Universität Stuttgart

Diplomarbeit

On the Lichnerowicz Laplace operator and its application to stability of spacetimes

Autor:
Peter-Simon Dieterich

Erstgutachter:
Prof. Dr. Uwe Semmelmann

Zweitgutachter:
Prof. Dr. Günter Wunner

vorgelegt am 13. Februar 2013

Contents

1 Introduction

The Lichnerowicz Laplacian Δ_L of a (semi)-Riemannian manifold (M, g) is a differential operator of order two acting on symmetric traceless 2-tensor fields $\mathcal{S}^2(M)$, which can be seen as infinitesimal deformations of the metric g, and describes the change of the Ricci tensor in terms of these infinitesimal deformations. If the metric g is Einstein and $h \in \mathcal{S}^2(M)$ is an eigentensor such that

$$\Delta_L h = 2\frac{s_g}{n} g, \quad \delta_g h = 0, \quad \operatorname{tr}_g h = 0, \tag{1.1}$$

where δ_g is the divergence, then the infinitesimal deformation h preserves the Einstein equation. Thus the spectrum of Δ_L is connected to the stability analysis of Einstein metrics.

The Lichnerowicz Laplacian also appears in stability analysis in physics, albeit in a different way. In higher dimensional gravity theories, solutions of the form $M = B \times M_q$, where B is a four-dimensional Lorentz manifold and M_q is a q-dimensional compact manifold with diameter of the order of the Planck length, are understood as ground states. Physical fields are then introduced as solutions to Eq. (1.1) and are interpreted as fluctuations around the ground state. The (warped) product structure of M allows to expand solutions to Eq. (1.1) in terms of eigentensors of the Lichnerowicz Laplacian $\tilde{\Delta}_L$ of M_q. This leads (among other cases) to scalar fields on B whose mass depends on the spectrum of $\tilde{\Delta}_L$. Most notably, the spectrum of the Lichnerowicz Laplacian on compact candidates M_q determines the mass spectrum of the effective four-dimensional theory. This possibly leads to particles with imaginary masses which are considered an instability in the case where B is Minkowski space.

Our goal in this thesis was to investigate the spectrum of the Lichnerowicz Laplacian on certain types of compact n-dimensional Einstein manifolds (M, g) with scalar curvature $s > 0$ and whether the smallest eigenvalue λ of the Lichnerowicz Laplacian violates the stability condition

$$\lambda \geq \frac{s}{n(n-1)} \left(4 - \frac{1}{4}(n-5)^2 \right) \tag{1.2}$$

which was established in [GH02, GHP02] for generalised Schwarzschild-Tangherlini spacetimes and Anti-de Sitter product spaces. We will call spaces violating Eq. (1.2) *physically unstable* in the context of the corresponding physical theories.

After giving a detailed introduction in Section 2 on how the Lichnerowicz Laplacian arises in the stability analysis of Einstein metrics and modern Kaluza-Klein theories, we will see in Section 3 how the Lichnerowicz Laplacian can be understood as the special case of a universal Weitzenböck formula and give a first lower bound on the eigenvalues, which was first found in [PP84].

For $n \leq 8$ the critical eigenvalue of Eq. (1.2) is positive and a non-trivial kernel of the Lichnerowicz Laplacian is a sufficient condition for physical instability. It is well known [PP84] that product manifolds admit symmetric 2-tensor fields that are parallel and therefore lie in the kernel of the Lichnerowicz Laplacian. In Section 4 we will give a proof of the following theorem using holonomy theory.

Theorem 4.4 ([PP84])**.** *Let (M^n, g) be a locally reducible connected Riemannian manifold. Then there exists an $h \in \mathcal{S}^2M$, $h \neq 0$ such that $\operatorname{tr} h = \delta h = 0$ and $\Delta_L h = 0$.*

Moreover, if $2 \leq n \leq 8$, then M is physically unstable.

On Kähler manifolds the complex structure J commutes with the Lichnerowicz Laplacian. We identify in Section 5 primitive harmonic J-invariant 2-forms with symmetric traceless 2-tensor fields that lie in the kernel of the Lichnerowicz Laplacian and are able to prove the following corollary.

Corollary 5.21. *If M is a compact Kähler manifold of dimension 4, 6 or 8 and $h^{1,1}(M) > 1$, then M is physically unstable.*

If $M = G/H$ is a Riemannian symmetric space of compact type, the Lichnerowicz Laplacian agrees with the Casimir operator. We show in Section 6 how the problem of determining the spectrum of the Casimir and, consequently, the Lichnerowicz Laplacian can be reduced to finding the irreducible G-representations whose restrictions to H contain the defining representation of $\mathrm{Sym}^2_0 M$. Stability and rigidity of Einstein metrics on Riemannian symmetric spaces has been determined by N. Koiso in [Koi80]. We use his results to prove the following theorem:

Theorem 6.11. *All irreducible Riemannian symmetric spaces of compact type are physically stable.*

Finally, in Section 7 we discuss the case when (M, g) is a compact spin manifold admitting Killing spinors. Following [Wan91], we will give a connection between the Rarita-Schwinger operator $D_{3/2}$ acting on spinor valued 1-forms and the Lichnerowicz Laplacian and show the following Corollary.

Corollary 7.13 ([GHP02])**.** *If (M, g) is a Riemannian spin manifold with positive scalar curvature admitting a non-zero Killing spinor, then M is physically stable.*

German translation of the introduction

Der Lichnerowicz Laplace Operator Δ_L einer (semi)-Riemannschen Mannigfaltigkeit (M, g) ist ein Differentialoperator zweiter Ordnung auf spurfreien symmetrischen 2-Tensorfeldern $\mathcal{S}^2(M)$, welche als infinitesimale Deformationen aufgefasst werden können, und beschreibt die Änderung des Ricci-Tensors unter diesen infinitesimalen Deformationen. Wenn die Metrik g Einstein ist und $h \in \mathcal{S}^2(M)$ ein Eigentensor ist, so dass

$$\Delta_L h = 2\frac{s_g}{n} g, \quad \delta_g h = 0, \quad \operatorname{tr}_g h = 0, \tag{1.1}$$

wobei δ_g die Divergenz bezeichnet, dann lässt die infinitesimale Deformation h die Einstein-Gleichungen invariant. Folglich spielt das Spektrum von Δ_L eine wichtige Rolle in der Stabilitätsuntersuchung von Einsteinmetriken.

Der Lichnerowicz Laplace Operator kommt auch in der Stabilitätsuntersuchung in der Physik vor, allerdings auf eine andere Art und Weise. In Gravitationstheorien in höheren Dimensionen untersucht man Lösungen der Form $M = B \times M_q$, wobei B eine vierdimensionale lorentzsche Mannigfaltigkeit und M_q eine q-dimensionale kompakte Mannigfaltigkeit in der Größenordnung der Plancklänge ist. Lösungen dieser Art werden als Grundzustand verstanden. Physikalische Felder werden dann als Lösungen von Gleichung (1.1) eingeführt und als Fluktuation oder Störung dieses Grundzustandes betrachtet. Die (verzerrte) Produktstruktur von M erlaubt es, Lösungen von Gleichung (1.1) in Eigenmoden bezüglich des Lichnerowicz Laplace Operators $\tilde{\Delta}_L$ auf M_q zu entwickeln. Dies führt unter anderem zu skalaren Feldern auf B deren Masse vom Spektrum von $\tilde{\Delta}_L$ abhängt. Das heißt also, dass das Spektrum des Lichnerowicz Laplace Operators auf M_q das Massenspektrum der effektiven vierdimensionalen Theorie bestimmt. Dies kann unter Umständen zu Teilchen mit imaginärer Masse führen, welche, im Falle dass B der Minkowskiraum ist, als Instabilität betrachtet werden.

Ziel dieser Arbeit war es, das Spektrum des Lichnerowicz Laplace Operators auf bestimmten Typen von kompakten n-dimensionalen Einsteinmannigfaltigkeiten (M, g) mit Skalarkrümmung $s > 0$ zu untersuchen und zu überprüfen, ob der kleinste Eigenwert λ des Lichnerowicz Laplace Operators die Stabilitätsbedingung

$$\lambda \geq \frac{s}{n(n-1)} \left(4 - \frac{1}{4}(n-5)^2 \right) \tag{1.2}$$

verletzt. Diese Bedingung wurde in [GH02, GHP02] für verallgemeinerte Schwarschild-Tangherlini Raumzeiten und Anti-de Sitter Produkträume hergeleitet. Wir nennen Räume, welche Gleichung (1.2) verletzen im Kontext der entsprechenden physikalischen Theorie *physikalisch instabil*.

Nach einer detaillierten Einführung über die Rolle des Lichnerowicz Laplace Operators in der Stabilitätsuntersuchung von Einsteinmetriken und modernen Kaluza-Klein Theorien in Abschnitt 2 werden wir in Abschnitt 3 sehen, wie der Lichnerowicz Laplace Operator als Spezialfall einer universellen Weitzenböck-Formel auftaucht und geben eine erste untere Schranke für die Eigenwerte, die von Page und Pope in [PP84] erstmals bewiesen wurde.

Für $n \leq 8$ ist der kritische Eigenwert von Gleichung (1.2) positiv und ein nicht-trivialer Kern des Lichnerowicz Laplace Operators ist eine hinreichende Bedingung für physikalische Instabilität. Es ist bekannt [PP84], dass Produktmannigfaltigkeiten symmetrische 2-Tensorfelder zulassen, die parallel sind und im Kern des Lichnerowicz Laplace Operators liegen. In Abschnitt 4 geben wir einen Beweis des folgenden Theorems mithilfe von Holonomietheorie.

Theorem 4.4 ([PP84])**.** *Sei (M^n, g) eine lokal reduzible Mannigfaltigkeit. Dann existiert ein $h \in \mathcal{S}^2 M$, $h \neq 0$, so dass, $\operatorname{tr} h = \delta h = 0$ und $\Delta_L h = 0$.*

Gilt außerdem $2 \leq n \leq 8$, dann ist M physikalisch instabil.

Auf Kählermannigfaltigkeiten kommutiert die komplexe Struktur J mit dem Lichnerowicz Laplace Operator. Wir identifizieren in Abschnitt 5 primitive harmonische J-invariante 2-Formen mit spurlosen symmetrischen 2-Tensorfeldern, die im Kern des Lichnerowicz Laplace Operators liegen und zeigen folgendes Korollar.

Korollar 5.21. *Ist M eine kompakte Kählermannigfaltigkeit der Dimension 4, 6 oder 8 und ist $h^{1,1}(M) > 1$, so ist M physikalisch instabil.*

Wenn $M = G/H$ ein Riemannscher symmetrischer Raum von kompaktem Typ ist, so stimmt der Lichnerowicz Laplace Operator mit dem Casimir Operator von G überein. Wir zeigen in Abschnitt 6, wie das Problem der Bestimmung des Spektrums des Casimirs und folglich auch des Lichnerowicz Laplace Operators darauf reduziert werden kann, die irreduziblen G-Darstellungen, deren Einschränkung auf H die definierende Darstellung von $\mathrm{Sym}_0^2 M$ enthalten, zu bestimmen. Stabilität von Einsteinmetriken auf Riemannschen symmetrischen Räumen von kompaktem Typ wurde von N. Koiso in [Koi80] untersucht. Wir zeigen, aufbauend auf seinen Ergebnissen, das folgende Theorem.

Theorem 6.11. *Alle irreduziblen Riemannschen symmetrischen Räume von kompaktem Typ sind physikalisch stabil.*

Schließlich diskutieren wir in Abschnitt 7 den Fall, in dem (M, g) eine kompakte Spin-Mannigfaltigkeit mit Killing-Spinoren ist. Wir zeigen , gemäß [Wan91], einen Zusammenhang zwischen dem Rarita-Schwinger Operator $D_{3/2}$, der auf spinorwertigen 1-Formen operiert, und dem Lichnerowicz Laplace Operator auf und beweisen folgendes Korollar.

Korollar 7.13 ([GHP02]). *Sei* (M, g) *eine Riemannsche Spin-Mannigfaltigkeit mit positiver Skalarkrümmung und Killing-Spinor. Dann ist* (M, g) *physikalisch stabil.*

2 Preliminaries

Let (M, g) be a closed connected Riemannian manifold together with its Levi-Civita connection ∇.

Conventions and Definitions. (1) Throughout this thesis, if not specified otherwise, we will be using the Einstein summation convention.

(2) We denote by $\Lambda^k M$ the bundle of k-forms on M, i.e. $\Lambda^k M = \Lambda^k(TM^*)$. Similarly, we define the bundle of symmetric k-tensors $\mathrm{Sym}^k M = \mathrm{Sym}^k(TM^*)$. We will sometimes implicitly identify the tangent and cotangent bundle via g and sometimes we will also implicitly use the metric to change the tensor type, e.g. we will identify $h \in \mathrm{Sym}^2 M$ with g-symmetric endomorphisms via $h(X) := h(X, e_i)e_i$ and vice versa.

If EM is a vector bundle over M, we will denote the *space of sections* by $\Gamma(EM)$. In particular, we will set $\mathfrak{X}(M) = \Gamma(TM)$, $\mathcal{S}^k(M) = \Gamma(\mathrm{Sym}^k M)$ and $\Omega^k(M) = \Gamma(\Lambda^k M)$ for the space of vector fields, symmetric k-tensor fields and differential k-forms, respectively.

(3) By a *Lorentz metric* g we understand a metric of signature $(1, n-1)$. A *Lorentz manifold* is a semi-Riemannian manifold equipped with a Lorentz metric.

(4) We define the Riemannian curvature tensor

$$\begin{aligned} R_{X,Y}Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \\ R(X, Y, Z, V) &= g(R_{X,Y}Z, V), \end{aligned}$$

the Ricci tensor

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \mapsto R_{Z,X}Y) = R(e_i, X, Y, e_i),$$

and the scalar curvature of (M, g)

$$s = \mathrm{tr}\,\mathrm{Ric} = \mathrm{Ric}(e_i, e_i)$$

for vector fields X, Y, Z and a local orthonormal frame $\{e_i\}$ on M. $\diamond$

Definition 2.1. The *Lichnerowicz Laplace* operator

$$\Delta_L = \nabla^*\nabla - \mathrm{Ric}_\star - 2\mathring{R} \tag{2.1}$$

is a second order elliptic differential operator acting on symmetric 2-tensor fields on M. Here $\mathrm{Ric}_\star$ is the induced action of $\mathrm{Ric} \in \mathrm{End}(TM)$ on $\mathrm{Sym}^2 M$ and $\mathring{R} \in \mathrm{End}(\mathrm{Sym}^2 M)$ is defined as $(\mathring{R}h)(X, Y) = R(e_i, X, Y, e_j)h(e_i, e_j)$. $\rtimes$

2.1 Deformations

Let (M, g) be a closed connected Riemannian manifold. Denote by $\mathcal{M}$ the manifold of Riemannian metrics on M and by $\mathcal{D}$ the group of diffeomorphisms of M which acts on $\mathcal{M}$ via pullback. We want to consider *deformations* of g, i.e. smooth curves $g(t)$ in $\mathcal{M}$ with $g(0) = g$.

If two metrics g and $\tilde{g}$ are in the same $\mathcal{D}$-orbit, then (M, g) and $(M, \tilde{g})$ share the same geometric properties. In physics, this would be interpreted as a different choice of gauge. Therefore the isometry classes of metrics are described by the quotient $\mathcal{M}/\mathcal{D}$ called the space of *Riemannian structures*.

We are thus only interested in deformations changing the geometric structure of the manifold. Using the slice theorem of D. Ebin (c.f. [Ebi70]) we get the existence of a real analytic submanifold S of $\mathcal{M}$, called the *slice* to the action of $\mathcal{D}$, which is normal to the $\mathcal{D}$-orbit of g at g.

On an infinitesimal level, since $\mathcal{M}$ is a positive cone in the space of symmetric 2-tensor fields $\mathcal{S}^2M$, we may identify the tangent space of $T_g\mathcal{M}$ of $\mathcal{M}$ at g with $\mathcal{S}^2M$; i.e. for a deformation $g(t)$ the element $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} g(t)$ is in $\mathcal{S}^2M$. We call such a tensor field an *infinitesimal deformation* of g.

Let $\delta^* : \mathcal{S}^kM \to \mathcal{S}^{k+1}M$ be the differential operator obtained by composing the covariant derivative $\nabla : \mathcal{S}^kM \to \mathcal{S}^1M \otimes \mathcal{S}^kM$ with symmetrization $\mathcal{S}^1M \otimes \mathcal{S}^kM \to \mathcal{S}^{k+1}M$. Its formal adjoint is called the divergence and is denoted by $\delta : \mathcal{S}^{k+1}M \to \mathcal{S}^kM$ which is just the restriction of the adjoint covariant derivative ∇^* to $\mathcal{S}^kM$. On 1-forms δ^* satisfies $\delta^*(X) = -\frac{1}{2}L_Xg$ where L denotes the Lie derivative.

Let EM be a vector bundle over M and $e, f \in \Gamma(EM)$. We define the *global scalar product* $\langle \cdot, \cdot \rangle : \Gamma(EM) \times \Gamma(EM) \to \mathbb{R}$ as

$$\langle e, f \rangle = \int_M g\,(e, f)\,\mu_g.$$

The following Lemma is an infinitesimal version of Ebin's slice theorem describing the tangent space of $\mathcal{M}$ at g.

Lemma 2.2. *The tangent space $T_g\mathcal{M} = \mathcal{S}^2M$ decomposes into*

$$\mathcal{S}^2M = \mathrm{Im}(\delta^*|_{\mathcal{S}^1(M)}) \oplus \mathrm{Ker}(\delta|_{\mathcal{S}^2(M)})$$

which is orthogonal with respect to the global scalar product and where $\mathrm{Im}(\delta^*|_{\mathcal{S}^1(M)})$ *is the tangent space of the $\mathcal{D}$-action.*

Sketch of proof. Let $g(t)$ be a deformation of g lying in the $\mathcal{D}$-orbit of g. Then $g(t) = \eta(t)^*g$ for a smooth family of diffeomorphisms $\eta(t)$ with $\eta(0) = \mathrm{id}$. Now let $p \in M$

then $\eta(t)(p)$ is a smooth curve in p and for each $p \in M$, $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \eta(t)(p) \in T_pM$. Thus $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \eta(t)$ may be identified with some vector field $X \in \mathfrak{X}(M)$. On the other hand, the flow ϕ_t of X is also a 1-parameter family of diffeomorphisms satisfying $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \phi_t = X$.

