result of J. Cheeger and D. Gromoll ([9, Theorem 6.65]) $(\tilde{M}, \tilde{g})$ is isometric to a Riemannian product $(\bar{M} \times \mathbb{R}^q, \bar{g} \times g_0)$, where g_0 is the canonical flat metric on $\mathbb{R}^q$ and $(\bar{M}, \bar{g})$ is a compact simply-connected manifold with positive Ricci curvature. In order to complete the classification of Kähler spin manifolds admitting nontrivial non-extremal Kählerian twistor spinors, one has to analyze the existence of such spinors on the product $(\bar{M} \times \mathbb{R}^q, \bar{g} \times g_0)$ and the action of the fundamental group of M on $\widetilde{M}$. In the special case of limiting manifolds for the even dimensional Kirchberg inequality, this classification was obtained by A. Moroianu, see Theorem 3.7.

In particular, Theorem 5.15 together with Proposition 4.7 answer a question raised by K.-D. Kirchberg, [41], about the description of all compact Kähler spin manifolds, whose square of the Dirac operator has the smallest eigenvalue of type r.

5.6 Weakly Bochner Flat Manifolds

All known examples of Kähler spin manifolds admitting special Kählerian twistor spinors have parallel Ricci form, being thus in particular weakly Bochner flat. The purpose of this section is to show conversely, that any spin weakly Bochner flat manifold admitting special Kählerian twistor spinors must have constant scalar curvature and thus, is described in Theorem 5.15.

We first recall that a Kähler manifold (M, g, J) is called *weakly Bochner flat* if its Bochner tensor (which is defined as the projection of the Weyl tensor onto the space of Kählerian curvature tensors) is co-closed. In [1, Proposition 1], the codifferential of the Bochner tensor is computed using the Matsushima identity and it is proven that a Kähler manifold is weakly Bochner flat if and only if the normalized Ricci form defined by

$$\tilde{\rho} := \rho - \frac{1}{2(m+1)} S\Omega \tag{5.27}$$

is a hamiltonian 2-form, *i.e.* it satisfies the following equation

$$\nabla_X \tilde{\rho} = \frac{1}{4(m+1)}(dS \wedge JX - d^c S \wedge X), \tag{5.28}$$

for all vector fields X.

Proposition 5.18. *Let (M, g, J) be a spin weakly Bochner flat manifold and $\varphi \in \Gamma(\Sigma_r M)$ (with $0 < r < m$) be a nontrivial anti-holomorphic (or holomorphic) Kählerian twistor spinor. Then the scalar curvature S of the metric g is constant.*

Proof: Let $\varphi \in \Gamma(\Sigma_r M)$ (with $0 < r < m$, $r \neq \frac{m}{2}$) be a nontrivial anti-holomorphic Kählerian twistor spinor (using the isomorphism $\mathfrak{j}$ the same argument holds for a holomorphic Kählerian twistor spinor). First we notice that using the projections onto $T^{(1,0)}M$ and $T^{(0,1)}M$, the equation (5.28) is equivalent to the following equations:

$$i\nabla_{X^+}\tilde{\rho} = \frac{1}{2(m+1)} X^+ \wedge (dS)^-, \tag{5.29}$$

$$i\nabla_{X^-}\tilde{\rho} = -\frac{1}{2(m+1)} X^- \wedge (dS)^+. \tag{5.30}$$

From (5.27) and (5.29) we obtain:

$$i\nabla_{X^+}\rho = i\nabla_{X^+}\tilde{\rho} + \frac{1}{2(m+1)} iX^+(S)\Omega = \frac{1}{2(m+1)}[X^+ \wedge (dS)^- + iX^+(S)\Omega].$$

Applying this equation to φ and using (4.39) we get

$$\begin{aligned} i\nabla_{X^+}\rho \cdot \varphi &= \frac{1}{2(m+1)}(X^+ \wedge (dS)^-) \cdot \varphi + \frac{1}{2(m+1)} iX^+(S)\Omega \cdot \varphi \\ &= \frac{1}{2(m+1)} X^+(S)\varphi + \frac{m-2r}{2(m+1)} X^+(S)\varphi = \frac{m-2r+1}{2(m+1)} X^+(S)\varphi. \end{aligned} \tag{5.31}$$

On the other hand, by (4.45) we have

$$i\nabla_{X^+}\rho \cdot \varphi = \frac{1}{2(2r+1)} X^+(S)\varphi. \tag{5.32}$$

Comparing the equations (5.31) and (5.32) we get

$$\frac{m-2r+1}{2(m+1)}X^+(S)\varphi = \frac{1}{2(2r+1)}X^+(S)\varphi,$$

which is equivalent to

$$\frac{r(m-2r)}{2(m+1)(2r+1)}X^+(S)\varphi = 0.$$

As $r \neq \frac{m}{2}$ and $r \neq 0$, it follows that $X^+(S) = 0$ at all points where φ does not vanish, thus proving that S must be constant. □

Appendix A

Auxiliary formulas

We collect here some formulas that we use in our computations. In the sequel (M^n, g) is a Riemannian manifold and $\{e_i\}_{i=\overline{1,n}}$ is a local orthonormal frame.

