Gruppenwirkungen und Kohomologie

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Contents

Chapter I.

Principal bundles and classifying spaces

1. Lie groups, Lie algebras, and the exponential map

This section merely introduces some notation and recollects some well–known facts from the theory of Lie groups. For proofs we refer to [17] and [10].

Definition 1.1. A *Lie group* is a group G endowed with the structure of a smooth manifold such that the group multiplication $G \times G \to G$, $(g, k) \mapsto gk$, is smooth.

Remark 1.2. By the inverse function theorem, the group inversion $G \to G$, $g \mapsto g^{-1}$, then necessarily is smooth as well.

Example 1.3.

- (i) The circle $S^1 \subseteq \mathbb{C}$ is a Lie group.
- (ii) Products of Lie groups are again Lie groups. Thus, the n–torus $T^n = \prod_{i=1}^{n} S^1$ is a Lie group.
- (iii) Further classical examples of Lie groups are matrix Lie groups:
  - a) $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, the invertible real and complex $(n \times n)$–matrices, respectively;
  - b) $\mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$, the set of orthogonal matrices;
  - c) $\mathrm{U}(n) \subseteq \mathrm{GL}(n, \mathbb{C})$, the set of unitary matrices;
  - d) $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$, the special real and complex groups.
- (iv) If V is an n–dimensional $\mathbb{K}$–vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), then after a choice of basis on V we may identify the group of $\mathbb{K}$–linear automorphisms $\mathrm{Iso}(V)$ of V with the group $\mathrm{GL}(n, \mathbb{K})$. Thus, $\mathrm{Iso}(V)$ becomes a Lie group too, and the Lie group structure obtained in this way is independent of the choice of basis.

A *Lie group homomorphism* between Lie groups G and K is a smooth map $f \colon G \to K$ which also is a homomorphism of groups. An *immersed Lie subgroup* of a Lie group G is an abstract subgroup $H \subseteq G$ admitting a Lie group structure for which the inclusion $i \colon H \to G$ is an immersion. If i is an embedding, then H is called an *(embedded) Lie subgroup* of G.

Example 1.4.

- (i) The subgroups of $\mathrm{GL}(n, \mathbb{K})$ given in example 1.3 are all embedded Lie subgroups.

(ii) Consider subgroups $S \subseteq T^2$ of the form $S = \{(e^{2\pi\mathrm{i}}, e^{2\pi\mathrm{i}\lambda t}) \,|\, t \in \mathbb{R}\}$ with λ an irrational number. These are immersed, but not embedded.

The following criterion is useful to check whether an (abstract) subgroup of a Lie group is embedded. It in particular implies that kernels of Lie group homomorphisms are embedded Lie subgroups.

Theorem 1.5. *Let G be a Lie group and $H \subseteq G$ be an (abstract) subgroup of G. If H is closed as a subspace of G, then H admits a unique smooth structure such that H becomes an embedded Lie subgroup.*

Associated with a Lie group G is the subspace $\mathfrak{g} \subseteq \Gamma(TG)$ consisting of all those vector fields X which are left–invariant, that is, for which the equality $(d\ell_g)_k(X(k)) = X(gk)$ holds for all $g, k \in G$. Here, $\ell_g \colon G \to G$ denotes the map $\ell_g(k) = gk$ and $(d\ell_g)_k \colon T_kG \to T_{gk}G$ is its differential. The condition that a vector field $X \in \mathfrak{g}$ be left invariant shows that X is determined by the value at any given point. Thus, as a real vector space $\mathfrak{g}$ is isomorphic to T_eG, the tangent space at the identity element $e \in G$, via the map $\mathfrak{g} \to T_eG$, $X \mapsto X(e)$.

Definition 1.6. A *(real) Lie algebra* is a real vector space V together with an $\mathbb{R}$–bilinear, skew–symmetric map $[\cdot,\cdot] \colon V \times V \to V$ satisfying the *Jacobi identity* for all $X, Y, Z \in V$:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Example 1.7.

(i) If G is a Lie group, then the space $\mathfrak{g}$ together with the restriction of the commutator $[\cdot,\cdot]$ of vector fields makes $\mathfrak{g}$ into an abstract Lie algebra and thus is referred to as the *Lie algebra of* or *associated to G*.

(ii) If A is an $\mathbb{R}$–algebra, define the commutator of elements $a, b \in A$ by $[a,b] := ab - ba$. Then $(A, [\cdot,\cdot])$ is a Lie algebra.

(iii) The previous example applies to the space of all real matrices $\mathbb{R}^{n\times n}$. Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, the inclusion induced map $T_{I_n}\mathrm{GL}(n,\mathbb{R}) \to T_{I_n}\mathbb{R}^{n\times n}$ is an isomorphism. Moreover, there is a canonical isomorphism of vector spaces $T_{I_n}\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, yielding an isomorphism of vector spaces $\mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n}$, which even is an isomorphism of Lie algebras.

If $f \colon G \to K$ is a morphism of Lie groups, then under the identification $\mathfrak{g} \cong T_eG$ there corredponds to df_e a unique $\mathbb{R}$–linear map $df \colon \mathfrak{g} \to \mathfrak{k}$ such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "df"] \arrow[d, "\cong"'] & \mathfrak{k} \arrow[d, "\cong"] \\
T_eG \arrow[r, "df_e"'] & T_eK
\end{tikzcd}$$

is commutative. We call df the *induced Lie algebra homomorphism.* This is indeed a homomorphism of (Lie) algebras: for all $X, Y \in \mathfrak{g}$ we have $df([X, Y]) = [dfX, dfY]$. In particular, if $H \subseteq G$ is an (possibly immersed) Lie subgroup and $i\colon H \to G$ denotes the inclusion, then di is injective, and hence we may identify $\mathfrak{h}$ with the subalgebra $di(\mathfrak{h}) \subseteq \mathfrak{g}$. Conversly, we have

Theorem 1.8. *Let G be a Lie group and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra (i. e. a subspace such that $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$). Then there exists a unique connected immersed Lie subgroup $H \subseteq G$ having Lie algebra $\mathfrak{h}$.*

A further property of left invariant vector fields is that they are complete. Thus, for every $X \in \mathfrak{g}$ there exists an integral curve $\gamma_X\colon \mathbb{R} \to G$ of X with $\gamma_X(0) = e$, and γ_X is actually a homomorphism from the Abelian Lie group $(\mathbb{R}, +)$ into G. The map $\exp\colon \mathfrak{g} \to G$, $X \mapsto \gamma_X(1)$, is smooth and called the *exponential map* of G. For fixed $X \in \mathfrak{g}$ the map $\mathbb{R} \to G$, $t \mapsto \exp(tX)$, is again a morphism of Lie groups.

Proposition 1.9. *The exponential map is natural. That is, for Lie groups G, K and every Lie group homomorphism $f\colon G \to K$ we have a commutative diagram*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "df"] \arrow[d, "\exp"'] & \mathfrak{k} \arrow[d, "\exp"] \\
G \arrow[r, "f"'] & K
\end{tikzcd}$$

The exponential map allows us to characterize the Lie subalgebra of an immersed Lie subgroup $H \subseteq G$:

Theorem 1.10. *We have $\mathfrak{h} = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \exp^G(tX) \in H\}$.*

Example 1.11. For a matrix $A \in \mathbb{R}^{n\times n}$ there is the matrix exponential

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Under the identification $\mathfrak{gl}(n, \mathbb{R}) \cong \mathbb{R}^{n\times n}$ the Lie group exponential $\exp\colon \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ becomes the matrix exponential $e^{(\cdot)}\colon \mathbb{R}^{n\times n} \to \mathrm{GL}(n, \mathbb{R})$, $A \mapsto e^A$. Thus, by the previous theorem and naturality of the exponential map, a Lie subgroup $H \subseteq \mathrm{GL}(n, \mathbb{R})$ has Lie algebra $\{A \in \mathbb{R}^{n\times n} \mid \forall t \in \mathbb{R} : e^{tA} \in H\}$.

2. Lie group actions and principal bundles

In this section we introduce Lie group actions and principal bundles. These concepts can be defined in the context of topological spaces and smooth manifolds, so in order to avoid endless repetitions, let us agree on the following convention: if both X and Y are smooth manifolds, we call a smooth map $X \to Y$ a morphism of smooth manifolds. If X and Y are topological spaces, then a morphism $X \to Y$ is just a continuous map. We shall use the term isomorphism to mean a diffeomorphism or homeomorphism, respectively.

Definition 2.1. Let G be a Lie group and X a topological space or a smooth manifold.

(i) A *(left) action* of G on X is a morphism $G \times X \to X$, $(g, x) \mapsto g.x$, subject to the following conditions:

  a) $e.x = x$ for all $x \in X$;

  b) $(gh).x = g.(h.x)$ for all $g, h \in G$ and all $x \in M$.

  The space X then is called a (left) G–space.

(ii) The *isotropy subgroup at* or the *stabilizer of* a point $x \in X$ is the subgroup

$$G_x := \{g \in G \,|\, g.x = x\}.$$

(iii) The *orbit* through a point $x \in X$ is the subset $G \cdot x = \{g.x \,|\, x \in X\}$ and the *orbit space* is the space $X/G = \{Gx \,|\, x \in X\}$ of all orbits.

(iv) The set of *fixed points* is the subspace $X^G = \{x \in X \,|\, \forall g \in G : g.x = x\}$.

Remark 2.2.

(i) We will always consider X/G as the topological space whose topology is the quotient (or final) topology with respect to the canonical projection $\pi \colon X \to X/G$. Recall that this topology is uniquely determined by the following condition: a map $f \colon X/G \to Z$ is continuous if and only if $f \circ \pi$ is continuous.

(ii) A morphism between G–spaces X and Y is a morphism $f \colon X \to Y$ which is G–*equivariant*. This means that $f(g.x) = g.f(x)$ holds for all $g \in G$ and all $x \in X$.

(iii) In a similar fashion one can define right actions and right G–spaces. These notions are essentially equivalent: if $X \times G \to X$, $(p, g) \mapsto p.g$, is a right action, then $G \times X \to X$, $(g, p) \mapsto p.g^{-1}$, is a left action and vice–versa.

Example 2.3.

(i) $\mathrm{GL}(n, \mathbb{K})$ smoothly acts on $\mathbb{K}^n$ by the rule $A.v = Av$ for all $A \in \mathrm{GL}(n, \mathbb{K})$ and all $v \in \mathbb{K}^n$, where the right hand side of the equation is the usual matrix product. Similarly, if V is a finite–dimensional $\mathbb{K}$–vector space, then $\mathrm{Iso}(V) \times V \to V$, $(L, v) \mapsto L(v)$, is a smooth left action.

(ii) Every Lie group G acts on itself in (at least) three ways. The *action by left–translation* is the left action given by $G \times G \to G$, $(g, k) \mapsto gk$. The *action by conjugation* is the left action determined by $(g, k) \mapsto gkg^{-1}$, and the *action by right–translation* is the right action sending (k, g) to kg.

(iii) The (special) orthogonal group $\mathrm{O}(n + 1)$ (respectively $\mathrm{SO}(n + 1)$) acts smoothly on $S^n \subseteq \mathbb{R}^{n+1}$ from the left by matrix multiplication.

As a further example, consider the 2–sphere $S^2 \subseteq \mathbb{C} \times \mathbb{R}$. Then $S^1 \subseteq \mathbb{C}$ acts in a smooth fashion on S^2 via $S^1 \times S^2 \to S^2$, $(z,(v,t)) \mapsto (zv,t)$. Observe that the orbit space S^2/S^1 is homeomorphic to $[0,1]$, with endpoints corresponding to the fixed points $(0,\pm 1) \in S^2$. Thus, the orbit space of an action might not be a manifold, even though the acting group is. Quotient spaces of *free* actions, that is, those actions for which G_x is the trivial subgroup for all x, are generally better behaved.

Theorem 2.4. *If a compact Lie group G acts freely and smoothly on a manifold M, then M/G can be endowed with a unique smooth structure such that the canonical projection $\pi\colon M \to M/G$ becomes a submersion. In particular, a map $f\colon M/G \to N$ is smooth if and only if $f \circ \pi$ is smooth.*

We will prove theorem 2.4 in the exercises. There we will also see that in this situation the map $\pi\colon M \to M/G$ is a principal G–bundle.

Definition 2.5. Let G be a Lie group.

(i) A *(locally trivial) principle G–bundle* consists of topological spaces (or manifolds) E, B and a morphism $\pi\colon E \to B$ with the following properties: G acts freely on E from the right and for every point $b \in B$ there exists an open neighborhood U of b admitting a *trivialization* of $E|_U := \pi^{-1}(U)$, that is, a G–equivariant isomorphism $\psi\colon E|_U \to U \times G$ making the diagram

$$\begin{tikzcd}
E|_U \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\mathrm{pr}_1"] \\
& U &
\end{tikzcd}$$

commute; here, pr_1 denotes the projection onto the first factor and we consider $U \times G$ a right G–space via the action $(U \times G) \times G \to U \times G$, $((x,h),g) \mapsto (x,hg)$. We call E the *total space*, B the *base*, and π the *projection* of the principal bundle $E \xrightarrow{\pi} B$.

(ii) A *morphism* between principal G–bundles $E \xrightarrow{\pi} B$ and $E_0 \xrightarrow{\pi_0} B_0$ is a pair (f,u) where $f\colon E \to E_0$ and $u\colon B \to B_0$ are morphisms, f is G–equivariant, and

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "\pi"'] & E_0 \arrow[d, "\pi_0"] \\
B \arrow[r, "u"'] & B_0
\end{tikzcd}$$

is a commutative diagram.

Remark 2.6. If $f\colon E \to E_0$ is a G–equivariant map between the total spaces of principal bundles $E \xrightarrow{\pi} B$ and $E_0 \xrightarrow{\pi_0} B_0$, then we can always define a set map $u\colon B \to B_0$ with $\pi_0 \circ f = u \circ \pi$, because the bundle projection of a principal bundle is surjective. The map u is actually a morphism, because principal bundles are locally trivial, and therefore it is customary to just denote a morphism of principal bundles (g,v) between $E \xrightarrow{\pi} B$ and $E_0 \xrightarrow{\pi_0} B_0$ by g. The map v is then said to be the map *induced* or *covered* by g.

Example 2.7.

(i) If H is a compact subgroup of a Lie group G, then the set of right cosets

$$G/H = \{gH \mid g \in G\}$$

is the same as the orbit space of the H–action on G determined by $H \times G \to G$, $(h, g) \mapsto gh^{-1}$. Thus, according to theorem 2.4, the canonical projection $\pi\colon G \to G/H$ is a smooth principal H–bundle. This holds more generally whenever H is a closed, not necessarily compact subgroup of G, cf. [17, Theorem 3.58]. We call G/H a *homogeneous space.*

(ii) Consider the S^1–action on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ given by $z.(z_0, \ldots, z_n) = (zz_0, \ldots, zz_n)$. This is a free, smooth action whose orbit space S^{2n+1}/S^1 is $\mathbb{C}P^n$. The principal S^1–bundle $\pi\colon S^{2n+1} \to \mathbb{C}P^n$ is the so–called Hopf bundle.

(iii) Suppose a Lie group G acts on a smooth manifold M and fix a point $p \in M$. The isotropy subgroup $G_p \subseteq G$ is closed, and so G/G_p is a smooth manifold. The orbit map $G/G_p \to M$, $gG_p \mapsto gp$, in an injective immersion, and thus its image, $G \cdot p$, is an immersed submanifold of M. We will only be interested in the case that G is compact. In this case G/G_p is compact too and $G \cdot p$ is an embedded submanifold, since $G/G_p \to M$ is an embedding.

(iv) Let $E \xrightarrow{\pi} B$ be a topological $\mathbb{K}$–vector bundle of rank r. The space

$$\mathrm{P}(E) = \{u\colon \mathbb{K}^r \to E_b \mid b \in B,\ u \text{ is a } \mathbb{K}\text{–linear isomorphism}\}$$

is the total space of a principal $\mathrm{GL}(r, \mathbb{K})$–bundle $\mathrm{P}(E) \xrightarrow{\tau} B$, the *frame bundle* of E. To see this, let $e_1, \ldots, e_r$ be the standard basis of $\mathbb{K}^r$ and note that we can consider $\mathrm{P}(E)$ as a subspace of the r–fold product $E^r = E \times \ldots \times E$ via the map $u \mapsto (u(e_1), \ldots, u(e_r))$. Then $\mathrm{P}(E)$ is a free right $\mathrm{GL}(r, \mathbb{K})$–space via the action $A.u \mapsto u \circ A$, where on the right hand side we identify the matrix $A \in \mathrm{GL}(r, \mathbb{K})$ with the linear map $\mathbb{K}^r \to \mathbb{K}^r$, $v \mapsto Av$. The bundle projection τ is defined by $\tau(u) = b$ if u is a map $\mathbb{K}^r \to E_b$. To show that this bundle is locally trivial, choose a trivialization $\psi\colon E|_U \to U \times \mathbb{K}^r$, $v \mapsto (\psi_1(v), \psi_2(v))$, of the vector bundle $E \xrightarrow{\pi} B$. Then the map

$$\begin{aligned} \psi_*\colon \mathrm{P}(E)|_U &\to U \times \mathrm{GL}(r, \mathbb{K}), \\ u &\mapsto \big(\tau(u), (\psi_2 \circ u(e_i))_{i=1,\ldots,r}\big), \end{aligned}$$

is an equivariant homeomorphism, hence a trivialization, and it follows that $\mathrm{P}(E) \xrightarrow{\tau} B$ is a topological principal bundle. In case that the vector bundle is smooth, we use the maps ψ_* to define a smooth structure on $\mathrm{P}(E)$, and in this way $\mathrm{P}(E) \xrightarrow{\tau} B$ becomes a smooth principal bundle.

Note that the fiber $E_b = \pi^{-1}(b)$ of a principal G–bundle $E \xrightarrow{\pi} B$ over a point $b \in B$ is precisely a G–orbit. In fact, choose a trivialization $\psi\colon E|_U \to U \times G$ over some open neighborhood U of b. We see that if E is smooth, then E_b is an embedded submanifold, because so is its diffeomorphic image $\{b\} \times G$. Now choose $x \in E_b$ with $\psi(x) = (b, e)$, where $e \in G$ is the identity element. Then the map $G \to E_b$, $g \mapsto xg$ is an isomorphism, since by G–equivariance $xg = \psi^{-1}(b, e)g = \psi^{-1}(b, g)$ is the composition of the isomorphisms $G \to \{b\} \times G$ and $\psi^{-1}|_{\{b\}\times G}\colon \{b\} \times G \to E_b$. This observation leads to the following

Proposition 2.8. *Every morphism between principal G–bundles over the same base B which induces the identity on the base is an isomorphism.*

Proof. Let $E \xrightarrow{\pi} B$ and $E_0 \xrightarrow{\pi_0} B$ be two principal G–bundles and $f\colon E \to E_0$ a morphism covering the identity. If $x, y \in E$ are such that $f(x) = f(y)$, then also $\pi(x) = \pi(y)$, since f preserves fibers, i. e. because $\pi_0 \circ f = \pi$. We have already seen that the fibers of $\pi\colon E \to B$ are exactly G–orbits. Thus, there exists $g \in G$ with $y = xg$, and hence also $f(x) = f(x)g$. But G–acts freely on E_0, whence $g = e$ and $x = y$. This proves injectivity.

As for surjectivity of f: given $x_0 \in E_0$, put $b := \pi_0(x_0)$ and choose any point x in E_b. Then $f(x) \in (E_0)_b$, and since $(E_0)_b = f(x) \cdot G$, we find $g \in G$ such that $f(x)g = x_0$. It follows that $f(xg) = x_0$, and f is surjective.

To see that f is an isomorphism, choose an open set $U \subseteq B$ over which both E and E_0 are trivial and define $h\colon U \times G \to U \times G$ by requiring that the diagram

$$\begin{tikzcd}
E|_U \arrow[r, "f"] \arrow[d, "\cong"'] & E_0|_U \arrow[d, "\cong"] \\
U \times G \arrow[r, "h"] & U \times G
\end{tikzcd}$$

be commutative. Note that h is G–equivariant, because so are f and trivializations of principal bundles. Hence, if $g_0\colon U \to G$ is the map with $h(x, e) = (x, g_0(x))$ for all $x \in U$, then also $h(x, g) = (x, g_0(x)g)$ for all $g \in G$ by G–equivariance. But this map is an isomorphism, with inverse given by $(x, g) \mapsto (x, (g_0(x))^{-1} \cdot g)$, whence f locally is an isomorphism. Since f is bijective, it must be a global isomorphism too. ☐

Given a morphism $f\colon X \to B$ into the base space of a principal G–bundle $E \xrightarrow{\pi} B$, the *pullback*

$$f^*E = \{(x, p) \in X \times E \mid f(x) = \pi(p)\}$$

together with the map $f^*E \to X$, $(x, p) \mapsto x$, is again principal G–bundle. To see this, note that the canonical free right G–action on $X \times E$ restricts to an action on f^*E. Moreover, if $\psi = (\psi_1, \psi_2)\colon E|_U \to U \times G$ is a trivialization of E over some open subset $U \subseteq B$, then $(f^*E)|_V \to V \times G$, $(x, p) \mapsto (x, \psi_2(p))$, is a trivialization over the open subset $V = f^{-1}(U)$, with inverse given by $(x, g) \mapsto (x, \psi^{-1}(f(x), g))$. As a consequence of proposition 2.8 we have

Corollary 2.9. *Let* $E \xrightarrow{\pi} B$ *and* $E_0 \xrightarrow{\pi_0} B_0$ *be principal* G*–bundles and suppose that* $f\colon E \to E_0$ *is a morphism of principal bundles.*

(i) If f *covers* u*, then* u^*E_0 *is canonically isomorphic to* $E \xrightarrow{\pi} B$*.*

(ii) If $E_1 \to B_1$ *is another principal* G*–bundle and* $v\colon B_0 \to B_1$ *is a morphism, then* $u^*(v^*E_1)$ *and* $(v \circ u)^*E_1$ *are canonically isomorphic.*

Proof. Note that $E \to u^*E_0$, $p \mapsto (\pi(p), f(p))$, is a morphism of principal bundles over the same base B and use proposition 2.8. This proves the first statement.

For the second, observe that $(v \circ u)^*E_1 \to u^*(v^*E_1)$, $(b, p) \mapsto (b, u(b), p)$ is a morphism of principal G–bundles. ☐

Another way of constructing new bundles out of principal bundles is as follows. Let $E \xrightarrow{\pi} B$ be a principal G–bundle and F a left G–space. Consider the free (left) action $(E \times F) \times G \to E \times F$, $((e, f), g) \mapsto (eg^{-1}, gf)$ and denote its orbit space by $E \times_G F := (E \times F)/G$. Write $\tau\colon E \times_G F \to B$ for the map $\tau([e, f]) = \pi(e)$; this is well–defined, because the fibers of π are exactly the G–orbits of the G–action on E. Then $E \times_G F \xrightarrow{\tau} B$ is called an *associated bundle*, the terminology being justified because of the following

Proposition 2.10. $E \times_G F \xrightarrow{\tau} B$ *is a locally trivial fiber bundle over* B *with typical fiber* F*. If* $\psi = (\psi_1, \psi_2)\colon E|_U \to U \times G$ *is a trivialization of* E *over some open subset* $U \subseteq B$*, then* $(E \times_G F)|_U \to U \times F$*,* $[e, f] \mapsto (\pi(e), \psi_2(e) \cdot f)$*, is a trivialization of* $E \times_G F$*.*

Proof. It suffices to prove that for each trivialization $\psi = (\psi_1, \psi_2)\colon E|_U \to U \times G$ of E the map $\psi_*\colon (E \times_G F)|_U \to U \times F$, $[e, f] \mapsto (\pi(e), \psi_2(e) \cdot f)$, is a homeomorphism and, if $\pi\colon E \to B$ is smooth, that for any two trivializations ψ, ϕ of E the map $\psi_* \circ (\phi_*)^{-1}$ is smooth. But $(\psi_*)^{-1}(b, f) = [\psi^{-1}(b, e), f]$ for all $b \in U$, $f \in F$, thus proving both statements. ☐

Example 2.11.

(i) Every (smooth or continuous) vector bundle over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is an associated bundle. To see this, choose a vector bundle $E \to B$ of rank k and consider

$$P(E) \times_{\mathrm{GL}(\mathbb{K},k)} \mathbb{K}^k \to E,\ [u, v] \mapsto u(v),$$

where $P(E)$ is the frame bundle of E and $\mathrm{GL}(\mathbb{K}, k)$ acts on $\mathbb{K}^k$ via the canonical left action, see example 2.7 and example 2.3.

(ii) Let H be a Lie subgroup of G and $E \to B$ a principal G–bundle. A principal H–bundle $Q \to B$ is a *reduction* of $E \to B$ if there is an H–equivariant map $f\colon Q \to E$ covering the identity, where we view E as a right H–space via the restriction of the principal G–action. The trivializations provided in proposition 2.10 show that $Q \times_H G$ is a principal G–bundle, and the map $Q \times_H G \to E$, $[x, g] \mapsto f(x)g$, is an isomorphism by proposition 2.8. Since $Q \to Q \times_H G$, $x \mapsto [x, e]$, is an embedding of topological spaces (and in the smooth case also of manifolds), we can identify Q with an H–invariant subspace of E intersecting each fiber E_b non–trivially. Conversly, if $Q \subseteq E$ is a subspace which is H–invariant and such that $Q \cap E_b \neq \emptyset$ for all b, then Q is a principal H–bundle and thus an H–reduction of E, as trivializations of E restrict to trivializations of Q.

3. Universal bundles

Let G be an arbitrary Lie group. We shall construct a principal G–bundle $EG \to BG$ such that any principal G–bundle $E \to B$ with suitable base space B admits a morphism of principal bundles $f\colon E \to EG$, and thus is isomorphic to the pullback u^*EG along the map u covered by f, cf. corollary 2.9.

The construction goes as follows. Recall that the *cone* of a topological space X is the quotient space $C(X) = (X \times [0,1])/X \times \{0\}$. The *join* of a family of topological spaces $(X_i)_{i\in I}$ is then the topological space whose underlying set is

$$X = *_{i\in I} X_i := \left\{ \left([x_i, t_i]\right)_{i\in I} \in \prod_{i\in I} C(X_i) \;\middle|\; t_i = 0 \text{ for almost all } i \text{ and } \sum_{i\in I} t_i = 1 \right\};$$

we will denote the element $([x_i, t_i])_{i\in I}$ of X by $\sum_i t_i x_i$. Then there are canonical maps $t_j\colon X \to [0,1]$, $\sum_i t_i x_i \mapsto t_j$ and $x_j\colon t_j^{-1}((0,1]) \to X_i$, $\sum_i t_i x_i \mapsto x_j$, and the topology on X is the smallest topology so that all the maps t_i and x_k are continuous. This topology is characterized by the following universal property: a map $f\colon Z \to X$ from a topological space Z is continuous if and only if $t_i \circ f\colon X \to [0,1]$ and $x_i \circ f\colon f^{-1}(t_i^{-1}((0,1])) \to X_i$ are continuous for all i. Note that in general, this topology is different from the subspace topology inherited from $\prod_{i\in I} C(X_i)$.

The *Milnor construction* now is $EG := *_{n\in\mathbb{N}} G$. There is a free right G–action

$$EG \times G \to EG, \quad \left(\sum_i t_i g_i, g\right) \mapsto \sum_i t_i (g_i g),$$

which is continuous by the universal property of the topology on EG. Denote by $p\colon EG \to BG := EG/G$ the canonical map into the quotient.

Proposition 3.1. *$EG \xrightarrow{p} BG$ is a topological principal G–bundle.*

Proof. We only need to show that $EG \xrightarrow{p} BG$ is locally trivial. The sets of the form $U = t_j^{-1}((0,1])$ cover EG and are G–invariant. Hence, $V = p(U)$ is open in BG and $EG|_V = U$. Moreover, $EG|_V$ is trivial with trivialization $\psi(\sum_i t_i g_i) = ([\sum_i t_i g_i], g_j)$, the inverse of which is $\psi^{-1}([\sum_i t_i g_i], g) = \sum_i t_i (g_i g_j^{-1} g)$. □

Example 3.2. Consider the set $\mathbb{R}^\infty = \{(t_j)_{j\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} \mid t_j = 0 \text{ for allmost all } j\}$ endowed with the final topology with respect to the inclusions $\mathbb{R}^n \subseteq \mathbb{R}^\infty$. That is, $U \subseteq \mathbb{R}^\infty$ is open if and only if $U \cap \mathbb{R}^n$ is open for all n. We can exhibit $E\mathbb{Z}_2$ as the infinite–dimensional sphere

$$S^{\infty} = \left\{ (t_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\infty} \;\middle|\; \sum_{j=0}^{\infty}(t_j)^2 = 1 \right\}$$

via the isomorphism $E\mathbb{Z}_2 \to S^{\infty}$, $\sum_i t_i z_i \mapsto ((-1)^{z_i}\sqrt{t_i})_{i\in\mathbb{N}}$. Thus, $B\mathbb{Z}_2 = S^{\infty}/\mathbb{Z}_2 =: \mathbb{R}P^{\infty}$ is the infinite–dimensional real–projective space.

In a similar way one can show that $ES^1 = S^{\infty}$ (use the map $\sum t_i z_i \mapsto (z_i\sqrt{t_i})_{i\in\mathbb{N}}$, where we consider S^1 as a subset of $\mathbb{C}$), and thus $BS^1 = \mathbb{C}P^{\infty}$ is the infinite–dimensional complex projective space.

The space BG is Hausdorff, but in general not paracompact, and so does not admit partitions of unity subordinate to an arbitrary cover. However, BG is numerable:

Definition 3.3. A principal G–bundle $E \xrightarrow{\pi} B$ is called *numberable* if there exists an open cover $\{U_k \,|\, k \in \mathbb{N}\}$ of B such that $E|_{U_k}$ is trivial for all $k \in \mathbb{N}$ and a partition of unity $(\xi_k)_{k\in\mathbb{N}}$ subordinate to that cover (i. e. $\operatorname{supp} \xi_k \subseteq U_k$ for all k).

Remark 3.4. Usually one does not require the index set in the definition of a numerable principal bundle to be $\mathbb{N}$, that is, a principal bundle $E \to B$ is said to be numerable if there exists an open cover $\mathcal{U} = \{U_i \,|\, i \in I\}$ admitting a subordinate partition of unity and such that $E|_{U_i}$ is trivial, see for example [9, Definition 9.2] or [16, Section 13.1]. However, by a theorem of Dold [6], in this case the bundle is also numerable in our sense, so for the applications we have in mind the two definitions are essentially equivalent. See also [9, Proposition 12.1] for a proof of this fact.