Finally, $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \eta(t)^* g = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \phi_t^* g$ which is by defininition $L_X g$, the Lie derivative of g along X. Thus $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \eta(t)^* g = -2\delta^*(X)$.

The decomposition of $\mathcal{S}^2(M)$ follows because the principal symbol of δ^* is injective. For details see [BE69] and [Ebi70]. □

2.2 Einstein deformations

A metric g is called *Einstein* if it satisfies the Einstein equation

$$\mathrm{Ric}_g = \frac{s_g}{n} g. \tag{2.2}$$

A (semi)-Riemannian manifold (M, g) is called *Einstein manifold* if the metric g is Einstein.

An *Einstein deformation* is a deformation $g(t)$ of an Einstein metric g which is orthogonal to the orbit of the $\mathcal{D}$-action such that each $g(t)$ satisfies Eq. (2.2) and the volume of $g(t)$ is constant in t. An *infinitesimal Einstein deformation* is the corresponding infinitesimal deformation.

Lemma 2.3. *The infinitesimal Einstein deformations are solutions $h \in \mathcal{S}^2 M$ of the system*

$$\Delta_L h = 2\frac{s_g}{n} g, \quad \delta_g h = 0, \quad \mathrm{tr}_g\, h = 0, \tag{1.1}$$

where Δ_L is the Lichnerowicz Laplacian.

Proof. The formulas for various curvature tensors in local coordinates show that the maps $g \mapsto R_g$ (resp. $g \mapsto \mathrm{Ric}_g$, $g \mapsto s_g$) are quasilinear second order differential operators from $\mathcal{M}$ to $\Gamma(\mathrm{Sym}^2(\Lambda^2 M))$ (resp. $\mathcal{S}^2 M$, $C^\infty(M)$). They are differentiable and for a given metric g the differentials at g are linear second order differential operators which we will denote by R'_g, Ric'_g and s'_g.

We need the following formulas for the differentials at g in the direction of h (c.f. [Bes87, chapter 1.K])

$$\begin{aligned}
\mathrm{Ric}'_g\, h &= \frac{1}{2}\Delta_L - \frac{1}{2}\,\mathrm{Hess}(\mathrm{tr}_g\, h) - \delta^*_g(\delta_g h), \\
s'_g h &= \Delta_g(\mathrm{tr}\, h) + \delta_g(\delta_g h) - g(\mathrm{Ric}_g, h), \\
\mathrm{vol}(M)'_g h &= -\frac{1}{2}\int_M (\mathrm{tr}_g\, h)\mu_g.
\end{aligned}$$

Taking into account a result of M. Obata [Oba62], which states that an Einstein metric other than the standard sphere is the unique metric with constant scalar curvature in its conformal class with normalized volume, we are only interested in infinitesimal deformations leaving the scalar curvature invariant. This leads, together with theory on elliptic operators,[1] to the decomposition

$$\mathcal{S}^2M = \left(\mathrm{Im}(\delta^*|_{\mathcal{S}^1(M)}) + C^\infty(M)\cdot g\right) \oplus \left(\delta_g^{-1}(0) \cap \mathrm{tr}_g^{-1}(0)\right),$$

which is orthogonal with respect to the global scalar product and where we have set $\delta_g^{-1}(0) = \mathrm{Ker}(\delta_g|_{\mathcal{S}^2M})$ and $\mathrm{tr}_g^{-1}(0) = \mathrm{Ker}(\mathrm{tr}_g|_{\mathcal{S}^2M})$.

The infinitesimal deformations of interest lie in the second component. They preserve the volume and the scalar curvature and are orthogonal to the orbit of the $\mathcal{D}$-action.

So let $g(t)$ be a deformation of g and $h \in \mathcal{S}^2M$ with $\mathrm{tr}\, h = \delta h = 0$ such that $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} g(t) = h$. Differentiating $\mathrm{Ric}_{g(t)} = (s_{g(t)}/n)g(t)$ at $t = 0$ yields

$$\mathrm{Ric}'_g h = \frac{s'_g h}{n} g + \frac{s_g}{n} h$$

which is equivalent to

$$\Delta_L h = 2\frac{s_g}{n} h. \qquad \square$$

Remark 2.4. Let (M, g) be an Einstein manifold. If $2\frac{s_g}{n}$ is not an eigenvalue of Δ_L, then g is not deformable, i.e. Einstein deformations do not exist. Consequently, its equivalence class $[g]$ in $\mathcal{M}/\mathcal{D}$ is isolated in the moduli space of Einstein structures. Such Einstein metrics g are called *rigid*.

Integrability of infinitesimal Einstein deformations into Einstein deformations is a complicated issue. In fact, N. Koiso discovered in [Koi82] that the symmetric metric g on $\mathbb{C}P^{2k} \times \mathbb{C}P^1$, $l \geq 2$ is rigid but is infinitesimally deformable. In other words there are solutions $h \in \mathcal{S}^2M$ to Eq. (1.1) which are not integrable. $\diamond$

2.3 Stability of Einstein metrics

Consider the total scalar curvature functional (also known as the *Einstein-Hilbert functional*)

$$S : \mathcal{M} \to \mathbb{R}, \quad S(g) = \int_M s_g \mu_g.$$

The following theorem is a classical result due to D. Hilbert [Hil24] and relates the solution of Einstein's equations to critical points of the variational problem of $S(g)$. Similar to variational problems in functional analysis we will give a definition for the stability of critical points of $S(g)$.

[1] For details, see [Bes87, Lemma 4.58]

Theorem 2.5. *For a compact connected Riemannian manifold (M, g) of dimension $n > 2$, the following properties are equivalent.*

(1) (M, g) *is Einstein,*

(2) g *is a critical point of* S *restricted to the set* $\mathcal{M}_1$ *of metrics of volume 1,*

(3) g *is a critical point of* S *restricted to the set* $\mathcal{N}_\mu$ *of metrics for which* $\mu_g = \mu$ *for a fixed volume element* μ *of* M.

Proof. The differentials of S and of μ_g in the direction of $h \in \mathcal{S}^2 M$ at g are given by

$$\begin{aligned} S'_g h &= \left\langle \frac{s_g}{2} g - \mathrm{Ric}_g, h \right\rangle \text{ and} \\ \mu'_g h &= -\frac{1}{2} \operatorname{tr}_g h, \end{aligned}$$

where $\langle \cdot, \cdot \rangle = \int_M g(\cdot, \cdot)\, \mu_g$ is the *global scalar product*.

Consequently,

$$\begin{aligned} T_g \mathcal{M}_1 &= \{ h \in \mathcal{S}^2 M \mid \langle h, g \rangle = 0 \} \text{ and} \\ T_g \mathcal{N}_\mu &= \{ h \in \mathcal{S}^2 M \mid \operatorname{tr}_g h = 0 \}. \end{aligned}$$

Then g is a critical point of S restricted to $\mathcal{M}_1$ (resp. $\mathcal{N}_\mu$) if and only if

$$S'_g h = \left\langle \frac{s_g}{2} g - \mathrm{Ric}_g, h \right\rangle = 0$$

for all $h \in T_g \mathcal{M}_1$ (resp. for all $h \in T_g \mathcal{N}_\mu$). In other words, if and only if the orthogonal projection of $\frac{s_g}{2} g - \mathrm{Ric}_g$ onto $T_g \mathcal{M}_1$ (resp. $T_g \mathcal{N}_\mu$) is zero. But since

$$\mathcal{S}^2 M = C^\infty(M) \cdot g \oplus \operatorname{tr}_g^{-1}(0)$$

and $T_g \mathcal{N}_\mu = \operatorname{tr}_g^{-1}(0) \subset T_g \mathcal{M}_1$, in both cases this implies $\mathrm{Ric}_g = fg$ for some $f \in C^\infty(M)$. It is well known that when this is the case, then f is constant and g is Einstein. □

Remark 2.6. Note that $S'_g h = \left\langle \frac{s}{n} g - \mathrm{Ric}, h \right\rangle$ for $h \in T_g \mathcal{N}_\mu$. ◇

Theorem 2.7 ([Koi79, Theorem 2.5])**.** *The Hessian* $(\operatorname{Hess} S)_g$ *has the form*

$$(\operatorname{Hess} S)_g(h, h) = -\frac{1}{2} \left\langle \Delta_L h - 2\frac{s_g}{n} h, h \right\rangle$$

for $h \in \delta_g^{-1}(0) \cap \operatorname{tr}_g^{-1}(0)$.

This motivates the following definition.

Definition 2.8. An Einstein metric g is called *stable* if $(\operatorname{Hess} S)_g$ restricted to $\delta_g^{-1}(0) \cap \operatorname{tr}_g^{-1}(0)$ is strictly negative or, equivalently, if the smallest eigenvalue of Δ_L is greater than $2\frac{s_g}{n}$. ⋊

Remark 2.9. Note that stability of g implies rigidity and non-deformability of g. ◇

2.4 Kaluza-Klein compactifications

First attempts to unify general relativity and electromagnetism were made by Kaluza [Kal21] and Klein [Kle26]. Their idea was that spacetime could be of higher dimension where the additional dimensions were at such length scales such that they are not directly observable.

This process of compactification, which was introduced by Kaluza and Klein and is usually termed *Kaluza-Klein compactification*, was at first done from dimension four to dimension five but can be generalized to higher dimensions.

2.4.1 The Kaluza-Klein model

A physical model describing general relativity and electromagnetism requires taking into account the electromagnetic potential A. The demand for a unified theory led Kaluza to consider an extended five dimensional spacetime (M, g) and a canonical projection $\pi : M \to B$ onto a four dimensional Lorentz spacetime $\mathbb{R}^4$. For the metric, consider the ansatz

$$g = \pi^* g' + (A + d\theta) \otimes (A + d\theta)$$

which is often referred to in the literature as the *Kaluza-Klein ansatz*. Here, g' is a Lorentz metric on $\mathbb{R}^4$, A is a 1-form on $\mathbb{R}^4$ and $\theta = x^0$ is the fifth coordinate on $\mathbb{R}^5$. Denote the remaining coordinates by x^i for $i = 1, \ldots, 4$. The idea to incorporate the 1-form A in the metric, which, of course, should in some way correspond to the 4-vector potential from electrodynamics, was inspired by the possibility that in five dimensions the Christoffel-symbols could somehow encode more than gravity in four dimensions. So certain assumptions on the form of the metric (e.g. requiring the $\mathbb{R}^4$-components g_{ij} of g for $i, j = 1, \ldots, 4$ to be independent of x^5) led to the Kaluza-Klein ansatz.

Adding a fifth non-compact dimension seems physically unreasonable as one would somehow expect this fifth dimension to have some measurable effect in our four dimensional spacetime. Oskar Klein suggested that the fifth coordinate is to be taken as periodic. In other words, the extra dimension has to be compact, i.e. a circle S^1 of length in the order of the Planck length. Consequently, π becomes a principal S^1-bundle.

Let $\pi : M \to B$ be a principal S^1-bundle over a (semi)-Riemannian manifold (B, g') with connection 1-form ω on M and metric g on M given by

$$g = \pi^* g' + \omega \otimes \omega. \tag{2.3}$$

Then we have the following theorem which is due to J. P. Bourguignon (for details, see [Bes87, FPII04]).

Theorem 2.10. *Any S^1-invariant (semi)-Riemannian metric g on the total space of a principal S^1-bundle for which the fibres are totally geodesic is of the form*

$$g = \pi^* g' + \omega \otimes \omega. \tag{2.3}$$

where ω is a connection 1-form over the bundle and g' is a (semi)-Riemannian metric on the base.

The metric g is Einstein with Einstein constant Λ if and only if

(1) the 2-form F' on B which pulls back to the curvature 2-form F of the connection ω is harmonic with constant norm $\|F'\| = \sqrt{\Lambda}$, and

(2) the Ricci tensor of B is given by $\mathrm{Ric}' - \Lambda g' = \frac{1}{2} F' \circ F'$,

where $(F' \circ F')(X, Y) := F'(X, e_i) F'(e_i, Y)$.

Thus if we start with a pure gravity theory on the total space M, (i.e. the metric of the total space is Einstein) our metric on the base satisfies the Einstein field equation

$$\mathrm{Ric}' - \Lambda g' = T$$

where $T = F' \circ F'$ is the *electromagnetic stress-energy tensor* and F' satisfies Maxwell's equations

$$\Delta F' = 0$$

where Δ is the Hodge-Laplacian.

2.4.2 Generalisation of the Kaluza-Klein ansatz

It is now natural to generalise the original Kaluza-Klein method to higher dimensions and non-abelian structure groups of the fibres, more specifically, we are now going to translate the Kaluza-Klein idea in the language of principal bundles and Riemannian submersions.

Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion with totally geodesic fibres. Let $\mathcal{V} = \ker d\pi$ denote the vertical distribution and let $\mathcal{H}$ be the horizontal distribution determined as the orthogonal complement of $\mathcal{V}$ via the metric g. Then the metric g restricted to the horizontal distribution is isometric to the metric g' on the base.

The invariant (2,1)-tensors A and T of the Riemannian submersion π introduced by O'Neill [O'N66] can be used to relate the curvature of B with that of M. Since the fibres π are by assumption totally geodesic T identically vanishes.

The tensor A is defined as

$$A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF, \quad \text{for vector fields } E \text{ and } F \text{ on } M$$

and satisfies $A_X Y = \frac{1}{2}v[X,Y]$ for horizontal vector fields X and Y, where we have denoted by h and v the projection onto the horizontal and vertical distribution of M, respectively. Moreover, A_E is a skew-symmetric endomorphism of TM and $A_E(\mathcal{V}) \subset \mathcal{H}$ and $A_E(\mathcal{H}) \subset \mathcal{V}$ for any vector field E on M.

The following proposition is a special case of O'Neills formulas [O'N66] and relates the Ricci curvature of M to the Ricci curvature of the base and the fibres via the tensor A. For details and proof we refer the reader to [Bes87] and [FPII04].

Proposition 2.11. *Let $\pi : (M^n, g) \to (B^m, g')$ be as above. We denote by* Ric, Ric' *and* $\widehat{\mathrm{Ric}}$ *the Ricci tensor of M, B and the fibres, respectively. And similarly, let s, s' and $\hat{s}$ denote the corresponding scalar curvatures. Then*

$$\mathrm{Ric}(U,V) = \widehat{\mathrm{Ric}}(U,V) + g(A_{X_i}U, A_{X_i}V), \tag{2.4a}$$

$$\mathrm{Ric}(X,Y) = \pi^* \mathrm{Ric}'(X,Y) - 2g(A_X U_j, A_Y U_j), \tag{2.4b}$$

$$\mathrm{Ric}(U,X) = g((\nabla_{X_i} A)_{X_i} X, U), \tag{2.4c}$$

and

$$\frac{m}{n} s = \pi^* s' - 2\|A\|^2, \tag{2.5a}$$

$$\frac{n-m}{n} s = \hat{s} + \|A\|^2, \tag{2.5b}$$

where U, V are vertical, X, Y are horizontal vector fields and X_i and U_j are local orthonormal frames of the horizontal and vertical distributions $\mathcal{H}$ and $\mathcal{V}$, respectively.

Corollary 2.12. *Let $\pi : (M^n, g) \to (B^m, g')$ be as above. Then (M, g) is Einstein if and only if*

$$\widehat{\mathrm{Ric}}(U,V) + g(A_{X_i}U, A_{X_i}V) = \frac{s}{n} g(U,V), \tag{2.6a}$$

$$\pi^* \mathrm{Ric}'(X,Y) - 2g(A_X U_j, A_Y U_j) = \frac{s}{n} g(X,Y), \tag{2.6b}$$

$$(\nabla_{X_i} A)_{X_i} = 0, \tag{2.6c}$$

where U, V are vertical, X, Y are horizontal and X_i and U_j are local orthonormal frames of the horizontal and vertical distributions $\mathcal{H}$ and $\mathcal{V}$, respectively.

In particular, if M is connected, the Einstein condition implies that $\|A\|$ and $\hat{s}$ are constant on the fibres. If any two fibres are isometric, they are constant on M and s' is also constant.

Remark 2.13. (1) Note that Eq. (2.4) can be interpreted as a decomposition of Ric into "block form". The horizontal part of Ric is related to the Ricci curvature of the base, the vertical part corresponds to the curvature of the fibres and the "off-diagonal part" corresponds to the Yang-Mills condition (see Item (5) below).

(2) If π is a totally geodesic Riemannian submersion (i.e. if γ is a geodesic on M then $\pi \circ \gamma$ is a geodesic on B) such that M is complete, simply connected and $\dim M > \dim B$, then M is a Riemannian product and π is the projection on one of the factors [Vil70].

(3) By a theorem of Hermann [Her60] a Riemannian submersion is a fibre bundle with structure group the isometry group of a fibre, if M is connected and the fibres are totally geodesic.

On the other hand, given $\pi : M \to B$ a principal G-bundle, a G-invariant metric $\hat{g}$ on G, a Riemannian metric g' on the base B and a connection 1-form θ on M, then by a theorem due to [Vil70] there exists one and only one Riemannian metric g on M such that π is a Riemannian submersion from (M, g) to (B, g').

(4) Assume that the Riemannian submersion π with totally geodesic fibres acts as the projection of a principal G-bundle with connection 1-form θ.