The conventions used for the curvature are the followings: the Riemannian curvature tensor is given by $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, and the Ricci tensor is $\mathrm{Ric}(X,Y) = \sum\limits_{i=1}^{n} R(e_i, X, Y, e_i)$.

If M is also endowed with a Kähler structure whose complex structure denoted by J, then the Ricci form is defined as $\rho(X,Y) = \mathrm{Ric}(JX,Y)$, or locally as

$$\rho = \sum_{i=1}^{n} \frac{1}{2} \mathrm{Ric}(e_i) \wedge Je_i.$$

The Ricci form is closed and its codifferential and Laplacian are:

$$\delta\rho = -\frac{1}{2} JdS = -\frac{1}{2} d^c S, \quad \Delta\rho = d\delta\rho = -\frac{1}{2} dd^c S = -i\partial\bar{\partial} S. \tag{A.1}$$

It is useful to introduce the following notation. We consider the local formula (3.5) defining the Dirac operator (and the formulas (3.6) defining D^- and D^+) and apply it to sections of different associated bundles (∇ is then the connection induced by the Levi-Civita connection). Thus, when applied to functions we get:

$$Df = df, \quad D^-(f) = \partial f, \quad D^+(f) = \bar{\partial} f.$$

Applying (3.5) to a vector field X we get the following endomorphisms of the spinor bundle:

$$D(X)\cdot := \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} X\cdot,$$

$$D^2(X)\cdot := \sum_{i,j=1}^{n} e_j \cdot e_i \cdot \nabla_{e_j}\nabla_{e_i} X \cdot .$$

On forms we have: $D = d + \delta$, $D^+ = \bar{\partial} + \partial^*$, $D^- = \partial + \bar{\partial}^*$. We may also extend the formula (3.5) of the Dirac operator to endomorphisms of the tangent bundle. For instance, for the Ricci tensor we define the following endomorphism of the spinor bundle:

$$D(\mathrm{Ric})(X)\cdot := e_i \cdot (\nabla_{e_i}\mathrm{Ric})(X) \cdot .$$

With the above notation, the following relations hold (for any tangent vector field X parallel at the point where the computations are done):

$$D^-(\mathrm{Ric})(X^+) = -i\nabla_{X^+}\rho - \frac{1}{2}X^+(S), \quad D^+(\mathrm{Ric})(X^+) = 0,$$

$$D^+(\mathrm{Ric})(X^-) = i\nabla_{X^-}\rho - \frac{1}{2}X^-(S), \quad D^-(\mathrm{Ric})(X^-) = 0,$$

$$D^+((dS)^-) = D^+(\partial S) = (\bar{\partial} + \partial^*)(\partial S) = -i\Delta\rho + \frac{1}{2}\Delta S,$$

$$D^-((dS)^-) = D^-(\partial S) = (\partial + \bar{\partial}^*)(\partial S) = -\partial\bar{\partial}^* S = 0.$$

For any tangent vector field X and any k-form ω, the following formulas for the Clifford contraction hold:

$$X \cdot \omega = X \wedge \omega - X \lrcorner\, \omega,$$

$$\omega \cdot X = (-1)^k (X \wedge \omega + X \lrcorner\, \omega), \quad X \cdot \omega = (-1)^k \omega \cdot X - 2X \lrcorner\, \omega.$$

Moreover, we have the following commutator relations:

$$D(\omega \cdot \varphi) = D\omega \cdot \varphi + (-1)^k \omega \cdot D\varphi - 2(e_i \lrcorner\, \omega) \cdot \nabla_{e_i}\varphi,$$

$$D^{\pm}(\omega \cdot \varphi) = D^{\pm}\omega \cdot \varphi + (-1)^k \omega \cdot D^{\pm}\varphi - 2(e_i^{\pm} \lrcorner\, \omega) \cdot \nabla_{e_i^{\mp}}\varphi.$$