Proposition 3.5. *The bundle $EG \xrightarrow{p} BG$ is numerable.*

Proof. We have already seen in the proof of proposition 3.1 that EG is trivial over $V_j = p(U_j)$, where $U_j = t_j^{-1}((0, 1])$, and we now construct a partition of unity subordinate to the cover $\mathcal{V} = \{V_j \,|\, j \in \mathbb{N}\}$. For $i \in \mathbb{N}$ consider the continuous function

$$w_i := \max\left(0, t_i - \sum_{j<i} t_j\right).$$

We have $w_i(b) = t_i(b)$ if i is the first index with $t_i(b) \neq 0$, so $\mathcal{W} = \{w_i^{-1}((0, \infty))\}_{i\in\mathbb{N}}$ is an open cover of BG. Moreover, if $w_i(b) > 0$, then also $t_i(b) > 0$, and therefore $\mathcal{W}$ refines $\mathcal{V}$. Let us show that $\mathcal{W}$ is locally finite. Thus, choose $b_0 \in BG$ arbitrarily and let $n \in \mathbb{N}$ be such that $t_k(b_0) = 0$ for all $k > n$. The set O of all $b \in BG$ such that $\sum_{j\leq n} t_j(b) > 1/2$ is an open neighborhood of b_0, as each t_j is continuous, and if $b \in O$ is arbitrary and $k > n$, then $w_k(b) = 0$, because otherwise $t_k(b) > \sum_{j\leq n} t_j(b) > 1/2$, which is impossible, since $\sum_i t_i = 1$. Therefore, O only meets finitely many sets in $\mathcal{W}$ and $\mathcal{W}$ is locally finite.

The family $(w_k/\sum_i w_i)_{k\in\mathbb{N}}$ is a partition of unity, but it is not subordinate to the cover $\mathcal{V}$. To achieve this, put $m := \max_{i\in\mathbb{N}} w_i$. This function is continuous, because the maximum of a finite number of real valued functions is continuous and $\mathcal{W}$ is locally finite. Also note that m is nowhere zero, because $\mathcal{W}$ covers BG. In particular, $w_i^{-1}((0,\infty)) = (w_i/m)^{-1}((0,\infty)$. Thus, possibly passing from w_i to w_i/m if necessary, we may assume that $w_i \leq 1$ and that for each $b \in BG$ there exists some i_b with $w_{i_b}(b) = 1$. Then let $\tau_i := \max(0, 2w_i - 1)$ and note that $\tau_i(b) > 0$ only if $w_i(b) > 1/2$. Hence

$$\operatorname{supp} \tau_i \subseteq w_i^{-1}([1/2,\infty)) \subseteq V_i.$$

Since $\tau_{i_b}(b) = 1$, the family $\{\operatorname{supp} \tau_i \mid i \in \mathbb{N}\}$ covers BG and is subordinate to the cover $\mathcal{V}$. It follows that $(\xi_i)_{i\in\mathbb{N}}$ with $\xi_j = \tau_j / \sum_i \tau_i$ is a partition of unity subordinate to $\mathcal{V}$. □

Recall that a *homotopy* between morphisms $f, g\colon X \to Y$ of topological spaces (or smooth manifolds) is a morphism $H\colon X \times [0,1] \to Y$, $(x,t) \mapsto h_t(x)$, such that $h_0 = f$ and $h_1 = g$. In this case, f and g are called *homotopic*. Being homotopic defines an equivalence relation on the set of all morphisms $X \to Y$ and we denote the set of all such equivalence classes by $[X,Y]$. The importance of the bundle $EG \to BG$ stems from the fact that it classifies numerable principal G–bundles over a fixed base B in terms of homotopy classes of maps $B \to BG$.

Theorem 3.6. *Denote by* $\operatorname{Bun}_G(B)$ *the set of isomorphism classes of numerable principle G–bundles. Then the map*

$$\begin{aligned} \Phi\colon [B, BG] &\to \operatorname{Bun}_G(B), \\ [f] &\mapsto [f^*EG], \end{aligned}$$

is a bijection. Any numerable principle G–bundle $E_0 \to B_0$ for which the assignment $[B, B_0] \to \operatorname{Bun}_G(B)$, $[f] \mapsto [f^*E_0]$, is a bijection is called a classifying *or* universal G–bundle *and B_0 is called a* classifying space*.*

One part of this statement is a theorem in its own right:

Theorem 3.7. *Let $E \xrightarrow{\pi} B$ be a numerable topological principal G–bundle and suppose that $f, g\colon X \to B$ are homotopic morphisms. Then f^*E and g^*E are isomorphic principal bundles.*

Proof. If $H\colon X \times [0,1] \to B$ is a homotopy from f to g, then $f = H \circ i_0$ and $g = H \circ i_0$, where $i_t\colon X \to X \times [0,1]$, $x \mapsto (x,t)$, and hence $f^*E \cong i_0^*(H^*E)$ and $g^*E \cong i_1^*(H^*E)$ by corollary 2.9. Since the pullback of a numerable bundle is again numerable, it thus suffices to consider the case $B = B_0 \times [0,1]$ and $f = i_0$, $g = i_1$.

Step 1. If E is trivial over $B_0 \times [0,t]$ and $B_0 \times [t,1]$, then E is trivial. Choose trivializations ψ and ρ of E over $B_0 \times [0,t]$ and $B_0 \times [t,1]$, respectively. Then

$$\begin{aligned} B_0 \times \{t\} \times G &\to B_0 \times \{t\} \times G, \\ (b,t,g) &\mapsto \psi(\rho^{-1}(b,t,g)), \end{aligned}$$

is a morphism of the trivial principal G–bundle $B_0 \times \{t\} \times G \to B_0$, and hence of the form $(b, t, g) \mapsto b(b, t, u(b)g)$ for some morphism $u \colon B_0 \to G$. Now E is trivialized by

$$B \times [0,1] \times G \to E,\ (b, s, g) \mapsto \begin{cases} \psi^{-1}(b, s, u(b)g), & s \le t, \\ \rho^{-1}(b, s, g), & s \ge t. \end{cases}$$

Step 2. There exists an open cover $\{U_\alpha\}_{\alpha \in A}$ *of* B_0*, indexed by a countable set* A*, and functions* $u_\alpha \colon B_0 \to [0,1]$ *such that* E *is trivial over* $U_\alpha \times [0,1]$*,* $\{\operatorname{supp} u_\alpha\}_{\alpha \in A}$ *is a locally finite cover of* B *subordinate to* $\{U_\alpha\}_{\alpha \in A}$*, and such that* $\max_{\alpha \in A} u_\alpha \equiv 1$*.*

This is the most technical part. Start with a cover $\{V_j \mid j \in \mathbb{N}\}$ of $B_0 \times [0,1]$ over which E is trivial and which admits a subordinate partition of unity $(\xi_j)_{j \in \mathbb{N}}$; this is possible, since we assume the bundle in question to be numerable. Now consider for a multi–index $\alpha \in \mathbb{N}^r$ the morphism

$$s_\alpha \colon B_0 \to [0,1],\ b \mapsto \prod_{i=1}^{r} \min_{t \in [\frac{i-1}{r}, \frac{i}{r}]} \xi_{\alpha_i}(b, t).$$

If we put $U_\alpha = (s_\alpha)^{-1}((0, \infty))$, then $U_\alpha \times [\frac{i-1}{r}, \frac{i}{r}]$ is contained in V_{α_i}, because ξ_{α_i} is supported in V_{α_i}. But E already is trivial over V_{α_i}, and so, by the first step, must also be trivial over $U_\alpha \times [0,1]$. To see that the $\{U_\alpha \mid \alpha \in A\}$, where $A = \bigcup_{r \ge 1} \mathbb{N}^r$, cover B_0, choose $b \in B_0$ and consider the strip $\{b\} \times [0,1]$. It is covered by the open sets $\xi_j^{-1}((0, \infty))$, so for r sufficiently large there will be indices $\alpha_1, \ldots, \alpha_r$ such that $\xi_{\alpha_j} > 0$ on $\{b\} \times [\frac{i-1}{r}, \frac{i}{r}]$, and then $s_\alpha(b) > 0$ too. Next, note that for fixed r the family $\{U_\alpha\}_{|\alpha| = r}$ is locally finite, where α ranges over all multi–indices of length r. In fact, the set of supports of the partition of unity $(\xi_j)_{j \in \mathbb{N}}$ is locally finite, so for a given $b \in B_0$ we know by compactness of $\{b\} \times [0,1]$ that there must be an open neighborhood W of b and a finite set of indices $J \subseteq \mathbb{N}$ such that ξ_j is non–zero on $W \times [0,1]$ only if $j \in J$. Thus, if s_α is non–zero on W for some α of length r, then necessarily $\alpha_1, \ldots, \alpha_r \in J$, and since there is only a finite number of ways to choose r indices out of the finite set J, we see that $U_\alpha \cap W$ can only be non–trivial for finitely many α of length r.

To obtain a family of functions whose support is locally finite, independet of r, we pass to the functions

$$t_\alpha := \max\left(0, s_\alpha - |\alpha| \sum_{|\beta| < |\alpha|} s_\beta\right);$$

note that the above expression is well–defined, as we just showed $\sum_{|\beta| = r} s_\beta$ is locally–finite. We also see that the family $\mathcal{T} = \{(t_\alpha)^{-1}((0, \infty))\}_{\alpha \in A}$ is subordinated to the cover $\{U_\alpha\}_{\alpha \in A}$. If $b_0 \in B_0$ is arbitrary, then we have already seen that there exists some α with $s_\alpha(b_0) > 0$, and we may assume that α has minimal length $|\alpha|$ among all β with $s_\beta(b_0) > 0$. Then $t_\alpha(b_0) = s_\alpha(b_0) > 0$ and $\mathcal{T}$ covers B_0. To prove that $\mathcal{T}$ is locally finite, define $2c = s_\alpha(b_0) > 0$ and choose r so large that that $rc > 1$ and $r > |\alpha|$. For every β with $|\beta| \ge r$ and all b in the open neighborhood $(s_\alpha)^{-1}((c, \infty))$ of b_0 we have $t_\beta(b) = 0$, since otherwise $s_\beta(b) > |\beta| s_\alpha(b) > 1$. Finally, the same argument as in the proof of proposition 3.5 shows that we may use the functions $(t_\beta)_{\beta\in A}$ to construct functions $(u_\alpha)_{\alpha\in A}$ for which $\operatorname{supp} u_\alpha \subseteq (t_\alpha)^{-1}((0,\infty))$ and such that for every point $b \in B_0$ there exists some α with $u_\alpha(b) = 1$.

Step 3. $(i_0)^*E \cong (i_1)^*E$. Let $\{U_\alpha\}_{\alpha\in A}$ and $(u_\alpha)_{\alpha\in A}$ be the cover and the functions constructed in the previous step. Let ψ_α be a trivialization of E over $U_\alpha\times[0,1]$ and define $h_\alpha\colon E\to E$ to be id_E on the complement of $\operatorname{supp} u_\alpha$ and on $E|_{U_\alpha\times[0,1]}$ by commutativity of the diagram

$$\begin{tikzcd}
E|_{U_\alpha} \arrow[r, "h_\alpha"] \arrow[d, "\psi_\alpha"] & E|_{U_\alpha\times[0,1]} \arrow[d, "\psi_\alpha"'] \\
U_\alpha\times[0,1]\times G \arrow[r, "{(b,t,g)\mapsto(b,\max(u_\alpha(b),t),g)}"'] & U_\alpha\times[0,1]\times G.
\end{tikzcd}$$

This gives a well–defined G–equivariant morphism, since u_α is supported in U_α. Also, for any point $p\in E$, say with $\pi(p)=(b,t)$, there exists some α with $u_\alpha(b)=1$ and hence $h_\alpha(p)\in E_{(b,1)}$. Moreover, the family of supports $\{\operatorname{supp} u_\alpha\}_{\alpha\in A}$ is locally finite, so there is a neighborhood W of b such that $h_\alpha|_{E|_{W\times[0,1]}} = \mathrm{id}_E$ for all but finitely many $\alpha\in A$. Therefore, the expression

$$h(p) := \prod_{j=0}^{\infty} h_{\alpha_j}(p) = (h_{\alpha_0}\circ h_{\alpha_1}\circ\ldots)(p)$$

is meaningful and defines a G–equivariant morphism $E\to E$, where we have fixed, once and for all, an enumeration $\mathbb{N}\to A$, $j\mapsto\alpha_j$, of A. By construction h covers the morphism $B\times[0,1]\to B\times[0,1]$, $(b,t)\mapsto(b,1)$, so

$$(i_0)^*E \cong E|_{B\times\{0\}} \xrightarrow{h} E|_{B\times\{1\}} \cong (i_1)^*E$$

is a morphism of principal bundles inducing the identity on the base B. Then cccording to proposition 2.8 it is an isomorphism. ☐

Proof of theorem 3.6. According to theorem 3.7, the assignment is well–defined. Now let us show that Φ is surjective. Thus, choose a numerable principal bundle $E\xrightarrow{\pi}B$, an open cover $\{U_i\mid i\in\mathbb{N}\}$ of B over which E is trivial, say with trivializations $E|_{U_i}\to U_i\times G$, $p\mapsto(\pi(p),\psi_i(p))$, as well as a partition of unity $(\xi_i)_{i\in\mathbb{N}}$ subordinate to this cover. Define a map $f\colon E\to EG$ by $f(p)=\sum_i\xi_i(\pi(p))\psi_i(p)$ and note that f is continuous by definition of the topology on EG. Moreover, f is G–equivariant, since the morphisms ψ_i are, being part of a trivialization of E. Pulling back EG along the morphism $u\colon B\to BG$ covered by f then produces a bundle isomorphic to $E\to B$, see corollary 2.9.

Now let $f,g\colon B\to BG$ be two morphisms such that f^*EG and g^*EG are isomorphic principal bundles. We will show in proposition 3.8 below that any two G–equivariant maps $E\to EG$ from a free G–space E are homotopic through a G–equivariant homotopy. Granted this result, the proof that f and g are homotopic is as follows. Let $L\colon g^*EG\to f^*EG$ be an isomorphism of principle G–bundles. Then the projection $B\times EG\to EG$

induces canonical G–equivariant maps $u\colon g^*EG \to EG$ and $v\colon f^*EG \to EG$. Since g^*EG is a free G–space and L is G–equivariant, u and $v \circ L$ are G–equivariant through a G–equivariant homotopy $(h_t)_{t\in[0,1]}$. Define, for $t \in [0,1]$, the map $k_t\colon B \to BG$ by $k_t(b) = p(h_t(b,x))$, where $x \in EG_{g(b)}$ is arbitrary and $p\colon EG \to BG$ is the bundle projection. The map does not depend on the particular choice of x, since h_t is G–equivariant. In particular, if $\psi\colon g^*EG|_U \to U \times G$ is a trivialization over some open neighborhood $U \subseteq BG$, then $k_t(b) = (p \circ h_t)(\psi^{-1}(b,e))$ is continous for all $(b,t) \in U \times [0,1]$, that is, $(k_t)_{t\in[0,1]}$ is a homotopy. But by construction of g^*EG and f^*EG we have $k_0 = g$ and $k_1 = f$, so f and g are homotopic. ☐

Proposition 3.8. *The identity on EG is G–equivariantly homotopic to the map*

$$\alpha\colon EG \to EG,\ \sum_i t_i g_i \mapsto (0,[g_0,t_0],0,[g_1,t_1],0,\dots).$$

In particular, EG is contractible and any two G–equivariant morphisms $E \to EG$ from a (necessarily free) G–space E are G–equivariantly homotopic.

Proof. Let E be a G–space and $f\colon E \to EG$ an arbitrary morphism, not necessarily G–equivariant, and let f_i be the i–th component of f. Thus, $f(p) = (f_0(p), f_1(p),\dots)$ for all $p \in E$. Let us also set $s[g,t] = [g,st]$ for all $s,t \in [0,1]$ and all $g \in G$.

We show that f is homotopic to $\alpha\circ f$. To this end, choose a strictly decreasing sequence $1 = a_0 > a_1 > a_2 > \dots a_n > 0$ with $\lim_{n\to\infty} a_n = 0$ and for each i a continuous function $\lambda_i\colon [a_{i+1},a_i] \to [0,1]$ such that $\lambda_i(a_i) = 0$, $\lambda_i(a_{i+1}) = 1$. Then define $h_i\colon E\times[a_{i+1},a_i] \to EG$ by

$$H_i(p,t) = (f_0(p),\dots,f_{i-1}(p),\lambda_i(t)f_i(p),\mu_i(t)f_i(p),\lambda_i(t)f_{i+1}(p),\mu_i(t)f_{i+1}(p),\dots),$$

where we have set $\mu_i := 1 - \lambda_i$. By definition of the topology on EG and since f is continuous, we see that h_i is continuous. Note that $H_i(\cdot,a_{i+1}) = H_{i+1}(\cdot,a_{i+1})$, and therefore these functions assemble to give a continuous map $H\colon E\times[0,1] \to EG$, $(p,t) \mapsto h_t(p)$ such that $H|_{E\times[a_{i+1},a_i]} = H_i$ and $h_0 = f$. Since $h_1 = \alpha\circ f$, it follows that f and $\alpha\circ f$ are homotopic. Observe that each h_t is G–equivariant if f is G–equivariant. In particular, α is G–equivariantly homotopic to the identity. This proves the first statement. To prove the remaining statements, let

$$\beta\colon EG \to EG,\ (e_i)_{i\in\mathbb{N}} \mapsto (e_0,0,e_1,0,\dots)$$

and observe that for any two G–equivariant morphisms $f,g\colon E \to EG$ the map

$$E\times[0,1] \to EG,\ p \mapsto (tg_0(p),(1-t)f_0(p),tg_1(p),(1-t)f_1(p),\dots)$$

is a G–equivariant homotopy connecting $\alpha\circ f$ with $\beta\circ g$. In particular, β is G–equivariantly homotopic to the identity too and thus f and g are G–equivariantly homotopic. If f and g are not necessarily equivariant, then the previous construction shows that f and g still are non–equivariantly homotopic, and thus EG is (non–equivariantly) contractible. ☐

The Milnor construction is functorial with respect to morphisms of Lie groups. That is, if $f\colon G \to K$ is a homomorphism of Lie groups, then there is bundle morphism

$$\begin{tikzcd} EG \arrow[r, "Ef"] \arrow[d] & EK \arrow[d] \\ BG \arrow[r, "Bf"] & BK \end{tikzcd}$$

where the undecorated vertical maps are the respective bundle projections. Explicitly, $Ef(\sum_i t_i g_i) = \sum_i t_i f(g_i)$. Note that the map is G–equivariant if we consider EK as a G–space via $EK \times G \to EK$, $(x, g) \mapsto x f(g)$.

For example, if H is a Lie subgroup of a Lie group G and $i\colon H \to G$ the inclusion, then the map Bi can be used to characterize the existence of H–reductions (cf. example 2.11).

Proposition 3.9. *Let $E \to B$ be a numerable principal G–bundle and $f\colon B \to BG$ its classifying map, that is, a morphism such that f^*EG and $E \to B$ are isomorphic. Then $E \to B$ admits an H–reduction if and only if f can be lifted, up to homotopy, to a map into BH, i. e. if and only if there exists a morphism $f_0\colon B \to BH$ such that*

$$\begin{tikzcd} & BH \arrow[d, "Bi"] \\ B \arrow[ur, "f_0"] \arrow[r, "f"'] & BG \end{tikzcd}$$

commutes up to homotopy.

Proof. Observe that we have a morphism of principal G–bundles

$$\begin{tikzcd} EH \times_H G \arrow[r, "{[x,g] \mapsto Ei(x)g}"] \arrow[d] & EG \arrow[d] \\ BH \arrow[r, "Bi"] & BG \end{tikzcd}$$

because Ei is H–equivariant. Thus, Bi is the classifying map of $EH \times_H G$ by corollary 2.9. Hence, if $f_0\colon B \to BH$ is a morphism such that f and $Bi \circ f_0$ are homotopic, then we have the following chain of isomorphisms of principal G–bundles

$$f^*EG \cong (f_0)^*(Bi^*EG) \cong (f_0)^*(EH \times_H G) \cong ((f_0)^*EH) \times_H G;$$

the first isomorphism is theorem 3.6, the second follows from corollary 2.9, and the last isomorphism is realized by the assignment $(b, [x, g]) \mapsto [(b, x), g]$ for all $b \in B$, $x \in EH$, and $g \in G$. But $(f_0)^*EH$ is a principal H–bundle and a reduction of $((f_0)^*EH) \times_H G$, so f^*EG and hence $E \to B$ admits an H–reduction.

Conversly, let $Q \to B$ be an H–reduction of $E \to B$ and $f_0\colon B \to BH$ the classifying map of $Q \to B$. The last two isomorphisms in the displayed equation above show that $Bi \circ f_0$ is the classifying map for $Q \times_H G \to B$. But $Q \times_H G \to B$ is isomorphic to $E \to B$ (cf. example 2.11), and the latter bundle also has classifying map f, so by theorem 3.6 f and $Bi \circ f_0$ must be homotopic. □

Proposition 3.10. *Let $E \xrightarrow{\pi} B$ be a numerable principal G–bundle and write $ME = *_{i\in\mathbb{N}}E$ for its infinite join. Then $ME \xrightarrow{q} MB = ME/G$ is a classifying bundle for G.*

Proof. First, assume that $E = B \times G$ is a trivial bundle. $ME \xrightarrow{q} MB$ still is locally trivial over the sets $V_j = q(U_j)$ with $U_j = (t_j)^{-1}((0,\infty))$ via the map $ME|_{V_j} \to V_j \times G$, $\sum_t i_e i \mapsto ([\sum_i t_i e_i], ej)$, so the proof of proposition 3.5 carries over verbatimly and shows that $ME \xrightarrow{q} MB$ is numerable in this case.

Now let E be completely arbitrary. By assumption, we find a cover $\{V_i | i \in \mathbb{N}\}$ of E over which E is trivial. Set $U_i := E|_{V_i}$. Then $U_i \xrightarrow{q} V_i$ is a trivial bundle and hence $M(U_i) \to M(V_i)$ is a numerable principal G–bundle. Let $\{V_{i,j} \,|\, j \in \mathbb{N}\}$ be a cover over which $M(U_i)$ is trivial and which admits a subordinate partition of unity. By definition of the topology on the join, the inclusion $M(U_i) \hookrightarrow ME$ is a homeomorphism onto its open image $\{\sum_k t_k e_k \,|\, \forall k\colon e_k \in U_i\}$, and since quotient maps onto orbit spaces are open, it follows that $M(V_i) \hookrightarrow MB$ too maps homeomorphically onto some open subset of MB. We can trivially extend a partition of unity on $M(V_i)$ which is subordinate to $V_{i,j}$ to a partition of unity on MB which still is subordinate to $V_{i,j} \subseteq MB$, and thus $ME \xrightarrow{q} MB$ is a numerable principle G–bundle.

Fix a point $p_0 \in E$ and consider the morphism

$$\alpha\colon BG \to MB, \quad \left[\sum_i t_i g_i\right] \mapsto \left[\sum_i t_i (p_0 g_i)\right].$$

According to corollary 2.9, $\alpha^* ME$ and $EG \to BG$ are isomorphic principal bundles. On the other hand, since $ME \xrightarrow{q} MB$ is numerable, it is classified by some map $\beta\colon MB \to BG$. Then $\beta\circ\alpha$ is a classifying map for $EG \to BG$, and hence must be homotopic to id_{BG} by theorem 3.6. Next, observe that the proof of proposition 3.8 still applies with ME in place of EG. Since β is covered by the canonical projection $\beta_0\colon \beta^* ME \to EG$, which is G–equivariant, and α too is covered by a G–equivariant morphism $\alpha_0\colon EG \to ME$, the morphism $\alpha_0 \circ \beta_0\colon ME \to ME$ is G–equivariantly homotopic to id_{ME}. This homotopy then descends to a homotopy between $\alpha \circ \beta$ and id_{BE}.

Thus, MB and BG are homotopy equivalent and $[B_0, BG] \to [B_0, MB]$, $[f] \mapsto [\alpha \circ f]$, is a bijection with inverse $[f] \mapsto [\beta \circ f]$. It follows that $ME \xrightarrow{q} MB$ is G–universal. □

Corollary 3.11. *If $H \subseteq G$ is a closed subgroup of a Lie group G, then $EG \to EG/H$ is a universal bundle for H.*

Proof. Just observe that $G \to G/H$ is a smooth principal bundle, cf. example 2.7. □

Example 3.12.

(i) We already know that $ES^1 = S^\infty$, see example 3.2. Since we can consider each cyclic group $\mathbb{Z}_p$ as a subgroup of S^1 it thus follows that the infinite–dimensional lens space $\ell_p = S^\infty/\mathbb{Z}_p$ is the classifying space for $\mathbb{Z}_p$ and that $S^\infty \to \ell_p$ is its universal bundle.

(ii) In the exercises we will show: if G and K are Lie groups with respective universal bundles $EG \to BG$ and $EK \to BK$, then the product of bundles $EG \times EK \to BG \times BK$ is universal for $G \times K$.

Chapter II.

Čech cohomology

1. Sheaves

Definition 1.1. A *category* $\mathcal{C}$ consists of

(i) a class of objects

(ii) a family of disjoint sets $\hom_{\mathcal{C}}(A, B)$, one for each pair A, B;

(iii) for each object A of $\mathcal{C}$ an element $1_A \in \hom_{\mathcal{C}}(A, A)$;

(iv) for each tripe of objects A, B, and C a function

$$\hom_{\mathcal{C}}(B, C) \times \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{C}}(A, C),\ (g, f) \mapsto g \circ f;$$

subject to the following conditions.

(a) for any two objects A, B of $\mathcal{C}$ and every morphism $f \in \hom_{\mathcal{C}}(A, B)$ we have $1_B \circ f = f$ and $f \circ 1_A = f$.

(b) for all objects A, B, C, and D of $\mathcal{C}$ and all morphisms $f \in \hom_{\mathcal{C}}(A, B)$, $g \in \hom_{\mathcal{C}}(B, C)$, and $h \in \hom_{\mathcal{C}}(C, D)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

We write $A \in \mathcal{C}$ to mean that A is an object of $\mathcal{C}$. A category is *small* if its underlying class is actually a set. The element 1_A is called an *identity*, and in general, an element f of $\hom_{\mathcal{C}}(A, B)$ is called a *morphism* or *arrow* and also written $f\colon A \to B$. If there exists $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$, then we say that f is an *isomorphism.*

Example 1.2.

(i) The category Sets is the category consisting of all sets and with $\hom_{\mathsf{Sets}}(X, Y)$ the set of all (set) maps $X \to Y$. The composition is the usual composition of maps and the identity element of a set X is the identity map id_X.

(ii) For a fixed commutative ring R, the category R–mod has as objects all R–modules and as morphisms the homomorphisms of R–modules. Similarly, R–alg is the category whose objects are R–algebras and with morphisms the homomorphisms of R–algebras.

(iii) The category of topological spaces has as objects topological spaces, morphisms are continuous maps.

(iv) The category of smooth manifolds has as objects smooth manifolds and as morphisms the smooth maps. Note that the notion of morphism and isomorphism we introduced earlier for topological spaces and smooth manifolds coincides with this categorical definition.

(v) The morphisms of a category need not be actual maps of sets. For example, if G is a group then we can consider G as a (small) category $\mathcal{C}$ with one object $*$ and with $\hom_{\mathcal{C}}(*,*) = G$. Composition is the group multiplication and the identity element of $\hom_{\mathcal{C}}(*,*)$ is the neutral element e of G.

(vi) If $\mathcal{C}$ is a category, then its *opposite category* $\mathcal{C}^{\mathrm{op}}$ has as objects the objects of $\mathcal{C}$. For objects A, B of $\mathcal{C}^{op}$ we let $\hom_{\mathcal{C}^{\mathrm{op}}}(A,B) = \hom_{\mathcal{C}}(B,A)$. The composition of morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathcal{C}^{op}$ is defined as $g \circ f := fg$, where on the right hand side the composition in $\mathcal{C}$ is used.

Definition 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

(i) A *(covariant) functor* $F\colon \mathcal{C} \to \mathcal{D}$ is an assignment associating to every object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$, and to every morphism $f\colon A \to B$ of $\mathcal{C}$ a morphism $F(f)\colon F(A) \to F(B)$ in such a way that

  a) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f\colon A \to B$ and $g\colon B \to C$ of $\mathcal{C}$,

  b) $F(1_A) = 1_{F(A)}$ for every object A of $\mathcal{C}$.

(ii) A *contravariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

(iii) A *natural transformation* $\eta\colon F \Rightarrow G$ between functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is an assignment associating to every object A of $\mathcal{C}$ a morphism $\eta_A\colon F(A) \to G(A)$ in such a way that

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

is commutative for all morphisms $f\colon A \to B$ of $\mathcal{C}$.

Example 1.4.

(i) If G and H are groups, each considered as a category with one element, then a functor $F\colon G \to K$ is just a homomorphism of groups.

(ii) Let $\mathcal{C}$ be the category of smooth manifolds. We can consider formation of the tangent bundle as a functor $T\colon \mathcal{C} \to \mathcal{C}$: namely, to a manifold M associate its tangent bundle TM and to a smooth map $f\colon M \to N$ the bundle map $T(f) = df$ induced by the differential of f.

(iii) Let k be a field. In this case the category k–mod is just the category k–vect of all k–vector spaces with morphisms the k–linear maps. We have a contravariant functor $^*\colon k$–$\mathsf{vect} \to k$–vect which takes a k–vector space V to its dual V^* and a k–linear map $f\colon V \to W$ to the dual map $f^*\colon W^* \to V^*$.

(iv) Let $\mathcal{C}$ be the category of Lie groups and $\mathcal{D}$ be the category of Lie algebras. We have a functor $\mathrm{Lie}\colon \mathcal{C} \to \mathcal{D}$ which takes a Lie group G to its Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ and a homomorphism of Lie groups $f\colon G \to K$ to its induced map $df\colon \mathfrak{g} \to \mathfrak{k}$. The statement that the exponential map of a Lie group is natural with respect to morphisms can be rephrased by saying that we can consider exp as a natural transformation $\mathrm{Lie} \Rightarrow \mathrm{id}_{\mathcal{C}}$: this just amounts to saying that for every morphism $f\colon G \to K$ the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\exp"] \arrow[d, "df"'] & G \arrow[d, "f"] \\
\mathfrak{k} \arrow[r, "\exp"] & K
\end{tikzcd}$$

commutes, which it does by proposition 1.9.