If we restrict A to horizontal vector fields, we get a 2-form that is closely related to the curvature 2-form $F^\theta = d\theta + [\theta, \theta]$. Recall that $\theta_p|_{\mathcal{V}_p}$ is a linear isometry onto the Lie algebra $\mathfrak{g}$ of G and the curvature 2-form acts as $F^\theta = -\theta(v[X, Y])$. Thus

$$(A_X Y)_p = -\frac{1}{2}(\theta_p|_{\mathcal{V}_p})^{-1}(F^\theta(X, Y)_p).$$

Using the skew-symmetry of A_E and the fact that A_E reverses the horizontal and vertical distribution for any vector field E on M it follows that $\|A\|^2 = \frac{1}{2}\|F^\theta\|^2$. Indeed,

$$\begin{aligned}
\|A\|^2 &= g(A_{X_i} X_j, A_{X_i} X_j) + g(A_{X_i} U_j, A_{X_i} U_j) \\
&= \frac{1}{4}\|F^\theta\|^2 + g(A_{X_i} X_k, U_j) g(A_{X_i} X_k, U_j) \\
&= \frac{1}{2}\|F^\theta\|^2.
\end{aligned}$$

(5) Eq. (2.6c) is also called the *Yang-Mills condition*. In fact, if π is a G-principal bundle, this condition only depends on the metric g' and the connection 1-form θ. An equivalent definition of a connection to be Yang-Mills is that the connection is a critical point of the functional

$$S_{YM}(\theta) = \frac{1}{2}\int_M \|F^\theta\|^2 \mu_g$$

whose Euler-Lagrange equations are $D_\theta^* F^\theta = 0$ where D_θ^* is the adjoint of the exterior covariant derivative D_θ. Since $D_\theta F^\theta = 0$, Yang-Mills connections are connections with harmonic curvature with respect to the exterior covariant derivative. ◇

To summarize Items (3) to (5), if we start with pure gravity on the total space of a principal G-bundle $\pi : M \to B$ whose fibres are totally geodesic, i.e. we take g as a critical point of the Einstein-Hilbert functional

$$S(g) = \int_M s_g \mu_g$$

and only vary the connection, the metric of the fibre and the metric of the base space, then our theory looks on B like gravity coupled to Yang-Mills theory for the gauge group G. In other words, using Eq. (2.5) we find that

$$s = \pi^* s' - \frac{1}{2}\|F^\theta\|^2 + \hat{s}$$

where F^θ is the curvature 2-form of the connection θ on M. That is to say, our action functional splits up into

$$S_M(g) = S_B(g') + S_{YM}(\theta) + S_{mod}(\hat{g}).$$

where S_B is the Einstein-Hilbert action on the base, S_{YM} is the action functional of Yang-Mills theory and S_{mod} is the action functional for the metric on the fibre.

2.5 Stability of spacetimes

In this section we will be looking at different physical theories and use the Kaluza-Klein approach explained above or variations thereof to gain an effective theory on a four-dimensional base manifold. We want to motivate the following definition.

Definition 2.14. Let (M, g) be a compact Riemannian Einstein manifold with positive curvature. The metric g is called *physically stable* if the smallest eigenvalue λ of the Lichnerowicz Laplace operator satisfies

$$\lambda \geq \frac{s}{n(n-1)}\left(4 - \frac{1}{4}(n-5)^2\right). \tag{1.2}$$

Conversely, if the smallest eigenvalue λ violates Eq. (1.2), then the metric is called *physically unstable*. ⋊

Of course, this definition does not apply to every physical theory. However, the condition for stability for various different theories happens to coincide with Eq. (1.2). We discuss examples in Sections 2.5.2 and 2.5.3.

First, however, we look at pure gravity on a Riemannian submersion with totally geodesic and compact fibres in Section 2.5.1. Stability turns out to be given if the metric is stable in the sense of Definition 2.8 but the approach is similar to Sections 2.5.2 and 2.5.3.

2.5.1 Pure gravity theory

Consider an $(4 + q)$ dimensional pure gravity theory with action given by

$$S(g) = \int_M s_g \mu_g$$

for Lorentz metrics.

What we are now interested in is an *effective theory* in four dimensions, that is, a low-energy limit of our ten dimensional theory.

To arrive at such a theory one considers *ground states* (M, g) that are solutions of Einstein's equations

$$\mathrm{Ric} = \frac{s}{n} g$$

and are of the form $M = B \times M_q$ where the base (B, g') is a p-dimensional Lorentz Einstein spacetime and $(M_q, \hat{g})$ is a q-dimensional compact Einstein manifold. We denote their scalar curvatures by s' and $\hat{s}$, respectively.

By a *fluctuation* of our ground state we understand a solution $h \in \Gamma(\mathrm{Sym}^2 M)$ of the wave equation

$$\Delta_L h = 2\frac{s_g}{n} g, \quad \delta_g h = 0, \quad \mathrm{tr}_g\, h = 0, \tag{1.1}$$

which is nothing else than an infinitesimal Einstein deformation of (M, g).

One now makes the ansatz (by slight abuse of notation)

$$\begin{aligned} h(U,V) &= \varphi\tilde{h}(U,V) \\ h(X,Y) &= 0 \\ h(X,U) &= 0 \end{aligned} \tag{2.7}$$

for $U, V \in TM_q$, $X, Y \in TB$ and $\varphi \in C^\infty(B)$ and $\tilde{h}$ is an eigentensor of the Lichnerowicz Laplacian $\tilde{\Delta}_L$ on M_q to the eigenvalue λ.

Now, the condition for h to be an infinitesimal Einstein deformation is equivalent to

$$\begin{aligned} \Delta_L h = 2\frac{s}{n}h \quad &\iff \quad \left(\nabla^*\nabla + 2\frac{s}{n} - 2\mathring{R}\right) h = 2\frac{s}{n}h \\ &\iff \quad \left(\nabla^*\nabla + \Delta_{M_q} - 2\mathring{R}\right)\varphi\tilde{h} = 0 \\ &\iff \quad \left(\Delta_B + \tilde{\Delta}_L - 2\frac{\hat{s}}{q}\right)\varphi\tilde{h} = 0 \\ &\iff \quad \Delta_B\varphi = -\left(\lambda - 2\frac{\hat{s}}{q}\right)\varphi \end{aligned}$$

which is the Klein-Gordon equation of a free field of mass $\sqrt{\lambda - 2\hat{s}/q}$ on B. Here, Δ_B and Δ_{M_q} denote the B and M_q part of $\nabla^*\nabla$, respectively. To avoid fields with imaginary mass or, equivalently, negative energy the compact manifold M_q has to be stable in the sense of Definition 2.8.

2.5.2 Anti-de Sitter product spacetimes

Definition 2.15. The Anti-de Sitter space $AdS_n \subset \mathbb{R}^{n+1}$ is the hypersurface

$$L^2 = -X_0^2 - X_1^2 + X_2^2 + \ldots + X_n^2$$

for some constant L, called the *length scale* of AdS_n. The standard metric on AdS_n is the metric induced from the flat metric of signature $(2,4)$

$$ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + \ldots + dX_n^2.$$

The induced metric is Einstein with scalar curvature given by

$$s_{AdS_n} = -\frac{n(n-1)}{L^2}. \qquad \rtimes$$

The Anti-de Sitter space AdS_n is the Lorentz analogon of the hyperbolic hypersurface in Euclidian $\mathbb{R}^{n+1}$.

Consider an $n = p + q$ dimensional gravity theory ($q \geq 2$) coupled to a q-form field strength $F \in \Omega^q(M)$ with equations of motion given by

$$\mathrm{Ric} = F \circ F - \frac{q-1}{n-2}\|F\|^2 \tag{2.8a}$$

$$\Delta F = 0 \tag{2.8b}$$

where $(F \circ F)(X, Y) := g(X \lrcorner F, Y \lrcorner F)$.

In [FR80] Freund and Rubin showed that this admits a direct product solution of the form $M = B \times M_q$ and $F = k\,\mathrm{vol}_{M_q}$ where (B, g') and $(M_q, \hat{g})$ are Einstein manifolds with scalar curvatures given by

$$\begin{aligned} s_B &= s' = -\frac{p(p-1)}{L^2}, \\ s_{M_q} &= \hat{s} = \frac{q(q-1)}{R^2}, \\ s_M &= s = \frac{q-p}{LR}, \end{aligned}$$

where we have chosen $k = \frac{(n-1)(q-1)}{(p-1)R^2}$ and $L = \frac{p-1}{q-1}R$ for some constant R. Note that M_q is necessarily compact.

We take (B, g') as an Anti-de Sitter space AdS_q with length scale L and M_q compact. Then the ansatz

$$h = \varphi\tilde{h}$$

where φ is a function on B and $\tilde{h}$ is an eigentensor on M_q to the eigenvalue λ (compare Eq. (2.7)) requires of φ to solve

$$\Delta\varphi = -\left(\lambda - 2'\frac{\hat{s}}{q}\right)\varphi$$

that is, φ corresponds to a scalar field on AdS_q with mass given by

$$m^2 = \lambda - 2\frac{\hat{s}}{q}.$$

Breitenlohner and Freedman showed in [BF82] that due to the negative scalar curvature of AdS_p scalar fields with imaginary mass are only considered a perturbative instability if

$$m^2L^2 < -\frac{(p-1)^2}{4} \iff \text{instability} \tag{2.9}$$

or, equivalently,

$$m^2 < -\frac{(q-1)^4}{4R^4} \iff \text{instability.} \tag{2.10}$$

Thus we get instability if the spectrum of the Lichnerowicz Laplacian Δ_L on M_q posseses eigenvalues satisfying

$$\lambda < \frac{s_{M_q}}{q(q-1)}\left(4 - \frac{1}{4}(q-5)^2\right). \tag{1.2}$$

2.5.3 Generalised Schwarzschild-Tangherlini spacetimes

A *generalised Schwarzschild-Tangherlini spacetime* is an n-dimensional Ricci-flat spacetime (M, g) with metric g given in local coordinates by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\tilde{s}^2$$

where $d\tilde{s}^2$ is an Einstein metric on a q-dimensional compact manifold B and $f(r) = (1 - (l/r)^{q-1})$, $l \in \mathbb{R}$. With this choice the metric g is Ricci-flat. For $B = S^2$ one recovers the classical Schwarzschild metric.

In [GHP02, GH02] Gibbons et al. established conditions under which these spacetimes produce a stable physical theory.

Following their approach, we are looking for solutions $h \in \mathcal{S}^2(M)$ of

$$\Delta_L h = 2\frac{s_g}{n}g, \quad \delta_g h = 0, \quad \operatorname{tr}_g h = 0. \tag{1.1}$$

We assume, similarly to Eq. (2.7), that h only deforms the metric on B, i.e. in local coordinates

$$h_{0a} = h_{1a} = 0.$$

where 0 and 1 are the t and r coordinates respectively.

A direct computation then yields

$$(\Delta_L h)_{0a} = (\Delta_L h)_{1a} = 0$$

and

$$\begin{aligned}(\Delta_L h)_{\alpha\beta} = {} & \frac{1}{r^2}(\tilde{\Delta}_L h)_{\alpha\beta} + \frac{1}{f}\frac{\partial^2}{\partial t^2}h_{\alpha\beta} - f\frac{\partial^2}{\partial r^2}h_{\alpha\beta} \\ & - \left(f' - f\frac{4-n}{r}\right)\frac{\partial}{\partial r}h_{\alpha\beta} - \frac{4f}{r^2}h_{\alpha\beta}\end{aligned}$$

where $\tilde{\Delta}_L$ is the Lichnerowicz Laplacian on B and $\alpha, \beta \in \{2, \dots, n-1\}$ are the indices corresponding to B.

Using the ansatz

$$h_{\alpha\beta}(t, r, \tilde{x}) = \tilde{h}_{\alpha\beta}(\tilde{x})r^2\varphi(r)e^{\omega t},$$

where $\tilde{\Delta}_L \tilde{h} = \lambda \tilde{h}$, and changing variables to Regge-Wheeler type

$$dr_\star = \frac{dr}{f}, \quad \Phi = r^{d/2}\varphi,$$

Eq. (1.1) can be recast as a Schrödinger equation

$$-\frac{\partial^2}{\partial r_\star^2}\Phi + V(r(r_\star))\Phi = -\omega^2\Phi,$$

with potential given by

$$V(r) = \frac{\lambda f}{r^2} + \frac{q-4}{2}\frac{f'f}{r} + (4 - \frac{1}{2}(q-5)^2)\frac{f^2}{r^2}.$$

The spacetime is considered unstable if there exist normalisable bounded solutions with negative energy $E = -\omega^2$. In [GHP02] it is shown that these exist if there are eigenvalues of the Lichnerowicz Laplacian on B satisfying

$$\lambda < \frac{\tilde{s}}{q(q-1)}\left(4 - \frac{1}{4}(q-5)^2\right). \tag{1.2}$$

3 The Lichnerowicz Laplacian

In this section we will give a definition of the Lichnerowicz Laplacian from a more abstract point of view.

3.1 The curvature endomorphism

Let (M^n, g) be a Riemannian manifold with Levi-Civita connection ∇ and holonomy group $\mathrm{Hol} = \mathrm{Hol}(M, g)$.

Definition 3.1. The *curvature operator* $\mathcal{R} : \Lambda^2 M \to \Lambda^2 M$ is defined by

$$g(\mathcal{R}(X \wedge Y), U \wedge V) = g(R_{X,Y}V, U)$$

for any vector fields X, Y, V and U on M. $\rtimes$

In a local orthonormal frame $\{e_i\}$, $\mathcal{R}$ takes the form

$$\mathcal{R}(e_i \wedge e_j) = \frac{1}{2}R_{ijrs}e_s \wedge e_r = -\frac{1}{2}e_k \wedge R_{e_i,e_j}e_k.$$

Using the identification of 2-vectors $X \wedge Y$ with $\mathrm{End}(TM)$ given by $(X \wedge Y)(Z) := g(X, Z)Y - g(Y, Z)X$, we find $\mathcal{R}(X \wedge Y)(Z) = -R_{X,Y}Z$.

For a general 2-form α a straightforward computations shows that

$$\mathcal{R}(\alpha)(X, Y) = -\frac{1}{2}R(e_i, e_j, X, Y)\alpha(e_i, e_j).$$

Applying the Bianchi identity yields

$$\mathcal{R}(\alpha)(X, Y) = R(e_i, X, Y, e_j)\alpha(e_i, e_j). \tag{3.1}$$

Recall that $\mathring{R} \in \mathrm{End}(\mathrm{Sym}^2 M)$ was defined as

$$\mathring{R}(h)(X, Y) = R(e_i, X, Y, e_j)h(e_i, e_j) \tag{3.2}$$

for $h \in \mathrm{Sym}^2 M$.

Definition 3.2. For a representation $\pi : \mathrm{Hol} \to \mathrm{Aut}(E)$, let EM be the associated vector bundle of π. Then there is a canonical fibre-wise action of $\mathfrak{hol}$ on $\mathrm{End}(EM)$ given and denoted by $A_\star := d\pi(A)$. We refer to this action as the *induced action.* $\rtimes$

Remark 3.3. If T is the defining representation of the tangent bundle TM and E is in the tensor algebra of T, (i.e. EM is a subbundle of the tensor bundle) then the induced action corresponds to the usual action of endomorphisms of TM on the tensor bundle. $\diamond$

Definition 3.4. For $\pi : \mathrm{Hol} \to \mathrm{Aut}(E)$ a representation, let EM be as in Definition 3.2. We define the *curvature endomorphism* $q(R) \in \mathrm{End}(EM)$ as

$$q(R) := -\frac{1}{2}(f_\alpha)_\star \mathcal{R}(f_\alpha)_\star$$

where $\{f_\alpha\}$ is an orthonormal basis of $\mathfrak{hol}$. ⋊

Lemma 3.5. *The Lichnerowicz Laplacian Δ_L satisfies*

$$\Delta_L h = \nabla^* \nabla h + q(R)h \tag{3.3}$$

for $h \in \mathcal{S}^2 M$.

In particular,

$$q(R) = -\mathrm{Ric}_\star - 2\mathring{R}$$

where $\mathring{R} \in \mathrm{End}(\mathrm{Sym}^2 M)$ is defined as $(\mathring{R}h)(X,Y) := R(e_i, X, Y, e_j)h(e_i, e_j)$.

Proof. Assume $\mathrm{Hol} = \mathrm{SO}(n)$ and let $h \in \mathrm{Sym}^2 M$ and $A \in \mathrm{End}(TM)$, then the induced action is given by

$$-A_\star h(X,Y) = h(AX,Y) + h(X,AY).$$

Therefore,

$$\begin{aligned} -2q(R)h(X,Y) = h(&\mathcal{R}(e_i \wedge e_j)(e_i \wedge e_j)X, Y) \\ &+ 2h((e_i \wedge e_j)X, \mathcal{R}(e_i \wedge e_j)Y) \\ &+ h(X, \mathcal{R}(e_i \wedge e_j)(e_i \wedge e_j)Y) \end{aligned}$$

where we have used $\mathcal{R}(e_i \wedge e_j)X \otimes (e_i \wedge e_j)Y = (e_i \wedge e_j)X \otimes \mathcal{R}(e_i \wedge e_j)Y$ due to symmetries of R.

But since

$$\mathcal{R}(e_i \wedge e_j)(e_i \wedge e_j)X = 2\,\mathcal{R}(X \wedge e_j)e_j = -2R_{X,e_j}e_j = -2\,\mathrm{Ric}(X)$$

the first and the third term can be simplified to $\mathrm{Ric}_\star h(X,Y)$.