If ω is a k-form and we consider the extended action of the complex structure J on forms as a derivation, $J(\omega) := Je_i \wedge e_i \lrcorner\, \omega$, then:

$$e_i^+ \cdot (e_i^- \lrcorner\, \omega) = e_i^+ \wedge (e_i^- \lrcorner\, \omega) - e_i^+ \lrcorner (e_i^- \lrcorner\, \omega) = \frac{1}{2}(k\omega - iJ(\omega)) - e_i^+ \lrcorner (e_i^- \lrcorner\, \omega).$$

If ω is a 2-form, then the above formula becomes:

$$e_i^+ \cdot (e_i^- \lrcorner \omega) = \frac{1}{2}(2\omega - iJ(\omega)) - \omega(e_i^-, e_i^+) = \frac{1}{2}(2\omega - iJ(\omega)) - \frac{1}{2} i\omega(Je_i, e_i).$$

In particular, for the Ricci form ρ, which is a $(1,1)$-form, $J(\rho) = 0$ and it follows:

$$e_i^+ \cdot (e_i^- \lrcorner \rho) = \rho + \frac{1}{2} i\mathrm{Ric}(e_i, e_i) = \rho + \frac{1}{2} iS,$$

$$e_i^+ \cdot (e_i^- \lrcorner \Delta\rho) = \Delta\rho - \frac{1}{2} i(\Delta\rho)(Je_i, e_i) = \Delta\rho + \frac{1}{2} i\Delta S.$$

Lemma A.1. *For any vector field X and a local orthonormal frame $\{e_i\}_{i=\overline{1,n}}$ parallel at the point where the computations are done, the following commutator rules hold:*

$$DX \cdot + X \cdot D = D(X) \cdot -2\nabla_X, \tag{A.2}$$

$$D^+X^+ \cdot + X^+ \cdot D^+ = D^+(X^+)\cdot,\ D^+X^- \cdot + X^- \cdot D^+ = D^+(X^-)\cdot -2\nabla_{X^-}, \tag{A.3}$$

$$D^-X^- \cdot + X^- \cdot D^- = D^-(X^-)\cdot,\ D^-X^+ \cdot + X^+ \cdot D^- = D^-(X^+)\cdot -2\nabla_{X^+}, \tag{A.4}$$

$$[\nabla_X, D] = -\frac{1}{2}\mathrm{Ric}(X) \cdot - e_i \cdot \nabla_{\nabla_{e_i} X}, \tag{A.5}$$

$$[\nabla_{X^+}, D^+] = -\frac{1}{2}\mathrm{Ric}(X^+)\cdot - e_i^+\cdot\nabla_{\nabla_{e_i^-}X^+},\ [\nabla_{X^-}, D^+] = -e_i^+\cdot\nabla_{\nabla_{e_i^-}X^-},\ \text{(A.6)}$$

$$[\nabla_{X^-}, D^-] = -\frac{1}{2}\mathrm{Ric}(X^-)\cdot - e_i^-\cdot\nabla_{\nabla_{e_i^+}X^-},\ [\nabla_{X^+}, D^-] = -e_i^-\cdot\nabla_{\nabla_{e_i^+}X^+},\ \text{(A.7)}$$

$$[D^2, X] = -\mathrm{Ric}(X)\cdot + D^2(X)\cdot - 2\nabla_{e_i}X\cdot\nabla_{e_i}, \qquad \text{(A.8)}$$

$$D(\mathrm{Ric}(X)\cdot) + \mathrm{Ric}(X)\cdot D = -2\nabla_{\mathrm{Ric}(X)} + D(\mathrm{Ric})(X)\cdot, \qquad \text{(A.9)}$$

$$\begin{aligned}[\nabla_X, D^2] = &- \frac{1}{2}D(\mathrm{Ric})(X)\cdot + \nabla_{\mathrm{Ric}(X)} - e_j\cdot e_i\cdot R_{e_j,\nabla_{e_i}X} - e_j\cdot e_i\cdot\nabla_{\nabla_{e_i}X}\nabla_{e_j}\\ &- e_j\cdot e_i\cdot\nabla_{\nabla_{e_j}\nabla_{e_i}X} - e_i\cdot\nabla_{\nabla_{e_i}X}D, \qquad \text{(A.10)}\end{aligned}$$