Any topological space X can be considered as a small category as follows. As objects we take the open subset U of X. The set of morphisms $V \to U$ consists of the inclusion map $\iota_{VU}\colon V \to U$, $x \mapsto x$, if $V \subseteq U$, and is empty otherwise,

Definition 1.5. Let X be a topological space, consideres as a category, and $\mathcal{C}$ a category.

(i) A *presheaf on X with values in $\mathcal{C}$* is a contravariant functor $F\colon X \to \mathcal{C}$. If $U \subseteq X$ and $V \subseteq X$ are open subsets with $V \subseteq U$ and $s \in F(U)$, then we write $\mathrm{res}_{VU} := F(\iota_{VU})$ and $s|_V := \mathrm{res}_{VU}(s)$. Elements of $F(U)$ are called *sections* over U.

(ii) A presheaf $\mathcal{F}\colon X \to \mathcal{C}$ is called a *sheaf*, if it satisfies the following axioms:

  (a) If $\{U_i \,|\, i \in I\}$ is a collection of open subsets of X and $s, t \in \mathcal{F}(U)$, where $U = \bigcup_{i\in I} U_i$, are two sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

  (b) Let $\{U_i \,|\, i \in I\}$ be a family of open subsets of X and $(s_i)_{i\in I}$ a family of section with $s_i \in \mathcal{F}(U_i)$ for all $i \in I$. If $(s_i)|_{U_i\cap U_j} = (s_j)|_{U_i\cap U_j}$ holds for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$, $U = \bigcup_{i\in I}$ with $s|_{U_i} = s_i$ for all $i \in I$.

Example 1.6. Let X be a topological space.

(i) Let R be a fixed ring and M an R–module. The *constant presheaf* $\underline{M}$ on X is the presheaf with

$$\underline{M}(U) = \begin{cases} M, & U \neq \emptyset, \\ \{0_R\}, & U = \emptyset \end{cases}$$

and which assigns to any inclusion $V \hookrightarrow U$ the identity map $M \to M$, unless $V = \emptyset$, in which case it assigns the unique map $M \to \{0_R\}$.

The constant presheaf $\underline{M}$ is not a sheaf: consider, for example, the case that $X = \{1, 2\}$ is a discrete space with two distinct points and let M be any R–module having at least two disctinct elements m and n. Consider m as an element in $\underline{M}(\{1\})$ and n as an element in $\underline{M}(\{2\})$. Since $\{1\} \cap \{2\} = \emptyset$, for $\underline{M}$ to be a sheaf, there must exist an element $t \in \underline{M}(X) = M$ with $t|_{\{1\}} = m$ and $t|_{\{2\}} = n$. However, $t|_{\{1\}} = t$ and $t|_{\{2\}} = t$, so this is impossible.

(ii) Let A be an arbitrary object of a category $\mathcal{C}$. The *locally constant sheaf* $\tilde{A}$ is the presheaf which assigns to every open subset U the set of all continuous maps $U \to A$, where A is viewed as a discrete space, and res_{VU} just restricts a map $U \to A$ to V. As the name suggests, $\tilde{A}$ is a sheaf: if $U = \bigcup_{i \in I} U_i$ and we are given continuous maps $s_i \colon U_i \to A$ such that s_i and s_j agree on their common domain $U_i \cap U_j$, define $s \colon U \to C$ by $s(x) = s_i(x)$ whenever $x \in U_i$. This is well–defined and continuous, since $s^{-1}(V) = \bigcup_{i \in I} (s_i)^{-1}(V)$.

(iii) More familiar examples of sheaves are the *sheaf of continuous functions with values in* Y, which assigns, for a fixed topological space Y, to an open subset $U \subseteq X$ the set of continuous maps $U \to Y$, or, if X and Y are smooth manifolds, the *sheaf of smooth maps with values in* Y, which assigns to U set of the smooth maps $U \to Y$.

(iv) If $f \colon X \to Y$ is a continuous map into a topological space Y and F is a presheaf on X, then f_*F, the *direct image* of F under f, is the presheaf on Y defined by

$$(f_*F)(U) = F(f^{-1}(U))$$

and by $(f_*F)(\iota_{VU}) = \iota_{f^{-1}(V)f^{-1}(U)}$.

Given presheaves F and F_0 on X, a *morphism of presheaves* $k \colon F \to F_0$ is a natural transformation $k \colon F \Rightarrow F_0$. If G is a presheaf on a topological space Y, then a *morphism between* (X, F) *and* (Y, G) is a pair (f, k) consisting of a continuous map $f \colon X \to Y$ and a morphism of presheaves $k \colon G \to f_*F$. Following [5], we will also call k an f–*cohomomorphism.*

Example 1.7. Let F be a presheaf on a topological space X and $U \subseteq X$ an open subset. Every open subset $V \subseteq U$ is also open in X, and thus we may define the *restriction of* F *to* U. This is the presheaf $(F|U)$ on U with $(F|U)(V) = F(V)$ and restriction maps the restriction maps of F. If $\iota \colon U \hookrightarrow X$ is the canonical inclusion, then there is a canonical ι–cohomomorphism $\iota^* \colon F \to \iota_*(F|U)$ which takes an open set W of X to the restriction map $\mathrm{res}_{U \cap W W} \colon F(W) \to F(U \cap W)$ of F.

2. Čech cohomology of covers

Definition 2.1. Let X be a topological space, R a (commutative, associative) ring and F a presheaf of R–modules such that $F(\emptyset) = \{0_R\}$. Denote by $\mathcal{P}(X)$ the power set of X.

(i) A *(Čech) cover of X* is an indexed open cover of X, i. e. map $\mathfrak{U} : I \to \mathcal{P}(X)$, $i \mapsto \mathfrak{U}_i$, such that $X = \bigcup_{i \in I} \mathfrak{U}_i$ and $I \neq \emptyset$.

(ii) A *p–simplex of $\mathfrak{U}$* is a tuple $\sigma\colon \{0, \dots, p\} \to I$, $k \mapsto \sigma_k$, i. e. an element of I^{p+1}. We also write $(\sigma_0, \dots, \sigma_p)$ in place of σ.

(iii) The *(Čech) p–cochains of a Čech cover $\mathfrak{U}$ with values in F*, $p \geq 0$, are

$$\check{C}^p(\mathfrak{U}; F) := \prod_{\sigma \in I^{p+1}} F(\mathfrak{U}_\sigma),$$

where $\mathfrak{U}_\sigma = \mathfrak{U}_{\sigma_0} \cap \dots \cap \mathfrak{U}_{\sigma_p}$.

(iv) The *p–th coboundary map* $\delta_p\colon \check{C}^p(\mathfrak{U}; F) \to \check{C}^{p+1}(\mathfrak{U}; F)$ is given by

$$(\delta_p c)(\sigma) = \sum_{j=0}^{p+1} (-1)^j \; c(\sigma|_{\{0,\dots,\widehat{j},\dots,p+1\}})\Big|_{\mathfrak{U}_\sigma}.$$

(v) The *Čech complex of $\mathfrak{U}$ with values in F* is $\check{C}^\bullet(\mathfrak{U}; F) := \bigoplus_{p \geq 0} \check{C}^p(\mathfrak{U}; F)$ together with $\delta := \bigoplus_{p \geq 0} \delta_p$. Elements in $\ker \delta$ and $\operatorname{im} \delta$ are called *cocycles* and *coboundaries*, respectively.

Remark 2.2. The p–cochains form an R–module with respect to pointwise addition and scalar multiplication.

Proposition 2.3. *The Čech complex $(\check{C}^\bullet(\mathfrak{U}; F), \delta)$ is a complex, i. e. $\delta^2 = 0$.*

Proof. Let $c \in \check{C}^p(\mathfrak{U}; F)$ be arbitrary. Then

$$\begin{aligned}
(\delta^2 c)(\sigma) &= \sum_{j=0}^{p+2} (-1)^j \; (\delta c)(\sigma|_{\{0,\dots,\widehat{j},\dots,p+2\}})\Big|_{\mathfrak{U}_\sigma} \\
&= \sum_{j=0}^{p+2} \sum_{r<j} (-1)^{r+j} \; c(\sigma|_{0,\dots,\widehat{r},\dots,\widehat{j},\dots,p+2\}})\Big|_{\mathfrak{U}_\sigma} + \\
&\quad \sum_{j=0}^{p+2} \sum_{s>j} (-1)^{s-1+j} \; c(\sigma|_{\{0,\dots,\widehat{j},\dots,\widehat{s},\dots,p+2\}})\Big|_{\mathfrak{U}_\sigma} \\
&= \sum_{j=0}^{p+2} \sum_{r<j} (-1)^{r+j} \; c(\sigma|_{\{0,\dots,\widehat{r},\dots,\widehat{j},\dots,p+2\}})\Big|_{\mathfrak{U}_\sigma} + \\
&\quad \sum_{s=0}^{p+2} \sum_{j<s} (-1)^{s-1+j} \; c(\sigma|_{\{0,\dots,\widehat{j},\dots,\widehat{s},\dots,p+2\}})\Big|_{\mathfrak{U}_\sigma} \\
&= 0.
\end{aligned}$$

□

Definition 2.4. The *p–th Čech cohomology group of* $\mathfrak{U}$ *with values in* F is

$$\check{\mathrm{H}}^p(\mathfrak{U}; F) := \frac{\ker(\delta) \cap \check{C}^p(\mathfrak{U}; F)}{\operatorname{im}(\delta) \cap \check{C}^p(\mathfrak{U}; F)}.$$

Remark 2.5. Note that, despite its name, $\check{\mathrm{H}}^p(\mathfrak{U}; F)$ actually is an R–module.

Proposition 2.6. *The canonical map* $F(X) \to \check{C}^0(\mathfrak{U}; F)$ *induced by restriction descends to a map* $\Phi\colon F(X) \to \check{\mathrm{H}}^0(\mathfrak{U}; F)$ *which is an isomorphism if* F *is a sheaf.*

Proof. Let $s \in F(X)$ and denote by $\hat{s} \in \check{C}^0(\mathfrak{U}; F)$ the cochain with $\hat{s}(i) = s|_{\mathfrak{U}_i}$. We have

$$(\delta\hat{s})(i, j) = \hat{s}(j)|_{\mathfrak{U}_{ij}} - \hat{s}(i)|_{\mathfrak{U}_{ij}} = (s|_{\mathfrak{U}_j})|_{\mathfrak{U}_{ij}} - (s|_{\mathfrak{U}_i})|_{\mathfrak{U}_{ij}} = 0,$$

so $\hat{s}$ is a cocycle and thus defines an element $\Phi(s) := [\hat{s}]$ in $\check{\mathrm{H}}^0(\mathfrak{U}, F)$.

Now assume that F is a sheaf. The map Φ is injective, for if $[\hat{s}]$ is the trivial element, then $\hat{s}$ is trivial, because there are no 0–boundaries. Thus, $\hat{s}(i) = s|_{\mathfrak{U}_i}$ vanishes for all $i \in I$, and then $s = 0$, because $\mathfrak{U}$ is a cover of X and F is a sheaf.

To show surjectivity of Φ, let $[c] \in \check{\mathrm{H}}^0(\mathfrak{U}; F)$ be arbitrary. Similarly to the computation above, the fact that c is a cocyle is equivalent to saying that $c(i)|_{\mathfrak{U}_{ij}} = c(j)|_{\mathfrak{U}_{ij}}$ for all $i, j \in I$. But then, using again that F is a sheaf and that $\mathfrak{U}$ is a cover of X, there must be a global section $s \in F(X)$ with $s|_{\mathfrak{U}_i} = c(i)$ for all $i \in I$, and so $\Phi(s) = [c]$. ☐

Example 2.7. In the exercises we will see that for a fixed base B every principal G–bundle $E \to B$ can be obtained by a gluing process. More precisely, given an (ordinary) open cover $\mathcal{U}$ of B and for ever pair $U, V \in \mathcal{U}$ with non–empty intersection a morphism $g_{VU}\colon U \cap V \to G$ suppose the following, so–called cocycle condition is satisfied: if U, V, and W are sets of $\mathcal{U}$ with $U \cap V \cap W \neq \emptyset$, then

$$g_{WU} = g_{WV} \cdot g_{VU}$$

holds on $U \cap V \cap W$. In this case one can construct a principal G–bundle $E \to B$ admitting trivializations $\psi_U\colon E|_U \to U \times G$ for each $U \in \mathcal{U}$ with transition functions

$$\begin{aligned} \psi_V \circ (\psi_U)^{-1}\colon (U \cap V) \times G &\to (U \cap V) \times G, \\ (x, g) &\mapsto (x, g_{VU}(x) \cdot g). \end{aligned}$$

If G is *Abelian*, this process is related to a particular Čech complex. Fix an open cover $\mathcal{U}$ of B and consider the Čech cover $\mathfrak{U}\colon \mathcal{U} \to \mathcal{P}(B)$, $U \mapsto U$. There is a canonical map

$$\Phi\colon \check{\mathrm{H}}^1(\mathfrak{U}; C_B(G)) \to \mathrm{Bun}_G(\mathcal{U}),$$

where $\mathrm{Bun}_G(\mathcal{U})$ is the set of isomorphism classes of (not necessarily numerable) principal G–bundles which are trivial over the sets of $\mathcal{U}$ and $C_B(G)$ is the sheaf of G–valued morphisms on B. The map Φ is given as follows. Any element of $\check{\mathrm{H}}^1(\mathfrak{U}; C_B(G))$ is represented by a cocyle $c \in \check{C}^1(\mathfrak{U}; C_B(G))$. This cocyle assigns to any pair of sets $U, V \in \mathcal{U}$ a morphism $c(U, V)\colon U \cap V \to B$, unless $U \cap V$ is empty. Define $\Phi([c])$ to be the bundle

E glued together using the transition functions $(g_{VU})_{VU}$ with $g_{VU} := c(U,V)$. In the exercises we will verify that this assignment is independent of the chosen representative c, that the cocycle condition amounts to c being a cocycle (hence the name), and that Φ actually is a bijection.

Definition 2.8. Suppose that we are given two covers $\mathfrak{U}\colon I \to \mathcal{P}(X)$ and $\mathfrak{V}\colon J \to \mathcal{P}(X)$ of X. We say that $\mathfrak{V}$ is a *refinement of* $\mathfrak{U}$, if there exists a function $\alpha\colon J \to I$, called *(refinement) projection*, such that $\mathfrak{V}_j \subseteq \mathfrak{U}_{\alpha(j)}$ for all $j \in J$.

Let $\alpha\colon J \to I$ be a refinement projection. It induces a map $\alpha_*\colon \check{C}^p(\mathfrak{U}; F) \to \check{C}^p(\mathfrak{V}; F)$, $c \mapsto \alpha_*(c)$, defined by

$$\alpha_*(c)(\sigma) = c(\alpha(\sigma_0), \dots, \alpha(\sigma_p))|_{\mathfrak{V}_\sigma}\,.$$

This is a map of cochain complexes, since

$$\begin{aligned}(\delta\alpha_*(c))(\sigma) &= \sum_{k=0}^{p+1}(-1)^k \cdot \alpha_*(c)(\sigma_0, \dots, \widehat{\sigma_k}, \dots, \sigma_{p+1})|_{\mathfrak{V}_\sigma}\\ &= \sum_{k=0}^{p+1}(-1)^k \cdot c(\alpha(\sigma_0), \dots, \widehat{\alpha(\sigma_k)}, \dots, \alpha(\sigma_{p+1}))\Big|_{\mathfrak{V}_\sigma}\\ &= (\delta c)(\alpha(\sigma_0), \dots, \alpha(\sigma_{p+1}))|_{\mathfrak{V}_\sigma}\\ &= \alpha_*(\delta c)(\sigma_0, \dots, \sigma_{p+1}),\end{aligned}$$

and so induces a map $\alpha_*\colon \check{\mathrm{H}}^\bullet(\mathfrak{U}; F) \to \check{\mathrm{H}}^\bullet(\mathfrak{V}; F)$ on the level of cohomology.

Proposition 2.9. *Let $\mathfrak{V}$ be a refinement of $\mathfrak{U}$. For any two refinement projections $\alpha, \beta\colon J \to I$ the induced maps $\check{\mathrm{H}}^\bullet(\mathfrak{U}; F) \to \check{\mathrm{H}}^\bullet(\mathfrak{V}; F)$ agree, i .e. $\alpha_* = \beta_*$.*

Proof. We define, for each $p \geq 0$ and every $0 \leq r \leq p$, maps

$$H_{p,r}\colon \check{C}^{p+1}(\mathfrak{U}; F) \to \check{C}^p(\mathfrak{V}; F) \text{ and } H_p := \sum_{j=0}^{p}(-1)^j H_{p,j},$$

where

$$H_{p,r}(c)(\sigma) = c(\alpha(\sigma_0), \dots, \alpha(\sigma_r), \beta(\sigma_r), \beta(\sigma_{r+1}), \dots, \beta(\sigma_p))|_{\mathfrak{V}_\sigma}\,.$$

These maps assemble to give a chain homotopy between α_* and β_*, i. e. we claim

$$\beta_* - \alpha_* = \delta \circ H_{p-1} + H_p \circ \delta$$

holds on $\check{C}^p(\mathfrak{V};F)$. To see this, let $c \in \check{C}^p(\mathfrak{V};F)$ be arbitrary. We compute

$$\begin{aligned}
&(\delta H_{p-1}c)(\sigma_0,\dots,\sigma_p)\\
&=\sum_{k=0}^{p}(-1)^k\ H_{p-1}(c)(\sigma_0,\dots,\widehat{\sigma_k},\dots,\sigma_p)|_{\mathfrak{V}_\sigma}\\
&=\sum_{k=0}^{p}\sum_{r<k}(-1)^{k+r}\ c(\alpha(\sigma_0),\dots,\alpha(\sigma_r),\beta(\sigma_r),\dots,\widehat{\beta(\sigma_k)},\dots,\beta(\sigma_p))\Big|_{\mathfrak{V}_\sigma}+\\
&\quad\sum_{k=0}^{p}\sum_{r>k}(-1)^{k+r-1}\ c(\alpha(\sigma_0),\dots,\widehat{\alpha(\sigma_k)},\dots,\alpha(\sigma_r),\beta(\sigma_r),\dots,\beta(\sigma_p))\Big|_{\mathfrak{V}_\sigma}
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&(H_p\delta c)(\sigma_0,\dots,\sigma_p)\\
&=\sum_{r=0}^{p}(-1)^r\ (\delta c)(\alpha(\sigma_0),\dots,\alpha(\sigma_r),\beta(\sigma_r),\dots,\beta(\sigma_p))|_{\mathfrak{V}_\sigma}\\
&=\sum_{r=0}^{p}\sum_{k\le r}(-1)^{r+k}\ c(\alpha(\sigma_0),\dots,\widehat{\alpha(\sigma_k)},\dots,\alpha(\sigma_r),\beta(\sigma_r),\dots,\beta(\sigma_p))\Big|_{\mathfrak{V}_\sigma}+\\
&\quad\sum_{r=0}^{p}\sum_{k\ge r}(-1)^{r+k+1}\ c(\alpha(\sigma_0),\dots,\alpha(\sigma_r),\beta(\sigma_r),\dots,\widehat{\beta(\sigma_k)},\dots,\beta(\sigma_p))\Big|_{\mathfrak{V}_\sigma}\\
&=\sum_{k=0}^{p}\sum_{r\ge k}(-1)^{r+k}\ c(\alpha(\sigma_0),\dots,\widehat{\alpha(\sigma_k)},\dots,\alpha(\sigma_r),\beta(\sigma_r),\dots,\beta(\sigma_p))\Big|_{\mathfrak{V}_\sigma}+\\
&\quad\sum_{k=0}^{p}\sum_{r\le k}(-1)^{r+k+1}\ c(\alpha(\sigma_0),\dots,\alpha(\sigma_r),\beta(\sigma_r),\dots,\widehat{\beta(\sigma_k)},\dots,\beta(\sigma_p))\Big|_{\mathfrak{V}_\sigma}.
\end{aligned}$$

Thus, adding up the two terms, we obtain

$$\begin{aligned}
(H_p\delta c+\delta H_{p-1}c)(\sigma_0,\dots,\sigma_p)&=\sum_{k=0}^{p}c(\alpha(\sigma_0),\dots,\alpha(\sigma_{k-1}),\beta(\sigma_k),\dots,\beta(\sigma_p))|_{\mathfrak{V}_\sigma}-\\
&\quad\sum_{k=0}^{p}c(\alpha(\sigma_0),\dots,\alpha(\sigma_k),\beta(\sigma_{k+1}),\dots,\beta(\sigma_p))|_{\mathfrak{V}_\sigma}\\
&=(\beta_*c)(\sigma_0,\dots,\sigma_p)-(\alpha_*c)(\sigma_0,\dots,\sigma_p),
\end{aligned}$$

hence the claim. □

3. Direct limits

Definition 3.1.

(i) A *directed category* is a non–empty category I with the following properties.

   a) For any two objects i, j of I there exists at most one morphism $i \to j$, indicated by writing $i \leq j$.

   b) If $i \leq j$ and $j \leq k$, then also $i \leq k$.

   c) If $i, j \in I$, then there exists an object k of I such that $i \leq k$ and $j \leq k$.

(ii) A directed category I is called a *directed set* if I is small.

(iii) A *cone* for a functor $A \colon I \to \mathcal{C}$, $i \mapsto A_i$, consists of an object C of $\mathcal{C}$ and for each object i of I a morphism $f_i \colon A_i \to C$ such that the diagram

$$\begin{tikzcd}
A_i \arrow[rr] \arrow[dr, "f_i"'] & & A_j \arrow[dl, "f_j"] \\
& C &
\end{tikzcd}$$

commutes for all objects i, j of I with $i \leq j$.

(iv) A *(direct) limit* of A is a cone $(L, (g_i)_{i\in I})$ for A with the following universal property: if $(C, (f_i)_{i\in I})$ is another cone for A, then there exists a unique morphism $\alpha \colon L \to C$ such that $f_i = \alpha \circ g_i$ for all $i \in I$. The situation is captured by requiring that the diagram

$$\begin{tikzcd}
A_i \arrow[rr] \arrow[dr, "g_i"] \arrow[ddr, bend right, "f_i"'] & & A_j \arrow[dl, "g_j"'] \arrow[ddl, bend left, "f_j"] \\
& L \arrow[d, dashed, "\exists!\alpha"] & \\
& C &
\end{tikzcd}$$

be commutative for all $i, j \in I$ with $i \leq j$.

Proposition 3.2. *Let I be a directed category and $A \colon I \to \mathcal{C}$ a functor. If $i, j \in I$ and $i \leq j$, write $A_{ji} := A(i \to j)$ for the image of the unique morphism $i \to j$ in I.*

(i) If $(L, (f_i)_{i\in i})$ and $(L', (f'_i)_{i\in I})$ are two limits of A, then there exists a unique morphism $\alpha \colon L \to L'$ such that

$$\begin{tikzcd}
A_i \arrow[d, "f_i"'] \arrow[dr, "f'_i"] & \\
L \arrow[r, "\alpha"'] & L'
\end{tikzcd}$$

commutes for all $i \in I$, and this map α is an isomorphism.

(ii) *If I is small and $\mathcal{C}$ is either the category* Sets *of sets, the category R–*mod *of R–modules, or the category R–*alg *of R–algebras, then*

$$\varinjlim_{i\in I} A_i := \left(\coprod_{i\in I} A_i\right) / \sim$$

is a limit of A, which we refer to as the *limit of A. Here, the equivalence relation $\sim$ is declared by $a_i \sim a_j$ if and only if there exists some $k \in I$ with $i \leq k$ and $j \leq k$ such that $A_{ki}(a_i) = A_{kj}(a_j)$.*

Proof.

(i) Since L' is a cone for A, the universal propert of L guarantees the existence of a unique morphism $\alpha\colon L \to L'$ such that $f_i' = \alpha \circ f_i$. We need to show that α is an isomorphism. Now, by the universal property of L' there exists a unique morphism $\beta\colon L' \to L$ with $f_i = \beta \circ f_i'$, and so, $\beta \circ \alpha$ is a morphism which satisfies

$$(\beta \circ \alpha) \circ f_i = \beta \circ f_i' = f_i$$

for all $i \in I$. But the identity morphism 1_L too satisfies $1_L \circ f_i = f_i$. Since by the universal property of L there is a unique such map, we see that $\beta \circ \alpha = 1_L$. A completely analgous argument shows that $\alpha \circ \beta = 1_{L'}$, that is, α is an isomorphism.

(ii) First, suppose A is a functor into Sets and let us prove that $\sim$ is an equivalence relation. Since A is a functor, $A_{ii} = \mathrm{id}_{A_i}$, and therefore $\sim$ is reflexive. Clearly, $\sim$ is symmetric, so suppose $a_i \sim a_j$ and $a_j \sim a_k$ for elements $i, j, k \in I$. This means that there exists an element $r \in I$ with $i \leq r$, $j \leq r$, and $A_{ri}(a_i) = A_{rj}(a_j)$ as well as an element $s \in I$ with $j \leq s$, $k \leq s$, and $A_{sj}(a_j) = A_{sk}(a_k)$. Since I is directed, there exists an element $t \in I$ such that $s \leq t$ and $r \leq t$, and because there exists at moste one morphism between any two objects of i and A is a functor, we have

$$\begin{aligned} A_{ti}(a_i) &= A_{tr}(A_{ri}(a_i)) \\ &= A_{tr}(A_{rj}(a_j)) \\ &= A_{tj}(a_j) \\ &= A_{ts}(A_{sj}(a_j)) \\ &= A_{ts}(A_{sk}(a_k) \\ &= A_{tk}(a_k). \end{aligned}$$

This is exactly the statement $a_i \sim a_k$, hence $\sim$ is transitive.

Let $\iota_j\colon A_j \hookrightarrow \coprod_{i\in I} A_i$ be the canonical inclusion map and $\pi\colon \coprod_{i\in I} A_i \to \varinjlim A_i$ the quotient map. We set $f_i := \pi \circ \iota_i$. It is an immediate consequence of the construction of $\varinjlim A_i$ that $f_j \circ A_{ji} = A_i$, so $\varinjlim A_i$ together with the maps $(f_i)_{i\in I}$ is a cone. That $\varinjlim A_i$ is universal is seen as follows. If $(C, (g_i)_{i\in I})$ is another cone, define $\overline{\alpha}\colon \coprod_{i\in I} A_i \to C$ by $\overline{\alpha}(a_i) = g_i(a_i)$ for $a_i \in A_i$. Since C is a cone, if $a_i \sim a_j$, then also $g_i(a_i) = g_j(a_j)$, and so $\overline{\alpha}(a_i) = \overline{\alpha}(a_j)$. Therefore, $\overline{\alpha}$ induces a unique map $\alpha\colon \varinjlim A_i \to C$ with $\overline{\alpha} = \alpha \circ \pi$. In particular, $\alpha \circ f_i = g_i$ for all $i \in I$, and if $\beta\colon \varinjlim A_i \to C$ is another map with $\beta \circ f_i = g_i$, then $\beta \circ \pi = \overline{\alpha}$, whence $\beta = \alpha$. Hence, $\varinjlim A_i$ is a limit of A.

Now if A is a functor into R–mod, then each A_i is an R–module and each morphism A_{ji} is R–linear. Therefore, $\varinjlim A_i$ can be endowed with a unique structure of an R–module making the maps f_i, which we constructed earlier, into morphisms of R–modules. For example, if $x, y \in \varinjlim A_i$, choose representatives $x = [a_i]$, $y = [a_j]$, and an element $k \in I$ with $i \leq k$ and $j \leq k$, which is possible since I is directed. Again it follows from the construction of $\varinjlim A_i$ that the element $[A_{ki}(a_i) + A_{kj}(a_j)]$ does not depend on the specific choice of k, and similarly we see that $x + y := [A_{ki}(a_i) + A_{kj}(a_j)]$ is independent of the choice of representatives for x or y. That $\varinjlim A_i$ is a limit in case A maps into R–alg is proved analogously.

□

Remark 3.3. Let I be a directed set and $A\colon I \to \mathcal{C}$ a functor into one of the categories Sets, R–mod, or R–alg. Let further $(L, (g_i)_{i\in I})$ be a limit of A.

(i) If A maps into R–mod or R–alg and if $x_i \in A_i$ is such that $g_i(x_i) = 0$, then it follows from the uniqueness statement in proposition 3.2 and our explicit construction of $\varinjlim A_i$ that there must be some $j \in I$ with $i \leq j$ and $A_{ji}(x_i) = 0$.

(ii) Every element $y \in L$ is contained in the image of some g_i. Indeed, we can once more appeal to the construction of the direct limit of A to conclude this; or we can note that the subset $L_0 := \bigcup_{i\in I} \operatorname{im}(g_i)$ of L together with the family $(\tilde{g}_i)_{i\in I}$ consisting of the morphisms $\tilde{g}_i := g_i\colon A_i \to L_0$ is a cone for A. Moreover, L_0 satisfies the universal property of a limit, because L does and we can restrict any morphism starting on L to L_0. Thus, there is a unique morphism $\alpha\colon L_0 \to L$ so that $\alpha \circ \tilde{g}_i = g_i$ holds for all $i \in I$, and this morphism is an isomorphism. But the inclusion $\iota\colon L_0 \hookrightarrow L$ satisfies $\iota \circ \tilde{g}_i = g_i$ as well, whence $\iota = \alpha$ must be this isomorphism and $L_0 = L$.

Example 3.4.

(i) Let F be a presheaf on a topological space X. The *stalk* of F at a point $x \in X$ is

$$F_x := \varinjlim_{U \ni x} F(U),$$

where the limit ranges over all open neighborhoods U of x.

(ii) If $(X_i)_{i\in I}$ is a family of topological spaces, index over a directed set I, then the set $X := \varinjlim X_i$ also is a limit in the category of topological spaces: just endow X with the final topology with respect to the canonical maps $f_i\colon X_i \to X$; this amounts to saying that a set $U \subseteq X$ is open if and only if $(f_i)^{-1}(U)$ is open for all $i \in I$. Then by construction, if Y is a cone for $(X_i)_{i\in I}$, the set map $\alpha\colon X \to Y$ whose existence is guaranteed by the universal property of the limit in Sets is continuous, and this proves that X indeed is a limit.