Finally, the second term

$$\begin{aligned} h((e_i \wedge e_j)X, \mathcal{R}(e_i \wedge e_j)Y) &= -h((e_i \wedge e_j)X, R_{e_i,e_j}Y) \\ &= -h(X^i e_j - X^j e_i, R_{e_i,e_j}Y) \\ &= -2h(e_i, R_{X,e_i}Y) =: 2\mathring{R}h(X,Y) \end{aligned}$$

concludes the proof. □

Remark 3.6. Using the curvature endomorphism we get a Laplace operator for any Hol-representation π. These Laplacians are called *universal Laplace operators*. For example in the case $E = \Lambda^k T$ we have the classical Weitzenböck formula $\Delta_\pi = \nabla^*\nabla + q(R) = dd^* + d^*d$ for the Hodge Laplacian operating on k-forms. In particular $q(R) = -\operatorname{Ric}_\star$ on 1-forms and $q(R) = -\operatorname{Ric}_\star - 2\mathcal{R}$ on 2-forms. ◇

3.2 Generalized gradients

The definition of the Lichnerowicz Laplacian and the universal Laplace operator is of course compatible with the reduction of the principal bundle from $\mathrm{SO}(n)$ to Hol.

Consider a representation $\pi : \mathrm{Hol} \to E$ of the holonomy group and the decomposition $E \otimes T = \bigoplus_{i=1}^r V_i$ into irreducible components and denote by $\mathrm{pr}_i : E \otimes T \to V_i$ the orthogonal projection onto the i-th component. We then define the *generalized gradients* P_i of the Levi-Civita connection ∇ as $P_i := \mathrm{pr}_i \circ \nabla$ for $i = 1, \dots, r$. For example, the rough Laplacian can be written as

$$\nabla^*\nabla = \sum_{i=1}^r P_i^* P_i.$$

In [SW10] Semmelmann and Weingart expressed the curvature endomorphism using the conformal weight operator

$$B_{a\otimes b}\psi := \mathrm{pr}_{\mathfrak{hol}}(a \wedge b)_\star \psi \in \mathrm{Hom}_{\mathfrak{hol}}(T \otimes T, \operatorname{End} E)$$

for $a, b \in T$, $\psi \in E$ and $\mathrm{pr}_{\mathfrak{hol}}$ the projection $\mathrm{pr}_{\mathfrak{hol}} : \Lambda^2 T \to \mathfrak{hol}$. The curvature endomorphism acting on a section ψ of EM can then be written as

$$q(R)\psi = B(\nabla^2\psi) = B_{e_i \otimes e_j} \nabla^2_{e_i, e_j} \psi.$$

The conformal weight operator B can be seen as an equivariant endomorphism of $E \otimes T$, thus acting as a scalar on irreducible components by Schur's lemma. Then $q(R)$ takes the form

$$q(R) = -\sum_{i=0}^r b_i P_i^* P_i$$

where the numbers b_i are the eigenvalues of B on the irreducible components V_i of $E \otimes T$ and the universal Laplace operator can be written as

$$\Delta_\pi = \sum_{i=1}^r (1 - b_i) P_i^* P_i.$$

In the case $\mathrm{Hol} = \mathrm{SO}(n)$ and $E = \mathrm{Sym}^2_0 T$ consider the irreducible decomposition

$$T \otimes \mathrm{Sym}^2_0 T = \mathrm{Sym}^3_0 T \oplus T \oplus \Gamma^{(2,1)}$$

with corresponding generalized gradients P_1, P_2 and P_3. Here $\Gamma^{(2.1)}$ is the irreducible representation corresponding to the highest weight $(2, 1, 0, \dots, 0)$ for a choice of roots and dominant weights as in Section 6.3. A computation of the eigenvalues (c.f. [SW10, Corollary 3.4]) of B yields $b_1 = 2$, $b_2 = -n$, $b_3 = -1$. Consequently,

$$\Delta_L = -P_1^* P_1 + (n+1) P_2^* P_2 + 2 P_3^* P_3.$$

Notation. Let $A, B \in \mathrm{End}(EM)$ for some vector bundle EM over a compact manifold M. We write

$$A \geq B \quad :\Longleftrightarrow \quad \langle h, Ah \rangle \geq \langle h, Bh \rangle \quad \text{for all } h \in \Gamma(EM). \qquad \diamond$$

Proposition 3.7 ([PP84, SW12])**.** *If Δ_L is restricted to traceless and divergence-free symmetric 2-tensors, then*

$$\Delta_L \geq 2q(R) \tag{3.4}$$

where equality is achieved for $h \in \mathcal{S}^2 M$, $\operatorname{tr} h = \delta h = 0$ if and only if $P_1 h = 0$.

Moreover, if (M, g) is a naturally reductive homogeneous manifold $M = G/H$ with G-invariant metric g, then the eigenvalues of $q(R)_x$ for $x \in M$ do not depend on x and

$$\Delta_L \geq 2\sigma, \tag{3.5}$$

where σ is the smallest eigenvalue of $q(R)$. If additionally (M, g) is Einstein, then $\sigma = 2s/n - 2\kappa$ and

$$\Delta_L \geq 4s/n - 4\kappa \tag{3.6}$$

where κ is the largest eigenvalue of $\mathring{R}$.

Proof. If h is divergence-free, then $P_2 h = 0$. Indeed, the map $\mathrm{pr}_2 : \nabla h \mapsto \delta h$ is equivariant. Thus

$$\Delta_L - 2q(R) = \nabla^* \nabla - q(R) = 3 P_1^* P_1 \geq 0.$$

In the case that $M = G/H$ is naturally reductive homogeneous, $q(R)$ is invariant under G acting by isometries and by [KN96, Chapter X, Theorem 2.6], $\nabla q(R) = 0$ for the canonical connection ∇ of M. Hence, the eigenvalues of $q(R)$ and $\mathring{R}$ are constant on M and

$$\langle h, q(R) h \rangle \geq \min_{h \in \mathcal{S}^2 M} \langle h, q(R) h \rangle \geq \sigma \|h\|^2 \quad \text{for all } h \in \mathcal{S}^2 M. \qquad \square$$

Remark 3.8. The projection P_1 corresponds to the symmetrization of ∇h. Tensors satisfying $P_1 h = 0$ are known as Stäckel tensors. $\diamond$

4 Parallel endomorphisms

It is well known [DNP84, DFG$^+$02] that Einstein product manifolds are physically unstable as they admit a trace- and divergence-free solution in the kernel of the Lichnerowicz Laplacian that corresponds to shrinking one factor and expanding the other while keeping the volume constant. In the following we will derive this result using holonomy theory.

Consider $h \in \mathrm{End}(TM)$ such that h is g-symmetric and $\nabla h \equiv 0$. On $\mathrm{End}(TM)$ the induced action of $A \in \mathrm{End}(TM)$ is given by

$$A_\star f = A \circ f - f \circ A, \quad \text{for } f \in \mathrm{End}(TM).$$

It follows, that $\mathcal{R}(X \wedge Y)_\star h = R_{Y,X} \circ h - h \circ R_{Y,X} = R_{Y,X} h$, which is identically zero, since h is parallel. Now, recall from Lemma 3.5 that $\Delta_L = \nabla^*\nabla + q(R)$ where $q(R) = -\frac{1}{2}(e_i \wedge e_j)_\star \mathcal{R}(e_i \wedge e_j)_\star$. We have thus just proved the following proposition:

Proposition 4.1. *Consider $h \in \mathrm{End}(TM)$ such that $\nabla h \equiv 0$. Then $\Delta_L h = 0$.*

Existence of parallel endomorphisms (or sections of vector bundles in general) is closely tied to holonomy theory, in particular to the decomposition into irreducible components of T_xM as a $\mathrm{Hol}_x(M,g)$-representation. The following proposition is a consequence of the *holonomy principle* [Bau09, Satz 5.3] and Schur's lemma.

Proposition 4.2. *Let (M,g) be a connected Riemannian manifold, $h \in \mathrm{End}(TM)$ g-symmetric and fix $x \in M$. Consider the direct sum decomposition of*

$$T_xM = E_1 \oplus \ldots \oplus E_r$$

into irreducible $\mathrm{Hol}_x(M,g)$-representations. Let $\mathcal{E}_k$ be the corresponding involutive distribution attained by parallel transport of E_k and denote by

$$\mathrm{pr}_k : TM = \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_r \to \mathcal{E}_k$$

the projection onto the k-th distribution.

Then $\nabla h = 0$ if and only if $h = \sum_{k=1}^r \lambda_k \,\mathrm{pr}_k$ for some $\lambda_k \in \mathbb{R}$.

Proof. Let h be a g-symmetric endomorphism with $\nabla h = 0$. Then by [Bau09, Satz 5.3] $\nabla h = 0$ is equivalent to h_x being invariant under the action of $\mathrm{Hol}_x(M,g)$ for $x \in M$. This action is the usual action on endomorphisms given by $(g \cdot h_x)(v) = g h_x(g^{-1}v)$ for $g \in \mathrm{Hol}_x(M,g)$, $v \in T_xM$ and invariance is equivalent to $h_x \circ g = g \circ h_x$ for $g \in \mathrm{Hol}_x(M,g)$. Consequently, h_x is a $\mathrm{Hol}_x(M,g)$ equivariant endomorphism and by Schur's lemma it is of the form $h_x = \sum_{k=1}^{r} \lambda_k (\mathrm{pr}_k)_x$ for some $\lambda_k \in \mathbb{R}$. Since h is the tensor field attained by parallel translation of h_x and the pr_k are parallel themselves, h is of the form $h = \sum_{k=1}^{r} \lambda_k \, \mathrm{pr}_k$.

The converse is clear. □

Remark 4.3. A special case of a parallel endomorphism on M is the metric g itself. It corresponds to setting $\lambda_k = 1$ for $k = 1, \ldots, r$.

We can use this to turn any parallel endomorphism $h = \sum_{k=1}^{r} \lambda_k \, \mathrm{pr}_k$ into a traceless one by setting $\tilde{h} = h - (\sum_{k=1}^{r} \lambda_k n_k) g$ where $n_k = \mathrm{rank}(\mathcal{E}_k)$. $\tilde{h}$ is divergence-free, traceless and satisfies $\Delta_L \tilde{h} = 0$. ◇

Theorem 4.4 ([PP84]). *Let (M^n, g) be a locally reducible connected Riemannian manifold. Then there exists an $h \in \mathcal{S}^2 M$, $h \neq 0$ such that $\mathrm{tr}\, h = \delta h = 0$ and $\Delta_L h = 0$.*

Moreover, if $2 \leq n \leq 8$, then M is physically unstable.

Proof. Since (M^n, g) is locally reducible we have a decomposition of the tangent bundle into parallel distributions $TM = \mathcal{E}_1 \oplus \ldots \oplus E_r$ with $r > 1$. Thus we can choose $\lambda_k \in \mathbb{R}$ such that $\sum_k \mathrm{rank}(\mathcal{E}_k) \lambda_k = 0$. Then $h = \sum_k \lambda_k \, \mathrm{pr}_k$ is a trace- and divergence-free solution in the kernel of the Lichnerowicz Laplacian. □

5 Kähler manifolds

In this section, after a short introduction on Kähler manifolds mostly following [Mor07] and [BG08], we will give an identification of primitive harmonic J-invariant 2-forms with symmetric 2-tensor fields in the kernel of the Lichnerowicz Laplacian.

5.1 Preliminaries

5.1.1 Complex manifolds

Let M be a smooth manifold of real dimension $2m$. We call an atlas of M *holomorphic* if for any two coordinate charts $u : U \to \mathbb{C}^m$ and $w : W \to \mathbb{C}^m$ the coordinate transition map $u \circ v^{-1}$ is biholomorphic. A coordinate chart of a holomorphic atlas is called a *holomorphic coordinate chart* of M. Any holomorphic atlas uniquely determines a maximal holomorphic atlas called a *complex structure* of M.

We say M is a *complex manifold* of dimension m if it comes equipped with a holomorphic atlas.

Definition 5.1. An endomorphism $J \in \mathrm{End}(TM)$ satisfying $J^2 = -\mathrm{id}$ is called an *almost complex structure*. We will refer to the pair (M, J) as an *almost complex manifold*.

⋊

Example 5.2. $\mathbb{R}^{2m}$ is an almost complex manifold in a canonical way. Denote by $\{e_1, \ldots, e_{2m}\}$ the standard basis and define $j_m \in \mathrm{End}(\mathbb{R}^{2m})$ via $j_m(e_i) = e_{i+m}$ for $i = 1, \ldots, m$ and $j_m^2 = -1$. Then j_m corresponds to complex scalar multiplication by i if we see $\mathbb{R}^{2m}$ as the underlying real vector space of $\mathbb{C}^m$. We will call j_m the *standard complex structure* of $\mathbb{R}^{2m}$. We drop the "almost", as in this case j_m comes indeed from a complex structure. In the standard basis it is of the form

$$j_m = \begin{pmatrix} 0 & -\mathrm{id}_m \\ \mathrm{id}_m & 0 \end{pmatrix}. \qquad \diamond$$

Remark 5.3. (1) Note that the dimension of an almost complex manifold is necessarily even. The almost complex structure turns TM into a complex vector bundle by defining complex multiplication on the fibres via J.

(2) A complex manifold is an almost complex manifold in a natural way. However, the converse is not true in general. ◇

5.1.2 Complexified tangent bundle

Let (M, J) be an almost complex manifold.

We would like to diagonalize the endomorphism J. To do this, we have to consider the complexified tangent bundle

$$TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}.$$

Sections of $TM_{\mathbb{C}}$ will be called *complex vector fields*. Any such complex vector field Z can be uniquely written as $Z = X + iY$, where X, Y are vector fields on M.

We will denote by $T^{1,0}M$ (resp. $T^{0,1}M$) the eigenbundle of J in $TM_{\mathbb{C}}$ corresponding to the eigenvalue i (resp. $-i$). These bundles have the form

$$\begin{aligned} T^{1,0}M &= \{X - iJX \mid X \in TM\} \\ T^{0,1}M &= \{X + iJX \mid X \in TM\} \end{aligned}$$

and

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M \tag{5.1}$$

The following theorem by Newlander and Nirenberg gives a sufficient condition for (M, J) being a complex manifold.

Theorem 5.4. *Let (M, J) be an almost complex manifold. The almost complex structure J comes from a complex structure if and only if $T^{0,1}M$ is integrable.*

Remark 5.5. (1) If M is a complex manifold, we may pick a specific type of coordinates. Consider a holomorphic chart (U, ϕ_U) and let $z_\alpha = x_\alpha + iy_\alpha$ be the α-th component of ϕ_U, $\alpha = 1, \dots, m$. Denote by $\{e_1, \dots, e_{2m}\}$ the standard basis of $\mathbb{R}^{2m}$ and let j_m be the standard (almost) complex structure such that $j_m(e_\alpha) = e_{\alpha+m}$. We can define J locally via $J = (\phi_U)_*^{-1} j_m$. This definition is independent of the chosen coordinate system since holomorphy of a map $f : \mathbb{C}^m \to \mathbb{C}^m$ is equivalent to $f_* \circ j_m = j_m \circ f_*$.

By definition,

$$\frac{\partial}{\partial x_\alpha} = (\phi_U)_*^{-1} e_\alpha \quad \frac{\partial}{\partial y_\alpha} = (\phi_U)_*^{-1} e_{m+\alpha}$$

and

$$J\left(\frac{\partial}{\partial x_\alpha}\right) = \frac{\partial}{\partial y_\alpha}.$$

We easily see that

$$\frac{\partial}{\partial z_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right), \quad \frac{\partial}{\partial \bar{z}_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right)$$

are local sections of $T^{1,0}M$ and $T^{0,1}M$, respectively. Moreover, they form a local basis for each point of U.

(2) The map $Z = X + iY \mapsto \bar{Z} = X - iY$ is an involution on $T_{\mathbb{C}}M$ and a complex conjugate linear isomorphism from $T^{1,0}M$ to $T^{0,1}M$.

Similarly, the maps

$$TM \to T^{1,0}M, X \mapsto X^{1,0} := X - iJX, \quad TM \to T^{0,1}M, X \mapsto X^{0,1} := X + iJX$$

are complex linear and conjugate complex linear vector bundle isomorphisms, respectively, if we consider TM together with J as a complex vector bundle. $\diamond$

5.1.3 Complexified exterior bundle

There is a similar decomposition for the complexified exterior bundle $\Lambda^k_{\mathbb{C}}M = \Lambda^k M \otimes_{\mathbb{R}} \mathbb{C}$.

We define two subbundles of $\Lambda^1_{\mathbb{C}}M = T^*_{\mathbb{C}}M$

$$\Lambda^{1,0}M := \{\xi \in \Lambda^1_{\mathbb{C}}M \mid \xi(Z) = 0 \ \ \forall Z \in T^{0,1}M\} = \{\omega - i\omega \circ J \mid \omega \in \Lambda^1 M\},$$
$$\Lambda^{0,1}M := \{\xi \in \Lambda^1_{\mathbb{C}}M \mid \xi(Z) = 0 \ \ \forall Z \in T^{1,0}M\} = \{\omega + i\omega \circ J \mid \omega \in \Lambda^1 M\}$$

and we also have

$$\Lambda^1_{\mathbb{C}}M = \Lambda^{1,0}M \oplus \Lambda^{0,1}M.$$

Let us denote by $\Lambda^{k,0}M$ (resp. $\Lambda^{0,k}M$) the k-th exterior product of $\Lambda^{1,0}M$ (resp. of $\Lambda^{0,1}M$) and by $\Lambda^{p,q}M$ the tensor product $\Lambda^{p,0}M \otimes \Lambda^{0,q}M$.

Using the following formula for exterior products of direct sums

$$\Lambda^k(V \oplus W) \cong \bigoplus_{i=0}^{k} \Lambda^i V \otimes \Lambda^{k-i} W$$

we finally get

$$\Lambda^k_{\mathbb{C}}M \cong \bigoplus_{p+q=k} \Lambda^{p,q}M.$$

Sections of $\Lambda^{p,q}M$ are called forms of type (p,q). The space of all forms of type (p,q) is denoted by $\Omega^{p,q}(M) \subset \Omega^{p+q}_{\mathbb{C}}(M)$.

It is easy to check that a k-form ω is of type $(k,0)$ if and only if $Z \lrcorner \omega = 0$ for all $Z \in T^{0,1}M$. More generally, $\omega \in \Omega^{p,q}(M)$ if and only if ω vanishes when applied to $p+1$ vectors from $T^{1,0}M$ or to $q+1$ vectors from $T^{0,1}M$.