Proof: We show these relations by straightforward computation using the well known elementary relations of spin geometry. For (A.2) and (A.3) we compute as follows:

$$\begin{aligned}D(X\cdot\varphi) &= e_i\cdot\nabla_{e_i}(X\cdot\varphi) = e_i\cdot\nabla_{e_i}X\cdot\varphi + e_i\cdot X\cdot\nabla_{e_i}\varphi\\ &= D(X)\cdot\varphi - X\cdot D(\varphi) - 2\nabla_X\varphi.\end{aligned}$$

$$\begin{aligned}D^+(X^+\cdot\varphi) &= e_i^+\cdot\nabla_{e_i^-}(X^+\cdot\varphi) = e_i^+\cdot\nabla_{e_i^-}X^+\cdot\varphi + e_i^+\cdot X^+\cdot\nabla_{e_i^-}\varphi\\ &= D^+(X^+)\cdot\varphi - X^+\cdot D^+(\varphi).\end{aligned}$$

$$\begin{aligned}D^+(X^-\cdot\varphi) &= e_i^+\cdot\nabla_{e_i^-}(X^-\cdot\varphi) = e_i^+\cdot\nabla_{e_i^-}X^-\cdot\varphi + e_i^+\cdot X^-\cdot\nabla_{e_i^-}\varphi\\ &= D^+(X^-)\cdot\varphi - X^-\cdot D^+(\varphi) - 2\nabla_{X^-}\varphi.\end{aligned}$$

By conjugating (A.3) we get (A.4).

For (A.5) we have:

$$\begin{aligned}\nabla_X(D\varphi) = \nabla_X(e_i \cdot \nabla_{e_i}\varphi) &= e_i \cdot R_{X,e_i}\varphi + e_i \cdot \nabla_{e_i}\nabla_X\varphi + e_i \cdot \nabla_{\nabla_X e_i - \nabla_{e_i}X}\varphi \\ &= -\frac{1}{2}\mathrm{Ric}(X)\cdot\varphi + D(\nabla_X\varphi) - e_i \cdot \nabla_{\nabla_{e_i}X}\varphi,\end{aligned}$$

and similarly for (A.6):

$$\begin{aligned}\nabla_{X^+}(D^+\varphi) &= \nabla_{X^+}(e_i^+ \cdot \nabla_{e_i^-}\varphi) \\ &= e_i^+ \cdot R_{X^+,e_i^-}\varphi + e_i^+ \cdot \nabla_{e_i^-}\nabla_{X^+}\varphi + e_i^+ \cdot \nabla_{\nabla_{X^+}e_i^- - \nabla_{e_i^-}X^+}\varphi \\ &= -\frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi + D^+(\nabla_{X^+}\varphi) - e_i^+ \cdot \nabla_{\nabla_{e_i^-}X^+}\varphi.\end{aligned}$$

$$\begin{aligned}\nabla_{X^-}(D^+\varphi) &= \nabla_{X^-}(e_i^+ \cdot \nabla_{e_i^-}\varphi) \\ &= e_i^+ \cdot R_{X^-,e_i^-}\varphi + e_i^+ \cdot \nabla_{e_i^-}\nabla_{X^-}\varphi + e_i^+ \cdot \nabla_{\nabla_{X^-}e_i^- - \nabla_{e_i^-}X^-}\varphi \\ &= D^+(\nabla_{X^-}\varphi) - e_i^+ \cdot \nabla_{\nabla_{e_i^-}X^-}\varphi.\end{aligned}$$

Again (A.7) is obtained by conjugating (A.6). For (A.8) we first compute:

$$\begin{aligned}&D(D(X)\cdot\varphi) = e_j \cdot \nabla_{e_j}(e_i \cdot \nabla_{e_i}X \cdot \varphi) \\ &= D^2(X)\cdot\varphi - e_i \cdot e_j \cdot \nabla_{e_i}X \cdot \nabla_{e_j}\varphi - 2\nabla_{e_i}X \cdot \nabla_{e_i}\varphi \\ &= D^2(X)\cdot\varphi + e_i \cdot \nabla_{e_i}X \cdot e_j \cdot \nabla_{e_j}\varphi + 2\langle e_j, \nabla_{e_i}X\rangle e_i \cdot \nabla_{e_j}\varphi - 2\nabla_{e_i}X \cdot \nabla_{e_i}\varphi \\ &= D^2(X)\cdot\varphi + D(X)\cdot D\varphi + 2e_i \cdot \nabla_{\nabla_{e_i}X}\varphi - 2\nabla_{e_i}X \cdot \nabla_{e_i}\varphi,\end{aligned}$$