(iii) We already encountered limits of topological spaces in the exercises: the infinite–dimensional Stiefel manifolds $V_k(\mathbb{K}^\infty)$ and the infinite–dimensional Grassmanians $\mathrm{Gr}_k(\mathbb{K}^\infty)$ are categorical limits of their finite–dimensional counterparts. More generally, if G is a Lie group, put $E_nG := *_{k=1}^{n} G$, $n \geq 1$, and observe that that the canonical inclusion $E_nG \hookrightarrow EG$ is a homeomorphism onto its image, by the universal property of the topology of the join. Since $E_1G \subseteq E_2G \subseteq \ldots$ is an increasing sequence and $EG = \bigcup_{n\geq 1} E_nG$, it follows that EG carries the final topology with respect to the inclusions $E_nG \hookrightarrow EG$. In fact, if $U \subseteq EG$ is a set such that $U \cap E_nG$ is open for all n, then by definition of the subspace topology we can choose $U_n \subseteq EG$ open with $U_n \cap E_nG = U \cap EG$. There is no loss of generality in assuming that $U_n \subseteq U_{n+1}$, because

$$\begin{aligned}(U_n \cup U_{n+1}) \cap E_nG &= (U_n \cap E_nG) \cup (U_{n+1} \cap E_nG) \\ &= (U \cap E_nG) \cup ((U_{n+1} \cap E_{n+1}G) \cap E_nG) \\ &= U \cap E_nG,\end{aligned}$$

and then we see that $U \cap E_kG = \bigcup_{n\geq 1}(U_n \cap E_kG)$ for all k, whence $U = \bigcup_{n\geq 1} U_n$ is open in EG. This observation is particularly useful if G is compact, since $E_nG \subseteq \prod_{i=1}^{n} C(G)$ carries the subspace topology of the n–fold product of the cone $C(G)$ in this case (denote by $\tilde{E}_n$ the set E_nG endowed with the subspace topology; being a closed subspace of a compact space, $\tilde{E}_n$ is compact, and since E_nG is Hausdorff, the continuous map $\mathrm{id}\colon \tilde{E}_n \to E_nG$ must be a homeomorphism).

Definition 3.5. A functor $T\colon J \to I$ between directed categories is called *cofinal*, if for every $i \in I$ there exists some $j \in J$ with $i \leq T(j)$. A subcategory K of I is called cofinal, if so is the inclusion functor $K \to I$ (a category $\mathcal{D}$ is a *subcategory* of a category $\mathcal{C}$ if every object of $\mathcal{D}$ is also an object of $\mathcal{C}$, if for all objects A, B of $\mathcal{D}$ we have $\hom_{\mathcal{D}}(A,B) \subseteq \hom_{\mathcal{C}}(A,B)$, and if the composition law in $\mathcal{D}$ is the restriction of the composition law in $\mathcal{C}$).

Proposition 3.6. *Let I be a directed category, $T\colon J \to I$ a cofinal functor, and $A\colon I \to \mathcal{C}$ a functor. Put $A_{ji} := A(i \to j)$ and suppose that $(C, (f_j)_{j\in J})$ is a cone for $A \circ T$.*

(i) For each $i \in I$ there exists a unique map $g_i\colon A_i \to C$ such that

$$\begin{tikzcd} A_i \arrow[r, "A_{T(j)i}"] \arrow[dr, "g_i"'] & A_{T(j)} \arrow[d, "f_j"] \\ & C \end{tikzcd}$$

commutes whenever $j \in J$ is such that $i \leq j$.

(ii) If $(C, (f_j)_{j\in J})$ is a limit for $A \circ T$, then $(C, (g_i)_{i\in I})$ is a limit for A.

Proof.

(i) Given $i \in I$ we can choose, by definition of a cofinal functor, an element $j \in J$ with $i \leq T(j)$. Put $g_i := f_j \circ A_{T(j)i}$. To show that g_i has the desired property, choose another element $k \in J$ with $i \leq T(k)$. Since J is directed, we find $\ell \in J$ such that $j \leq \ell$ and $k \leq \ell$. Then consider the diagram

$$\begin{tikzcd}
 & A_{T(j)} \arrow[dr, "A_{T(\ell)T(j)}"'] \arrow[drr, bend left, "f_j"] & & \\
A_i \arrow[ur, "A_{T(j)i}"] \arrow[rr, "A_{T(\ell)i}"] \arrow[dr, "A_{T(k)i}"'] & & A_{T(\ell)} \arrow[r, "f_\ell"] & C \\
 & A_{T(k)} \arrow[ur, "A_{T(\ell)T(k)}"] \arrow[urr, bend right, "f_k"'] & &
\end{tikzcd}$$

It commutes, because there is at most one morphism between any two objects of I and J, so that $A_{T(\ell)i} = A_{T(\ell)T(k)} \circ A_{T(k)i}$ as well as $A_{T(\ell)i} = A_{T(\ell)T(j)} \circ A_{T(j)i}$, and because C is a cone. Since the composition of the upper two vertical maps is precisely g_i, it thus follows that

$$\begin{tikzcd}
A_i \arrow[r, "A_{T(k)i}"] \arrow[dr, "g_i"'] & A_{T(k)} \arrow[d, "f_k"] \\
 & C
\end{tikzcd}$$

commutes as well. That g_i is uniquely determined follows from the commutativity of the above diagram for $k = j$.

(ii) By construction $(C, (g_i)_{i \in I})$ is a cone for A. Let $(D, (h_i)_{i \in I})$ be another cone for A and put $k_j := h_{T(j)}$ for all $j \in J$. Then $(D, (k_j)_{j \in J})$ is a cone for $A \circ T$, whence by the universal property of the limit C for $A \circ T$ there exists a unique morphism $\alpha \colon C \to D$ such that $k_j = \alpha \circ f_j$ for all $j \in J$. Now given $i \in I$, pick $j \in J$ with $i \leq T(j)$. Then the diagram

$$\begin{tikzcd}
A_i \arrow[r, "A_{T(j)i}"] \arrow[rr, bend left, "g_i"] \arrow[dr, "h_i"'] & A_{T(j)} \arrow[r, "f_j"] \arrow[d, "k_j"'] & C \arrow[dl, "\alpha"] \\
 & D &
\end{tikzcd}$$

commutes, and this shows that $(C, (g_i)_{i \in I})$ satisfies the universal propery of a limit, because any other morphism $\beta \colon C \to D$ satisfying $h_i = \beta \circ g_i$ for all $i \in I$ must be equal to α by the universal property of C for $A \circ T$.

□

Remark 3.7. Note that the converse statement is also true: if $(L, (f_i)_{i\in I})$ is a limit for A, then $(L, (k_j)_{j\in J})$ with $k_j := f_{T(j)}$ also is a limit for $A \circ T$. In fact, if $(C, (h_j)_{j\in J})$ is a cone for $A \circ T$, then according to proposition 3.6 we may extend C to a cone $(C, (g_i)_{i\in I})$ for A, that is, $g_{T(j)} = h_j$ for all $j \in J$, because $g_{T(j)} = h_j \circ A_{T(j)T(j)}$ and $A_{T(j)T(j)} = \mathrm{id}$. By the universal property of L there is a unique map $\alpha\colon L \to C$ with $\alpha \circ f_i = g_i$ for all $i \in I$. In particular, $\alpha \circ k_j = h_j$ for all $j \in J$. It remains to show that α is unique. So let $\beta\colon L \to C$ be another morphism with $\beta \circ k_j = h_j$ for all $j \in J$ and pick $i \in I$. By cofinality of T, there exists $T(j) \geq i$, $j \in J$, and then

$$\beta \circ f_i = \beta \circ f_{T(j)} \circ A_{T(j)i} = \beta \circ k_j \circ A_{T(j)i} = h_j \circ A_{T(j)i} = g_i.$$

The universal property of L for A now implies $\alpha = \beta$, so L is also a limit for $A \circ T$.

Proposition 3.8. *Let I be a directed set and $A, B, C\colon I \to R$–mod be functors having limits $(L_A, (g_{A,i})_{i\in I})$, $(L_B, (g_{B,i})_{i\in I})$, and $(L_C, (g_{C,i})_{i\in I})$, respectively. Suppose further that for each $i, j \in I$ with $i \leq j$ we are given a commutative diagram*

$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[d, "A_{ji}"] & B_i \arrow[r, "\beta_i"] \arrow[d, "B_{ji}"] & C_i \arrow[d, "C_{ji}"] \\ A_j \arrow[r, "\alpha_j"] & B_j \arrow[r, "\beta_j"] & C_j \end{tikzcd}$$

whose rows are exact. Then so is the sequence

$$L_A \xrightarrow{\alpha} L_B \xrightarrow{\beta} L_C.$$

with α, β determined by $\alpha \circ g_{A,i} = g_{B,i} \circ \alpha_i$ and $\beta \circ g_{B,i} = g_{C,i} \circ \beta_i$ for all $i \in I$.

Proof. If we let $\gamma = 0$ or $\gamma = \beta \circ \alpha$, then the diagram

$$\begin{tikzcd} A_i \arrow[d, "g_{A,i}"'] \arrow[dr, "g_{C,i}\circ\beta_i\circ\alpha_i"] & \\ L_A \arrow[r, "\gamma"'] & L_C \end{tikzcd}$$

commutes for all $i \in I$. However, by the universal property of L_A there can be only one such map γ, whence $\beta \circ \alpha = 0$. This shows that $\operatorname{im} \alpha$ is contained in $\ker \beta$.

To prove the converse inclusion, it suffices to show that every element $y \in \ker \beta$ can be written as $g_{B,j}(x)$ for some $x \in \ker \beta_j$. Now we already know from remark 3.3 that there exists some $i \in I$ and some $x_i \in B_i$ with $y = g_{B,i}(x_i)$. Thus,

$$0 = \beta(y) = \beta(g_{B,i}(x_i)) = g_{C,i}(\beta_i(x_i)).$$

But by the same remark we also know that this is possible only if there exists some $j \geq i$ such that $C_{ji}(\beta_i(x_i)) = 0$. Since $C_{ji}(\beta_i(x_i)) = \beta_j(B_{ji}(x_i))$ and $g_{B,j}(B_{ji}(x_i)) = g_{B,i}(x_i)$, it follows that $x = B_{ji}(x_i)$ is contained in the kernel of β_j and maps onto y under $g_{B,j}$. □

Remark 3.9. The previous proposition can be slightly generalized as follows. Let I be a directed category such that

$$\begin{tikzcd}
A_i \arrow[r, "\alpha_i"] \arrow[d, "A_{ji}"] & B_i \arrow[r, "\beta_i"] \arrow[d, "B_{ji}"] & C_i \arrow[d, "C_{ji}"] \\
A_j \arrow[r, "\alpha_j"] & B_j \arrow[r, "\beta_j"] & C_j
\end{tikzcd}$$

commutes for all $i, j \in I$. Suppose that $T\colon J \to I$ is a cofinal functor from a directed *set* J and further suppose that the rows of the diagram above are exact whenever $i, j \in \operatorname{im} T$. Then the sequence $L_A \to L_B \to L_C$ is exact as well, where L_A, L_B, and L_C are arbitrary limits (which exist by proposition 3.2 and proposition 3.6). Indeed, by remark 3.7 L_A, L_B, and L_C are also limits for $A \circ T$, $B \circ T$, and $C \circ T$, respectively, thus, as J is a set,

$$L_A \xrightarrow{\alpha'} L_B \xrightarrow{\beta'} L_C$$

is exact by proposition 3.8, where α' and β' are the maps determined by $\alpha' \circ g_{A,T(j)} = g_{B,T(j)} \circ \alpha_{T(j)}$ and $\beta' \circ g_{B,T(j)} = g_{C,T(j)} \circ \beta_{T(j)}$ for all $j \in J$. But since L_A, L_B, and L_C are cones and T is cofinal, we must have $\alpha' \circ g_{A,i} = g_{B,i} \circ \alpha_i$ and $\beta' \circ g_{B,i)} = g_{C,i} \circ \beta_i$ for all $i \in I$, and so $\alpha' = \alpha$ and $\beta' = \beta$.

4. Čech cohomology

Note that if $\mathfrak{U}\colon I \to \mathcal{P}(X)$ and $\mathfrak{V}\colon J \to \mathcal{P}(X)$ are two Čech covers on X, then there is a common refinement $\mathfrak{W}\colon I \times J \to \mathcal{P}(X)$ declared by $\mathfrak{W}_{i,j} := \mathfrak{U}_i \cap \mathfrak{V}_j$. Thus, the category $\mathcal{C}_X$ whose objects are the Čech covers of X and whose morphism set $\hom_{\mathcal{C}_X}(\mathfrak{U}, \mathfrak{V})$ consists of exactly one element if and only if $\mathfrak{V}$ refines $\mathfrak{U}$, is a directed category. Moreover, for F and p fixed, we have a functor $\check{\mathrm{H}}^p(-; F)\colon \mathcal{C}_X \to R\text{–}\mathsf{mod}$, sending a Čech cover $\mathfrak{U}$ to $\check{\mathrm{H}}^p(\mathfrak{U}; F)$ and sending the unique morphism $\mathfrak{U} \to \mathfrak{V}$ to the unique morphism $\check{\mathrm{H}}^p(\mathfrak{U}; F) \to \check{\mathrm{H}}(\mathfrak{V}; F)$ induced by any refinement projection.

Proposition 4.1. *The set of all Čech covers* $\mathfrak{U}\colon X^k \to \mathcal{P}(X)$*, with the convention* $X^0 := \{*\}$*, is a cofinal subset in* $\mathcal{C}_X$*.*

Proof. Let C be the set in question. As already noted, if $\mathfrak{U}\colon X^k \to \mathcal{P}(X)$ and $\mathfrak{V}\colon X^\ell \to \mathcal{P}(X)$ are two Čech covers, then $\mathfrak{W}\colon X^{k+\ell} = X^k \times X^\ell \to \mathcal{P}(X)$, $(x, y) \mapsto \mathfrak{U}_x \cap \mathfrak{V}_y$, is a Čech cover refining $\mathfrak{U}$ and $\mathfrak{V}$. Therefore, C is directed. To prove that it is cofinal, let $\mathfrak{V}\colon J \to \mathcal{P}(X)$ be a Čech cover. Since $\mathfrak{V}$ is a cover of X, we find a function $\alpha\colon X \to J$ such that $x \in \mathfrak{V}_{\alpha(x)}$ holds for all $x \in X$. Then $\mathfrak{U} := \mathfrak{V} \circ \alpha$ lies in C and refines $\mathfrak{V}$. □

Definition 4.2. The *p–th Čech cohomology group with values in F* is

$$\check{\mathrm{H}}^p(X; F) := \varinjlim_{\mathfrak{U}} \check{\mathrm{H}}^p(\mathfrak{U}; F),$$

where $\mathfrak{U}$ ranges over the directed set of all Čech covers $\mathfrak{V}\colon X^k \to \mathcal{P}(X)$, $k \geq 0$.

Proposition 4.3. *$\check{\mathrm{H}}^p(X;F)$ is a limit for $\check{\mathrm{H}}^p(-;F)$.*

Proof. By proposition 4.1 the set C of all Čech covers $\mathfrak{U}\colon X^k \to \mathcal{P}(X)$ is cofinal in $\mathcal{C}_X$, so the claim follows from proposition 3.6, because $\check{\mathrm{H}}^p(X;F)$ is a limit of $\check{\mathrm{H}}^p(-;F)|_C$. □

Example 4.4 (Dimension axiom)**.** Suppose that $X = \{x\}$ consists of one point only. In this case, $F(\{x\}) = F_x$ and we claim that

$$\check{\mathrm{H}}^p(X;F) = \begin{cases} F_x, & p = 0, \\ 0, & p > 0. \end{cases}$$

To see this, first note that every Čech cover of X is refined by the cover $\mathfrak{U}\colon X \to \mathcal{P}(X)$ with $\mathfrak{U}_x = \{x\}$, whence $\check{\mathrm{H}}^p(\mathfrak{U};F) \to \check{\mathrm{H}}^p(X;F)$ is an isomorphism. Next, observe that there is exactly one p–simplex σ of $\mathfrak{U}$, namely, the constant simplex $\sigma_{x,p} \equiv x$. Since by assumption $F(\emptyset) = 0$, any p–cochain c thus is determined by the value it takes on the simplex $\sigma_{x,p}$. Now, if p is odd, then

$$(\delta c)(\sigma_{x,p+1}) = \sum_{j=0}^{p+1} (-1)^j c(\sigma_{x,p}) = c(\sigma_{x,p}),$$

so in particular $\check{\mathrm{H}}^p(\mathfrak{U};F) = 0$ for odd p, because there are no non–trivial p–cocycles. Similarly, we see that if p is even or 0, then $(\delta c)(\sigma_{x,p+1}) = 0$, whence $\delta = 0$ and every p–cochain is a cocyle. However, if p is even and $p > 0$, then according to the previous computation $\delta\colon \check{C}^{p-1}(\mathfrak{U};F) \to \check{C}^p(\mathfrak{U};F)$ is surjective, and so $\check{\mathrm{H}}^p(\mathfrak{U};F) = 0$ for such p, and $\check{\mathrm{H}}^0(\mathfrak{U};F) = F_x$.

Example 4.5. If F is a sheaf on X, then by proposition 2.6 the canonical map $F(X) \to \check{\mathrm{H}}^0(\mathfrak{U};F)$ is an isomorphism for every Čech cover $\mathfrak{U}$ of X and hence remains an isomorphism upon passing to the limit by proposition 3.8: $F(X) \cong \check{\mathrm{H}}^0(X;F)$. The same is true if $F = \underline{M}$ is a constant presheaf and X is connected. Indeed, if $\mathfrak{U}$ is a cover of X defined on the index set I and $c \in \check{C}^0(\mathfrak{U};\underline{M})$ is a cocycle, let $I_m \subseteq I$ be the set of indices i such that $c(i) = m$, where $m \in M$. Then $X = \bigcup_{m\in M}(\bigcup_{i\in I_m} \mathfrak{U}_i)$ is a disjoint union, for if $U_i \cap U_j$ is non–empty and if $i \in I_m$, $j \in I_n$, then, since c is closed,

$$m = c(i)|_{\mathfrak{U}_i\cap\mathfrak{U}_j} = c(j)|_{\mathfrak{U}_i\cap\mathfrak{U}_j} = n.$$

Since X is connected, we thus have $I = I_m$ for some $m \in M$. Hence $[c]$ is contained in the image of the canonical morphism $F(X) \to \check{\mathrm{H}}^0(\mathfrak{U};\underline{M})$. Since the latter is injective anyway, it follows that it induces an isomorphism $F(X) \to \check{\mathrm{H}}^0(X;\underline{M})$.

5. Induced maps

Let X, Y be topological spaces and F, G presheaves on X, Y, respectively. Further suppose that we are given a morphism $(f,k)\colon (X,F) \to (Y,G)$. Every Čech cover $\mathfrak{V}\colon J \to \mathcal{P}(Y)$ of Y gives rise to Čech cover

$$f^{-1}\mathfrak{V}\colon J \to \mathcal{P}(X),\ j \mapsto f^{-1}(\mathfrak{V}_j),$$

of X and, since a p–simplex σ of $f^{-1}\mathfrak{V}$ also is a p–simplex of $\mathfrak{V}$, thus also to a map

$$\begin{aligned}(f,k)^*\colon \check{C}^p(\mathfrak{V};G) &\to \check{C}^p(f^{-1}\mathfrak{V};F),\\ c &\mapsto \big(\sigma \mapsto k(c(\sigma))\big).\end{aligned}$$

This map is a chain map, because for any simplex σ of $f^{-1}\mathfrak{V}$

$$\begin{aligned}(\delta(f,k)^*c)(\sigma) &= \sum_{j=0}^{p+1}(-1)^j\ ((f,k)^*c)(\sigma|_{\{0,\dots,\widehat{j},\dots,p+1\}})\Big|_{(f^{-1}\mathfrak{V})_\sigma}\\ &= \sum_{j=0}^{p+1}(-1)^j\ k\big(c(\sigma|_{\{0,\dots,\widehat{j},\dots,p+1\}})\big)\Big|_{(f^{-1}\mathfrak{V})_\sigma}\\ &= \sum_{j=0}^{p+1}(-1)^j k\left(\ c(\sigma|_{\{0,\dots,\widehat{j},\dots,p+1\}})\Big|_{\mathfrak{V}_\sigma}\right)\\ &= (f,k)^*(\delta c)(\sigma).\end{aligned}$$

Thus, $(f,k)^*$ induces a map $\check{\mathrm{H}}^p(\mathfrak{V};G) \to \check{\mathrm{H}}^p(f^{-1}\mathfrak{V};F)$ which composed with the map into the limit gives a map $\check{\mathrm{H}}^p(\mathfrak{V};G) \to \check{\mathrm{H}}^p(X;F)$. Note that induced maps are natural with respect to refinement projections; that is to say, if $\mathfrak{W}$ refines $\mathfrak{V}$ with refinement projection α, then the diagram

$$\begin{tikzcd}
\check{C}^p(\mathfrak{V};G) \arrow[r, "\alpha_*"] \arrow[d, "(f,k)^*"'] & \check{C}^p(\mathfrak{W};G) \arrow[d, "(f,k)^*"] \\
\check{C}^p(f^{-1}\mathfrak{V};F) \arrow[r, "\alpha_*"] & \check{C}^p(f^{-1}\mathfrak{W};F)
\end{tikzcd}$$

is commutative, hence remains commutative when passing to cohomology. Thus, we can consider $\check{\mathrm{H}}^p(X;F)$ as a cone for the functor $\check{\mathrm{H}}^p(-;G)$ defined on the category of all Čech covers of Y, and taking limits over Čech covers of Y we obtain a map, again denoted $(f,k)^*$ or just f^*, so that the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{V};G) \arrow[r, "(f,k)^*"] \arrow[d] & \check{\mathrm{H}}^p(f^{-1}\mathfrak{V};F) \arrow[d] \\
\check{\mathrm{H}}^p(Y;G) \arrow[r, "f^*"] & \check{\mathrm{H}}^p(X;F)
\end{tikzcd}$$

commutes for all Čech covers $\mathfrak{V}$ of Y.

Example 5.1 (Additivity)**.** If $X = \coprod_{a\in A} X_a$ is a topological sum, i. e. a disjoint union of open subspaces, and $i_a\colon X_a \hookrightarrow X$ denotes the inclusion, then

$$\prod_{a\in A}(i_a)^*\colon \check{\mathrm{H}}^p(X;F) \to \prod_{a\in A}\check{\mathrm{H}}^p(X_a;F|_{X_a})$$

is an isomorphism for all p. This can be seen as follows. Denote by $\mathcal{C}$ the subcategory of $\mathcal{C}_X$ consisting all Čech covers $\mathfrak{U}\colon I \to \mathcal{P}(X)$ with the property that for every index $i \in I$ there exists some $a \in A$ with $\mathfrak{U}_i \subseteq X_a$. Such covers can be assembled from Čech covers $\mathfrak{U}_a\colon I_a \to \mathcal{P}(X_a)$, $i \mapsto \mathfrak{U}_{a,i}$, by declaring

$$\mathfrak{U}\colon \coprod_{a\in A} I_a \times \{a\} \to \mathcal{P}(X),\ (i,a) \mapsto \mathfrak{U}_{a,i},$$

and then $i_a^{-1}\mathfrak{U}$ simultaneously refines $\mathfrak{U}_a$ for all $a \in A$, since a refinement projection $\coprod_{b\in A} I_b \times \{b\} \to I_a$ is given by sending (i,a) to i and (j,b) to any element of I_a if $a \neq b$, as $(i_a)^{-1}\mathfrak{U}_{(j,b)} = X_a \cap \mathfrak{U}_{j,b}$ is empty in this case. Thus, since maps into limits are eventually surjective (cf. remark 3.3), we can find for a given element y in $\prod_a \check{\mathrm{H}}^p(X_a; F|_{X_a})$ a Čech cover $\mathfrak{U}$ in $\mathcal{C}$ such that y is contained in the image of the map

$$\prod_{a\in A} \check{\mathrm{H}}^p(i_a^{-1}\mathfrak{U}; F|_{X_a}) \to \prod_{a\in A} \check{\mathrm{H}}^p(X_a; F|_{x_a}),$$

where each component is given by the map into its respective limit.

Likewise, if $x \in \check{\mathrm{H}}^p(X;F)$ maps to 0 under $\prod_{a\in A}(i_a)^*$, then there exists a Čech cover $\mathfrak{U}$ of X so that x lies in the image of $\check{\mathrm{H}}^p(\mathfrak{U};F) \to \check{\mathrm{H}}^p(X;F)$, because each X_a is open in X and $\mathcal{C}$ thus is cofinal. Using that an element maps to zero under the limit map $\check{\mathrm{H}}^p(i_a^{-1}\mathfrak{U}; F|_{X_a}) \to \check{\mathrm{H}}^p(X_a; F|_{X_a})$ if and only if it is eventually zero, we see that it will suffice to show that

$$\prod_{a\in A}(i_a)^*\colon \check{\mathrm{H}}^p(\mathfrak{U};F) \to \prod_{a\in A} \check{\mathrm{H}}^p(i_a^{-1}\mathfrak{U}; F|_{X_a})$$

is an isomorphism for all Čech covers $\mathfrak{U}\colon I \to \mathcal{P}(X)$ in $\mathcal{C}$.

To this end, decompose I as $I = \coprod_{a\in A} I_a$ in such a way that $\mathfrak{U}_i \subseteq X_a$ for all $i \in I_a$ and all $a \in A$. To show injectivity, let $c \in \check{C}^p(\mathfrak{U};F)$ be a cocycle such that $i_a^*([c]) = 0$ for all $a \in A$. By definition we can find, for each a, an element $d_a \in \check{C}^{p-1}(i_a^{-1}\mathfrak{U};F)$ with $\delta(d_a) = i_a^*(c)$, where $d_a = 0$ and $\delta = 0$ if $p = 0$. Also note that if σ is a p–simplex of $\mathfrak{U}$, then either $\sigma(0),\dots,\sigma(p) \in I_a$ for some $a \in A$ or $\mathfrak{U}_\sigma = \emptyset$, as the X_b are mutually disjoint. Hence, if $\mathfrak{U}_\sigma \neq \emptyset$ and $\sigma(0) \in I_a$, then $F(\mathfrak{U}_\sigma) = (F|_{X_a})((i_a^{-1}\mathfrak{U})_\sigma)$, so we may define $d \in \check{C}^{p-1}(\mathfrak{U};F)$ by $d(\sigma) = d_a(\sigma)$, whenever $\sigma(0) \in I_a$. It follows that for every p–simplex σ with $\mathfrak{U}_\sigma \neq \emptyset$ there is some a such that

$$\begin{aligned}
(\delta d)(\sigma) &= \sum_{j=0}^{p}(-1)^j\ d(\sigma|_{\{0,\dots,\widehat{j},\dots,p\}})\Big|_{\mathfrak{U}_\sigma}\\
&= \sum_{j=0}^{p}(-1)^j\ d_a(\sigma|_{\{0,\dots,\widehat{j},\dots,p\}})\Big|_{(i_a^{-1}\mathfrak{U})_\sigma}\\
&= (\delta d_a)(\sigma)\\
&= (i_a)^*(c)(\sigma)\\
&= c(\sigma)|_{X_a\cap\mathfrak{U}_\sigma}\\
&= c(\sigma),
\end{aligned}$$

so $c = \delta d$ is exact and $[c] = 0$. Thus, the map in question is injective.

Its surjectivity is verified similarly. Given p–cocycles c_a of $i_a^{-1}\mathfrak{U}$ for every a, define $c \in \check{C}^p(\mathfrak{U}; F)$ by $c(\sigma) = c_a(\sigma)$ if σ is a p–simplex with $\sigma(0) \in I_a$. Then $(i_a)^*(c) = c_a$ by construction, and a similar computation as above shows that for every $p+1$–simplex σ of $\mathfrak{U}$ with $\sigma(0) \in I_a$ we have $(\delta c)(\sigma) = (\delta c_a)(\sigma) = 0$. Therefore, c is a cocycle and surjectivity follows.

Remark 5.2. If in the previous example A was a finite set, then the conclusion would follow at once from remark 3.9, since in this case $\prod_{a\in A} \check{\mathrm{H}}^p(X_a; F|_{X_a})$ is a limit for the functor sending a Čech cover $\mathfrak{U}$ to $\prod_{a\in A} \check{\mathrm{H}}^p(i_a^{-1}\mathfrak{U}; F|_{X_a})$, so that $\prod_a (i_a)^*$ is an isomorphism if so is the map $\prod_a (i_a)^* \colon \check{\mathrm{H}}^p(\mathfrak{U}; F) \to \check{\mathrm{H}}^p(i_a^{-1}\mathfrak{U}; F|_{X_a})$ for all Čech covers $\mathfrak{U}$ of $\mathcal{C}$. For infinite A, however, this is no longer true, because limits do not commute with arbitrary products in general.

6. Relative cohomology

Let $f \colon X \to Y$ be a morphism of topological spaces and G a presheaf on Y. The *pullback* or *inverse image* of G is a presheaf on X, defined by

$$(f^*G)(U) = \varinjlim_{W \supseteq f(U)} G(W),$$

where the limit is taken over all open subsets $W \subseteq Y$ that contain $f(U)$. If $A \subseteq X$ is a subspace, F is a presheaf on X, and $\iota \colon A \hookrightarrow X$ is the inclusion map, we also write $F|_A$ instead of $\iota^* F$. Note that if A is open this definition agrees, up to canonical isomorphism, with the restriction of the presheaf F defined earlier, cf. example 1.7.

By a *Čech cover of the pair* (X, A) we shall mean a tuple $(\mathfrak{U}, \mathfrak{U}_A)$ consisting of a Čech cover $\mathfrak{U} \colon I \to \mathcal{P}(X)$ and a Čech cover $\mathfrak{U}_A \colon I_A \to \mathcal{P}(X)$ such that $I_A \subseteq I$ and $(\iota^{-1}\mathfrak{U})|_{I_A} = \mathfrak{U}_A$; that is, for every element $i \in I_A$ we have $\mathfrak{U}_i \cap A = (\mathfrak{U}_A)_i$. Note that $\mathfrak{U}_A$ canonically refines $\iota^{-1}\mathfrak{U}$ via the refinement projection $I_A \to I$, $i \mapsto i$.

Definition 6.1. The *p–th relative Čech cochain group of* (X, A) with respect to a Čech cover $(\mathfrak{U}, \mathfrak{U}_A)$ and with coefficients in F is

$$\check{C}^p(\mathfrak{U}, \mathfrak{U}_A; F) := \ker\left(h_{\mathfrak{U},\mathfrak{U}_A} \colon \check{C}^p(\mathfrak{U}; F) \xrightarrow{\iota^*} \check{C}^p(\iota^{-1}\mathfrak{U}; F|_A) \to \check{C}^p(\mathfrak{U}_A; F|_A)\right).$$

It is useful to spell out what it means for a cochain $c \in \check{C}^p(\mathfrak{U}; F)$ to be in the kernel of $h_{\mathfrak{U},\mathfrak{U}_A}$: this is the case if and only if $(\iota^*)_{\mathfrak{U}_\sigma}(c(\sigma)) = 0$ for all p–simplices σ of $\mathfrak{U}$ with $\sigma_0, \ldots, \sigma_p \in I_A$, where $\iota^* \colon F \to \iota_*(F|_A)$ is the canonical ι–cohomomorphism.