The following statement will be needed later.

Lemma 5.6. *Let $\omega \in \Omega^{2k}_{\mathbb{C}}(M)$. Then ω is of type (k,k) if and only if $J_\star \omega = \omega$, where $A_\star$ denotes the induced action of $A \in \mathrm{End}(TM)$ on $\Lambda^{2k}_{\mathbb{C}} M$.*

Proof. We will show the statement for the case $k = 1$.

Let $X, Y \in T_{\mathbb{C}}M$. Decompose $X = X^{1,0} + X^{0,1}$ and $Y = Y^{1,0} + Y^{0,1}$ according to Eq. (5.1). Then a straightforward computation shows that $(J_\star\omega)(X,Y) := \omega(JX, JY) = \omega(X,Y)$ if and only if

$$\omega(X^{1,0}, Y^{1,0}) + \omega(X^{0,1}, Y^{0,1}) = 0.$$

Since the components can be chosen independently, this is true if and only if each summand is zero which is equivalent to $\omega \in \Lambda^{1,1}M$. □

Remark 5.7. For a complex structure J and a local holomorphic coordinate system $z_\alpha = x_\alpha + iy_\alpha$ we can extend the exterior derivative by $\mathbb{C}$-linearity to $dz_\alpha = dx_\alpha + idy_\alpha$ and $d\bar{z}_\alpha = dx_\alpha - idy_\alpha$. They form a local basis for $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$ respectively and are dual to the bases defined in Remark 5.5. A local basis of $\Omega^{p,q}(M)$ is given by

$$\{dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}, i_1 < \ldots < i_p, j_1 < \ldots < j_q\}. \qquad \diamond$$

The *Nijenhuis tensor*

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

is a (2,1)-tensor that can be interpreted as a kind of torsion of the almost complex structure J.

The following proposition summarizes conditions under which J is indeed a complex structure.

Proposition 5.8. *Let J be the almost complex structure of a real manifold M. The following statements are equivalent:*

(i) *J is a complex structure.*

(ii) *$T^{0,1}M$ is integrable, i.e. $[Z_1, Z_2] \in T^{0,1}M$ for $Z_1, Z_2 \in T^{0,1}M$.*

(iii) *$d(\Omega^{1,0}(M)) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$.*

(iv) *$d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$.*

(v) *$N_J = 0$.*

5.1.4 Kähler manifolds

Definition 5.9. (1) Let (M, g, J) be an almost complex Riemannian manifold. We say that the metric g is *Hermitian* if

$$g(JX, JY) = g(X, Y)$$

for all vector fields X, Y on M. An (almost) complex manifold together with a Hermitian metric is called an *(almost) Hermitian manifold*.

(2) On an almost Hermitian manifold (M, g, J) we define the *fundamental 2-form* of the Hermitian metric g as $\Omega(X, Y) := g(JX, Y)$.

(3) Let $E \to M$ be a complex vector bundle of rank k. A *Hermitian structure* h on E is a smooth field of Hermitian products on the fibres of E. E together with h is called a *Hermitian vector bundle*.

$\rtimes$

Remark 5.10. Let (M, g, J) be an almost Hermitian manifold. Then the complex bilinear extension of g to $T_{\mathbb{C}}M$ satisfies

$$\begin{aligned} g(\bar{Z}_1, \bar{Z}_2) &= \overline{g(Z_1, Z_2)}, \\ g(Z_1, Z_2) &= 0 \qquad \text{for all } Z_1, Z_2 \in T^{1,0}M, \\ g(Z, \bar{Z}) &> 0 \qquad \text{unless } Z = 0. \end{aligned}$$

That is, $h(Z_1, Z_2) := g(Z_1, \bar{Z}_2)$ defines a Hermitian structure on $T_{\mathbb{C}}M$. $\diamond$

Definition 5.11. Let g be a Hermitian metric on an almost complex manifold (M, J). Then g is called a *Kähler metric* if J is a complex structure and the fundamental 2-form Ω is closed.

An almost Hermitian manifold together with a Kähler metric is called a *Kähler manifold*. $\rtimes$

There are various equivalent definitions which we will summarize in the following proposition.

Proposition 5.12. *Let (M, g, J) be a $2m$-dimensional almost Hermitian manifold with fundamental 2-form Ω and Levi-Civita connection ∇. Then the following are equivalent:*

(1) g is a Kähler metric.

(2) $N_J = 0$ and $d\Omega = 0$.

(3) $\nabla J = 0$.

(4) The holonomy group of M is a subgroup of $U(m)$.

5.2 Primitive forms

For this section, assume (M, g, J, Ω) is a Kähler manifold of real dimension $2n$.

We define two real operators acting on differential forms

$$L : \Lambda^k M \to \Lambda^{k+2} M, \quad \omega \mapsto \Omega \wedge \omega = \frac{1}{2} e_i \wedge Je_i \wedge \omega$$

and its formal adjoint Λ satisfying

$$\Lambda : \Lambda^{k+2} M \to \Lambda^k M, \quad \eta \mapsto \frac{1}{2} Je_i \lrcorner e_i \lrcorner \eta$$

and define their action on $\Lambda^*_{\mathbb{C}} M$ by linear extension.

Remark 5.13. These operators, more precisely their complex extensions to $\Lambda^*_{\mathbb{C}} M$, and their commutator $H := [\Lambda, L]$ span a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. One can check that $[H, \Lambda] = 2\Lambda$ and $[H, L] = -2\Lambda$. So Λ, L and H correspond to the standard basis vectors of $\mathfrak{sl}_2(\mathbb{C})$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \diamond$$

Definition 5.14. We say $\alpha \in \Lambda^k M$ is *primitive* if $\Lambda(\alpha) = 0$.

Since Λ^* has constant rank on $\Lambda^k M$ its kernel defines a subbundle, the *bundle of primitive forms of degree* k denoted by $(\Lambda^k M)_0$. $\rtimes$

Lemma 5.15. *A 2-form ω is primitive if and only if $g(\Omega, \omega) = 0$.*

Proof. This follows from the duality of L and Λ (more specifically of the duality of the interior and exterior product):

$$\Lambda(\omega) = g(\Lambda(\omega), 1) = g(\omega, \Omega \wedge 1) = g(\omega, \Omega). \qquad \square$$

5.3 Decompositions of the exterior bundle

Let (M, g) be a $2n$-dimensional compact almost complex Riemannian manifold. Then J acts on $\Lambda^2 M$ as an involution, yielding the splitting

$$\Lambda^2 M = \Lambda^+_J M \oplus \Lambda^-_J M \tag{5.2}$$

into eigenspaces of J corresponding to the eigenvalue 1 and -1, respectively.

Their complexifications relate to the decomposition of the complexified exterior bundle $\Lambda^2_{\mathbb{C}}M = \Lambda^{2,0}M \oplus \Lambda^{1,1}M \oplus \Lambda^{0,2}M$ as follows

$$\Lambda^+_J M \otimes \mathbb{C} = \Lambda^{1,1}M, \tag{5.3}$$
$$\Lambda^-_J M \otimes \mathbb{C} = \Lambda^{2,0}M \oplus \Lambda^{0,2}M. \tag{5.4}$$

Henceforth, let M be almost Hermitian of dimension four with fundamental form Ω. Then the Hodge-de Rham operator $*$ of a Riemannian metric also acts as an involution on $\Lambda^2 M$. This induces a splitting of the bundle $\Lambda^2 M$ into so-called self-dual and anti-self-dual parts

$$\Lambda^2 M = \Lambda^+_g M \oplus \Lambda^-_g M. \tag{5.5}$$

The splittings (5.2) and (5.5) relate as follows:

$$\Lambda^+_J M = \mathbb{R}\Omega \oplus \Lambda^-_g M, \tag{5.6}$$
$$\Lambda^+_g M = \mathbb{R}\Omega \oplus \Lambda^-_J M, \tag{5.7}$$
$$\Lambda^-_g M \cap \Lambda^-_J M = 0. \tag{5.8}$$

We want to relate these facts to de Rham cohomology and harmonic forms. Denote by $H^k_{dR}(M,\mathbb{R})$ the k-th de Rham cohomology group of M (i.e. closed k-forms modulo exact forms) and by $\mathcal{H}^k(M)$ the space of harmonic k-forms.

Since the Laplacian on forms commutes with $*$, Eq. (5.5) also holds for $\mathcal{H}^2(M)$, thus

$$H^2_{dR}(M,\mathbb{R}) \cong \mathcal{H}^2(M) = \mathcal{H}^+_g(M) \oplus \mathcal{H}^-_g(M).$$

Let $\mathcal{Z}^2(M)$ denote the closed 2-forms and $\mathcal{Z}^\pm_g(M) := \mathcal{Z}^2(M) \cap \Omega^\pm_g(M)$. Recall that a form α is co-closed if and only if $*\alpha$ is closed and thus $\mathcal{Z}^\pm_g(M) = \mathcal{H}^\pm_g(M)$. Define

$$H^\pm_g(M) = \{a \in H^2_{dR}(M,\mathbb{R}) \mid \exists \alpha \in \mathcal{Z}^\pm_g : [\alpha] = a\}$$

and clearly $H^\pm_g(M) \cong \mathcal{H}^\pm_g(M)$. Set $b^\pm_2(M) := \dim(H^\pm_g(M))$ then $b_2(M) = b^+_2(M) + b^-_2(M)$ where $b_2(M)$ is the second Betti number.

Lemma 5.16. *Let (M, g, J, Ω) be a four dimensional almost Hermitian manifold. The primitive forms in Λ^+_J are the anti-self dual forms, i.e.*

$$(\Lambda^+_J M)_0 = \Lambda^-_g M.$$

Moreover,

$$\mathcal{H}^+_J(M)_0 = \mathcal{H}^-_g(M),$$

where $\mathcal{H}^+_J(M)_0$ denotes the primitive J-invariant harmonic 2-forms.

Proof. The first claim follows from Lemma 5.15 and Eq. (5.6). The second claim is obvious. ☐

In the Kähler case, the complexifications of the cohomology subgroups $H_J^{\pm}(M)$ relate to Dolbeault cohomology analogously to Eqs. (5.3) and (5.4):

$$H_J^+(M) \otimes \mathbb{C} \cong H_J^{1,1}(M) \tag{5.9}$$

$$H_J^-(M) \otimes \mathbb{C} \cong H_J^{2,0}(M) \oplus H_J^{0,2}(M). \tag{5.10}$$

Corollary 5.17. *Let (M, g, J, Ω) be a four dimensional Kähler manifold. Then*

$$h^{1,1}(M) = b_2^-(M) - 1$$

where $h^{p,q}(M) = \dim_{\mathbb{C}}(H^{p,q}(M, \mathbb{C}))$ are the Hodge numbers *of the Riemannian structure.*

5.4 Harmonic J-invariant 2-forms and the Lichnerowicz Laplacian

The one-to-one correspondence of real symmetric J-invariant bilinear forms and $\Lambda_J^+ M$ is well known [Bes87, Paragraph 2.26]. In the following we will show that in the Kähler case this also holds for harmonic J-invariant 2-forms and symmetric 2-tensors the kernel of the Lichnerowicz Laplacian.

Definition 5.18. Let (M, g, J) be an almost complex manifold. We define the bundle morphism $j : \Lambda_J^+ M \to \mathrm{Sym}^2 M$ which maps $\alpha \in \Lambda_J^+ M$ to $\beta \in \mathrm{Sym}^2 M$ given by $\beta(X, Y) = \alpha(X, JY)$.

If we identify α via the metric as a skew-symmetric endomorphism of the tangent bundle, then the corresponding symmetric endomorphism of $j(\alpha)$ is given by $\alpha \circ J$. ⋊

It is straightforward to check that if α is primitive then $j(\alpha)$ is traceless.

Lemma 5.19. *Let (M, g, J, Ω) be a Kähler manifold.*

The bundle morphism j commutes with the universal Laplacian in the sense that

$$(\Delta_L \circ j)(\alpha) = (j \circ \Delta)(\alpha), \quad \text{for } \alpha \in \Omega_J^+ M \tag{5.11}$$

where Δ is the Hodge-Laplacian on 2-forms.

Proof. Recall that $\Delta_L = \nabla^*\nabla - \mathrm{Ric}_\star - 2\mathring{R}$ and $\Delta = dd^* + d^*d = \nabla^*\nabla - \mathrm{Ric}_\star - 2\mathcal{R}$ on 2-forms. Using $\nabla J = 0$ it easily follows that $[\nabla^*\nabla, j]\alpha = [\mathrm{Ric}_\star, j]\alpha = 0$. Finally, $(j \circ \mathcal{R}) = (\mathring{R} \circ j)$ follows from Eqs. (3.1) and (3.2). ☐

Proposition 5.20. *Let (M, g, J, Ω) be a compact Kähler manifold. If $h^{1,1}(M) > 1$, then the Lichnerowicz Laplacian Δ_L acting on traceless divergence-free symmetric endomorphisms has a non-zero kernel.*

In particular, this is the case if M is of dimension four and if $b_2^- > 0$.

Proof. Using Lemma 5.15 we see that

$$\mathcal{H}_J^+(M) = \mathbb{R}\Omega \oplus (\mathcal{H}_J^+(M))_0.$$

By Eq. (5.9) the real dimension of $\mathcal{H}_J^+(M)$ corresponds to $h^{1,1}(M)$ and $h^{1,1}(M) > 1$ thus guarantees the existence of $0 \neq \alpha \in \mathcal{H}_J^+(M)_0$. Then $\beta := j(\alpha)$ defines a symmetric and traceless $(0, 2)$-tensor field. Computing the divergence of β yields

$$(\delta\beta)(X) = -\nabla_{e_i}\alpha(e_i, JX) = (\delta\alpha)(JX) = 0,$$

since α is harmonic and thus co-closed. By Lemma 5.19

$$\Delta_L\beta = (\Delta_L \circ j)\alpha = (j \circ \Delta)\alpha = j(\Delta\alpha) = 0.$$

The four dimensional case follows from Corollary 5.17. □

Corollary 5.21. *If M is a compact Kähler manifold of dimension 4, 6 or 8 and $h^{1,1}(M) > 1$, then M is physically unstable.*

Example 5.22. (1) The standard example of a four dimensional Kähler manifold is the complex projective space $\mathbb{C}P^2$. We have $b_2^+(\mathbb{C}P^2) = 1$ and $b_2^-(\mathbb{C}P^2) = 0$. By a result of Donaldson [Don83] and Freedman [Fre82] all simply connected 4-manifolds with positive definite intersection form are homeomorphic to the k-fold connected sum $\mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2$. Now if M is a simply connected 4 dimensional Kähler manifold with $b_2^- = 0$, then its intersection form is positive definite and M is homeomorphic to $\mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2$.

(2) The *blowup* of a point p in $\mathbb{C}^n$ is obtained by replacing a neighborhood U of p with $\tilde{U} = \{(z, l) \in U \times \mathbb{C}P^{n-1} \mid z \in l\}$. Then the projection $\pi : \tilde{U} \to U$, $(z, l) \mapsto z$ is biholomorphic on $U \setminus \{p\}$ and the fibre over p is just $\mathbb{C}P^{n-1}$. This operation can also be applied to manifolds. In terms of n-dimensional complex oriented smooth manifolds the blowup $\tilde{M}$ of M at $p \in M$ corresponds to the connected sum $\tilde{M} = M \# \overline{\mathbb{C}P^n}$ where the overline indicates the choice of the opposite orientation with respect to the standard one. Blowing up a manifold increases the Hodge numbers (for details, we refer to [GH94])

$$h^{i,i}(\tilde{M}) = h^{i,i}(M) + 1, \quad i > 0,$$

in particular $h^{1,1}(\tilde{M}) = h^{1,1}(M) + 1$, thus by Corollaries 5.17 and 5.21 we can conclude that physically stable Kähler 4-manifolds are not blowups.

The del Pezzo surfaces P_k are the blowup of k points in general position on $\mathbb{C}P^2$ for $1 \leq k \leq 8$. However, P_k only admits Einstein-Kähler metrics for $k \geq 3$.

(3) The product $S^2 \times S^2$ has Betti numbers $b_2^+ = b_2^- = 1$ and is thus unstable. ◇

Remark 5.23. We say the first Chern class $c_1(M)$ is *positive* (resp. *negative*) if it can be represented by a $(1,1)$-form whose corresponding symmetric 2-tensor via j is positive (resp. negative). The Ricci form $\rho := j(\mathrm{Ric})$ of a compact Kähler manifolds represents the first Chern class $c_1(M)$. Thus a necessary condition for the existence of Einstein-Kähler metrics is that $c_1(M)$ can be represented by a $(1,1)$-form. In fact, the Calabi conjecture, which was proved by Yau in [Yau78], states that any $(1,1)$-form representing $c_1(M)$ is the Ricci-form of some Kähler metric on M. This implies the existence of Ricci-flat metrics if $c_1(M) = 0$. If $c_1(M)$ is negative, existence of Kähler-Einstein metrics has been independently proved by Aubin [Aub76] and Yau [Yau78]. However, the case where $c_1(M) > 0$ and, consequently, $s > 0$ is more subtle. In fact, it turns out that $c_1(M) > 0$ is not a sufficient condition for the existence of Einstein-Kähler metrics. For example, the del Pezzo surfaces P_k do not have any Kähler-Einstein metrics for $k = 1, 2$. ◇

6 Symmetric spaces

In this section we will present a representation theoretical method to compute the eigenvalues of the Casimir operator acting on sections of associated bundles over homogeneous spaces. If the homogeneous space is symmetric, the Casimir operator will turn out to be the universal Laplacian which will allow us to compute the eigenvalues of the Lichnerowicz Laplacian acting on sections of symmetric 2-tensors.