which implies

$$\begin{aligned}D^2(X\cdot\varphi) &\overset{(A.2)}{=} - D(X\cdot D\varphi) + D(D(X)\cdot\varphi) - 2D(\nabla_X\varphi) \\ &\overset{(A.2)}{=} X\cdot D^2\varphi - D(X)\cdot D\varphi + 2\nabla_X(D\varphi) + D^2(X)\cdot\varphi + D(X)\cdot D\varphi \\ &\quad + 2e_i\cdot\nabla_{\nabla_{e_i}X}\varphi - 2\nabla_{e_i}X\cdot\nabla_{e_i}\varphi - 2D(\nabla_X\varphi) \\ &\overset{(A.5)}{=} X\cdot D^2\varphi - \mathrm{Ric}(X)\cdot\varphi + D^2(X)\cdot\varphi - 2\nabla_{e_i}X\cdot\nabla_{e_i}\varphi.\end{aligned}$$

The relation (A.9) is just (A.2) with X replaced by $\mathrm{Ric}(X)$. For the last relation, (A.10), we have:

$$
\begin{aligned}
\nabla_X(D^2\varphi) &\overset{(A.5)}{=} D(\nabla_X(D\varphi)) - \frac{1}{2}\mathrm{Ric}(X)\cdot D\varphi - e_i\cdot\nabla_{\nabla_{e_i}X}D\varphi \\
&= D^2(\nabla_X\varphi) - \frac{1}{2}D(\mathrm{Ric}(X)\cdot\varphi) - e_j\cdot e_i\cdot R_{e_j,\nabla_{e_i}X}\varphi - e_j\cdot e_i\cdot\nabla_{\nabla_{e_i}X}(\nabla_{e_j}\varphi) \\
&\quad - e_j\cdot e_i\cdot\nabla_{\nabla_{e_j}\nabla_{e_i}X}\varphi - \frac{1}{2}\mathrm{Ric}(X)\cdot D\varphi - e_i\cdot\nabla_{\nabla_{e_i}X}D\varphi \\
&\overset{(A.9)}{=} D^2(\nabla_X\varphi) - \frac{1}{2}D(\mathrm{Ric})(X)\cdot\varphi + \nabla_{\mathrm{Ric}(X)}\varphi - e_j\cdot e_i\cdot R_{e_j,\nabla_{e_i}X}\varphi \\
&\quad - e_j\cdot e_i\cdot\nabla_{\nabla_{e_i}X}(\nabla_{e_j}\varphi) - e_j\cdot e_i\cdot\nabla_{\nabla_{e_j}\nabla_{e_i}X}\varphi - e_i\cdot\nabla_{\nabla_{e_i}X}D\varphi.
\end{aligned}
$$

We notice that these commutator relations become simpler if $X \in \Gamma(\mathrm{TM})$ is parallel at the point where the computations are made. □

By straightforward computations we get:

Lemma A.2. *The following formulas for contractions hold, when considered as endomorphisms of the spinor bundle restricted to the subbundle $\Sigma_r M$:*

$$e_i^+\cdot e_i^- = -2r, \quad e_i^-\cdot e_i^+ = -2(m-r), \tag{A.11}$$

$$e_i\cdot R_{e_i,Y} = \frac{1}{2}\mathrm{Ric}(Y), \quad e_i^-\cdot R_{e_i^+,Y} == \frac{1}{4}[\mathrm{Ric}(Y) + i\rho(Y)], \tag{A.12}$$

$$e_i\cdot\mathrm{Ric}(e_i) = -S,\ e_i^-\cdot\mathrm{Ric}(e_i^+) = \frac{1}{2}[e_i\cdot\mathrm{Ric}(e_i) - i\cdot e_i\cdot\mathrm{Ric}(Je_i)] = -\frac{S}{2} - i\rho, \tag{A.13}$$

$$e_i^+\cdot Ric^2(e_i^-) = \frac{1}{2}[e_i\cdot Ric^2(e_i) - iJ(e_i)\cdot Ric^2(e_i)] = -\frac{1}{2}tr(Ric^2) + i\rho_2. \tag{A.14}$$

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Index of Notation

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen - noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde.

Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Uwe Semmelmann betreut worden.

Teilpublikationen:
Die in Abschnitten 1.1 und 1.2, beziehungsweise in Kapiteln 4 und 5 dargestellten Resultate wurden von mir zum Teil bereits auf dem Preprint-server *arXiv.org* (`http://arxiv.org/`) in den Preprints `math/0908.2413v1` bzw. `math/0812.3315v1` veröffentlicht.

____________________ ____________________

(Ort, Datum) (Mihaela Veronica Pilca)