Since induced maps and refinement projections are chain maps, it follows at once that $\check{C}^\bullet(\mathfrak{U}, \mathfrak{U}_A; F)$ is a *subcomplex* of $\check{C}^\bullet(\mathfrak{U}; F)$, i. e. the coboundary δ restricts to an endomorphism on $\check{C}^\bullet(\mathfrak{U}, \mathfrak{U}_A; F)$. Thus, it makes sense to consider the cohomology of this subcomplex, whose p–th graded component

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathfrak{U}_A; F) := \frac{\check{C}^p(\mathfrak{U}, \mathfrak{U}_A; F) \cap \ker\delta}{\check{C}^p(\mathfrak{U}, \mathfrak{U}_A; F) \cap \operatorname{im}\delta}$$

is the *p–th relative cohomology group of* (X,A) *with respect to the cover* $(\mathfrak{U},\mathfrak{U}_A)$.

Our goal now is to make the relative cohomology groups independent of a specific cover, and we proceed analogously as in the absolute case. Thus, if $(\mathfrak{V},\mathfrak{V}_A)$ is another Čech cover of (X,A), with $\mathfrak{V}$ defined on the index set J and $\mathfrak{V}_A$ defined on the index set J_A, then a *(refinement) projection* is a refinement projection $\alpha\colon J\to I$ from $\mathfrak{V}$ to $\mathfrak{U}$ with $\alpha(J_A)\subseteq I_A$ and whose restriction $\alpha|_{J_A}\colon J_A\to I_A$ is a refinement projection from $\mathfrak{V}_A$ to $\mathfrak{U}_A$. If a refinement projection exists, we say that $(\mathfrak{V},\mathfrak{V}_A)$ *refines* $(\mathfrak{U},\mathfrak{U}_A)$. In this case, the induced chain map $\alpha_*\colon \check{C}^p(\mathfrak{U};F)\to\check{C}^p(\mathfrak{V};F)$ restricts to a chain map $\alpha_*\colon \check{C}^p(\mathfrak{U},\mathfrak{U}_A;F)\to\check{C}^p(\mathfrak{V},\mathfrak{V}_A;F)$. To see this, just recall that induced maps are natural with respect to refinement projections, so we have a commutative diagram

$$\begin{tikzcd}
\check{C}^p(\mathfrak{U};F) \arrow[r, "\iota^*"] \arrow[d, "\alpha_*"'] & \check{C}^p(\iota^{-1}\mathfrak{U};F|_A) \arrow[r] \arrow[d, "\alpha_*"'] & \check{C}^p(\mathfrak{U}_A;F|_A) \arrow[d, "(\alpha|_{J_A})_*"'] \\
\check{C}^p(\mathfrak{V};F) \arrow[r, "\iota^*"] & \check{C}^p(\iota^{-1}\mathfrak{V};F|_A) \arrow[r] & \check{C}^p(\mathfrak{V}_A;F|_A).
\end{tikzcd}$$

Alternatively, we can appeal to the explicit characterization of the elements in the kernel of $h_{\mathfrak{U},\mathfrak{U}_A}$ (respectively $h_{\mathfrak{V},\mathfrak{V}_A}$) given earlier. Using this characterization, we see that the operator $H_p\colon \check{C}^{p+1}(\mathfrak{U};F)\to\check{C}^p(\mathfrak{V};F)$ defined in the proof of proposition 2.9 restricts to a map $\check{C}^{p+1}(\mathfrak{U},\mathfrak{U}_A;F)\to\check{C}^p(\mathfrak{V},\mathfrak{V}_A;F)$. Hence, we have

Proposition 6.2. *Let* $(\mathfrak{V},\mathfrak{V}_A)$ *be a refinement of* $(\mathfrak{U},\mathfrak{U}_A)$. *Any two refinement projections induce the same map* $\check{\mathrm{H}}^p(\mathfrak{U},\mathfrak{U}_A;F)\to\check{\mathrm{H}}^p(\mathfrak{V},\mathfrak{V}_A;F)$.

Therefore, we may consider $\check{\mathrm{H}}^p(-,-;F)$ as a functor on the category $\mathcal{C}_{X,A}$ whose objects are the Čech covers of (X,A) and with one arrow $(\mathfrak{U},\mathfrak{U}_A)\to(\mathfrak{V},\mathfrak{V}_A)$ if and only if $(\mathfrak{V},\mathfrak{V}_A)$ refines $(\mathfrak{U},\mathfrak{U}_A)$.

Proposition 6.3. *The category of all Čech covers of* (X,A) *is directed and the set of all Čech covers* $(\mathfrak{U},\mathfrak{U}_A)$ *of the form* $\mathfrak{U}\colon X^k\to\mathcal{P}(X)$ *and* $\mathfrak{U}_A\colon A^k\to\mathcal{P}(A)$ *is cofinal.*

Proof. If $(\mathfrak{U},\mathfrak{U}_A)$ and $(\mathfrak{V},\mathfrak{V}_A)$ are Čech covers of (X,A) defined on pairs of index sets (I,I_A) and (J,J_A), respectively, then the Čech cover $(\mathfrak{W},\mathfrak{W}_A)$ defined on the pair of index sets $(I\times J, I_A\times J_A)$ and given by $\mathfrak{W}_{i,j}:=\mathfrak{U}_i\cap\mathfrak{V}_j$ and $(\mathfrak{W}_A)_{r,s}:=(\mathfrak{U}_A)_r\cap(\mathfrak{V}_A)_s$ refines both $(\mathfrak{U},\mathfrak{U}_A)$ and $(\mathfrak{V},\mathfrak{V}_A)$, with refinement projections the canonical projections $I\times J\to I$ and $I\times J\to J$.

The proof that the set of covers of (X,A) whose underlying pair of index sets is of the form (X^k,A^k) is cofinal is almost the same as in the absolute case, the only modification being this: if $(\mathfrak{U},\mathfrak{U}_A)$ is a Čech cover of (X,A), then, as $\mathfrak{U}_A$ covers A, we may first choose a function $\alpha_A\colon A\to I_A$ such that $a\in(\mathfrak{U}_A)_{\alpha(a)}$ for all $a\in A$ and extend α to a function $\alpha\colon X\to I$ such that $x\in\mathfrak{U}_{\alpha(x)}$ for all $x\in X-A$. Since $(\iota^{-1}\mathfrak{U}_A)|_{I_A}=\mathfrak{U}_A$, it follows that $(\mathfrak{U}\circ\alpha,\mathfrak{U}_A\circ\alpha_A)$ refines $(\mathfrak{U},\mathfrak{U}_A)$ and is of the desired form. □

Definition 6.4. The *p–th Čech cohomology group of* (X,A) with values in F is

$$\check{\mathrm{H}}^p(X,A;F) := \varinjlim_{(\mathfrak{U},\mathfrak{U}_A)} \check{\mathrm{H}}^p(\mathfrak{U},\mathfrak{U}_A;F),$$

where the limit ranges over all Čech covers $(\mathfrak{U}, \mathfrak{U}_A)$ of (X, A) with $\mathfrak{U}$ defined on X^k and $\mathfrak{U}_A$ defined on A^k for some integer $k \geq 0$.

It is immediate from proposition 6.3 and proposition 3.6 that $\check{\mathrm{H}}^p(X, A; F)$ is a limit for $\check{\mathrm{H}}^p(-, -; F)$ on $\mathcal{C}_{X,A}$. However, $\check{\mathrm{H}}^p(X, \emptyset; F)$ also is a limit for $\check{\mathrm{H}}^p(-; F)$ on $\mathcal{C}_X$. In fact, if $\mathfrak{U}\colon I \to \mathcal{P}(X)$ is a Čech cover of X, then $\mathfrak{U}_0\colon I \to \mathcal{P}(\emptyset)$, $i \mapsto \emptyset$, is a cover of $\emptyset$ and $(\mathfrak{U}, \mathfrak{U}_0)$ is a cover of $(X, \emptyset)$. This shows that the functor $T\colon \mathcal{C}_{X,\emptyset} \to \mathcal{C}_X$ taking a Čech cover $(\mathfrak{U}, \mathfrak{U}_A)$ of (X, A) to $\mathfrak{U}$ is cofinal. On the other hand, for every cover $(\mathfrak{U}, \mathfrak{U}_\emptyset)$ of $(X, \emptyset)$ the complexes $\check{C}^\bullet(i^{-1}\mathfrak{U}; F|_\emptyset)$ and $\check{C}^\bullet(\mathfrak{U}_A; F|_\emptyset)$ are trivial, and so $\check{C}^\bullet(\mathfrak{U}, \mathfrak{U}_A; F)$ is just the absolute Čech cochain complex $\check{C}^\bullet(\mathfrak{U}; F)$. In particular, $\check{\mathrm{H}}^p(\mathfrak{U}, \mathfrak{U}_\emptyset; F) = \check{\mathrm{H}}^p(\mathfrak{U}; F)$. Therefore, $\check{\mathrm{H}}^p(-, -; F)$ coincides with the functor $\check{\mathrm{H}}^p(-; F) \circ T$, and since $\check{\mathrm{H}}^p(X, \emptyset; F)$ is a limit of the former, it must also be a limit for $\check{\mathrm{H}}^p(-; F)$ by remark 3.7.

We now would like to relate the relative cohomology $\check{\mathrm{H}}^\bullet(\mathfrak{U}, \mathfrak{U}_A) = \check{\mathrm{H}}^\bullet(\mathfrak{U}, \mathfrak{U}_A; F)$ to the absolute cohomologies $\check{\mathrm{H}}^\bullet(\mathfrak{U}) = \check{\mathrm{H}}^\bullet(\mathfrak{U}; F)$ and $\check{\mathrm{H}}^\bullet(\mathfrak{U}_A) = \check{\mathrm{H}}^\bullet(\mathfrak{U}_A; F|_A)$. To this end, recall that in the exercises we have shown that every short exact sequence of cochain complexes induces a long exact sequence in cohomology. But while the sequence

$$0 \to \check{C}^p(\mathfrak{U}, \mathfrak{U}_A) \to \check{C}^p(\mathfrak{U}) \xrightarrow{h} \check{C}^p(\mathfrak{U}_A)$$

is always exact for all $p \geq 0$, the map $h = h_{\mathfrak{U},\mathfrak{U}_A}$ need not be surjective, as is demonstrated by the following example.

Example 6.5. In $X = \mathbb{R}$ consider the subspace $A = \mathbb{R} - \{0\}$ and the locally constant sheaf $F = \widetilde{\mathbb{Z}}$. As a cover of (X, A), take $\mathfrak{U} = \{\mathbb{R}\}$ and $\mathfrak{U}_A = \{\mathbb{R} - \{0\}\}$. Then h is not surjective in degree $p = 0$: we have a commuative diagram

$$\begin{tikzcd}
\check{C}^0(\mathfrak{U}) \arrow[r, "h"] \arrow[d, "\cong"'] & \check{C}^0(\mathfrak{U}_A) \arrow[d, "\cong"] \\
F(\mathbb{R}) \arrow[r] & F(\mathbb{R} - \{0\}),
\end{tikzcd}$$

where the lower horizontal map is given by restriction and the vertical maps evaluate a cochain c on the unique 0–simplex of $\mathfrak{U}$, respectively $\mathfrak{U}_A$. The horizontal maps are not surjective, because the locally constant function $s \in F(\mathbb{R} - \{0\})$ which equals -1 on $(-\infty, 0)$ and 1 on $(0, \infty)$ does not extend to a locally constant function on $\mathbb{R}$.

To circumenvent this problem, we follow [13] and replace $\check{C}^p(\mathfrak{U}_A)$ by $\mathrm{im}(h_{\mathfrak{U},\mathfrak{U}_A})$ in the sequence above. By construction, we obtain a short exact sequence of complexes

$$0 \to \check{C}^\bullet(\mathfrak{U}, \mathfrak{U}_A) \to \check{C}^\bullet(\mathfrak{U}) \xrightarrow{h_{\mathfrak{U},\mathfrak{U}_A}} \mathrm{im}\, h_{\mathfrak{U},\mathfrak{U}_A} \to 0$$

and thus also a long exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & \check{\mathrm{H}}^p(\mathfrak{U}, \mathfrak{U}_A) \arrow[r] & \check{\mathrm{H}}^p(\mathfrak{U}) \arrow[r, "h_{\mathfrak{U},\mathfrak{U}_A}"] & \mathrm{H}^p(\mathrm{im}\, h_{\mathfrak{U},\mathfrak{U}_A}) \arrow[r, "\partial_{\mathfrak{U},\mathfrak{U}_A}"] & \check{\mathrm{H}}^{p+1}(\mathfrak{U}, \mathfrak{U}_A) \arrow[r] & \ldots
\end{tikzcd}$$

with $\partial_{\mathfrak{U},\mathfrak{U}_A}$ the connecting homomorphism and $\mathrm{H}^\bullet(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A})$ the cohomology of the subcomplex $\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}$ of $\check{C}^\bullet(\mathfrak{U}_A)$. Note that the assignment $(\mathfrak{U},\mathfrak{U}_A) \mapsto \operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}$ is functorial, since we already saw that $h_{\mathfrak{U},\mathfrak{U}_A}$ is natural with respect to refinement projections of pairs. Moreover, this functor admits a limit $\mathrm{H}^p(\operatorname{im} h_{X,A})$ by proposition 3.2, since there is a cofinal set of coverings of (X, A) by proposition 6.3. Thus, there is a canonical map $j\colon \mathrm{H}^p(\operatorname{im} h_{X,A}) \to \check{\mathrm{H}}^p(A)$ making the diagram

$$\begin{tikzcd}
\mathrm{H}^p(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}) \arrow[r] \arrow[d] & \check{\mathrm{H}}^p(\mathfrak{U}_A) \arrow[d] \\
\mathrm{H}^p(\operatorname{im} h_{X,A}) \arrow[r, "j"] & \check{\mathrm{H}}^p(A)
\end{tikzcd}$$

commute for all Čech covers $(\mathfrak{U},\mathfrak{U}_A)$.

Theorem 6.6. *Let X be a topological space, F a preasheaf on X, and $A \subseteq X$ a subspace. For every integer $p \geq 0$ there exists a morphism $\partial = \partial_p\colon \check{\mathrm{H}}^p(\operatorname{im} h_{X,A}) \to \check{\mathrm{H}}^{p+1}(X, A)$ with the property that for all Čech covers $(\mathfrak{U},\mathfrak{U}_A)$ of (X, A) we have the following commutative ladder with exact rows:*

$$\begin{tikzcd}
\ldots \arrow[r] & \check{\mathrm{H}}^p(\mathfrak{U},\mathfrak{U}_A) \arrow[r] \arrow[d] & \check{\mathrm{H}}^p(\mathfrak{U}) \arrow[r, "h_{\mathfrak{U},\mathfrak{U}_A}"] \arrow[d] & \mathrm{H}^p(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}) \arrow[r, "\partial_{\mathfrak{U},\mathfrak{U}_A}"] \arrow[d] & \check{\mathrm{H}}^{p+1}(\mathfrak{U},\mathfrak{U}_A) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & \check{\mathrm{H}}^p(X, A) \arrow[r] & \check{\mathrm{H}}^p(X) \arrow[r, "q"] & \mathrm{H}^p(\operatorname{im} h_{X,A}) \arrow[r, "\partial"] & \check{\mathrm{H}}^{p+1}(X, A) \arrow[r] & \ldots
\end{tikzcd}$$

Here, the map q is such that $\iota^*\colon \check{\mathrm{H}}^p(X) \to \check{\mathrm{H}}^p(A)$ factors as $\iota^* = j \circ q$. The bottom row is called the long exact sequence of the pair (X, A) *and ∂ is its* connecting homomorphism*.*

Proof. Consider the functor $T\colon \mathcal{C}_{X,A} \to \mathcal{C}_X$ sending a cover $(\mathfrak{U},\mathfrak{U}_A)$ to $T(\mathfrak{U},\mathfrak{U}_A) = \mathfrak{U}$. It is cofinal: if $\mathfrak{U}$ is a Čech cover of X and $\iota\colon A \to X$ denotes the inclusion, then $(\mathfrak{U}, \iota^{-1}\mathfrak{U})$ is a Čech cover of (X, A) mapping onto $\mathfrak{U}$ under T. Next, observe that by remark 3.7 $\check{\mathrm{H}}^p(X)$ is a limit for $\check{\mathrm{H}}^p(-) \circ T$. Therefore, we obtain maps $q = q_p$, $k = k_p$, and $\partial = \partial_p$ making the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{U},\mathfrak{U}_A) \arrow[r] \arrow[d] & \check{\mathrm{H}}^p(\mathfrak{U}) \arrow[r, "h_{\mathfrak{U},\mathfrak{U}_A}"] \arrow[d] & \mathrm{H}^p(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}) \arrow[r, "\partial_{\mathfrak{U},\mathfrak{U}_A}"] \arrow[d] & \check{\mathrm{H}}^{p+1}(\mathfrak{U},\mathfrak{U}_A) \arrow[d] \\
\check{\mathrm{H}}^p(X, A) \arrow[r, "k"] & \check{\mathrm{H}}^p(X) \arrow[r, "q"] & \mathrm{H}^p(\operatorname{im} h_{X,A}) \arrow[r, "\partial"] & \check{\mathrm{H}}^{p+1}(X, A)
\end{tikzcd}$$

commute for all covers $(\mathfrak{U},\mathfrak{U}_A)$ of (X, A). Exactness of the ladder now is a consequence of proposition 3.8. For example, to see that

$$\begin{tikzcd}
\mathrm{H}^p(\operatorname{im} h_{X,A}) \arrow[r, "\partial"] & \check{\mathrm{H}}^{p+1}(X, A) \arrow[r] & \check{\mathrm{H}}^{p+1}(X)
\end{tikzcd}$$

is exact, note that

$$\mathrm{H}^p(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}) \xrightarrow{\partial_{\mathfrak{U},\mathfrak{U}_A}} \check{\mathrm{H}}^{p+1}(\mathfrak{U},\mathfrak{U}_A) \longrightarrow \check{\mathrm{H}}^{p+1}(\mathfrak{U})$$

is exact for all pairs $(\mathfrak{U},\mathfrak{U}_A)$, because this is a part of a long exact sequence, and apply proposition 3.8. It remains to show that ι^* factors through q. But this is immediate from the commutative diagrams

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{U}) \arrow[r, "\iota^*"] \arrow[d, equal] & \check{\mathrm{H}}^p(\iota^{-1}\mathfrak{U}) \arrow[r] & \check{\mathrm{H}}^p(\mathfrak{U}_A) \arrow[d, equal] \\
\check{\mathrm{H}}^p(\mathfrak{U}) \arrow[r, "h_{\mathfrak{U},\mathfrak{U}_A}"] \arrow[d] & \mathrm{H}^p(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}) \arrow[r] \arrow[d] & \check{\mathrm{H}}^p(\mathfrak{U}_A) \arrow[d] \\
\check{\mathrm{H}}^p(X) \arrow[r, "q"] & \mathrm{H}^p(\operatorname{im} h_{X,A}) \arrow[r, "j"] & \check{\mathrm{H}}^p(A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{U}) \arrow[r, "\iota^*"] \arrow[d] & \check{\mathrm{H}}^p(\iota^{-1}\mathfrak{U}) \arrow[r] \arrow[dr] & \check{\mathrm{H}}^p(\mathfrak{U}_A) \arrow[d] \\
\check{\mathrm{H}}^p(X) \arrow[rr, "\iota^*"] & & \check{\mathrm{H}}^p(A)
\end{tikzcd}$$

together with the uniqueness statement about maps from the limit $\check{\mathrm{H}}^p(X)$. □

Example 6.7. Here are two cases in which we can take $\mathrm{H}^p(\operatorname{im} h_{X,A}) = \check{\mathrm{H}}^p(A)$.

(i) If $F = \underline{M}$ is a constant presheaf, then $h_{\mathfrak{U},\mathfrak{U}_A}$ is surjective, since we can trivially extend any cochain $c \in \check{C}^p(\mathfrak{U}_A)$ to a cochain $\overline{c} \in \check{C}^p(\mathfrak{U})$. In particular, $\mathrm{H}^p(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A}) = \check{\mathrm{H}}^p(\mathfrak{U}_A)$. In the proof of theorem 6.6 we already observed that the functor sending a Čech cover (X, A) to A is cofinal, and thus we see that also $\check{\mathrm{H}}^p(A)$ is a limit for the functor $(\mathfrak{U},\mathfrak{U}_A) \mapsto \mathrm{H}^p(\operatorname{im} h_{\mathfrak{U},\mathfrak{U}_A})$.

(ii) If A is an open subset of X, then the covers $(\mathfrak{U},\mathfrak{U}_A)$ of (X, A) such that $(\mathfrak{U}_A)_i \subseteq A$ for all i are cofinal. For if $(\mathfrak{U},\mathfrak{U}_A)$ is an arbitrary cover of (X, A), defined on the pair of index sets (I, I_A), let $J := I \cup (I_A \times \{\infty\})$, $J_A := I_A \times \{\infty\}$, and define a cover $\mathfrak{V}$ of X on the index set J by

$$\mathfrak{V}_j := \begin{cases} \mathfrak{U}_j, & j \in I, \\ \mathfrak{U}_i \cap A, & j = (i, \infty) \in I_A \times \{\infty\}. \end{cases}$$

There is a canonical refinement projection α from $\mathfrak{V}$ to $\mathfrak{U}$ given by $\alpha(i) = i$ and $\alpha(i,\infty) = i$ for all $i \in I$, and this refinement projection shows that $(\mathfrak{V}, \mathfrak{V}|_{J_A})$ refines $(\mathfrak{U},\mathfrak{U}_A)$.

Now if $(\mathfrak{U}, \mathfrak{U}_A)$ is a cover of (X, A) defined on the pair of index sets (I, I_A) and if we have $(\mathfrak{U}_A)_i \subseteq A$ for all $i \in I$, then we can extend any $c \in \check{C}^p(\mathfrak{U}_A)$, as in the case of constant coefficients, to a cochain $\overline{c} \in \check{C}^p(\mathfrak{U})$: just note that the canonical ι–cohomomorphism $\iota^*\colon F \to \iota_*(F|_A)$ yields isomorphisms $(\iota^*)_U\colon F(U) \to (\iota_*(F|_A)(U)$ for all open subsets $U \subseteq A$, so we may define

$$\overline{c}(\sigma) := \begin{cases} \left((\iota^*)_{\mathfrak{U}_\sigma}\right)^{-1}(c(\sigma)), & \sigma_0, \ldots, \sigma_p \in I_A, \\ 0, & \text{else.} \end{cases}$$

Therefore, $h_{\mathfrak{U},\mathfrak{U}_A}$ is surjective and we can set $\mathrm{H}^p(\operatorname{im} h_{X,A}) = \check{\mathrm{H}}^p(A)$.

There is also a notion of morphism of pairs of spaces: if X, Y are topological spaces with presheaves F and G, respectively, and if $A \subseteq X$, $B \subseteq Y$ are subspaces, then a morphism $(f, k)\colon (X, A, F) \to (Y, B, G)$ is a morphism $(f, k)\colon (X, F) \to (Y, G)$ with $f(A) \subseteq B$. Such a morphism induces a map $\check{\mathrm{H}}^p(Y, B) \to \check{\mathrm{H}}^p(X, A)$ as follows. If $(\mathfrak{V}, \mathfrak{V}_B)$ is a cover of (Y, B) defined on a pair of index sets (J, J_B), then $(f^{-1}\mathfrak{V}, (f|_A)^{-1}\mathfrak{V}_B)$ is a cover of (X, A) defined on the same pair of index sets: indeed, $(f|_A)^{-1}\mathfrak{V}_B$ is a Čech cover of A, and for all $j \in J_B$ we have

$$(f^{-1}\mathfrak{V})_j \cap A = f^{-1}(\mathfrak{V}_j) \cap A = (f|_A)^{-1}(\mathfrak{V}_j \cap B) = (f|_A)^{-1}((\mathfrak{V}_B)_j) = ((f|_A)^{-1}\mathfrak{V}_B)_j.$$

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\check{C}^p(\mathfrak{V}) \arrow[r, "h_{\mathfrak{V},\mathfrak{V}_B}"] \arrow[d, "f^*"] & \operatorname{im} h_{\mathfrak{V},\mathfrak{V}_B} \arrow[r] \arrow[d, "(f|_A)^*"] & \check{C}^p(\mathfrak{V}_B) \arrow[d, "(f|_A)^*"'] \\
\check{C}^p(f^{-1}\mathfrak{V}) \arrow[r, "h_{\mathfrak{f}^{-1}\mathfrak{V},(f|_A)^{-1}\mathfrak{V}_B}"'] & \operatorname{im} h_{\mathfrak{f}^{-1}\mathfrak{V},(f|_A)^{-1}\mathfrak{V}_B} \arrow[r] & \check{C}^p((f|_A)^{-1}\mathfrak{V}_B)
\end{tikzcd}$$

and this shows that the chain map $f^*\colon \check{C}^p(\mathfrak{V}) \to \check{C}^p(f^{-1}\mathfrak{V})$ restricts to a chain map $f^*\colon \check{C}^p(\mathfrak{V}, \mathfrak{V}_B) \to \check{C}^p(f^{-1}\mathfrak{V}, (f|_A)^{-1}\mathfrak{V}_B)$, which in turn induces a map on the level of cohomology. We define $f^*\colon \check{\mathrm{H}}^p(Y, A) \to \check{\mathrm{H}}^p(X, A)$ to be the unique map making the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{V}, \mathfrak{V}_B) \arrow[r, "f^*"] \arrow[d] & \check{\mathrm{H}}^p(f^{-1}\mathfrak{V}, (f|_A)^{-1}\mathfrak{V}_B) \arrow[d] \\
\check{\mathrm{H}}^p(Y, B) \arrow[r, "f^*"] & \check{\mathrm{H}}^p(X, A)
\end{tikzcd}$$

commute for all covers $(\mathfrak{V}, \mathfrak{V}_B)$ of (Y, A); here, as usual, the undecorated vertical maps are the maps into the limit.

Proposition 6.8. *The long exact sequence of pairs is natural with respect to maps of pairs of spaces, i. e. if $f\colon (X, A) \to (Y, B)$ is a map of pairs of spaces, then we have a* *commutative ladder:*

$$\begin{tikzcd}
\dots \arrow[r] & \check{\mathrm{H}}^p(Y,B) \arrow[r] \arrow[d, "f^*"] & \check{\mathrm{H}}^p(Y) \arrow[r, "q"] \arrow[d, "f^*"] & \mathrm{H}^p(\operatorname{im} h_{Y,B}) \arrow[r, "\partial"] \arrow[d, "(f|_A)^*"] & \check{\mathrm{H}}^{p+1}(Y,B) \arrow[r] \arrow[d, "f^*"] & \dots \\
\dots \arrow[r] & \check{\mathrm{H}}^p(X,A) \arrow[r] & \check{\mathrm{H}}^p(X) \arrow[r, "q"] & \mathrm{H}^p(\operatorname{im} h_{X,A}) \arrow[r, "\partial"] & \check{\mathrm{H}}^{p+1}(X,A) \arrow[r] & \dots
\end{tikzcd}$$

Proof. This follows at once from the naturality of long exact sequences in cohomology induced from short exact sequences of complexes. Consider, for example, for any Čech cover $(\mathfrak{V}, \mathfrak{V}_B)$ of (Y, A) the diagram

$$\begin{tikzcd}
& \mathrm{H}^p(\operatorname{im} h_{\mathfrak{V},\mathfrak{V}_B}) \arrow[rr, "\partial"] \arrow[dl] \arrow[dd, "(f|_A)^*"] & & \check{\mathrm{H}}^{p+1}(\mathfrak{V},\mathfrak{V}_B) \arrow[dl] \arrow[dd, "f^*"] \\
\mathrm{H}^p(\operatorname{im} h_{Y,B}) \arrow[rr, "\partial"] \arrow[dd, "(f|_A)^*"'] & & \check{\mathrm{H}}^p(Y,B) \arrow[dd, "f^*"] & \\
& \mathrm{H}^p(\operatorname{im} h_{f^{-1}\mathfrak{V},(f|_A)^{-1}\mathfrak{V}_B}) \arrow[rr, "\partial"] \arrow[dl] & & \check{\mathrm{H}}^{p+1}(f^{-1}\mathfrak{V},(f|_A)^{-1}\mathfrak{V}_B) \arrow[dl] \\
\mathrm{H}^p(\operatorname{im} h_{X,A}) \arrow[rr, "\partial"'] & & \check{\mathrm{H}}^{p+1}(X,A) &
\end{tikzcd}$$

The bottom and top horizontal faces of this diagram commute by theorem 6.6, and the left and right vertical faces commute by construction of induced maps. Since the back face of this diagram commutes due to naturality of long exact sequences, so must the two maps $\mathrm{H}^p(\operatorname{im} h_{\mathfrak{V},\mathfrak{V}_B}) \to \check{\mathrm{H}}^{p+1}(X, A)$ obtained by composing the limit map $\mathrm{H}^p(\operatorname{im} h_{\mathfrak{V},\mathfrak{V}_B}) \to \check{\mathrm{H}}^p(\operatorname{im} h_{Y,B})$ with either $\partial \circ (f|_A)^*$ or $f^* \circ \partial$. Then the uniqueness statement in the universal property of $\check{\mathrm{H}}^p(\operatorname{im} h_{Y,B})$ implies that the front face must commute as well. Commutativity of the other parts of the ladder is concluded similarly. ☐

7. Excision

Let X be a topological space and F a preasheaf on X. As in the previous section, we consider all subspaces A of X with the restricted presheaf $F|_A$ and suppress coefficients in the notation.

Theorem 7.1. *Let $Z \subseteq X$ be a non–empty subspace admitting an open neighborhood W such that $\overline{W}$ (closure of W in X) is contained in $\mathring{A}$ (interior of A in X). Then the inclusion induced map $\check{\mathrm{H}}^\bullet(X, A) \to \check{\mathrm{H}}^\bullet(X - Z, A - Z)$ is an isomorphism.*

Proof. Let $\mathcal{D}_X$ be the subcategory of all Čech covers $(\mathfrak{U}, \mathfrak{U}_A)$ of (X, A) with the following properties: $(\mathfrak{U}, \mathfrak{U}_A)$ refines the cover $(\{X - \overline{W}, \mathring{A}\}, \{A - \overline{W}, \mathring{A}\})$ and if (I, I_A) is the pair of index sets on which $(\mathfrak{U}, \mathfrak{U}_A)$ is defined, then $\mathfrak{U}_i \subseteq \mathring{A}$ only holds if $i \in I_A$. This is a cofinal subcategory, because by assumption X is covered by $X - \overline{W}$ and $\mathring{A}$, and because if $(\mathfrak{U},\mathfrak{U}_A)$ is such that $\mathfrak{U}_i$ is contained in $\mathring{A}$ for some $i \notin I_A$, then $(\mathfrak{U}|_{I-\{i\}},\mathfrak{U}_A|_{I-\{i\}})$ still is a cover of (X,A), since A already is covered by all sets $\mathfrak{U}_j$ with $j \in I_A$.