6.1 Casimir operator

Let G be a compact semi-simple Lie group with Killing form B. Since G is compact B is negative definite, so $g = -B$ becomes an invariant Riemannian metric on G. The *Casimir operator* $\mathrm{Cas}_\pi \in \mathrm{End}\, V$ of a representation $\pi : G \to V$ is defined as

$$\mathrm{Cas}_\pi := -\sum_i \pi_*(X_i)^2$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{g}$ relative to $g = -B$ on $\mathfrak{g}$. Note that if we chose $\{Y_i\}$ to be orthonormal with respect to the Killing form B then $\mathrm{Cas}_\pi = \sum_i \pi_*(Y_i)^2$.

The Casimir operator is independent of the choice of a g-orthonormal basis for $\mathfrak{g}$ and commutes with $\pi(\mathfrak{g})$ and, by Schur's lemma, acts as a scalar on irreducible representations. Let V be an irreducible representation with highest weight λ and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ be the half-sum of the positive roots. Then this scalar is given by $(\lambda + 2\rho, \lambda) = \|\lambda+\rho\|^2 - \|\lambda\|^2$. For details and proofs see [Hum73, section 6.2] or [GW09, section 3.3.2]. Note that since λ is dominant, the Casimir is positive.

Moreover, on compact naturally reductive homogeneous spaces $M = G/H$ the Casimir operator corresponds to a G-invariant second order differential operator which is reflected in the following

Lemma 6.1 ([MS10])**.** *Let G be a compact semi-simple Lie group and $H \leq G$ a compact subgroup such that $M = G/H$ is a naturally reductive homogeneous space with the Riemannian metric induced by $-B$ and denote by ∇ the canonical homogeneous connection on $M = G/H$. Let $\pi : H \to E$ be a H-representation and let $EM := G \times_\pi E$ be the associated vector bundle over M.*

If we consider the space of sections $\Gamma(EM)$ as a G-representation via the left-regular representation l, then the differential operator $\Delta_\pi = \nabla^*\nabla + q(R)$ acts on $\Gamma(EM)$ as $\Delta_\pi = \mathrm{Cas}^G_l$.

Remark 6.2. The canonical homogeneous connection ∇ coincides with the Levi-Civita connection *only* in the case when G/H is a symmetric space. $\diamond$

Proof. Since M is naturally reductive there is an $Ad(H)$-invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad \text{and} \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

The canonical homogeneous connection $A \in \Omega^1(G, \mathfrak{h})$ in the principal H-bundle $G \to G/H$ is given by $A = \mathrm{pr}_{\mathfrak{h}} \circ \mu_G$ where $\mu_G \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form. As a consequence the horizontal distribution is $T^h_g G = dL_g(\mathfrak{m})$.

Let $s \in \Gamma(EM)$ be a section and denote by $\bar{s} \in C^\infty(G, E)^H$ the corresponding H-equivariant function. We can then write the covariant derivative $\nabla = \nabla^A$ at the origin $o = eH$ as $(\nabla_X s)_{eH} = [e, X^*(\bar{s})_e]$ for $X \in T_o M$. But the horizontal lift X^* of X at e is simply given by the identification of $\mathfrak{m}$ with $T_o M \cong \mathfrak{m}$ via $d\pi$. The directional derivative $X(\bar{s})$ is just minus the differential of the left-regular representation $X(\bar{s}) = -l_*(X)\bar{s}$. Thus if $\{e_i\}$ denotes a g-orthonormal basis in $\mathfrak{m}$, the rough Laplacian $\nabla^* \nabla$ translates into the sum $-l_*(e_i) l_*(e_i) = \mathrm{Cas}^G_l - \mathrm{Cas}^H_l$. Since $\Delta = \nabla^* \nabla + q(R)$ it remains to show that $q(R) = \mathrm{Cas}^H_l = \mathrm{Cas}^H_\pi$.

Since the Killing form B and, consequently, the metric is invariant under automorphisms of $\mathfrak{g}$, the isotropy representation is orthogonal. We assert that the differential $\lambda_* : \mathfrak{h} \to \mathfrak{so}(\mathfrak{m}) \cong \Lambda^2 \mathfrak{m}$ of the isotropy representation is given by $\lambda_*(A) = \frac{1}{2} e_i \wedge [A, e_i]$ for any $A \in \mathfrak{h}$. To see this, recall that $(X \wedge Y)(Z) = g(X, Z)\, Y - g(Y, Z)\, X$. Then

$$\begin{aligned}
(e_i \wedge [A, e_i])(X) &= g(e_i, X)\, [A, e_i] - g(X, [A, e_i])\, e_i \\
&= [A, X] + B(X, [A, e_i]) e_i \\
&= [A, X] + B([X, A], e_i) e_i = 2[A, X]
\end{aligned}$$

for $X \in \mathfrak{m}$, which proves the assertion.

The curvature operator $\mathcal{R} : \Lambda^2 \mathfrak{m} \to \Lambda^2 \mathfrak{m}$ can be expressed as

$$\mathcal{R}(X \wedge Y) = -\frac{1}{2} e_i \wedge R_{X,Y} e_i = \frac{1}{2} e_i \wedge [[X, Y]_{\mathfrak{h}}, e_i] = \lambda_*([X, Y]_{\mathfrak{h}})$$

for any $X, Y \in \mathfrak{m}$ where we have used that $R_{X,Y} = -\lambda_*([X, Y]_{\mathfrak{h}})$ and where the subscript $\mathfrak{h}$ denotes the projection onto $\mathfrak{h}$ (for details see [KN96, Chapter 10.2]).

Consider the bundle of orthonormal frames $P_{\mathrm{SO}(\mathfrak{m})}$ of $M = G/H$. Then any $\mathrm{SO}(\mathfrak{m})$ representation $\tilde{\pi}$ defines a H-representation via $\pi = \tilde{\pi} \circ \lambda$. Moreover, the principal H-bundle $G \to G/H$ can be seen as a λ-reduction of $P_{\mathrm{SO}(\mathfrak{m})}$, i.e. we can identify

$$EM = P_{\mathrm{SO}(\mathfrak{m})} \times_{\tilde{\pi}} E = G \times_\pi E.$$

Let $\{f_k\}$ denote a g-orthonormal basis of $\mathfrak{h}$. Then by definition of $q(R)$ we have

$$\begin{aligned} q(R) &= -\frac{1}{2}\tilde{\pi}_*(e_i \wedge e_j)\tilde{\pi}_*(\mathcal{R}(e_i \wedge e_j)) = -\frac{1}{2}\tilde{\pi}_*(e_i \wedge e_j)(\tilde{\pi} \circ \lambda)_*([e_i, e_j]_{\mathfrak{h}}) \\ &= \frac{1}{2}\tilde{\pi}_*(e_i \wedge e_j)\pi_*(f_k)B([e_i, e_j], f_k) \\ &= \frac{1}{2}\tilde{\pi}_*(e_i \wedge e_j)\pi_*(f_k)B(e_j, [e_i, f_k]) \\ &= -\frac{1}{2}\tilde{\pi}_*(e_i \wedge [f_k, e_i])\pi_*(f_k) = -\pi_*(f_k)\pi_*(f_k) = \mathrm{Cas}^H_\pi \,. \end{aligned}$$

We have thus shown that $q(R) \in \mathrm{End}(EM)$ acts fibre-wise as Cas^H_π. Let $Z \in \mathfrak{h}$, $g \in G$ and $f \in C^\infty(G, E)^H$. Then, by equivariance, $\pi_*(Z)f(g) = -Z(f)_g = (l_*(Z)f)(g)$. Hence, $q(R) = \mathrm{Cas}^H_\pi = \mathrm{Cas}^H_l$ and $\Delta_\pi = \mathrm{Cas}^G_l$. □

Corollary 6.3. *Every Riemannian symmetric manifold (M^n, g) of compact type is physically stable if $n \geq 9$.*

Proof. Since the Casimir is positive and $\lambda_{\mathrm{crit}} \leq 0$ for $n \geq 9$ the claim immediately follows. □

6.2 Induced representations, Frobenius reciprocity and the Peter-Weyl theorem

Let G be a compact Lie group with closed subgroup H associated to the homogeneous space $M = G/H$. Consider a finite-dimensional H-representation (E, π) and let $EM = G \times_\pi E$ be the associated vector bundle.

We are interested in a decomposition into isotypical summands of the infinite dimensional H-representation of smooth sections $\Gamma(EM)$.

Definition 6.4. Let G be a compact Lie group and consider a not necessarily finite dimensional G-representation V of a vector space V. The finite-dimensional subrepresentations of V generate a subrepresentation V_s of V. V_s is called the *locally finite part* of V. V is called *locally finite* if $V = V_s$. ⋊

Definition 6.5. For compact Lie groups $H \subset G$ and a finite-dimensional H-representation E, the *induced G-representation* $\mathrm{Ind}^G_H E$ is defined as

$$\begin{aligned} \mathrm{Ind}^G_H E :&= C^\infty(G, E)^H \\ &= \{f : G \to E \mid f(gh) = h^{-1}f(g) \text{ for } h \in H, g \in G\}. \end{aligned}$$

Endowed with the supremum-norm, $(\mathrm{Ind}^G_H E, \|\cdot\|_\infty)$ becomes a Banach-space and the left-regular G-action $(gf)(u) = f(g^{-1}u)$ is continuous. ⋊

Note that $\operatorname{Ind}_H^G E$ is an infinite-dimensional G-representation unless H has finite index in G. In any case, the locally finite part of $iE_s := (\operatorname{Ind}_H^G E)_s$ admits a decomposition into isotypical summands ([BtD85, Proposition III.1.7])

$$iE_s \cong \bigoplus_{\lambda \in \hat{G}} \Gamma_\lambda \otimes \operatorname{Hom}_G(\Gamma_\lambda, \operatorname{Ind}_H^G E),$$

where $\hat{G}$ are the dominant integral weights of G.

By the generalized Peter-Weyl theorem iE_s is dense in $\operatorname{Ind}_H^G E$ (for details, see [BtD85, Chapter III.5]).

Let E be an H-representation and U a G-representation. Frobenius reciprocity states that any H-module homomorphism $\varphi : E \to U$ extends uniquely to a G-module homomorphism $\tilde{\varphi} : \operatorname{Ind} E \to U$, i.e.

$$\operatorname{Hom}_H(E, \operatorname{Res}_H^G U) \cong \operatorname{Hom}_G(\operatorname{Ind}_H^G E, U),$$

where $\operatorname{Res}_H^G U$ is the restriction of U to an H-representation.

Combining Frobenius reciprocity with the isotypical decomposition and the fact that the space of sections $\Gamma(EM)$ is isomorphic to $C^\infty(G, E)^H$ we get the decomposition

$$C^\infty(EM) \cong \bigoplus_{\lambda \in \hat{G}} \Gamma_\lambda \otimes \operatorname{Hom}_H(\operatorname{Res}_H^G \Gamma_\lambda, E). \tag{6.1}$$

Now $\operatorname{Res}_H^G \Gamma_\lambda$ decomposes as an H-representation into $\operatorname{Res}_H^G \Gamma_\lambda = \bigoplus_{\bar{\lambda} \in \hat{H}} \Gamma_{\bar{\lambda}}$. Without loss of generality we will henceforth assume that E itself is an irreducible H-representation. Then, by Schur's lemma, $\operatorname{Hom}_H(\operatorname{Res}_H^G \Gamma_\lambda, E)$ is just the multiplicity of E in the decomposition $\operatorname{Res}_H^G \Gamma_\lambda$.

We have thus reduced the problem of determining the spectrum of Δ_π acting on $C^\infty(EM)$ to finding the irreducible G-representations whose restrictions to H contain E. This can for example be done by using so-called *branching rules* as is illustrated in the following example.

6.3 Spectrum of the Lichnerowicz-Laplacian on the sphere

As an example, we will consider the sphere $M = S^{2n} = \mathrm{SO}(2n+1)/\mathrm{SO}(2n)$. We set $r = [n/2]$ and choose our root system and its partial order in such a way that the dominant weights of $\mathfrak{so}(n)$ satisfy

$$\begin{aligned} \lambda_1 \geq \ldots \geq \lambda_r \geq 0, &\quad \text{for } n \text{ odd and} \\ \lambda_1 \geq \ldots \geq \lambda_{r-1} \geq |\lambda_r| &\quad \text{for } n \text{ even.} \end{aligned}$$

Lemma 6.6 (Branching rule, [FH91]). *Consider the complex Lie algebras $\mathfrak{so}(2n) \subset \mathfrak{so}(2n+1)$ and the irreducible representation Γ_λ of $\mathfrak{so}(2n+1)$ given by the highest weight vector $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$.*

Then the restriction

$$\mathrm{Res}^{\mathfrak{so}(2n+1)}_{\mathfrak{so}(2n)}(\Gamma_\lambda) = \bigoplus_{\bar{\lambda}} \Gamma_{\bar{\lambda}}$$

decomposes into irreducible $\mathfrak{so}(2n)$-representations $\Gamma_{\bar{\lambda}}$ for highest weight vectors $\bar{\lambda}$ satisfying

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \ldots \geq \bar{\lambda}_{n-1} \geq \lambda_n \geq |\bar{\lambda}_n|.$$

Example 6.7. (1) Consider the trivial representation $E = \mathbb{C}$ with highest weight $\bar{\lambda} = 0$. Then by Lemma 6.6 the irreducible $\mathfrak{so}(2n+1)$ representations containing E are given by the highest weights $\lambda_k = (k, 0, \ldots, 0)$, $k \in \mathbb{N}_0$. The corresponding irreducible representations are $\Gamma_{\lambda_k} = \mathrm{Sym}^k_0(\mathbb{C}^{2n+1})$. Note that sections of this bundle are just functions on M. The eigenvalues of the Casimir operator and thus of the Laplacian on functions are given by

$$\mu_k := g\left(\lambda_k, \lambda_k + 2\rho\right) = k(2n-1+k).$$

(2) The standard representation $E = \mathbb{C}^{2n}$ of $\mathfrak{so}(2n)$ correponds to the highest weight $\bar{\lambda} = (1, 0, \ldots, 0)$. In this case we have two series of admissible highest weights of $\mathfrak{so}(2n+1)$-representations:

$$\begin{aligned} &\lambda^1_k = (k, 0, 0, \ldots, 0), \quad \mu_k = k(2n-1+k), \\ &\lambda^2_k = (k, 1, 0, \ldots, 0), \quad \mu_k = k(2n-1+k) + 2n - 2, \quad k \geq 1. \end{aligned}$$

The eigenspaces of the first series actually come from eigenfunctions of the Laplacian on $C^\infty(M)$. This is simply due to the fact that the differential d commutes with the Laplace operator on forms $\Delta = dd^* + d^*d$ and thus $\Delta df = \mu_k \, df$ if $\Delta f = \mu_k f$. Also, recall that for $\mathfrak{so}(m)$ we have $E \cong E^*$ via g as representations.

(3) The bundle of 2-forms corresponds to the representation $E = \Lambda^2\mathbb{C}^{2n}$ with highest weight $\bar{\lambda} = (1, 1, 0, \ldots, 0)$. Then the admissible highest weights with eigenvalues are

$$\begin{aligned} &\lambda^1_k = (k, 1, 0, 0, \ldots, 0), \\ &\lambda^2_k = (k, 1, 1, 0, \ldots, 0), \quad \mu_k = (k+2)(2n-1+k), \quad k \geq 1. \end{aligned}$$

Again, we note that the differential d maps Laplace eigenspaces on vector fields into Laplace eigenspaces of 2-forms.

(4) In the interesting case where $E = \mathrm{Sym}^2_0(\mathbb{C}^{2n})$ and $\bar{\lambda} = (2, 0, \dots, 0)$ Δ_π is the Lichnerowicz Laplacian and there are three series of admissible highest weights λ^i_k with eigenvalues μ^i_k:

$$\begin{aligned} \lambda^1_k &= (k, 0, 0, \dots, 0), \quad \mu_k = k(2n - 1 + k), \quad k \geq 2 \\ \lambda^2_k &= (k, 1, 0, \dots, 0), \quad \mu_k = k(2n - 1 + k) + 2n - 2, \quad k \geq 2 \\ \lambda^3_k &= (k, 2, 0, \dots, 0), \quad \mu_k = k(2n - 1 + k) + 2(2n - 1), \quad k \geq 2. \end{aligned}$$ ◇

The following proposition shows the nature of the first two series of irreducible representations arising in Example 6.7.(4).

Proposition 6.8 ([SW12, Corollary 5.1])**.** *Let (M, g) be an irreducible locally symmetric space of compact type. Then any Laplace eigenspace on functions or vector fields is mapped into an eigenspace of the Lichnerowicz Laplacian on symmetric 2-tensors for the same eigenvalue. For functions this map is given by $f \mapsto \mathrm{Hess}(f)$ and for vector fields by $X \mapsto L_X g$.*

Furthermore, the image of each map is a subspace of $\mathrm{Im}(\delta^*)$.

Proof. The first claim follows from [SW12, Theorem 4.2] which states that the Laplacian commutes with the covariant derivative ∇ on symmetric spaces.

A direct computation shows that $L_{df^*} g = 2\,\mathrm{Hess}(f)$. □

We conclude this example with the following corollary and refer to [Bou09] for a comprehensive computation of the eigenvalues with multiplicities of the Lichnerowicz Laplacian on the sphere.

Corollary 6.9. *The eigenvalues of the Lichnerowicz operator on S^n acting on symmetric, traceless and divergence-free 2-tensors are given by $\mu_k = k(n - 1 + k) + 2n - 2$ for $k \geq 2$. It follows that S^n is physically stable for all $n \geq 2$.*

6.4 An algorithm to compute the smallest eigenvalue

Let $(\mathfrak{g}, \mathfrak{k})$ be the Lie algebras of a simply connected and irreducible symmetric space $M = G/K$ of compact type. Then $\mathfrak{g}$ is real simple, $\mathfrak{g}_\mathbb{C}$ is a complex simple Lie algebra and $\mathfrak{k}$ is a maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is the isotropy representation of $\mathfrak{h}$. Set $E = \mathrm{Sym}^2_0 \mathfrak{m}$.

Let $\omega_1, \dots, \omega_n$ be the fundamental weights of $\mathfrak{g}$. Let μ be a dominant weight. Then $\mu = \sum_i k_i \omega_i$. Define the *height* $\mathrm{ht}(\mu)$ of a dominant weight μ as $\mathrm{ht}(\mu) := \sum_i k_i$. We recall that the eigenvalue of the Casimir operator on the irreducible representation with highest weight μ is given by $\lambda_\mu = (\mu + 2\rho, \mu)$. It easily follows that for dominant weights μ_1 and μ_2 such that $\mathrm{ht}(\mu_1) < \mathrm{ht}(\mu_2)$ we have $\lambda_{\mu_1} < \lambda_{\mu_2}$.

By Eq. (6.1) the irreducible $\mathfrak{g}$ representation Γ_μ is an eigenspace of Δ_L if and only if $\mathrm{Hom}_H(\mathrm{Res}^G_H \Gamma_\mu, E) \neq 0$, i.e. $\mathrm{Res}^G_H \Gamma_\mu$ and E have common irreducible $\mathfrak{h}$ factors.

Our algorithm will inductively construct $\mathfrak{g}$-representations Γ_μ for linear combinations μ of the fundamental weights of $\mathfrak{g}$ to a given height and check for common factors of $\mathrm{Res}^G_H \Gamma_\mu$ and E using branching rules.

Example 6.10. The algorithm was implemented with the computer algebra software sage [S+12] for the symmetric space $M = G_2/\mathrm{SO}(4)$. The source code can be found in Listings 1 and 2 of the Appendix. The decomposition $\mathfrak{g}_2 = \mathfrak{so}(4) \oplus \mathfrak{m}$ can be constructed following [Mur65, Théorème 1] and is illustrated in Fig. 1. It follows that the highest weight of $\mathfrak{m}$ is given by $3\tilde{\omega}_1 + \tilde{\omega}_0$ where $\tilde{\omega}_i$ are the fundamental weights corresponding to the simple roots α_0 and α_1 of $\mathfrak{so}(4)$.

The output of the algorithm is the following:

```
Lowest height weights:  [(2, 0, -2), (3, -1, -2), (4, -2, -2)]
Corresponding eigenvalues:  [28, 42, 60]
Lowest eigenvalue:  28
```

We can conclude from the proof of Theorem 6.11 and Table 1 in the following section that the inequality (3.4) is not sharp in the case of $M = G_2/\mathrm{SO}(4)$ and, consequently, that $G_2/\mathrm{SO}(4)$ does not admit Stäckel tensors. ◇

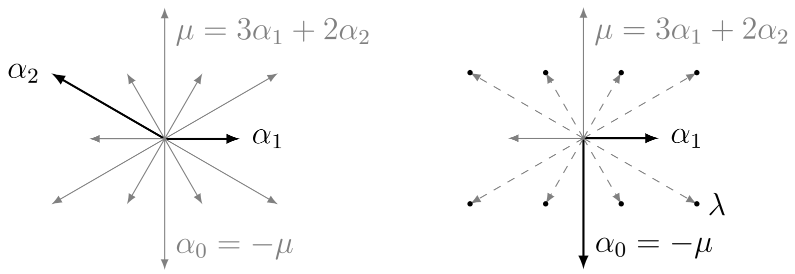

Figure 1: In the left diagram we see the root system of $\mathfrak{g}_2$. The maximal root is $\mu = 3\alpha_1 + 2\alpha_2$. In the right diagram the root system of $\mathfrak{so}(4)$ with simple roots α_1 and $\alpha_0 = -\mu$ is depicted. The weight spaces of the isotropy representation $\mathfrak{m}$ are shown as black dots. They correspond to the remaining root spaces. Its highest weight is given by $\lambda = -\alpha_2$.

6.5 General Riemannian symmetric spaces of compact type

Riemannian symmetric spaces were first classified by Élie Cartan in 1926 [Car26]. Let $M = G/K$ be a simply-connected irreducible Riemannian symmetric space of compact type. Then M is of one of the following types (c.f. [Bes87]):

(I) G is simple.

(II) There exists a simply-connected compact simple Lie Group H such that $G = H\times H$ and $K = H$ where H is understood as the diagonal subgroup of $H \times H$.

Spaces of type II are already known by the classification of compact simple Lie groups. The classification of type I is a consequence of the classification of the real simple Lie groups and their maximal compact subgroups.

Type	G/K		n	s/n	κ	λ_{crit}	λ_{min}	stable
A_1	SU(2)×SU(2)/SU(2)	$A_1\times A_1/A_1$	3	2	-1	3	12	yes
A_2	SU(3)×SU(3)/SU(3)	$A_2\times A_2/A_2$	8	3	3	0.75	0	no
AI_1	SU(3)/SO(3)	A_2/B_1	5	3	1.5	3	6	yes
AII	SU(4)/Sp(2)	A_3/C_2	5	4	2	4	8	yes
$AIII$	SU(2)/S(U(1)×U(1))	A_1/T	2	2	-2	3.5	16	yes
	SU(3)/S(U(2)×U(1))	$A_1/A_1\times T$	4	3	1	3.75	8	yes
	SU(4)/S(U(3)×U(1))	$A_1/A_2\times T$	6	4	1	3	12	yes
	SU(5)/S(U(4)×U(1))	$A_1/A_3\times T$	8	5	1	1.25	16	yes
	SU(4)/S(U(2)×U(2))	$A_1/A_1\times A_1\times T$	8	4	2	1	8	no
BI	SO(5)/SO(3)×SO(2)	$B_2/B_1\times T$	6	3	2	2.25	4	no
CII	Sp(2)/Sp(1)×Sp(1)	$C_2/C_1\times C_1$	4	6	-1	7.5	28	yes
	Sp(3)/Sp(2)×Sp(1)	$C_3/C_2\times C_1$	8	8	4	2	16	no
G	G_2/SO(4)	$G_2/A_1\times A_1$	8	12	6	3	24	yes

Table 1: Values for the lower bound λ_{min} of eigenvalues of the Lichnerowicz Laplacian for Riemannian symmetric spaces of dimension $n < 9$ which are not spheres. The last column shows whether the space is stable in the sense of Definition 2.8 and should not be confused with physical stability.

In his paper [Koi80], N. Koiso used the method of classification described in [Mur65] involving the classification of involutive automorphisms of the complexified Lie algebra of G to determine which Riemannian symmetric spaces of compact type are rigid or mathematically stable by computing the eigenvalues of $\mathring{R}$ on the irreducible components of $\mathrm{Sym}^2 T$ as a K-representation. This method automatically gives s/n as this is the eigenvalue of $\mathring{R}$ on the trace component $\mathbb{R} \cdot g$ of $\mathrm{Sym}^2 T$.

We will be using his results to compute a lower bound for the eigenvalues of the Lichnerowicz Laplacian using the inequality we found in Proposition 3.7 to prove the following theorem.

Theorem 6.11. *All irreducible Riemannian symmetric spaces of compact type are physically stable.*

Proof. Let (M, g) be a Riemannian symmetric space of compact type such that $M = G/K$ for Lie groups G and K. We know from Proposition 3.7 that

$$\Delta_L \geq 4s/n - 4\kappa \tag{3.6}$$

where κ is the largest eigenvalue of $\mathring{R}$.

In table Table 1 we have computed the lower bound $\lambda_{\mathrm{min}} := 4s/n - 4\kappa$ for all irreducible Riemannian symmetric spaces of compact type whose dimension n is smaller than 9 and are not spheres. By comparison with the values of λ_{crit} which also can be found in the table and by Corollaries 6.3 and 6.9 we can conclude that all Riemannian symmetric spaces of compact type, except for spaces of type A_2, are physically stable.

For type A_2, following [KN62, Koi80], the smallest eigenvalue of Δ_L has been computed to be $16/3$. Hence, $\Delta_L \geq 16/3 > \lambda_{\mathrm{crit}}$ and A_2 is physically stable. $\square$

Remark 6.12. Although not listed in Table 1, it turns out that on the spheres the Inequality (3.4) is sharp and, as a consequence, admits Stäckel tensors. $\diamond$

7 Spin manifolds

In this section we will be considering Riemannian spin manifolds admitting Killing spinors. Following [Wan91], we will establish a connection between the Rarita-Schwinger operator $D_{3/2}$ and the Lichnerowicz Laplacian and derive a result of [GHP02] that all Riemannian spin manifolds with positive scalar curvature admitting Killing spinors are physically stable. Interestingly, the lower bound turns out to be exactly the critical value of the physical stability criterion.

7.1 Preliminaries

Let (M^n, g) be a compact Riemannian spin manifold. Let P be the $\mathrm{Spin}(n)$-principal bundle. Then the spinor bundle S is given as the associated vector bundle $S = P \times_{Spin(n)} \Delta_n$, where Δ_n is the spinor representation. On S we will denote by ∇ the connection induced by the Levi-Civita connection of M which we also denote by ∇. The Clifford multiplication $\mu : S \otimes TM \to S$ is the bundle morphism defined on the fibers as the Clifford multiplication on the spinor module.

Definition 7.1. The operator $D = \mu \circ \nabla : \Gamma(S) \xrightarrow{\nabla} \Gamma(S \otimes TM) \xrightarrow{\mu} \Gamma(S)$ is called the *Dirac operator* of S. ⋊

Remark 7.2. (1) Locally, we have $D\psi = e_i \cdot \nabla_{e_i}\psi$ for $\psi \in S$

(2) One can generalize the definition of the Dirac operator on the spinor bundle to arbitrary Clifford bundles. Clifford bundles W are complex vector bundles with Clifford multiplication $\mu : W \otimes TM \to W$ such that

   (i) each fibre is a Clifford module,
   (ii) the Clifford multiplication is skew-symmetric with respect to the induced metric on W,
   (iii) the induced connection ∇^W is metric and
   (iv) $\nabla^W(X \cdot w) = (\nabla X) \cdot w + X \cdot \nabla^W w$ for all $X \in \Gamma(TM)$ and $w \in \Gamma(W)$.

   Generalized Dirac operators are elliptic formally self-adjoint first order differential operators and thus have a discrete spectrum on compact manifolds. ◇

Definition 7.3. Let $\psi \in \Gamma(S)$, $\psi \neq 0$. If ψ satisfies the field equation

$$\nabla_X \psi = \mu X \cdot \psi$$

for a constant $\mu \in \mathbb{C}$ and for all vector fields X, then ψ is called a *Killing spinor* and μ is called its *Killing number*. ⋊

Manifolds admitting non-trivial Killing spinors require certain geometric conditions and are in some sense complementary to the types of manifolds we discussed in previous chapters:

Proposition 7.4 ([BFGK91]). *Let (M^n, g) be a connected Riemannian spin manifold with non-trivial Killing spinor ψ and Killing number μ. Then the following statements hold.*

(1) (M^n, g) is Einstein with scalar curvature $s = 4\mu^2 n(n-1)$. In particular, μ is either real or purely imaginary.

(2) If $\mu \neq 0$, then M is not Kähler.

(3) If $\mu \neq 0$, then M is locally irreducible.

(4) If M is a symmetric space of compact type and $\mu > 0$, then M has constant sectional curvature $4\mu^2$ and is conformally flat. Moreover, the universal cover $\tilde{M}$ is a round sphere.

7.2 The Rarita-Schwinger operator

We will consider spinor-valued 1-forms, i.e. sections $\Gamma(S \otimes T_{\mathbb{C}}M)$. The bundle $S \otimes T_{\mathbb{C}}M$ is called the *twisted spinor bundle*. The operator

$$D_T : \Gamma(S \otimes T^*_{\mathbb{C}}M) \xrightarrow{\nabla} \Gamma(S \otimes T^*_{\mathbb{C}}M \otimes T^*M) \xrightarrow{\mu \otimes \mathrm{id}} \Gamma(S \otimes T^*_{\mathbb{C}}M)$$

is called the *twisted Dirac operator*. Locally, $(D_T\psi)(X) = e_i \cdot \nabla_{e_i}(\psi(X)) = D(\psi(X))$.

Lemma 7.5. *The twisted Dirac operator D_T is formally self-adjoint.*

Proof. The bundle $S \otimes T_{\mathbb{C}}M$ with Clifford multiplication $X \cdot (\phi \otimes \omega) = (X \cdot \phi) \otimes \omega$ for $X \in TM$, $\phi \in S$ and $\omega \in T_{\mathbb{C}}M$ is a Clifford bundle. Then the twisted Dirac operator corresponds to the Dirac operator of the twisted bundle and thus is formally self-adjoint. □

Similarly as in Section 3.2, we have a decomposition of the twisted spinor bundle into associated bundles of Spin-representations given by

$$S \otimes T_{\mathbb{C}}M \cong S \oplus \ker(\mu).$$

We will use the same notation and choice of root system as in Section 6.3. The highest weight corresponding to the bundle $\ker(\mu)$ is $\gamma = \frac{1}{2}(3, 1, \ldots, 1)$ for n odd. If

n is even, then $\ker(\mu)$ corresponds to the sum of irreducible representations of highest weight $\gamma^\pm = \frac{1}{2}(3, 1, \ldots, \pm 1)$. Note that γ and $\gamma^\pm$ is the sum of the weights of the spin representation and of the standard representation of $\mathfrak{so}(n)$.

The bundle $\ker(\mu)$ is also denoted by $S_{3/2}$. In the even case, $\ker(\mu) = S^+_{3/2} \oplus S^-_{3/2}$ where $S^\pm_{3/2}$ are the associated bundles of the irreducible representations of weight $\gamma^\pm$.

Lemma 7.6. *The embedding $\iota : S \to S \otimes T_{\mathbb{C}}M$ is given by $\iota(\phi)(X) = -1/nX \cdot \phi$ for $X \in T_{\mathbb{C}}M$. Then $p_1 = \iota \circ \mu$ and $p_2 = 1 - \iota \circ \mu$ are the projections onto $\iota(S)$ and $\ker(\mu)$.*

Theorem 7.7 ([Sem95, Satz E.5])**.** *If (M, g) is Einstein, then D_T^2 leaves the decomposition $S \otimes T_{\mathbb{C}}M = S \oplus \ker(\mu)$ invariant. Moreover,*

$$D_T^2(\iota(\phi)) = (\iota(D^2\phi) - \frac{s}{n}\iota(\phi)) + \frac{1}{n}(\mathrm{Ric} - \frac{s}{n}\,\mathrm{id}_{T_{\mathbb{C}}M}) \cdot \phi$$

where the first summand and the second summand lie in $\iota(S)$ and $\ker(\mu)$, respectively.

Definition 7.8. The *Rarita-Schwinger operator* $D_{3/2}$ is given by

$$D_{3/2} = p_2 \circ D_T \circ p_2. \qquad \rtimes$$

In the following section we will construct special sections of $S_{3/2}$ with the help of Killing spinors. This method was first used by Wang [Wan91] in relation with metric deformations and establishes a connection between the Rarita-Schwinger operator $D_{3/2}$ and the Lichnerowicz Laplacian.

7.3 Killing spinors and the Lichnerowicz Laplacian

Define $\Psi^{(h,\sigma)}(X) = h(X) \cdot \sigma$, for a g-symmetric $h \in \mathrm{End}(M)$ and Killing-spinor σ. Then $\Psi^{(h,\sigma)} \in \Gamma(S \otimes T^*_{\mathbb{C}}M)$ and locally $\Psi^{(h,\sigma)} = h(e_i) \cdot \sigma \otimes e_i^*$.

Theorem 7.9 ([Wan91, GHP02])**.** *Let (M, g) be a Riemannian spin manifold admitting a non-zero Killing spinor σ with Killing number μ. Suppose h is a g-symmetric, traceless endomorphism of TM satisfying $\delta(h) = 0$. Furthermore, assume that its corresponding symmetric 2-tensor is an eigentensor of the Lichnerowicz Laplacian Δ_L to the eigenvalue λ. Let $\Psi = \Psi^{(h,\sigma)}$ be defined as above. Then*

$$(D_T - \mu)^2\Psi = \lambda\Psi - \Lambda\left(4 - \frac{1}{4}(n-5)^2\right)\Psi,$$

where $\Lambda := 4\mu^2 = \frac{s}{n(n-1)}$.

If additionally M is of positive scalar curvature, then

$$\lambda \geq \Lambda\left(4 - \frac{1}{4}(n-5)^2\right).$$

We will do some basic computations needed for the proof in the following lemmata.

Lemma 7.10. *With $\Psi^{(h,\sigma)}$ defined as above for g-symmetric $h \in \mathrm{End}(M)$ and with $\Theta \in \Gamma(S \otimes T^*_{\mathbb{C}}M)$ defined as $\Theta(X) = e_i \cdot (\nabla_{e_i} h)(X) \cdot \sigma$ we have*

(1) $\mu(\Psi^{(h,\sigma)}) = -\operatorname{tr}(h)\sigma$ and

(2) $\delta(\Psi^{(h,\sigma)}) = -\delta^\nabla(h)\sigma + \mu \operatorname{tr}(h)\sigma$,

(3) $(D_T\Psi^{(h,\sigma)})(X) = \Theta(X) + \mu(n-2)\Psi^{(h,\sigma)}(X)$,

where δ is the adjoint covariant derivative and δ^∇ is the codifferential.