Now let $(\mathfrak{U},\mathfrak{U}_A) \in \mathcal{D}_X$ be a cover defined on the pair of index sets (I,I_A) and extend $(\mathfrak{U},\mathfrak{U}_A)$ to a cover $(\overline{\mathfrak{U}},\overline{\mathfrak{U}}_A)$ defined on the index set $(\overline{I},\overline{I}_A) = (I \cup \{*\}, I_A \cup \{*\})$ for some point $* \notin I$ by declaring $\overline{\mathfrak{U}}_* = W$. We claim that the canonical refinement projection $\alpha\colon I \hookrightarrow \overline{I}$ induces an isomorphism $\alpha_*\colon \check{C}^p(\overline{\mathfrak{U}},\overline{\mathfrak{U}}_A) \to \check{C}^p(\mathfrak{U},\mathfrak{U}_A)$. In fact, α_* certainly is surjective, because we can trivially extend any cochain $d \in \check{C}^p(\mathfrak{U},\mathfrak{U}_A)$ to a cochain in $\check{C}^p(\overline{\mathfrak{U}},\overline{\mathfrak{U}}_A)$. Thus, let $c \in \check{C}^p(\overline{\mathfrak{U}},\overline{\mathfrak{U}}_A)$ be such that $\alpha_*(c) = 0$ and let σ be a p–simplex of $\overline{\mathfrak{U}}$. If none of the vertices $\sigma_0,\dots,\sigma_p$ equals $*$, then

$$0 = (\alpha_* c)(\sigma) = c(\sigma)|_{\mathfrak{U}_\sigma} = c(\sigma).$$

On the other hand, if $\sigma_i = *$, then, since $\overline{\mathfrak{U}}_{\sigma_i} = W$ and $(\mathfrak{U},\mathfrak{U}_A)$ is a cover in $\mathcal{D}_X$, we have $\overline{\mathfrak{V}}_\sigma = \emptyset$ unless $\overline{\mathfrak{U}}_{\sigma_j} \subseteq \mathring{A}$ for all j, that is, unless $\sigma_j \in \overline{I}_A$ for all j. But then $c(\sigma) = 0$ holds anyway, because c is a relative cochain, and therefore α_* is injective.

A completely analogous argument shows that the subcategory $\mathcal{D}_{X-Z}$ of all Čech covers $(\mathfrak{W},\mathfrak{W}_{A-Z})$ of $(X-Z,A-Z)$ which refine the cover $(\{X-\overline{W},\mathring{A}-Z\},\{A-\overline{W},\mathring{A}-Z\})$ and which are defined on pairs of index sets (K,K_{A-Z}) with $\mathfrak{W}_k \subseteq X-\overline{W}$ unless $k \in K_{A-Z}$ is cofinal. Moreover, the extension $(\overline{\mathfrak{W}},\overline{\mathfrak{W}}_{A-Z})$ of such a cover $(\mathfrak{W},\mathfrak{W}_{A-Z})$ to the index set $(\overline{K},\overline{K}_{A-Z}) = (K \cup \{*\}, K_{A-Z} \cup \{*\})$ by the set $W-Z$ admits a canonical refinement projection inducing an isomorphism $\check{C}^p(\overline{\mathfrak{W}},\overline{\mathfrak{W}}_{A-Z}) \to \check{C}^p(\mathfrak{W},\mathfrak{W}_{A-Z})$.

Next, let us show that for every cover $(\mathfrak{U},\mathfrak{U}_A) \in \mathcal{D}_X$ the map $\iota^*\colon \check{C}^p(\mathfrak{U},\mathfrak{U}_A) \to \check{C}^p(\iota^{-1}\mathfrak{U},\iota^{-1}\mathfrak{U}_A)$ induced by the inclusion of pairs $\iota\colon (X-Z,A-Z) \to (X,A)$ is an isomorphism. Indeed, given $d \in \check{C}^p(\iota^{-1}\mathfrak{U},\iota^{-1}\mathfrak{U}_A)$ and a p–simplex σ of $\mathfrak{U}$, suppose that there is at least one vertex σ_i with $\sigma_i \notin I_A$. Then $\mathfrak{U}_{\sigma_i} \subseteq X-\overline{W}$ and hence $\mathfrak{U}_\sigma$ too is contained in $X-\overline{W}$. But this is an open subset of X which does not meet Z, so the canonical ι–cohomomorphism ι^* induces an isomorphism

$$(\iota^*)_{\mathfrak{U}_\sigma}\colon F(\mathfrak{U}_\sigma) \to \iota_*(F|_{X-Z})(\mathfrak{U}_\sigma).$$

Defining $c(\sigma) \in F(\mathfrak{U}_\sigma)$ by $(\iota^*)_{\mathfrak{U}_\sigma}(c(\sigma)) = d(\sigma)$ and putting $c(\sigma) = 0$ if all vertices of σ are contained in I_A, we see that $\iota^*(c) = d$, because d vanishes on simplices σ of $\iota^{-1}\mathfrak{U}$ whose vertices are all contained in I_A. This shows that ι^* is surjective. Injectivity is concluded similarly: if we suppose that $\iota^*(c) = 0$ holds for a cochain $c \in \check{C}^p(\mathfrak{U},\mathfrak{U}_A)$, then for a p–simplex σ such that $\sigma_i \notin I_A$ for some i and $\mathfrak{U}_\sigma \neq \emptyset$, the map $(\iota^*)_{\mathfrak{U}_\sigma}$ is an isomorphism, whence $0 = \iota^*(c)(\sigma) = (\iota^*)_{\mathfrak{U}_\sigma}(c(\sigma))$ necessarily implies $c(\sigma) = 0$; and on simplices σ with $\sigma_0,\dots,\sigma_p \in I_A$ the relative cochain c vanishes anyway.

The proof now follows thus. Let $\overline{\mathcal{D}}_X$ denote the subcategory of $\mathcal{D}_X$ consisting of those covers $(\mathfrak{U},\mathfrak{U}_A)$ for which there is some index i with $(\mathfrak{U}_A)_i = W$. This is a directed subcategory, for if $(\mathfrak{U},\mathfrak{U}_A)$ and $(\mathfrak{V},\mathfrak{V}_A)$ are two covers in $\overline{\mathcal{D}}_X$ and $(\mathfrak{W},\mathfrak{W}_A)$ is a refinement of these covers, not necessarily contained in $\overline{\mathcal{D}}_X$, then $(\overline{\mathfrak{W}},\overline{\mathfrak{W}}_A)$ still is a refinement of both $(\mathfrak{U},\mathfrak{U}_A)$ and $(\mathfrak{V},\mathfrak{V}_A)$, and is contained in $\overline{\mathcal{D}}_X$. Indeed, if $(\mathfrak{U},\mathfrak{U}_A)$ is defined on (I,I_A) and $(\mathfrak{W},\mathfrak{W}_A)$ is defined on (K,K_A), and if $\alpha\colon J \to I$ is a refinement projection, then we can extend α to $\overline{K}$ by setting $\alpha(*) = i$ whenever $i \in I_A$ is an index with $\mathfrak{U}_i = W$, and this shows that $(\mathfrak{U},\mathfrak{U}_A)$ is refined by $(\overline{\mathfrak{W}},\overline{\mathfrak{W}}_A)$. Note that this also shows that the functor $T\colon \mathcal{D}_X \to \overline{\mathcal{D}}_X$ taking $(\mathfrak{U}, \mathfrak{U}_A)$ to $(\overline{\mathfrak{U}}, \overline{\mathfrak{U}}_A)$ is cofinal. In a similar way we see that the functor $S\colon \mathcal{D}_{X-Z} \to \overline{\mathcal{D}}_{X-Z}$ taking $(\mathfrak{W}, \mathfrak{W}_{A-Z})$ to $(\overline{\mathfrak{W}}, \overline{\mathfrak{W}}_{A-Z})$ is cofinal in the subcategory $\overline{\mathcal{D}}_{X-Z}$ of covers $(\mathfrak{W}, \mathfrak{W}_{A-Z})$ in $\mathcal{D}_{X-Z}$ with $(\mathfrak{W}_{A-Z})_k = W - Z$ for some index k. Now since

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\overline{\mathfrak{U}}, \overline{\mathfrak{U}}_A) \arrow[rr] \arrow[dr] & & \check{\mathrm{H}}^p(\mathfrak{U}, \mathfrak{U}_A) \arrow[dl] \\
& \check{\mathrm{H}}^p(X, A) &
\end{tikzcd}$$

is a commutative diagram and we have shown that the upper horizontal map is an isomorphism, it follows that $\check{\mathrm{H}}^p(X, A)$ is a limit for $\check{\mathrm{H}}^p(-,-)|_{\overline{\mathcal{D}_X}} \circ T$. By proposition 3.6, $\check{\mathrm{H}}^p(X, A)$ then also is a limit for $\check{\mathrm{H}}^p(-,-)|_{\overline{\mathcal{D}_X}}$. Similarly, we see that $\check{\mathrm{H}}^p(X-Z, A-Z)$ is a limit for $\check{\mathrm{H}}^p(-,-)|_{\overline{\mathcal{D}_{X-Z}}}$, and therefore we can consider $\iota^*\colon \check{\mathrm{H}}^p(X, A) \to \check{\mathrm{H}}^p(X-Z, A-Z)$ as the unique map making the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{U}, \mathfrak{U}_A) \arrow[r, "\iota^*"] \arrow[d] & \check{\mathrm{H}}^p(\iota^{-1}\mathfrak{U}, \iota^{-1}\mathfrak{U}_A) \arrow[d] \\
\check{\mathrm{H}}^p(X, A) \arrow[r, "\iota^*"] & \check{\mathrm{H}}^p(X - Z, A - Z)
\end{tikzcd}$$

commute for all Čech covers $(\mathfrak{U}, \mathfrak{U}_A) \in \overline{\mathcal{D}}_X$. Thus, by proposition 3.8, to conclude that ι^* is an isomorphism, we merely need to show that the functor $\overline{\mathcal{D}}_X \to \overline{\mathcal{D}}_{X-Z}$ taking $(\mathfrak{U}, \mathfrak{U}_A)$ to $(\iota^{-1}\mathfrak{U}, \iota^{-1}\mathfrak{U}_A)$ is cofinal. So suppose $(\mathfrak{W}, \mathfrak{W}_{A-Z})$ is a cover in $\overline{\mathcal{D}}_{X-Z}$, defined on a pair of index sets (K, K_{A-Z}), and choose for each $k \in K$ an open subset $U_k \subseteq X$ with $\mathfrak{W}_k = U_k - Z$. Since $(\mathfrak{W}, \mathfrak{W}_{A-Z})$ is an object of $\mathcal{D}_{X-Z}$, we may assume that U_k is chosen such that $U_k \subseteq X - \overline{W}$ or $U_k \subseteq \mathring{A}$ for all k, and that the latter inclusion only holds if $k \in K_A$. Moreover, if $* \in K_{A-Z}$ is such that $\mathfrak{W}_* = W - Z$, we may assume that $U_* = W$, and then the cover $(\mathfrak{U}, \mathfrak{U}_A)$ defined on (K, K_{A-Z}) and determined by $\mathfrak{U}_k = U_k$ is contained in $\overline{\mathcal{D}}_X$ and satisfies $(\iota^{-1}\mathfrak{U}, \iota^{-1}\mathfrak{U}_A) = (\mathfrak{W}, \mathfrak{W}_{A-Z})$. $\square$

Remark 7.2. In singular cohomology the conclusion of theorem 7.1 already holds if Z and A are subsets of X such that $\overline{Z} \subseteq \mathring{A}$, see for example [8, Section 3.1]. The following example shows that Čech cohomology in general fails to satisfy excision for such triples, even for constant coefficients and if all spaces are compact (however, if the spaces are also assumed to be Hausdorff, then theorem 7.1 does holds under these weaker assumptions, see example 7.3). Consider the unit interval $[-1, 1]$ with its standard topology $\mathcal{O}$ inherited from $\mathbb{R}$. We define a new topology τ on $[-1, 1]$ by declaring a subset $U \subseteq [-1, 1]$ to be open if $U \in \mathcal{O}$ and one of the following conditions is satisfied:

(i) $U \subseteq (-1, 1)$,

(ii) $U = [-1, 1]$ or $U = [-1, 0) \cup (0, 1]$,

(iii) $U \subseteq [-1, 1)$ and there exists $\varepsilon > 0$ such that $[-1, 0) \cup (0, \varepsilon) \subseteq U$, or

(iv) $U \subseteq (-1,1]$ and there exists $\varepsilon > 0$ such that $(-\varepsilon, 0) \cup (0,1] \subseteq U$.

Let us check that this really defines a topology on $[-1,1]$. Thus, let $(U_i)_{i\in I}$ be a collection of τ–open subsets and put $U := \bigcup_{i\in I} U_i$. If $-1, 1 \in U$, then necessarily $[-1,0)\cup(0,1] \subseteq U$ and U is τ–open. If -1 is contained in U but not 1, then there must some $\varepsilon > 0$ with $[-1,0) \cup (0,\varepsilon) \subseteq U$, and since $U \subseteq [-1,1)$, U is again open. Similarly, we see that U is open if $1 \in U$ and $-1 \notin U$. Finally, if neither 1 nor -1 is an element of U, then U is open because each U_i is open in $[-1,1]$.

To see that τ is closed under finite intersections, let $U, V \in \tau$. If $U = [-1,1]$, there is nothing to show. If $U = [-1,0) \cup (0,1]$, then $U \cap V = V - \{0\}$ is τ–open, and the same is true if $U \subseteq (-1,1)$, because then also $U \cap V \subseteq (-1,1)$. Therefore, we may assume that either 1 or -1 is contained in U, but not both. By the same reasoning, there is no loss of generality in assuming that either 1 or -1 is contained in V, but $\{-1,1\} \subsetneq V$. If $U \cap V \subseteq (-1,1)$, then this intersection is τ–open. Otherwise, say, if $1 \in U \cap V$, there are $\varepsilon, \delta > 0$ such that $(-\varepsilon,0)\cup(0,1] \subseteq U$ and $(-\delta,0)\cup[0,1] \subseteq V$. Taking $\gamma := \min(\varepsilon,\delta)$, we see that $U \cap V \subseteq (-1,1]$ containes $(-\gamma,0)\cup(0,1]$ and hence is τ–open.

We denote the topological space $[-1,1]$ with the topology τ by X. Put $Z = \{-1,1\}$ and $A = [-1,0) \cup (0,1]$. The set A is open in X and $Z = X - (-1,1)$ is closed, so certainly $\overline{Z} \subseteq \mathring{A}$. Next, observe that X is not Hausdorff. In fact, if U, V, and W are open neighborhoods of -1, 0, and 1, respectively, then $U \cap V \cap W$ is non–empty. This implies that for any non–trivial Abelian group G, $\check{\mathrm{H}}^1(X,A) = \check{\mathrm{H}}^1(X,A;\underline{G})$ is trivial. To see this, let us first note that covers of (X,A) of the form $(\mathfrak{U},\mathfrak{U}_A)$ with $I_A = \{r,s\}$ and $\mathfrak{U}_r = [-1,0)\cup(0,\varepsilon)$ and $\mathfrak{U}_s = (-\varepsilon,0)\cup(0,1]$ are cofinal. For if $(\mathfrak{U},\mathfrak{U}_A)$ is an arbitrary cover of (X,A), defined on the pair of index sets (I,I_A), then there must be indices $r,s \in I_A$ with $-1 \in \mathfrak{U}_r$ and $1 \in \mathfrak{U}_s$. By construction of the topology on X, there then exists $\varepsilon > 0$ such that $[-1,0)\cup(0,\varepsilon) \subseteq \mathfrak{U}_r$ and $(-\varepsilon,0)\cup(0,1] \subseteq \mathfrak{U}_s$, whence the claim. Furthermore, we can assume that $\mathfrak{U}_i \subseteq A$ only if $i \in I_A = \{r,s\}$, since otherwise $(\mathfrak{U}|_{I-\{i\}},\mathfrak{U}_A)$ still is a cover of (X,A) and refines $(\mathfrak{U},\mathfrak{U}_A)$.

Now choose a cocycle $c \in \check{C}^1(\mathfrak{U},\mathfrak{U}_A)$ for some such cover $(\mathfrak{U},\mathfrak{U}_A)$ of (X,A) and put

$$d(i) := \begin{cases} c(r,i), & \text{if } \mathfrak{U}_r \cap \mathfrak{U}_i \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for all $i \in I$, where $r \in I_A$ is the index such that $-1 \in \mathfrak{U}_r$. The 0–cochain d is an element of $\check{C}^0(\mathfrak{U},\mathfrak{U}_A)$, because c vanishes on 1–simplices σ whose both vertices σ_0, σ_1 are contained in $I_A = \{r,s\}$. In particular, on such simplices we have $c(\sigma) = (\delta d)(\sigma)$, and we claim that this identity remains true for all simplices $\sigma = (i,j)$. The observation to make here is that $\mathfrak{U}_\sigma$ always intersects U_r non–trivially, because $\mathfrak{U}_i$ (respectivelly $\mathfrak{U}_j$) either is a neighborhood of 0 or it is contained in A, whence $i = r$ or $i = s$ by choice of I. Thus, $\underline{G}(\mathfrak{U}_\sigma \cap \mathfrak{U}_r) = G$ and since c is closed, it follows that

$$(\delta d)(\sigma) = d(j) - d(i) = c(r,j) - c(r,i) = c(i,j) - (\delta c)(r,i,j) = c(\sigma).$$

In other words, c is exact and $\check{\mathrm{H}}^1(X,A) = 0$.

However, $X - Z$ is just $(-1, 1)$ with its standard topology, and $A - Z = (-1, 0) \cup (0, 1)$ too carries the topology inherited from $\mathbb{R}$. By the long exact sequence of the pair $(X - Z, A - Z)$ we then have an exact sequence

$$\check{\mathrm{H}}^0(X - Z) \to \check{\mathrm{H}}^0(A - Z) \to \check{\mathrm{H}}^1(X - Z, A - Z).$$

Since $X - Z$ is connected and $A - Z$ has two connected components, the above map $G = \check{\mathrm{H}}^0(X - Z) \to \check{\mathrm{H}}^0(A - Z) = G \oplus G$ is not onto, and hence $\check{\mathrm{H}}^1(X - Z, A - Z)$ is non–trivial. In particular, $\check{\mathrm{H}}^1(X, A) \to \check{\mathrm{H}}^1(X - Z, A - Z)$ cannot be surjective.

Example 7.3. Suppose that X is *normal*, that is, any two disjoint closed sets can be separated by open neighborhoods. In this case, if $Z, A \subseteq X$ are such that $\overline{Z} \subseteq \mathring{A}$, then $\check{\mathrm{H}}^\bullet(X, A) \to \check{\mathrm{H}}^\bullet(X - Z, A - Z)$ is an isomorphism. To see this, note that by assumption $\overline{Z}$ does not meet the boundary $\partial A = \overline{A} - \mathring{A}$ of A. Since X is normal, we thus can find open neighborhoods W of $\overline{Z}$ and V of ∂A which are disjoint. Intersecting W with $\mathring{A}$ if necessary, we may assume that $W \subseteq \mathring{A}$, and since $X - V$ is closed, we see that $\overline{W} \subseteq X - V$; in particular, $\overline{W}$ does not meet ∂A. Therefore,

$$\overline{W} \subseteq \overline{A} - \partial A = \mathring{A},$$

and so the assumptions of theorem 7.1 are satisfied.

8. Homotopy invariance

Let X be a topological space, $A \subseteq X$ a subspace, and fix a presheaf F on X. Let $q\colon X \times [0, 1] \to X$ be the canonical projection map and put $\iota_t\colon X \to X \times [0, 1]$, $x \mapsto (x, t)$, for $t = 0, 1$. Since q is an open map, we can consider the presheaf $F \times [0, 1]$ on $X \times [0, 1]$ given by $(F \times [0, 1])(U) = F(q(U))$ for all open subsets $U \subseteq X \times [0, 1]$; up to canonical isomorphism, this is just the pullback q^*F. Observe that ι_t induces a canonical morphism $(X, F) \to (X \times [0, 1], F \times [0, 1])$, for if $U \subseteq X \times [0, 1]$ is an open subset, then $(\iota_t)^{-1}(U) \subseteq q(U)$, so we have a restriction map $F(q(U)) \to F((\iota_t)^{-1}(U))$. These restriction maps assemble to give the desired morphism.

Theorem 8.1. *The maps* $\check{\mathrm{H}}^\bullet(X \times [0, 1], A \times [0, 1]; F \times [0, 1]) \to \check{\mathrm{H}}^\bullet(X, A; F)$ *induced by* ι_0 *and* ι_1 *coincide.*

The proof makes use of the following

Lemma 8.2. *Let G be a presheaf on a topological space Y, $B \subseteq Y$ a subspace, and $(\mathfrak{U}, \mathfrak{U}_B)$ a Čech cover of (Y, B). Every cocycle $c \in \check{C}^p(\mathfrak{U}, \mathfrak{U}_B; G)$ is cohomologous to a cocycle that vanishes on simplices containing two adjacent equal indices; that is, there exists a cocycle $d \in \check{C}^p(\mathfrak{U}, \mathfrak{U}_B; G)$ representing the same cohomology class as c and such that $d(\sigma) = 0$ if $\sigma(k) = \sigma(k + 1)$ for some $k = 0, \ldots, p$.*

Proof. Let I be the index set on which $\mathfrak{U}$ is defined. Given an index $i \in I$, denote by $i^{\otimes r}$ the element $(i, \ldots, i) \in I^r$. Then every q–simplex σ admits a decomposition

$\sigma = ((\sigma_0)^{\otimes a_0}, \ldots, (\sigma_m)^{\otimes a_m})$ for certain indices $\sigma_0, \ldots, \sigma_m \in I$ and integers $a_0, \ldots, a_m \geq 1$ and $m \geq 0$ in such a way that $\sigma_k \neq \sigma_{k+1}$ for all $k = 0, \ldots, m-1$. Let us further write

$$S^q_\ell = \left\{((i_0)^{\otimes a_0}, \ldots, (i_m)^{\otimes a_m}) \in I^{q+1} \,|\, \exists k : a_k \geq \ell\right\}$$

for the set of simplices containing at least ℓ adjacent equal indices and

$$\check{C}^q_\ell = \{c \in \check{C}^q(\mathfrak{U}, \mathfrak{U}_B) \,|\, \forall \sigma \in S^q_\ell : c(\sigma) = 0\}$$

for the set of q–cochains vanishing on such simplices. Notice that for trivial reasons $\check{C}^q_{q+2} = \check{C}^q(\mathfrak{U}, \mathfrak{U}_B)$, so to prove the lemma, we only need to show that for all $\ell \geq 2$ every cocycle in $\check{C}^q_{\ell+1}$ is cohomologous to a cocycle in $\check{C}^q_\ell$.

The first observation to make is that the boundary operator on q–cochains reads

$$\begin{aligned}
&\delta(c)((i_0)^{\otimes a_0}, \ldots, (i_m)^{\otimes a_m}) \\
&= \sum_{t=0}^{m} (-1)^{a_0+\ldots+a_{t-1}} \sum_{j=0}^{a_t-1} (-1)^j \cdot c((i_0)^{\otimes a_0}, \ldots, (i_t)^{\otimes a_t - 1}, \ldots, (i_m)^{\otimes a_m}) \\
&= \sum_{t\colon a_t \text{ odd}} (-1)^{a_0+\ldots+a_{t-1}} \cdot c((i_0)^{\otimes a_0}, \ldots, (i_t)^{\otimes a_t - 1}, \ldots, (i_m)^{\otimes a_m}).
\end{aligned}$$

In particular, if ℓ is even and $c \in \check{C}^p_{\ell+1}$ is a cocycle, then c necessarily is an element of $\check{C}^p_\ell$: indeed, if $\sigma = ((\sigma_0)^{\otimes a_0}, \ldots, (\sigma_m)^{\otimes a_m})$ is a p–simplex and $a_k = \ell$, say, then

$$0 = \delta(c)((\sigma_0)^{\otimes a_0}, \ldots, (\sigma_k)^{\otimes \ell+1}, \ldots, (\sigma_m)^{\otimes a_m}) = (-1)^{a_0+\ldots+a_{k-1}} \cdot c(\sigma),$$

since c is closed and vanishes on simplices with $\ell + 1$ adjacent equal indices. Henceforth, we shall suppose that ℓ is odd. Then consider for each $k \geq 0$ the operator $T_k \colon \check{C}^p(\mathfrak{U}, \mathfrak{U}_B) \to \check{C}^{p-1}(\mathfrak{U}, \mathfrak{U}_B)$ defined on a $(p-1)$–simplex $\tau = ((\tau_0)^{\otimes b_0}, \ldots, (\tau_n)^{\otimes b_n})$ with $\tau_t \neq \tau_{t+1}$ by

$$(T_k c)(\tau) = \begin{cases} (-1)^{b_0+\ldots+b_{k-1}} \cdot c((\tau_0)^{\otimes b_0}, \ldots, (\tau_k)^{\otimes b_k+1}, \ldots, (\tau_n)^{\otimes b_n}), & k \leq n, \\ 0, & \text{else.} \end{cases}$$

Since for a p–simplex σ only those indices with a_t odd contribute to $(\delta T_k c)(\sigma)$ and $\ell+1$ is even, it follows that δT_k maps $\check{C}^p_{\ell+1}$ into itself. To conclude the lemma, let us further denote by $D_k \subseteq \check{C}^p_{\ell+1}$ those cochains that vanish on p–simplices $\sigma = ((\sigma_0)^{\otimes a_0}, \ldots, (\sigma_m)^{\otimes a_m})$ such that $a_j = \ell$ for some $j = 0, \ldots, k-1$. We claim that if $c \in D_k$ is closed, then $c - \delta T_k c$ is contained in D_{k+1}. This will prove the lemma, because $D_0 = \check{C}^p_{\ell+1}$ and $D_p = \check{C}^p_\ell$.

So suppose we are given a cocycle $c \in D_k$ and let σ be a p–simplex such that $a_j = \ell$ for some $j = 0, \ldots, k$. If $j < k$, then

$$\begin{aligned}
0 &= (T_k \delta c)(\sigma) \\
&= (-1)^{a_j+\ldots+a_{k-1}} \cdot c((\sigma_0)^{\otimes a_0}, \ldots, (\sigma_j)^{\otimes \ell-1}, \ldots, (\sigma_k)^{\otimes a_k+1}, \ldots, (\sigma_m)^{\otimes a_m}) \\
&= -(\delta T_k c)(\sigma)
\end{aligned}$$

because c is closed and contained in D_k. If $j = k$, then

$$(\delta T_k c)(\sigma) = (-1)^{a_0+\ldots+a_{k-1}} \cdot (T_k c)((\sigma_0)^{\otimes a_0}, \ldots, (\sigma_k)^{\otimes \ell - 1}, \ldots, (\sigma_m)^{\otimes a_m}),$$

because c vanishes whenever $\ell + 1$ consecutive indices are equal. But by construction

$$(T_k c)((\sigma_0)^{\otimes a_0}, \ldots, (\sigma_k)^{\otimes \ell - 1}, \ldots, (\sigma_m)^{\otimes a_m}) = (-1)^{a_0+\ldots+a_{k-1}} \cdot c(\sigma),$$

whence we have $(c - \delta T_k c)(\sigma) = 0$ in this case too. □

Proof of theorem 8.1. We treat the absolute case $A = \emptyset$, since the relative case only requires minor modifications. Choose a cofinal family $(\mathfrak{S}_\ell)_{\ell \geq 0}$ of Čech covers of $[0, 1]$ with the following properties:

(i) $\mathfrak{S}_\ell$ is defined on the index set $\{1, \ldots, 2^\ell\}$;

(ii) $0 \in (\mathfrak{S}_\ell)_1$, $1 \in (\mathfrak{S}_\ell)_{2^\ell}$, and $(\mathfrak{S}_\ell)_m \cap (\mathfrak{S}_\ell)_{m+1} \neq \emptyset$;

(iii) $(\mathfrak{S}_{\ell+1})_{2k}, (\mathfrak{S}_{\ell+1})_{2k+1} \subseteq (\mathfrak{S}_\ell)_k$;

(iv) $(\mathfrak{S}_\ell)_k \subseteq (\mathfrak{S}_{\ell_0})_{k_0}$ only if $\ell_0 \geq \ell$.

For example, such a family can be constructed recursively by starting with the cover $\mathfrak{S}_0 = \{[0, 1]\}$ and then writing $(\mathfrak{S}_\ell)_k$ as a union of two open intervals $(\mathfrak{S}_{\ell+1})_{2k}$ and $(\mathfrak{S}_{\ell+1})_{2k+1}$ of diameter $3/4 \cdot \operatorname{diam}(\mathfrak{S}_\ell)_k$. Note that we have a canonical refinement projection $\tau_{\ell,\ell+1}$ from $\mathfrak{S}_{\ell+1}$ to $\mathfrak{S}_\ell$ defined by $\tau_{\ell,\ell+1}(2k - 1) = \tau_{\ell,\ell+1}(2k) = k$ for all $k = 1, \ldots, 2^\ell$ and hence also for each $n > 1$ a refinement projection from $\mathfrak{S}_{\ell+n}$ to $\mathfrak{S}_\ell$

$$\tau_{\ell,\ell+n} := \tau_{\ell,\ell+1} \circ \tau_{\ell+1,\ell+2} \circ \ldots \circ \tau_{\ell+n-1,\ell+n}.$$

Having fixed this family, let $\mathfrak{V}$ be a Čech cover of X defined on some index set J and without any non–trivial inclusion relations, that is, such that $X_i \subseteq X_j$ only if $i = j$. Consider the subcategory $\mathcal{D}(\mathfrak{V})$ of Čech covers $\mathfrak{U}$ of $X \times [0, 1]$ such that $\mathfrak{U}$ is defined on an index set $I \subseteq J \times \mathbb{Z}_{\geq 0}$, for each $j \in J$ there exists an integer $\ell(j) = \ell_{\mathfrak{U}}(j) \geq 1$ with $(j, m) \in I$ if and only if $m \in \{1, \ldots, 2^{\ell(j)}\}$, and such that $\mathfrak{U}_{j,m} = \mathfrak{V}_j \times (\mathfrak{S}_{\ell(j)})_m$. Observe that by definition of $\mathcal{D}(\mathfrak{V})$ the cover $\mathfrak{V}$ canonically refines both $(\iota_0)^{-1}\mathfrak{U}$ and $(\iota_1)^{-1}\mathfrak{U}$ via the refinement projections $\alpha, \beta \colon J \to I$ with $\alpha(j) = (j, 1)$ and $\beta(j) = (j, 2^{\ell(j)})$. We will show that $\alpha_* \circ (\iota_0)^*$ and $\beta_* \circ (\iota_1)^*$ induce the same map $\check{\mathrm{H}}^p(\mathfrak{U}; F \times [0, 1]) \to \check{\mathrm{H}}^p(\mathfrak{V}; F)$. This implies the claim, because the category $\mathcal{D}$ whose objects are the objects of $\mathcal{D}(\mathfrak{V})$ for all possible choices of covers $\mathfrak{V}$ of X is a cofinal subcategory, since every Čech cover of $X \times [0, 1]$ admits a refinement by a cover in $\mathcal{D}(\mathfrak{V})$ for some cover $\mathfrak{V}$ of X without non–trivial inclusion relations, $X \times [0, 1]$ being endowed with the product topology and $(\mathfrak{S}_\ell)_{\ell \geq 1}$ being cofinal.