Proof. Using the definition of Clifford multiplication $\mu(\psi) = e_i \cdot \psi(e_i)$, one gets

$$\mu(\Psi^{(h,\sigma)}) = e_i \cdot h(e_i) \cdot \sigma = g\left(h(e_i), e_j\right) e_i \cdot e_j \cdot \sigma = -\operatorname{tr}(h)\sigma.$$

Secondly,

$$\begin{aligned} \delta(\Psi^{(h,\sigma)}) &= -\nabla_{e_i}(h(e_i) \cdot \sigma) = -(\nabla_{e_i} h)(e_i) \cdot \sigma - \mu h(e_i) \cdot e_i \cdot \sigma \\ &= -\delta^\nabla(h) \cdot \sigma + \mu \operatorname{tr}(h)\sigma. \end{aligned}$$

And lastly another straightforward computation shows

$$\begin{aligned} (D_T\Psi^{(h,\sigma)})(X) &= e_i \cdot \nabla_{e_i}(h(X) \cdot \sigma) \\ &= e_i \cdot (\nabla_{e_i} h)(X) \cdot \sigma + \mu e_i \cdot h(X) \cdot e_i \cdot \sigma \\ &= \Theta(X) + \mu(n-2)\Psi^{(h,\sigma)}(X), \end{aligned}$$

where we have used the identity $e_i \cdot e_j \cdot e_i = (n-2)e_j$. □

Lemma 7.11. *If $h \in \mathrm{End}(M)$ is g-symmetric and $\delta h = \operatorname{tr} h = 0$, then*

(1) $\delta(\Psi^{(h,\sigma)}) = \mu(\Psi^{(h,\sigma)}) = 0$, i.e. $\Psi^{(h,\sigma)} \in \Gamma(S_{3/2})$, and

(2) $\delta(\Theta) = \mu(\Theta) = 0$.

Proof. Part 1 follows immediately from Lemma 7.10.

As h is traceless and g-symmetric, so is its covariant derivative $\nabla_X h$. Thus,

$$\begin{aligned} \mu(\Theta) &= e_j \cdot e_i \cdot (\nabla_{e_i} h)(e_j) \cdot \sigma \\ &= \sum_{i \neq j} e_j \cdot e_i \cdot (\nabla_{e_i} h)(e_j) \cdot \sigma - (\nabla_{e_i} h)(e_i) \cdot \sigma \\ &= \sum_{i \neq j} g\left((\nabla_{e_i} h)(e_j), e_k\right) e_j \cdot e_i \cdot e_k \cdot \sigma = 0 \end{aligned}$$

and similarly,

$$\begin{aligned}\delta(\Theta) &= \nabla_{e_j}(\Theta(e_j)) = \nabla_{e_j}(e_i \cdot (\nabla_{e_i} h)(e_j) \cdot \sigma) \\ &= e_i \cdot (\nabla_{e_j}\nabla_{e_i} h)(e_j) \cdot \sigma + \mu e_i \cdot (\nabla_{e_i} h)(e_j) \cdot e_j \cdot \sigma \\ &= e_i \cdot \nabla_{e_i}(\delta h) \cdot \sigma = 0. \qquad \square\end{aligned}$$

The following lemma was first proved in [Wan91, Proposition 2.4] and is the last step we need to be able to show our theorem. However, we will skip the proof of the lemma as it involves some tedious computations.

Lemma 7.12. *With $\Theta \in \Gamma(S \otimes T^*_{\mathbb{C}} M)$ defined as in Lemma 7.10 and $\mathrm{tr}(h) = \delta^\nabla(h) = 0$,*

$$(D_T\Theta)(X) = \mu(4-n)\Theta(X) - 4\mu^2 n \Psi^{(h,\sigma)}(X) + (\Delta_L h)(X) \cdot \sigma.$$

Proof of the theorem. Let $\Psi = \Psi^{(h,\sigma)}$. We will compute $(D_T - \mu)^2\Psi$. By Lemma 7.10,

$$\begin{aligned}(D_T - \mu)^2\Psi &= (D_T - \mu)(\Theta + \mu(n-3)\Psi) \\ &= D_T\Theta + \mu(n-3)D_T\Psi - \mu\Theta - \mu^2(n-3)\Psi.\end{aligned}$$

Now, after using Lemma 7.12, the terms involving Θ cancel out. The remaing terms are

$$(D_T - \mu)^2\Psi = -\mu^2(16 - (n-5)^2)\Psi + (\Delta_L h) \cdot \sigma$$

where we interpret $(\Delta_L h) \cdot \sigma$ as a spinor valued 1-form. Finally, using $4\mu^2 = \frac{s}{n(n-1)}$ from Proposition 7.4 and $\Delta_L h = \lambda h$ by assumption, we find

$$(D_T - \mu)^2\Psi = -\frac{s}{n(n-1)}(4 - \frac{1}{4}(n-5)^2)\Psi + \lambda\Psi$$

and the first claim follows.

Since D_T is formally self-adjoint and $s \geq 0$ implies $\mu \in \mathbb{R}$, $(D_T - \mu)^2$ is positive. Therefore

$$\lambda \geq \Lambda\left(4 - \frac{1}{4}(n-5)^2\right). \qquad \square$$

Corollary 7.13 ([GHP02])**.** *If (M, g) is a Riemannian spin manifold with positive scalar curvature admitting a non-zero Killing spinor, then M is physically stable.*

7.4 Examples

We close this chapter with some examples of manifolds admitting Killing spinors. Killing spinors on odd-dimensional manifolds are closely related to Einstein-Sasaki structures, i.e. contact structures with a certain regularity condition. In [BFGK91], the authors prove existence of Killing spinors on simply connected Einstein-Sasaki manifolds.

Theorem 7.14 ([BFGK91])**.** *Let M be a simply connected Sasaki-Einstein manifold. Then M admits non-trivial real Killing spinors. In particular,*

(i) if M is of dimension $4m+1$, then M admits exactly one Killing spinor for each Killing number $\mu = \pm\frac{1}{2}$,

(ii) if M is of dimension $4m+3$, then M admits at least two Killing spinors for one of the Killing numbers $\mu = \pm\frac{1}{2}$.

The key to the proof is considering the connection $\nabla^{\pm}_X \Psi = \nabla_X \Psi \pm \frac{1}{2} X \cdot \Psi$ on S and certain subbundles of the spinor bundle S that are parallel with respect to $\nabla^{\pm}$. It turns out that $\nabla^{\pm}$ is flat on these subbundles and thus admits constant sections which are the Killing spinors.

One can realize simply connected Einstein-Sasaki manifolds as certain S^1-bundles over Einstein-Kähler manifolds. This construction method dates back to Kaluza [Kal21] and Klein [Kle26] who were trying to unify electromagnetism with general relativity by considering a five dimensional spacetime which is a S^1-bundle over the four dimensional spacetime.

We will give a brief sketch of this method and refer to [BFGK91, Chapter 4] for further details.

Proposition 7.15. *Let (M^{2m}, g, J, Ω) be an Einstein-Kähler manifold of dimension $2m$ with positive scalar curvature normalized to $s = 4m(m+1)$ and denote by $c_1(M) \in H^2(M,\mathbb{Z})$ its first Chern-class. Recall that isomorphism classes of S^1-bundles over M are in one-to-one correspondence with elements of $H^2(M,\mathbb{Z})$. Consider the S^1-bundle $\pi : P \to M$ with Chern class $c_1(\pi) = \frac{1}{A} c_1(M)$ where $A \in \mathbb{Z}$ maximal and $\eta \in \Omega^1(P, i\mathbb{R})$ a connection on P such that the curvature form $d\eta$ of P satisfies $d\eta = \frac{2(m+1)}{A} i\Omega$.*

Then by defining a metric on P by

$$\bar{g} = \pi^* g - \frac{A^2}{(m+1)^2} \eta \otimes \eta,$$

P becomes a simply connected Einstein-Sasakian manifold with spin structure.

An element of $H^2(M,\mathbb{Z})$ is said to be *positive* (resp. *negative*) if it can be represented by a real J-invariant 2-form such that its associated symmetric bilinear form is positive (resp. negative). Since $c_1(M)$ is represented by the Ricci form $c_1(M)$ is necessarily positive.

The only four-dimensional Kähler manifolds with positive first Chern class are analytically equivalent to $S^2 \times S^2$ ($\cong \mathbb{C}P^1 \times \mathbb{C}P^1$), $\mathbb{C}P^2$ or to one of the del Pezzo surfaces P_k (i.e. $\mathbb{C}P^2$ blown up in k points in general position) for $1 \leq k \leq 8$. It can be shown (see, e.g., [BFGK91]) that if a metric on $S^2 \times S^2$ or $\mathbb{C}P^2$ is Kähler-Einstein, then it is the standard metric. On P_1 and P_2 there are no Kähler-Einstein metrics. On P_k for $3 \leq k \leq 8$ existence of families of Kähler-Einstein metrics was shown by Tian and Yau [TY87].

The resulting five dimensional Sasaki-Einstein manifolds turn out to be $V_2(\mathbb{R}^4) \to S^2 \times S^2$, $S^5 \to \mathbb{C}P^2$ and $M_k^5 \to P_k$ for $3 \leq k \leq 8$. Here, M_k^5 is diffeomorphic to the k-fold connected sum of $S^2 \times S^3$ and $V_2(\mathbb{R}^4) = \mathrm{SO}(4)/\mathrm{SO}(2) \cong S^3 \times S^2$ is the Stiefel manifold of oriented orthonormal 2-frames in $\mathbb{R}^4$. The S^1-structure on $V_2(\mathbb{R}^4)$ is given by interpreting $S^2 \times S^2$ as the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$ of 2-planes in $\mathbb{R}^4$ and taking the map $V_2(\mathbb{R}^4) \to S^2 \times S^2$ as the projection onto $\mathrm{Gr}_2(\mathbb{R}^4)$.

The bundles $S^3 \times S^2$ and S^5 are examples of homogeneous Sasaki-Einstein manifolds. In fact, it turns out that there is a one-to-one correspondence between complex generalized flag manifolds and simply connected homogeneous Einstein-Sasaki manifolds. Moreover, every homogeneous Einstein-Sasaki manifold is of the form $M/\mathbb{Z}_k$ for M homogeneous Einstein-Sasaki, simply connected and $k \geq 1$ [BG08, Theorem 11.1.13].

8 Appendix

Listing 1: Main routine

```
load "functions.sage"

###
### G2/SO(4)
###

G2 = WeylCharacterRing("G2");
A1xA1 = WeylCharacterRing("A1xA1");

#Symmetric square of the isotropic representation without
#trivial factor
S20 = A1xA1(1,-1,1,-1) + A1xA1(1,-1,3,-3) + A1xA1(0,0,2,-2);

rho = G2.space().rho();
fw = G2.fundamental_weights();

results = find_reps(G2,A1xA1,"extended",S20,max_height=8);

if len(results) == 0:
  print "No results found"
else:
  eigenvals = [ w.inner_product(w+2*rho) for w in results ];
  print "G2/SO(4):"
  print "Lowest height weights: ", results;
  print "Corresponding eigenvalues: ", eigenvals;
  print "Lowest eigenvalue: ", min(eigenvals);
```

Listing 2: Definitions of functions

```
def getLinearCombination( fw, coeff ):
  fwlist = fw.list();
  result = fw.first()*0;
  for i in range(0,fw.cardinality()):
    result += coeff[i]*fwlist[i];
```

```
  return result;

def convertTuplesToInt( tuples ):
  result = [];
  for tup in tuples:
    if tup.length() == 0:
      result.append(0);
    else:
      result.append(tup[0]);
  return result;

def has_nonempty_intersection( list1, list2 ):
  for elm in list1:
    if elm in list2:
      return true;
  return false;

def find_reps(G,K,rule,S20,max_height):
  fw = G.fundamental_weights();
  results = [];
  bFound = false;
  for k in range(1, max_height+1):
    # Generate all linear combinations of height k
    linear_combinations = PartitionTuples(k, fw.cardinality())
    print "Searching at height", k, "with", linear_combinations.
       cardinality(), "possibilities"
    for lc in linear_combinations:
      lc = convertTuplesToInt(lc);
      w = getLinearCombination(fw, lc);

      #Branch the constructed G-representation
      pi = G(w).branch(K, rule);
      if has_nonempty_intersection(S20.monomials(), pi.monomials()):
        bFound = true;
        results.append(w);
    if bFound:
      break;
  return results;
```

References

[Aub76] T. Aubin, *Equations du type Monge-Ampere sur les varietes kahleriennes compactes*, C. R. Acad. Sci. Paris **283** (1976), 119–121.

[Bau09] H. Baum, *Eichfeldtheorie: eine Einführung in die Differentialgeometrie auf Faserbündeln*, Springer, Berlin, 2009.

[BE69] M. Berger and D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Diff. Geom. **3** (1969), no. 3, 379.

[Bes87] A. L. Besse, *Einstein manifolds*, Vol. 20, Springer-Verlag, 1987.

[BF82] P. Breitenlohner and D. Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, Phys. Lett. B **115** (1982), no. 3, 197–201.

[BFGK91] H. Baum, T. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner, 1991.

[BG08] C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford university press, 2008.

[Bou09] M. Boucetta, *Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on* S^n, Osaka J. Math. **46** (2009), no. 1, 235–254.

[BtD85] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Springer, New York, 1985.

[Car26] É. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926), no. 55, 214–264.

[DFG$^+$02] O. DeWolfe, D. Z. Freedman, S. S. Gubser, G. T. Horowitz, and I. Mitra, *Stability of* $\mathrm{AdS}_p \times M_q$ *compactifications without supersymmetry*, Phys. Rev. D (3) **65** (2002), no. 6, 064033, 16. MR1918467 (2003e:81146)

[DNP84] M. J. Duff, B. Nilsson, and C. Pope, *The criterion for vacuum stability in Kaluza-Klein supergravity*, Phys. Lett. B **139** (1984), no. 3, 154–158.

[Don83] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Diff. Geom. **18** (1983), no. 2, 279–315.

[Ebi70] D. Ebin, *The manifold of Riemannian metrics*, Proc. symp. ams, 1970, pp. 11–40.

[FH91] W. Fulton and J. Harris, *Representation Theory: A First Course*, Corrected, Springer, 1991.

[FPII04] M. Falcitelli, A. M. Pastore, S. Ianus, and S. Ianuș, *Riemannian submersions and related topics*, World Scientific Publishing Company Incorporated, 2004.

[FR80] P. G. O. Freund and M. A. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. B **97** (1980), no. 2, 233–235.

[Fre82] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), no. 3, 357–453.

[GH02] G. Gibbons and S. A. Hartnoll, *Gravitational instability in higher dimensions* (2002), available at `arXiv:hep-th/0206202`.

[GH94] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Vol. 197, Wiley Online Library, 1994.

[GHP02] G. W. Gibbons, S. A. Hartnoll, and C. N. Pope, *Bohm and Einstein-Sasaki Metrics, Black Holes and Cosmological Event Horizons* (2002), available at `arXiv:hep-th/0208031`.

[GW09] R. Goodman and N. R. Wallach, *Symmetry, representations, and invariants*, Vol. 255, Springer Verlag, 2009.

[Her60] R. Hermann, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle*, Proc. Amer. Math. Soc. (1960), 236–242.

[Hil24] D. Hilbert, *Die Grundlagen der Physik*, Math. Ann. **92** (1924), no. 1, 1–32.

[Hum73] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1973.

[Kal21] T. Kaluza, *Zum Unitätsproblem der Physik*, Sitz. Preuss. Akad. Wiss. Phys. Math. Kl. **1** (1921), 966.

[Kle26] O. Klein, *Quantentheorie und fünfdimensionale Relativitätstheorie*, Z. Phys. **37** (1926), no. 12, 895–906.

[KN62] S. Kaneyuki and T. Nagano, *On the first Betti numbers of compact quotient spaces of complex semi-simple Lie groups by discrete subgroups* (1962).

[KN96] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Wiley-Interscience, 1996.

[Koi79] N. Koiso, *On the second derivative of the total scalar curvature*, Osaka J. Math. **16** (1979), no. 2, 413–421.

[Koi80] ———, *Rigidity and stability of Einstein metrics. The case of compact symmetric spaces*, Osaka J. Math. **17** (1980), no. 1, 51–73.

[Koi82] ———, *Rigidity and infinitesimal deformability of Einstein metrics*, Osaka J. Math. **19** (1982), no. 3, 643–668.

[Mor07] A. Moroianu, *Lectures on Kähler Geometry*, Cambridge University Press, 2007.

[MS10] A. Moroianu and U. Semmelmann, *The Hermitian Laplace operator on nearly Kähler manifolds*, Comm. Math. Phys. **294** (2010), no. 1, 251–272.

[Mur65] S. Murakami, *Sur la classification des algebres de Lie reelles et simples*, Osaka J. Math. **2** (1965), 291–307.

[Oba62] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), no. 3, 333–340.

[O'N66] B. O'Neill, *The fundamental equations of a submersion.*, Michigan Math. J. **13** (1966), no. 4, 459–469.

[PP84] D. N. Page and C. Pope, *Stability analysis of compactifications of D=11 supergravity with SU(3)×SU(2)×U(1) symmetry*, Phys. Lett. B **145** (1984), no. 5, 337–341.

[Sem95] U. Semmelmann, *Komplexe Kontaktstrukturen und Kählersche Killingspinoren*, Ph.D. Thesis, 1995.

[S$^+$12] W. A. Stein et al., *Sage Mathematics Software (Version 5.3)*, The Sage Development Team, 2012. `http://www.sagemath.org`.

[SW10] U. Semmelmann and G. Weingart, *The Weitzenböck machine*, Compos. Math. **146** (2010), no. 02, 507–540.

[SW12] ______, *The universal Laplace operator*, 2012. Preprint.

[TY87] G. Tian and S. T. Yau, *Kähler-Einstein metrics on complex surfaces with* $c_1 > 0$, Comm. Math. Phys. **112** (1987), no. 1, 175–203.

[Vil70] J. Vilms, *Totally geodesic maps*, J. Diff. Geom. **4** (1970), 73–79.

[Wan91] M. Y. Wang, *Preserving parallel spinors under metric deformations*, Indiana Univ. Math. J. **40** (1991), 815–844.

[Yau78] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation, I*, Comm. pure appl. math. **31** (1978), no. 3, 339–411.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich meine Diplomarbeit mit dem Titel *On the Lichnerowicz Laplacian and its application to stability of spacetimes* selbstständig und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe und dass ich alle Stellen, die ich wörtlich oder sinngemäß aus Veröffentlichungen entnommen habe, als solche kenntlich gemacht habe. Die Arbeit hat bisher in gleicher oder ähnlicher Form oder auszugsweise noch keiner Prüfungsbehörde vorgelegen.

Stuttgart, den 13. Februar 2013

(Peter-Simon Dieterich)