Thus, fix a cover $\mathfrak{V}$ of X, defined on some index set J and without non–trivial inclusion relations. To begin with, note that if $\mathfrak{W} \in \mathcal{D}(\mathfrak{V})$ refines $\mathfrak{U} \in \mathcal{D}(\mathfrak{V})$, then there exists a canonical refinement projection $\gamma_{\mathfrak{U},\mathfrak{W}}$ from $\mathfrak{W}$ to $\mathfrak{U}$. Indeed, in this case for every index (j, m) there is an index (j_0, m_0) such that $\mathfrak{W}_{j,m} \subseteq \mathfrak{U}_{j_0,m_0}$. In particular, $\mathfrak{V}_j \subseteq \mathfrak{V}_{j_0}$, which by choice of $\mathfrak{V}$ is only possible if $j = j_0$, and $(\mathfrak{S}_{\ell_{\mathfrak{W}}(j)})_m \subseteq (\mathfrak{S}_{\ell_{\mathfrak{U}}(j)})_{m_0}$ which is only possible if $\ell_{\mathfrak{W}}(j) \geq \ell_{\mathfrak{U}}(j)$ for all j. Therefore, we may set

$$\gamma_{\mathfrak{U},\mathfrak{W}}(j,m) = \begin{cases} (j,m), & \ell_{\mathfrak{U}}(j) = \ell_{\mathfrak{W}}(j), \\ (j, \tau_{\ell_{\mathfrak{U}}(j),\ell_{\mathfrak{W}}(j)}(m)), & \ell_{\mathfrak{U}}(j) < \ell_{\mathfrak{W}}(j). \end{cases}$$

For the remainder of the proof, we also fix a cover $\mathfrak{U} \in \mathcal{D}(\mathfrak{V})$. Our goal now is to construct an operator $H\colon \check{C}^\bullet(\mathfrak{U}; F \times [0,1]) \to \check{C}^\bullet(\mathfrak{V}; F)$ which satisfies the identity $\alpha_* \circ (\iota_0)^* - \beta_* \circ (\iota_1)^* = \delta H + H\delta$ on the subspace $\check{C}^\bullet_0(\mathfrak{U}; F \times [0,1])$ of all cochains c which vanish on simplices with at least two adjacent equal indices. Since by lemma 8.2 every cocycle c is cohomologous to an element of $\check{C}^\bullet_0(\mathfrak{U}; F \times [0,1])$, this will give the desired result. Thus, let $c \in \check{C}^\bullet(\mathfrak{U}; F \times [0,1])$ be a $(p+1)$–cochain and let σ be a p–simplex of $\mathfrak{V}$. Further suppose that $\mathfrak{W} \in \mathcal{D}(\mathfrak{V})$ is a cover refining $\mathfrak{U}$ and satisfying $\ell_{\mathfrak{W}}(\sigma_0) = \ldots = \ell_{\mathfrak{W}}(\sigma_p) = r$ for some integer $r \geq 1$. By construction of the presheaf $F \times [0,1]$ and because $(\mathfrak{S}_r)_m \cap (\mathfrak{S}_r)_{m+1} \neq \emptyset$, we see that

$$(F \times [0,1])(\mathfrak{W}_{\sigma_0,m} \cap \ldots \cap \mathfrak{W}_{\sigma_j,m} \cap \mathfrak{W}_{\sigma_j,m+1} \cap \ldots \cap \mathfrak{W}_{\sigma_p,m+1}) = F(\mathfrak{V}_\sigma),$$

for all $j = 0, \ldots, p$ and all $m = 1, \ldots, 2^r$. Therefore, it makes sense to set

$$(H_{\mathfrak{W}}c)(\sigma) := \sum_{m=1}^{2^r-1} \sum_{j=0}^{p} (-1)^j \big((\gamma_{\mathfrak{U},\mathfrak{W}})_* c\big)\big((\sigma_0, m), \ldots, (\sigma_j, m), (\sigma_j, m+1), \ldots, (\sigma_p, m+1)\big).$$

Note that a cover $\mathfrak{W}$ with the required properties always exists: indeed, pick any integer $r \geq r(\sigma) := \max\{\ell_{\mathfrak{U}}(\sigma_0), \ldots, \ell_{\mathfrak{U}}(\sigma_p)\}$ and define $\mathfrak{W} = \mathfrak{W}(\sigma, r) \in \mathcal{D}(\mathfrak{V})$ by

$$\mathfrak{W}_{j,m} = \begin{cases} \mathfrak{V}_j \times (\mathfrak{S}_r)_m, & j \in \{\sigma_0, \ldots, \sigma_p\}, \\ \mathfrak{U}_{j,m}, & \text{else.} \end{cases}$$

In general, the expression $(H_{\mathfrak{W}}c)(\sigma)$ depends on the chosen cover $\mathfrak{W}$, whence we let $(Hc)(\sigma) := (H_{\mathfrak{W}(\sigma,r(\sigma))}c)(\sigma)$. If, however, c is a $(p+1)$–cochain in $\check{C}^\bullet_0(\mathfrak{U}; F \times [0,1])$, we claim that $(H_{\mathfrak{W}}c)(\sigma)$ is independent of the specific cover chosen. To see this, first note that the definition only depends on the sets $\mathfrak{W}_{\sigma_j,m}$, which, in turn, only depend on the number $\ell_{\mathfrak{W}}(\sigma_0) = \ldots = \ell_{\mathfrak{W}}(\sigma_p) = r$. Hence, it suffices to show that the definition of $(H_{\mathfrak{W}}c)(\sigma)$ yields the same result for $\mathfrak{W} = \mathfrak{W}(\sigma, r)$ and all integers $r \geq r(\sigma)$. We begin with the choices $r = r(\sigma)$ and $r = r(\sigma) + 1$. Thus, let $\mathfrak{W} = \mathfrak{W}(\sigma, r(\sigma))$ and

$\mathfrak{T} = \mathfrak{W}(\sigma, r(\sigma)+1)$. Using that $\mathfrak{T}$ refines $\mathfrak{W}$ and that $\gamma_{\mathfrak{U},\mathfrak{T}} = \gamma_{\mathfrak{U},\mathfrak{W}} \circ \gamma_{\mathfrak{W},\mathfrak{T}}$, we compute

$$\begin{aligned}
(H_{\mathfrak{T}}c)(\sigma) &= \sum_{k=1}^{2^r}\sum_{j=0}^{p}(-1)^j\big((\gamma_{\mathfrak{U},\mathfrak{T}})_*c\big)\big((\sigma_0,2k-1),\dots,(\sigma_j,2k-1),(\sigma_j,2k),\dots,(\sigma_p,2k)\big)+\\
&\quad\sum_{k=1}^{2^r-1}\sum_{j=0}^{p}(-1)^j\big((\gamma_{\mathfrak{U},\mathfrak{T}})_*c\big)\big((\sigma_0,2k),\dots,(\sigma_j,2k),(\sigma_j,2k+1),\dots,(\sigma_p,2k+1)\big)\\
&= \sum_{k=1}^{2^r}\sum_{j=0}^{p}(-1)^j\big((\gamma_{\mathfrak{U},\mathfrak{W}})_*c\big)\big((\sigma_0,k),\dots,(\sigma_j,k),(\sigma_j,k),\dots,(\sigma_p,k)\big)+\\
&\quad\sum_{k=1}^{2^r-1}\sum_{j=0}^{p}(-1)^j\big((\gamma_{\mathfrak{U},\mathfrak{W}})_*c\big)\big((\sigma_0,k),\dots,(\sigma_j,k),(\sigma_j,k+1),\dots,(\sigma_p,k+1)\big)\\
&= (H_{\mathfrak{W}}c)(\sigma)
\end{aligned}$$

by definition of the refinement projection $\tau_{r,r+1}$ and because c vanishes on simplices with two adjacent equal indices. Inductively we then see that $(H_{\mathfrak{W}(\sigma,r)}(c)) = (H_{\mathfrak{W}(\sigma,r+s)}(c))$ for all $s \geq 0$. We are ready to prove the identity $\beta_* \circ (\iota_1)^* - \alpha_* \circ (\iota_0)^* = \delta \circ H + H \circ \delta$ on $\check{C}_0^\bullet(\mathfrak{U}; F \times [0,1])$. To this end, let $c \in \check{C}_0^\bullet(\mathfrak{U}; F \times [0,1])$ be a p–cochain, σ a p–simplex of $\mathfrak{V}$, and $\mathfrak{W} = \mathfrak{W}(\sigma, r(\sigma))$. A computation similarly to that in the proof of proposition 2.9 shows that

$$\begin{aligned}
&(\delta H_{\mathfrak{W}}c)(\sigma) + (H_{\mathfrak{W}}\delta c)(\sigma)\\
&= \sum_{m=1}^{2^{r(\sigma)}-1}\big((\gamma_{\mathfrak{U},\mathfrak{W}})_*c\big)((\sigma_0,m+1),\dots,(\sigma_p,m+1)) - \big((\gamma_{\mathfrak{U},\mathfrak{W}})_*c\big)((\sigma_0,m),\dots,(\sigma_p,m))\\
&= (\beta_*(\iota_1)^*c)(\sigma) - (\alpha_*(\iota_0)^*c)(\sigma),
\end{aligned}$$

and since we have just seen that $(\delta Hc)(\sigma) = (\delta H_{\mathfrak{W}}c)(\sigma)$, the claimed identity holds. □

Let Y be a topological space and G a presheaf on Y. We say that two morphisms $f, g\colon (X,F) \to (Y,G)$ are *homotopic*, if there is a morphism $H\colon (X \times [0,1], F \times [0,1]) \to (Y,G)$ such that $f = H \circ \iota_0$ and $g = H \circ \iota_1$ as morphisms $(X,F) \to (Y,G)$. As an immediate consequence of theorem 8.1 we have

Corollary 8.3. *If $f, g\colon (X,F) \to (Y,G)$ are homotopic morphisms, then $f^* = g^*$.*

Example 8.4. For us, the most important case of the previous corollary is that of constant coefficients: if $f, g\colon X \to Y$ are two homotopic continuous maps, then for any R–module M the induced maps $\check{\mathrm{H}}^\bullet(Y; \underline{M}) \to \check{\mathrm{H}}^\bullet(X; \underline{M})$ coincide. In particular, if X and Y are homotopy equivalent, then $\check{\mathrm{H}}^\bullet(Y; \underline{M}) \cong \check{\mathrm{H}}^\bullet(X; \underline{M})$. Specializing even more, we see that any space X which is homotopy equivalent to a point has

$$\check{\mathrm{H}}^p(X; \underline{M}) = \begin{cases} M, & p = 0,\\ 0, & p > 0,\end{cases}$$

by example 4.4.

Combining the previous theorem with the long exact sequence of a pair yields

Theorem 8.5. *Let X be a normal space and $F = \underline{M}$ a constant presheaf on X. If A is a closed subspace of X which is a (strong) deformation retract of some open neighborhood U of A, then, if we denote by $\pi\colon X \to X/A$ the canonical quotient map and by $\iota\colon A \hookrightarrow X$ the inclusion, there is an exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \check{\mathrm{H}}^0(X/A, A/A) \arrow[r] & \check{\mathrm{H}}^0(X) \arrow[r, "\iota^*"] & \check{\mathrm{H}}^0(A) \arrow[r] & \check{\mathrm{H}}^1(X/A) \arrow[r, "\pi^*"] & \check{\mathrm{H}}^1(X) \arrow[r, "\iota^*"] & \check{\mathrm{H}}^1(A) \arrow[r] & \dots
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & \check{\mathrm{H}}^p(X/A) \arrow[r, "\pi^*"] & \check{\mathrm{H}}^p(X) \arrow[r, "\iota^*"] & \check{\mathrm{H}}^p(A) \arrow[r] & \check{\mathrm{H}}^{p+1}(X/A) \arrow[r] & \dots
\end{tikzcd}$$

Proof. Since we are considering coefficients in a constant presheaf, the long exact sequence of the pair (X, A) reads

$$\dots \to \check{\mathrm{H}}^p(X, A) \to \check{\mathrm{H}}^p(X) \xrightarrow{\iota^*} \check{\mathrm{H}}^p(A) \xrightarrow{\partial_{(X,A)}} \check{\mathrm{H}}^{p+1}(X, A) \to \dots$$

Then consider the commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(X/A, A/A) \arrow[d, "\pi^*"] & \check{\mathrm{H}}^p(X/A, U/A) \arrow[l] \arrow[r] \arrow[d, "\pi^*"] & \check{\mathrm{H}}^p(X/A - A/A, U/A - A/A) \arrow[d, "\pi^*"'] \\
\check{\mathrm{H}}^p(X, A) & \check{\mathrm{H}}^p(X, U) \arrow[l] \arrow[r] & \check{\mathrm{H}}^p(X - A, U - A).
\end{tikzcd}$$

All horizontal maps are induced by inclusions of pairs of spaces and are isomorphisms: the horizontal maps on the right hand side are so by excision, cf. example 7.3, and the maps on the left portion of the diagram are because A and U are homotopy equivalent, see corollary 8.3. But the map of pairs $\pi\colon (X - A, U - A) \to (X/A - A/A, U/A - A/A)$ is, by definition of X/A, a homeomorphism, so all vertical maps in the above diagram must be isomorphisms too. Now the long exact sequence of the pair $(X/A, A/A)$ shows that $\check{\mathrm{H}}^\bullet(X/A, A/A) \cong \check{\mathrm{H}}^\bullet(X/A)$ for all $p > 0$, because A/A has trivial cohomology in positive degrees by example 4.4, and therefore the sequence in question is exact for all $p > 0$. That the beginning portion is exact as well follows because the long exact sequence of pairs is natural, so that

$$\begin{tikzcd}
 & \check{\mathrm{H}}^0(X/A, A/A) \arrow[r] \arrow[d, "\cong"', "\pi^*"] & \check{\mathrm{H}}^0(X/A) \arrow[d, "\pi^*"] & \\
0 \arrow[r] & \check{\mathrm{H}}^0(X, A) \arrow[r] & \check{\mathrm{H}}^0(X) \arrow[r] & \check{\mathrm{H}}^0(A)
\end{tikzcd}$$

is a commutative diagram whose lower row is exact. □

Example 8.6. Let $D^n \subseteq \mathbb{R}^n$, $n \geq 1$, be the closed ball of length 1, centered around the origin. The $(n-1)$–sphere $S^{n-1} \subseteq D^n$ is closed and a deformation retract of an open neighborhood (take, for example, $D^n - \{0\}$), so the previous theorem applies. Note that

D^n/S^{n-1} is homeomorphic to S^n and that D^n is homotopy equivalent to a point, and therefore the exact sequence of theorem 8.5 yields the exact sequences (with coefficients in $\underline{M}$)

$$0 = \check{\mathrm{H}}^p(S^n, *) \to \check{\mathrm{H}}^p(D^n) \to \check{\mathrm{H}}^p(S^{n-1}) \to \check{\mathrm{H}}^{p+1}(S^n) \to 0;$$

note that $\check{\mathrm{H}}^p(S^n, *) = 0$, because S^n is connected. We show by induction on n that

$$\check{\mathrm{H}}^p(S^n) \cong \begin{cases} M, & p = 0, n \\ 0, & \text{else.} \end{cases}$$

For the induction base $n = 1$, fix an identification $\varphi\colon \check{\mathrm{H}}^0(D^1) \cong M$. Since $i_x\colon \{x\} \hookrightarrow D^1$ induces an isomorphism for every point $x \in D^1$, we obtain induced identifications $\varphi \circ ((i_x)^*)^{-1}\colon \check{\mathrm{H}}^0(\{x\}) \to M$. On the other hand, $S^0 = \{-1, 1\}$, so by example 5.1 the inclusion maps $\{\pm 1\} \hookrightarrow S^0$ induce an isomorphism $\check{\mathrm{H}}^p(S^0) \cong \check{\mathrm{H}}^p(\{-1\}) \oplus \check{\mathrm{H}}^p(\{1\})$ which combined with our chosen identifications makes the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^0(D^1) \arrow[r] \arrow[d, "\cong"'] & \check{\mathrm{H}}^0(S^0) \arrow[d, "\cong"'] \\
M \arrow[r, "\Delta\colon m \mapsto (m,m)"] & M \oplus M
\end{tikzcd}$$

commute. Thus, $\check{\mathrm{H}}^1(S^1) \cong (M \oplus M)/\Delta(M) \cong M$, and since both D^n and S^0 have trivial cohomology in positive degrees, also $\check{\mathrm{H}}^p(S^1) = 0$ for $p > 2$. As S^1 is connected and hence $\check{\mathrm{H}}^0(S^1) \cong M$ by example 4.5, the induction base is established.

Now consider the case $n > 1$. Then $\check{\mathrm{H}}^0(D^n) \to \check{\mathrm{H}}^0(S^{n-1})$ is an isomorphism, and hence $\check{\mathrm{H}}^p(S^n) \to \check{\mathrm{H}}^{p+1}(S^{n+1})$ too is an isomorphism. Using the induction hypothesis and that S^n is connected, we hence see that $\check{\mathrm{H}}^p(S^n)$ is trivial unless $p = 0$ or $p = n$, in which case $\check{\mathrm{H}}^p(S^n) \cong M$.

Example 8.7. We can also use theorem 8.5 to compute $\check{\mathrm{H}}^\bullet(\mathbb{C}P^n) = \check{\mathrm{H}}^\bullet(\mathbb{C}P^n; \underline{M})$. The claim here is that

$$\check{\mathrm{H}}^p(\mathbb{C}P^n) \cong \begin{cases} M, & p \text{ even,} \\ 0, & p \text{ odd.} \end{cases}$$

In fact, we shall show that for all $n \geq 1$ the embedding

$$f\colon \mathbb{C}P^{n-1} \to \mathbb{C}P^n,\ [z_0 : \ldots : z_{n-1}] \mapsto [z_0 : \ldots : z_{n-1} : 0],$$

induces isomorphisms $f^*\colon \check{\mathrm{H}}^p(\mathbb{C}P^n) \to \check{\mathrm{H}}^p(\mathbb{C}P^{n-1})$ in degrees $p < 2n$. The proof is again by induction on $n \geq 1$, the induction base being true because $\mathbb{C}P^1$ is homeomorphic to S^2. Now observe that $\operatorname{im}(f) \subseteq \mathbb{C}P^n$ is a deformation retract of $\mathbb{C}P^n - \{[0 : \ldots : 0 : 1]\}$ and that the surjective morphism

$$D^{2n} \subseteq \mathbb{C}^n \to \mathbb{C}P^n,\ z = (z_0, \ldots, z_{n-1}) \mapsto \left[z_0 : \ldots : z_{n-1} : \sqrt{1 - |z|^2}\right],$$

induces a homeomorphism $S^{2n} \cong D^{2n}/S^{2n-1} \to \mathbb{C}P^n/\operatorname{im}(f)$. Thus, we have exact sequences

$$0 = \check{\mathrm{H}}^{2k+1}(\mathbb{C}P^n/\operatorname{im}(f)) \to \check{\mathrm{H}}^{2k+1}(\mathbb{C}P^n) \to \check{\mathrm{H}}^{2k+1}(\operatorname{im}(f)) = 0,$$

so $\check{\mathrm{H}}^\bullet(\mathbb{C}P^n)$ vanishes in odd degrees. Also, for $k > 0$, we have exact sequences

$$0 \to \check{\mathrm{H}}^{2k}(S^{2n}) \to \check{\mathrm{H}}^{2k}(\mathbb{C}P^n) \to \check{\mathrm{H}}^{2k}(\operatorname{im}(f)) \to 0,$$

and this sequence shows that for $k < n$ we have $\check{\mathrm{H}}^{2k}(\mathbb{C}P^n) \cong \check{\mathrm{H}}^{2k}(\operatorname{im}(f))$, that is, that f^* is an isomorphism in degrees $p < 2n$; and that $\check{\mathrm{H}}^{2n}(\mathbb{C}P^n) \cong M$. Since the map f factors through the inclusion $\operatorname{im}(f) \to \mathbb{C}P^n$, the claim follows.

9. Tautness and change of coefficients

Let X be a topological space and $\eta\colon F \to G$ a morphism of presheaves on X. We can consider η as a cohomorphism for the identity morphism $\mathrm{id}_X\colon X \to X$, and so obtain a morphism $(\mathrm{id}_X, \eta)\colon (X, G) \to (X, F)$. Write $\eta_* = (\mathrm{id}_X, \eta)^*\colon \check{\mathrm{H}}^\bullet(X; F) \to \check{\mathrm{H}}^\bullet(X; G)$ for the induced morphism on the level of cohomology.

Theorem 9.1. *Every short exact sequence* $0 \to F \xrightarrow{\eta} F' \xrightarrow{\mu} F'' \to 0$ *of presheaves on* X *induces a long exact sequence*

$$\ldots \to \check{\mathrm{H}}^p(X; F) \xrightarrow{\eta_*} \check{\mathrm{H}}^p(X; F') \xrightarrow{\mu_*} \check{\mathrm{H}}^p(X; F'') \to \check{\mathrm{H}}^{p+1}(X; F) \xrightarrow{\eta_*} \ldots$$

Proof. For ever cover $\mathfrak{U}$ of X we have a short exact sequence of chain complexes

$$0 \to \check{\mathrm{H}}^\bullet(\mathfrak{U}; F) \xrightarrow{\eta_*} \check{\mathrm{H}}^\bullet(\mathfrak{U}; F') \xrightarrow{\mu_*} \check{\mathrm{H}}^\bullet(\mathfrak{U}; F'') \to 0$$

and hence also a corresponding long exact sequence in cohomology:

$$\ldots \to \check{\mathrm{H}}^p(\mathfrak{U}; F) \xrightarrow{\eta_*} \check{\mathrm{H}}^p(\mathfrak{U}; F') \xrightarrow{\mu_*} \check{\mathrm{H}}^p(\mathfrak{U}; F'') \to \check{\mathrm{H}}^{p+1}(\mathfrak{U}; F) \xrightarrow{\eta_*} \ldots$$

Passing to the limit then gives the desired result. □

Proposition 9.2. *Let* X *be a paracompact space and* F *a presheaf on* X *all of whose stalks* F_x *are trivial. Then* $\check{\mathrm{H}}^\bullet(X; F) = 0$.

Proof. By definition, every open cover of X admits a refinement by a locally finite open cover, so it suffices to show that for every such cover $\mathfrak{U}$ every cocycle $c \in \check{C}^p(\mathfrak{U}; F)$ eventually becomes zero. To this end, let $x \in X$ and let σ be a p–simplex such that $x \in \mathfrak{U}_\sigma$. Since $F_x = 0$ and hence $c(\sigma)|_x = 0$, we see that there must be an open neighborhood V of x such that already $c(\sigma)|_V = 0$. But $\mathfrak{U}$ is locally finite, so there is an open neighborhood V_x of x meeting only finitely many $\mathfrak{U}_i$. In particular, the set of p–simplices σ with $\mathfrak{U}_\sigma \cap V_x \neq \emptyset$ is finite. Shrinking V_x if necessary, we thus may assume that $c(\sigma)|_{V_x} = 0$ for all simplices σ, because for those simplices σ not meeting V_x we have $F(\mathfrak{U}_\sigma \cap V_x) = 0$ anyway. Now let $\mathfrak{W}$ be a cover of X refining both $\mathfrak{U}$ and the cover $\{V_x \,|\, x \in X\}$, and α a refinement projection from $\mathfrak{U}$ to $\mathfrak{W}$. If τ is a p–simplex of $\mathfrak{W}$, let $x \in X$ be a point such that $\mathfrak{W}_\tau \subseteq V_x$. Then $c(\alpha(\tau_0), \ldots, \alpha(\tau_p))|_{V_x} = 0$ by construction, and hence also $(\alpha_* c)(\tau) = c(\alpha(\tau))|_{\mathfrak{W}_\tau} = 0$. □

Corollary 9.3. *Let X be a paracompact space and $\eta\colon F \to G$ a morphism of presheaves on X inducing isomorphisms $\eta_x\colon F_x \to G_x$ for all $x \in X$. Then also $\eta_*\colon \check{\mathrm{H}}^\bullet(X;F) \to \check{\mathrm{H}}^\bullet(X;G)$ is an isomorphism.*

Proof. Consider the presheaves $\ker\eta$, $\operatorname{im}\eta$, and $\operatorname{coker}\eta$ on X given by $(\ker\eta)(U) := \ker\eta_U$, $(\operatorname{im}\eta)(U) := \operatorname{im}\eta_U$, and $(\operatorname{coker}\eta)(U) := \operatorname{coker}\eta_U$ for all open subsets $U \subseteq X$. Then η factors through the inclusion $\iota\colon \operatorname{im}\eta \hookrightarrow G$, giving a morphism of presheaves $\overline{\eta}\colon F \to \operatorname{im}\eta$, and we obtain an exact sequence of presheaves

$$0 \to \operatorname{im}\eta \xrightarrow{\iota} G \to \operatorname{coker}\eta \to 0$$

However, since $\eta_x\colon F_x \to G_x$ is an isomorphism, it follows that every stalk $(\operatorname{coker}\eta)_x$ must be trivial: any element in $(\operatorname{coker}\eta)_x$ arises as the restriction $c|_x$ of some element $c \in (\operatorname{coker}\eta)(U)$ for some open neighborhood U of x. By definition, $c = c' + \operatorname{im}\eta_U$ for some element $c' \in G(U)$, and since η_x is an isomorphism, after possibly shrinking U we see that there is an element $d \in F(U)$ such that $\eta_U(d)|_x = (c')|_x$. Shrinking U once more if necessary, we may assume that $\eta_U(d) = c'$, and so $c = 0$. Therefore, $(\operatorname{coker}\eta)_x = 0$. Now according to proposition 9.2, the long exact sequence induced by this short exact sequence of presheaves (theorem 9.1) reduces to isomorphisms $\iota_*\colon \check{\mathrm{H}}^\bullet(X;\operatorname{im}\eta) \to \check{\mathrm{H}}^\bullet(X;G)$. In a similar way we see that the short exact sequence of presheaves

$$0 \to \ker\eta \to F \xrightarrow{\overline{\eta}} \operatorname{im}\eta \to 0.$$

induces an isomorphism $(\overline{\eta})_*\colon \check{\mathrm{H}}^\bullet(X;F) \to \check{\mathrm{H}}^\bullet(X;\operatorname{im}\eta)$, and as $\eta = \iota \circ \overline{\eta}$, also η^* must be an isomorphism. $\square$

Corollary 9.4. *Let X be a paracompact space, F a presheaf on X, and $A \subseteq X$ a closed subspace. Consider the presheaf F^A on X given by*

$$(F^A)(U) := \begin{cases} F(U), & A \cap U \neq \emptyset, \\ 0, & \textit{else.} \end{cases}$$

Then the canonically induced morphism $\check{\mathrm{H}}^\bullet(X;F^A) \to \check{\mathrm{H}}^\bullet(A;F|_A)$ is bijective.

Proof. Let G be a presheaf on A and denote by $\iota\colon A \hookrightarrow X$ the canonical inclusion. We claim that the canonical morphism $\check{\mathrm{H}}^\bullet(X;\iota_*G) \to \check{\mathrm{H}}^\bullet(A;G)$ which is induced by ι and the identity cohomomorphism $k\colon \iota_*G \to \iota_*G$ is bijective. In fact, the functor taking a Čech cover $\mathfrak{U}$ of X to the cover $\iota^{-1}\mathfrak{U}$ of A is cofinal, because any cover $\mathfrak{V}$ of A can be extended to a cover of X by adding the set $X - A$. Moreover, for every Čech cover $\mathfrak{U}$ of X the morphism

$$\check{\mathrm{H}}^p(\mathfrak{U};\iota_*G) \to \check{\mathrm{H}}^p(\iota^{-1}\mathfrak{U};G)$$

induced by (ι,k) is an isomorphism, since for a simplex σ of $\mathfrak{U}$ we have $(\iota_*G)(\mathfrak{U}_\sigma) = G((\iota^{-1}\mathfrak{U})_\sigma)$. Hence, $\check{\mathrm{H}}^\bullet(X;\iota_*G) \to \check{\mathrm{H}}^\bullet(A;G)$ is an isomorphism as well. Applying this reasoning to $G = F|_A$, we see that the morphism $\check{\mathrm{H}}^\bullet(X;\iota_*(F|_A)) \to \check{\mathrm{H}}^\bullet(A;F|_A)$ is an isomorphism. But the canonical morphism of presheaves $F^A \to \iota_*(F|_A)$ is a stalkwise isomorphism, because A is closed. Furthermore $\iota\colon (A, F|_A) \to (X, F^A)$ factors through the morphism $(X, \iota_*(F|_A)) \to (X, F^A)$, so by corollary 9.3 the composite map $\check{\mathrm{H}}^\bullet(X; F^A) \to \check{\mathrm{H}}^\bullet(A; F|_A)$ is an isomorphism too. □

Theorem 9.5 (Tautness)**.** *Let X be a paracompact Hausdorff space, F a presheaf on X, and A a closed subspace of X. Then the inclusion induced morphism*

$$\varinjlim_N \check{\mathrm{H}}^p(N; F|_N) \to \check{\mathrm{H}}^p(A; F|_A)$$

is an isomorphism for all p, where the limit ranges over all closed neighborhoods N of A (that is, closed subspaces N whose interiors contain A).

Proof. For every closed neighborhood N of A we have, by corollary 9.4, a commuative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(X; F^N) \arrow[r] \arrow[d, "\cong"'] & \check{\mathrm{H}}^p(X; F^A) \arrow[d, "\cong"] \\
\check{\mathrm{H}}^p(N; F|_N) \arrow[r] & \check{\mathrm{H}}^p(A; F|_A)
\end{tikzcd}$$

so it will suffice to show that $\varinjlim_N \check{\mathrm{H}}^p(X; F^N) \to \check{\mathrm{H}}^p(X; F^A)$ is an isomorphism. Consider the presheaf G on X given by

$$G(U) := \varinjlim_N F^N(U),$$

where the limit again ranges over all closed neighborhoods N of A. For every such neighborhood N denote by $\kappa_N\colon F^N \to G$ the limit map. Our first claim is that for every locally finite Čech cover $\mathfrak{U}$ of X the morphism κ making the diagrams

$$\begin{tikzcd}
\check{C}^p(\mathfrak{U}; F^N) \arrow[d] \arrow[rd, "(\kappa_N)_*"] & \\
C^p := \varinjlim_M \check{C}^p(\mathfrak{U}; F^M) \arrow[r, "\kappa"] & \check{C}^p(\mathfrak{U}; G)
\end{tikzcd}$$

commute for all closed neighborhoods N of A is in fact an isomorphism. Surjectivity of κ is easily verified: if $c \in \check{C}^p(\mathfrak{U}; G)$ is arbitrary and σ is a p–simplex of $\mathfrak{U}$, then by definition of G there is some closed neighborhood N of A such that $(\kappa_N)_{\mathfrak{U}_\sigma}(d_\sigma) = c(\sigma)$ for some element $d_\sigma \in F^N(\mathfrak{U}_\sigma)$. Considering each such element d_σ as an element in $F(\mathfrak{U}_\sigma)$, we can define a cochain $d \in \check{C}^p(\mathfrak{U}; F)$ by $d(\sigma) := d_\sigma$, and this cochain satisfies $(\kappa_X)_*(d) = c$, because κ_X factors through κ_N for every closed neighborhood N of A. Now let us show that κ is injective. According to remark 3.3, this will be the case if we can show that given a closed neighborhood N of A every cochain $c \in \check{C}^p(\mathfrak{U}; F^N)$ with $(\kappa_N)_*(c) = 0$ eventually becomes zero. To see this, let Σ be those p–simplices σ with

$c(\sigma) \neq 0$ and define $M := N - (\bigcup_{\sigma\in\Sigma} \mathfrak{U}_\sigma)$. This is a closed set and we claim that it is a neighborhood of A. Pick $a \in A$ and choose an open neighborhood V of a meeting only finitely many sets $\mathfrak{U}_i$; this is possible, because $\mathfrak{U}$ is assumed locally finite. In particular, the subset $\Sigma' \subseteq \Sigma$ of simplices σ with $V \cap \mathfrak{U}_\sigma \neq \emptyset$ is finite. Since $(\kappa_N)_*(c) = 0$ there exists for each $\sigma \in \Sigma'$ a closed neighborhood N_σ of A such that $c(\sigma)$ maps to zero under the canonical map $F^N(\mathfrak{U}_\sigma) \to F^{N_\sigma}(\mathfrak{U}_\sigma)$; and because $c(\sigma) \neq 0$ this is only possible if $F^{N_\sigma}(\mathfrak{U}_\sigma) = 0$, that is, if $\mathfrak{U}_\sigma$ does not meet N_σ. Then $W = N \cap V \cap \bigcap_{\sigma\in\Sigma'} N_\sigma$ is an intersection of finitely many neighborhoods of a, whence W is a neighborhood of a as well which by construction does not meet $\bigcup_{\sigma\in\Sigma} \mathfrak{U}_\sigma$. Thus, $W \subseteq M$ and a is an interior point. Furthermore, $F^M(\mathfrak{U}_\sigma) = 0$ for all $\sigma \in \Sigma$, so c is mapped to the zero cochain under the map $\check{C}^p(\mathfrak{U}; F^N) \to \check{C}^p(\mathfrak{U}; F^M)$ induced by the canonical map $F^N \to F^M$. It follows that κ is injective.

Next, let us show that G and F^A have the same stalks. Namely, if $a \in A$, then for every closed neighborhood N of a and every open neighborhood U of a we have $F^N(U) = F^A(U) = F(U)$, whence $G_a = (F^A)_a = F_a$ in this case. If $x \notin A$, then, since X is paracompact Hausdorff and hence normal (see e. g. [4, Theorem I.12.5]), we find open neighborhoods V of x and W of A intersecting trivially. Then $N := X - V$ is a closed neighborhood of A, because $W \subseteq N$, and $F^N(V) = 0$. Hence also $G(V) = 0$ and consequently $G_x = 0$. Therefore, the canonical map $G \to F^A$ is a stalkwise isomorphism and the induced map $\check{\mathrm{H}}^p(X; G) \to \check{\mathrm{H}}^p(X; F^A)$ is an isomorphism by corollary 9.3.

The remainder of the proof is mostly algebraic. First, observe that the boundary maps $\delta_N \colon \check{C}^p(\mathfrak{U}; F^N) \to \check{C}^{p+1}(\mathfrak{U}; F^N)$ induce a differential δ on $C^\bullet$ with respect to which κ as well as the limit maps $\check{C}^p(\mathfrak{U}; F^N) \to C^p$ become cochain maps. Next, note that the exact sequences

$$0 \to \ker \delta_N \to \check{C}^p(\mathfrak{U}; F^N) \xrightarrow{\delta_N} \operatorname{im} \delta_N \to 0$$

remain exact upon passage to the limit, so the maps $\varinjlim_N \ker \delta_N \to \ker \delta$ and $\varinjlim_N \operatorname{im} \delta_N \to \operatorname{im} \delta$ induced by the maps into the limit C^p are in fact isomorphisms. Therefore, the composition

$$\varinjlim_N \check{\mathrm{H}}^p(\mathfrak{U}; F^N) \xrightarrow{\sim} \mathrm{H}^p(C^\bullet, \delta) \xrightarrow{\sim} \check{\mathrm{H}}^p(\mathfrak{U}; G)$$

which is just the limit map induced by the canonical maps $\check{\mathrm{H}}^p(\mathfrak{U}; F^N) \to \check{\mathrm{H}}^p(\mathfrak{U}; G)$, is an isomorphism. The claim of the theorem now follows upon noticing that $\varinjlim_N \check{\mathrm{H}}^p(X; F^N)$ is a limit for the functor $\mathfrak{U} \mapsto \varinjlim_N \check{\mathrm{H}}^p(\mathfrak{U}; F^N)$. □

10. Mayer–Vietoris

Another way of computing the cohomology of a space is in terms of a cover by two subsets. In the following, we only consider cohomology with values in a constant presheaf $\underline{M}$.

Theorem 10.1. *Let X be a topological space and $A, B \subseteq X$ subsets whose interiors cover X, that is, such that $X = \mathring{A} \cup \mathring{B}$. Further suppose that at least one of the following conditions is satisfied:*

(i) A and B are closed; or

(ii) X is normal.

Then the inclusion induced maps $k_A\colon \check{\mathrm{H}}^\bullet(X,A) \to \check{\mathrm{H}}^\bullet(B, A\cap B)$ and $k_B\colon \check{\mathrm{H}}^\bullet(X,B) \to \check{\mathrm{H}}^\bullet(A, A\cap B)$ are isomorphisms and the Mayer–Vietoris sequence

$$\ldots \to \check{\mathrm{H}}^p(X) \xrightarrow{(i_1)^*\oplus(i_2)^*} \check{\mathrm{H}}^p(A) \oplus \check{\mathrm{H}}^p(B) \xrightarrow{(j_1)^*-(j_2)^*} \check{\mathrm{H}}^p(A\cap B) \xrightarrow{\partial} \check{\mathrm{H}}^{p+1}(X) \to \ldots$$

is exact. The map ∂ is the composition $\partial = q_{(X,A)} \circ (k_A)^{-1} \circ \partial_{(B,A\cap B)}$, where $\partial_{(B,A\cap B)}$ is the connecting morphism in the long exact sequence of the pair $(B, A\cap B)$ and $q_{(X,A)}$ is induced by the inclusion of the relative subcomplex of the pair (X,A). All other maps appearing in the sequence above are induced by inclusions.

Proof. In order to prove that k_A is an isomorphism, we just need to observe that $\overline{X-B} \subseteq \mathring{A}$: if $x \in \overline{X-B}$ was not contained in $\mathring{A}$, then, as the interiors of A and B cover X, necessarily $x \in \mathring{B}$, which is impossible, because $\mathring{B}$ is an open set not meeting $X-B$. Now under the assumptions of the theorem the pair $(A, X-B)$ satisfies the requirements of excision (theorem 7.1), because either X is normal, and then $(A, X-B)$ is excisive by example 7.3, or B is closed, and then $X-B$ is open. In any case, k_A is an isomorphism, and for the same reasons k_B is too.

Exactness at $\check{\mathrm{H}}^p(A) \oplus \check{\mathrm{H}}^p(B)$. Suppose that $a \in \check{\mathrm{H}}^p(X)$ and $b \in \check{\mathrm{H}}^p(B)$ are such that $a|_{A\cap B} = b|_{A\cap B}$, where we write $a|_{A\cap B}$ instead of $(j_1)^*(a)$ and similarly for $(j_2)^*(b)$. By naturality of long exact sequences of pairs, we have a commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(X,A) \arrow[r, "q_{(X,A)}"] \arrow[d, "k_A"'] & \check{\mathrm{H}}^p(X) \arrow[r, "(i_1)^*"] \arrow[d, "(i_2)^*"'] & \check{\mathrm{H}}^p(A) \arrow[r, "\partial_{(X,A)}"] \arrow[d, "(j_1)^*"] & \check{\mathrm{H}}^{p+1}(X,A) \arrow[d, "k_A"] \\
\check{\mathrm{H}}^p(B, A\cap B) \arrow[r, "q_{(B,A\cap B)}"'] & \check{\mathrm{H}}^p(B) \arrow[r, "(j_2)^*"'] & \check{\mathrm{H}}^p(A\cap B) \arrow[r, "\partial_{(B,A\cap B)}"'] & \check{\mathrm{H}}^{p+1}(B, A\cap B)
\end{tikzcd}$$

with exact rows. As k_A is an isomorphism, we have $a \in \ker \partial_{(X,A)}$ and there exists an element $x_0 \in \check{\mathrm{H}}^p(X)$ such that $(x_0)|_A = a$. In particular, $((x_0)|_B - b)|_{A\cap B} = 0$, and so we find $x_1 \in \operatorname{im} q_{(X,A)}$ with $(x_1)|_B = (x_0)|_B - b$. Setting $x := x_0 + x_1$, we see that

$$x|_A = (x_0)|_A = a \text{ and } x|_B = (x_0)|_B + (x_1)|_B = b;$$

in other words, the pair (a,b) is contained in the image of $(i_1)^* \oplus (i_2)^*$. The converse inclusion is immediate.

Exactness at $\check{\mathrm{H}}^p(X)$. Let $x \in \check{\mathrm{H}}^p(X)$ be such that $x|_A = 0$ and $x|_B = 0$, and consider the commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^{p-1}(A \cap B) \arrow[d, "\partial_{(B,A\cap B)}"] & & \\
\check{\mathrm{H}}^{p}(B, A \cap B) \arrow[r, "q_{(B,A\cap B)}"] \arrow[d, "(k_A)^{-1}"] & \check{\mathrm{H}}^{p}(B) & \\
\check{\mathrm{H}}^{p}(X, A) \arrow[r, "q_{(X,A)}"] & \check{\mathrm{H}}^{p}(X) \arrow[u, "(i_2)^*"'] \arrow[r, "(i_1)^*"] & \check{\mathrm{H}}^{p}(A)
\end{tikzcd}$$

whose rows are parts of the long exact sequences of the pairs (X, A) and $(B, A \cap B)$. Since $x|_A = 0$, we find $y \in \check{\mathrm{H}}^p(B, A \cap B)$ such that $x = q_{(X,A)}(k_A(y))$, and since the diagram commutes, we see that $q_{(B,A\cap B)}(y) = x|_B = 0$. Therefore, $y \in \operatorname{im} \partial_{(B,A\cap B)}$, and then $x \in \operatorname{im} \partial$ too. Conversly, if we start with an element $z \in \check{\mathrm{H}}^{p-1}(A \cap B)$, then the above diagram readily shows that $\partial(z)|_A = 0$, because $(i_1)^* \circ q_{(X,A)} = 0$ and also $\partial(z)|_B = 0$, because $q_{(B,A\cap B)} \circ \partial_{(B,A\cap B)} = 0$.

Exactness at $\check{\mathrm{H}}^p(A \cap B)$. As a preliminary result, let us show that the map

$$\begin{aligned} \varphi \colon \check{\mathrm{H}}^p(X, A) \oplus \check{\mathrm{H}}^p(X, B) &\to \check{\mathrm{H}}^p(X, A \cap B) \\ (x, y) &\mapsto x|_{(X,A\cap B)} + y|_{(X,A\cap B)}, \end{aligned}$$

is an isomorphism. Injectivity is easily seen to hold, for if $\varphi(x, y) = 0$, then also

$$0 = \varphi(x, y)|_{(B,A\cap B)} = x|_{(B,A\cap B)} + y|_{(B,A\cap B)} = k_A(x),$$

because the inclusion induced map $\check{\mathrm{H}}^p(X, B) \to \check{\mathrm{H}}^p(B, A \cap B)$ factors through the inclusion induced map $0 = \check{\mathrm{H}}^p(B, B) \to \check{\mathrm{H}}^p(B, A \cap B)$. Since k_A is an isomorphism, $x = 0$. In a similar fashion one shows that $y = 0$. To see that φ is surjective, note that for any $x \in \check{\mathrm{H}}^p(X, A \cap B)$ we can find elements $a \in \check{\mathrm{H}}^p(X, A)$ and $z \in \check{\mathrm{H}}^p(X, B)$ such that $x|_{(A,A\cap B)} = b|_{(A,A\cap B)}$ and $x|_{(B,A\cap B)} = a|_{(B,A\cap B)}$, since k_A and k_B are isomorphisms. Then $z = x - a|_{(X,A\cap B)} - b|_{(X,A\cap B)}$ satisfies $z|_{(A,A\cap B)} = 0$ and $z|_{(B,A\cap B)} = 0$, and so it will suffice to show that $z = 0$.

Thus, assume $z \in \check{\mathrm{H}}^p(X, A \cap B)$ is such that $z|_{(A,A\cap B)} = 0$ and $z|_{(B,A\cap B)} = 0$, and consider the commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^{p-1}(A) \arrow[r, "\partial_{(X,A)}"] \arrow[d] & \check{\mathrm{H}}^{p}(X, A) \arrow[d] & \check{\mathrm{H}}^{p}(X, B) \arrow[dl] \arrow[d] \\
\check{\mathrm{H}}^{p-1}(A \cap B) \arrow[r, "\partial_{(X,A\cap B)}"] \arrow[d, "\partial_{(A,A\cap B)}"'] & \check{\mathrm{H}}^{p}(X, A \cap B) \arrow[r] \arrow[dl] \arrow[d] & \check{\mathrm{H}}^{p}(X) \arrow[d] \\
\check{\mathrm{H}}^{p}(A, A \cap B) & \check{\mathrm{H}}^{p}(B, A \cap B) \arrow[r] & \check{\mathrm{H}}^{p}(B)
\end{tikzcd}$$

It shows that also $z|_B = 0$ and hence there exists $x_B \in \check{\mathrm{H}}^p(X, B)$ such that $z|_X = x_B|_X$. In particular, z and $x_B|_{(X,A\cap B)}$ only differ by some element $\partial_{(X,A\cap B)}(u_B)$ with $u_B \in$

$\check{\mathrm{H}}^{p-1}(A \cap B)$. Similarly, we see that $z = x_A|_{(X,A\cap B)} + \partial_{(X,A\cap B)}(u_A)$ for some element $x_A \in \check{\mathrm{H}}^p(X,A)$ and some element $u_A \in \check{\mathrm{H}}^{p-1}(A \cap B)$. Therefore,

$$\partial_{(A,A\cap B)}(u_A) = \partial_{(X,A\cap B)}(u_A)|_{(A,A\cap B)} = z|_{A,A\cap B)} - x_A|_{(A,A\cap B)} = 0,$$

because $x_A \in \check{\mathrm{H}}^p(X,A)$. This means that u_A lifts to an element $w_A \in \check{\mathrm{H}}^{p-1}(A)$, i. e. $w_A|_{(A\cap B)} = u_A$. Similarly, we see that u_B lifts to an element $w_B \in \check{\mathrm{H}}^{p-1}(B)$. Now we compute

$$\varphi\big(x_A + \partial_{(X,A)}(w_A), -x_B - \partial_{(X,B)}(w_B)\big) = z - z = 0,$$

and since φ is injective, this is only possible if $x_A + \partial_{(X,A)}(w_A) = 0$. But then $z = 0$ too, and hence φ is surjective.

Exactness of the Mayer–Vietoris sequence at $\check{\mathrm{H}}^p(A \cap B)$ is now concluded as follows. Given $x \in \check{\mathrm{H}}^p(A \cap B)$, let a and b be elements such that $\partial_{(X,A\cap B)}(x) = \varphi(a,b)$. Then

$$\begin{aligned}\partial(x) &= \left(q_{(X,A)} \circ (k_A)^{-1} \circ \partial_{(B,A\cap B)}\right)(x)\\ &= \left(q_{(X,A)} \circ (k_A)^{-1}\right)\left(\partial_{(X,A\cap B)}(x)|_{(B,A\cap B)}\right)\\ &= q_{(X,A)}\big(a|_{(B,A\cap B)}\big)\\ &= a|_X.\end{aligned}$$

Similarly, if we set $\Delta := q_{(X,B)} \circ (k_B)^{-1} \circ \partial_{(A,A\cap B)}$, then one computes $\Delta(x) = b|_X$. But $\operatorname{im}\partial_{(X,A\cap B)} = \ker q_{(X,A\cap B)}$ by the long exact sequence of the pair $(X, A\cap B)$, so

$$0 = \partial_{(X,A\cap B)}(x)|_X = a|_X + b|_X,$$

and $\partial = -\Delta$. Because $\partial \circ (j_2)^* = 0$ and $\Delta \circ (j_1)^* = 0$, it follows that $(j_1)^* - (j_2)^*$ has image contained in $\ker\partial$. Conversly, if $x \in \check{\mathrm{H}}^p(A \cap B)$ is such that $\partial(x) = 0$, then $a|_X = b|_X = 0$. Then consider the commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(X) \arrow[dr] & \check{\mathrm{H}}^p(B) \arrow[d] & & \\
\check{\mathrm{H}}^p(A) \arrow[r] & \check{\mathrm{H}}^p(A\cap B) \arrow[r, "\partial_{(A,A\cap B)}"] \arrow[d, "\partial_{(B,A\cap B)}"'] \arrow[dr, "\partial_{(X,A\cap B)}"'] & \check{\mathrm{H}}^{p+1}(A, A\cap B) & \\
 & \check{\mathrm{H}}^{p+1}(B, A\cap B) & \check{\mathrm{H}}^{p+1}(X, A\cap B) \arrow[l] \arrow[u] \arrow[dr] & \\
 & & & \check{\mathrm{H}}^{p+1}(X)
\end{tikzcd}$$

The diagonal in this diagram is exact, so from $a|_X = 0$ and $b|_X = 0$ we see that we find elements $a', b' \in \check{\mathrm{H}}^p(A \cap B)$ such that $\partial_{(X,A\cap B)}(a') = a|_{(X,A\cap B)}$ and $\partial_{(X,A\cap B)}(b') =$

$b|_{(X,A\cap B)}$. But since the columns of this diagram are exact too, the element $x-a'$ must lift to an element $u \in \check{\mathrm{H}}^p(B)$, because

$$\partial_{(B,A\cap B)}(x-a') = \partial_{(X,A\cap B)}(x-a')|_{(B,A\cap B)} = a|_{(B,A\cap B)} + b|_{(B,A\cap B)} - a|_{(B,A\cap B)} = 0.$$

In the same way we see that $x-b'$ lifts to an element $v \in \check{\mathrm{H}}^p(A)$, and a similar reasoning shows that $x-a'-b'$ lifts to an element $z \in \check{\mathrm{H}}^p(X)$. In total we have

$$(j_1)^*(v - z|_A) - (j_2)^*(-u) = x - b' - (x - a' - b') + x - a' = x. \qquad \square$$

11. Cup product

Within this section F is a presheaf of R–algebras on X, that is, for every open subset $U \subseteq X$ we assume that $F(U)$ is not only an R–module, but also an (associative) R–algebra and that all restriction maps are morphisms of R–algebras. In this situation it is possible to define a ring structure on $\check{\mathrm{H}}^\bullet(X;F)$ as follows. Let $\mathfrak{U}$ be a Čech cover of X and $c \in \check{C}^p(\mathfrak{U})$, $d \in \check{C}^q(\mathfrak{U})$. Define a new cochain $c \smile d \in \check{C}^{p+q}(\mathfrak{U})$, the *cup product* of c and d, by

$$(c \smile d)(\sigma) = c(\sigma|_{\{0,\dots,p\}})|_{\mathfrak{U}_\sigma} \cdot d(\sigma|_{\{p,\dots,p+q\}})|_{\mathfrak{U}_\sigma},$$

where the multiplication is taken in the ring $F(\mathfrak{U}_\sigma)$.

Proposition 11.1. *The coboundary map δ is an anti–derivation with respect to the cup product: for $c \in \check{C}^p(\mathfrak{U})$, $d \in \check{C}^q(\mathfrak{U})$ we have*

$$\delta(c \smile d) = (\delta c) \smile d + (-1)^p \cdot c \smile (\delta d).$$

Proof. We compute (omitting restrictions)

$$\begin{aligned}((\delta c) \smile d)(\sigma) &= \sum_{j=0}^{p+1} (-1)^j c(\sigma|_{\{0,\dots,\hat{j},\dots,p+1\}}) \cdot d(\sigma|_{\{p+1,\dots,p+q+1\}}) \\ &= \sum_{j=0}^{p} (-1)^j (c \smile d)(\sigma_0,\dots,\widehat{\sigma_j},\dots,\sigma_{p+q+1}) + \\ &\quad (-1)^{p+1} c(\sigma|_{\{0,\dots,p\}}) \cdot d(\sigma|_{\{p+1,\dots,p+q+1\}})\end{aligned}$$

and

$$\begin{aligned}(c \smile (\delta d))(\sigma) &= (-1)^p \cdot \sum_{j=p}^{p+q+1} (-1)^j c(\sigma|_{\{0,\dots,p\}}) \cdot d(\sigma|_{\{p,\dots,\hat{j},\dots,p+q+1\}}) \\ &= c(\sigma|_{\{0,\dots,p\}}) \cdot d(\sigma|_{\{p+1,\dots,p+q+1\}}) + \\ &\quad (-1)^p \cdot \sum_{j=p+1}^{p+q+1} (-1)^j (c \smile d)(\sigma_0,\dots,\hat{j},\dots,p+q+1).\end{aligned}$$

Adding up the two terms we see that the left and right hand side of the claimed identity give the same result when evaluated on σ. $\square$

The previous proposition shows tha the cup product $c \smile d$ of two cocycles c and d is again a cocycle. Moreover, the cup product is R–bilinear, and so the cohomology class $c \smile d$ only depends on the cohomology class of c and d, because $c \smile (\delta e) = \pm\delta(c \smile e)$, as c is a cocycle, and similarly $(\delta f) \smile d = \pm\delta(f \smile d)$.

Also observe that if $A \subseteq X$ is a subspace and if $(\mathfrak{U}, \mathfrak{U}_A)$ is a cover of (X, A), defined on the pair of index sets (I, I_A), then the cup product reduces to a well–defined map

$$\smile \colon \check{C}^p(\mathfrak{U}) \times \check{C}^q(\mathfrak{U}, \mathfrak{U}_A) \to \check{C}^{p+q}(\mathfrak{U}, \mathfrak{U}_A),$$

because if c is a p–cocycle, d is a relative q–cocycle, and σ is a $(p+q)$–simplex with vertices $\sigma_0, \dots, \sigma_{p+q} \in I_A$, then in $(F|_A)(\mathfrak{U}_\sigma \cap A)$

$$\begin{aligned}(c \smile d)(\sigma)|_{\mathfrak{U}_\sigma \cap A} &= \big(c(\sigma|_{\{0,\dots,p\}})|_{\mathfrak{U}_\sigma} \cdot d(\sigma|_{\{p,\dots,p+q\}})|_{\mathfrak{U}_\sigma}\big)\,|_{\mathfrak{U}_\sigma \cap A} \\ &= c(\sigma|_{\{0,\dots,p\}})|_{\mathfrak{U}_\sigma \cap A} \cdot d(\sigma|_{\{p,\dots,p+q\}})|_{\mathfrak{U}_\sigma \cap A} \\ &= 0;\end{aligned}$$

here we use that also $(F|_A)$ canonically is a presheaf of rings on A in such a way that the limit map $F(U) \to (F|_A)(U \cap A)$ is a morphism of rings for every open set $U \subseteq X$. In total, we see that the cup product descends to a cup product

$$\smile \colon \check{\mathrm{H}}^p(\mathfrak{U}) \times \check{\mathrm{H}}^q(\mathfrak{U}, \mathfrak{U}_A) \to \check{\mathrm{H}}^{p+q}(\mathfrak{U}, \mathfrak{U}_A)$$

which in turn induces a cup product $\smile \colon \check{\mathrm{H}}^p(X) \times \check{\mathrm{H}}^q(X, A) \to \check{\mathrm{H}}^{p+q}(X, A)$, because of the following

Proposition 11.2. *Let I and J be directed categories, and $A \colon I \to R$–mod and $B \colon I \to R$–mod functors having limits L_A and L_B, respectively. Form the category $I \times J$ whose objecs are pairs of objects (i, j) with $i \in I$ and $j \in J$ and make this is a directed category by the rule $(i, j) \le (i_0, j_0)$ if and only if $i \le i_0$ and $j \le j_0$. Then the functor $A \times B \to R$–mod taking (i, j) to $A_i \times B_j$ has limit $L_A \oplus L_B = L_A \times L_B$.*

Proof. Suppose we are given a cone $((h_{i,j})_{i,j}, C)$ for $A \times B$ and fix $j \in J$. Use the universal property of L_A to obtain a morphism making

$$\begin{tikzcd} A_i \arrow[r] \arrow[drr] & A_i \oplus B_j \arrow[r, "h_{i,j}"] & C \\ & & L_A \arrow[u, "f_j"] \end{tikzcd}$$

commute for all $i \in I$, where the diagonal map is the limit map and the left horizontal map is the canonical inclusion of R–modules $A_i \to A_i \oplus 0 \subseteq A_i \bigoplus B_j$. We claim that this morphism is independent of the choice of j: indeed, if $j \le k$, then we have for each

$i \in I$ the commutative diagram

$$\begin{tikzcd}
 & & A_i \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
L_A \arrow[drr, "f_j"'] & A_i \oplus B_j \arrow[rr] \arrow[dr, "h_{i,j}"] & & A_i \oplus B_k \arrow[dl, "h_{i,k}"'] & L_A \arrow[dll, "f_k"] \\
 & & C & &
\end{tikzcd}$$

and so by the uniqueness statement in the universal property of L_A we have $f_j = f_k$. Since J is directed, it follows that $f = f_j$ is independent of the choice of j. In a similar fashion we obtain a morphism $g = g_i\colon L_B \to C$. Let then $(a_i, b_j) \in A_i \oplus B_j$ be arbitrary and denote by $a \in L_A$ and $b \in L_B$ the images of a_i and b_i under the limit maps $A_i \to L_A$ and $B_j \to L_B$, respectively. Then we have

$$(f \oplus g)(a, b) = f(a) + g(b) = f_j(a) + g_i(b) = h_{i,j}(a_i, 0) + h_{i,j}(0, b_j) = h_{i,j}(a_i, b_j),$$

and this means that

$$\begin{tikzcd}
A_i \oplus B_j \arrow[r, "h_{i,j}"] \arrow[dr] & C \\
 & L_A \oplus L_B \arrow[u, "f\oplus g"']
\end{tikzcd}$$

commutes for all (i, j). □

Applying this proposition to the categories $\mathcal{C}_X$ and $\mathcal{C}_{X,A}$ of Čech covers of X and (X, A), respectively, we see that the functor $\mathcal{C}_X \times \mathcal{C}_{X,A} \to R$–$\mathsf{mod}$ sending $(\mathfrak{U}, (\mathfrak{V}, \mathfrak{V}_A))$ to $\check{\mathrm{H}}^p(\mathfrak{U}) \times \check{\mathrm{H}}^q(\mathfrak{V}, \mathfrak{V}_A)$ has limit $\check{\mathrm{H}}^p(X) \times \check{\mathrm{H}}^q(X, A)$. But the functor $\mathcal{C}_{X,A} \to \mathcal{C}_X \times \mathcal{C}_{X,A}$ sending $(\mathfrak{W}, \mathfrak{W}_A)$ to $(\mathfrak{W}, (\mathfrak{W}, \mathfrak{W}_A)$ is cofinal, since if given $(\mathfrak{U}, (\mathfrak{V}, \mathfrak{V}_A)$, then $(\mathfrak{U}, \mathfrak{U} \cap A)$ also is a cover of (X, A) and we can simultaneously refine this cover and $(\mathfrak{V}, \mathfrak{V}_A)$, as $\mathcal{C}_{X,A}$ is directed. Therefore, $\check{\mathrm{H}}^p(X) \times \check{\mathrm{H}}^q(X, A)$ also is a limit for the functor sending a cover $(\mathfrak{U}, \mathfrak{U}_A)$ of (X, A) to $\check{\mathrm{H}}^p(\mathfrak{U}) \times \check{\mathrm{H}}^q(\mathfrak{U}, \mathfrak{U}_A)$. By a similar reasoning, there also is a cup product $\smile\colon \check{\mathrm{H}}^p(X, A) \times \check{\mathrm{H}}^q(X) \to \check{\mathrm{H}}^{p+q}(X, A)$.

The cup product is compatible with morphisms $(f, k)\colon (X, F) \to (Y, G)$, provided that G is also a presheaf R–algebras and that the cohomomorphism $k\colon G \to f_*F$ is a cohomomorphism of R–algebras.

Proposition 11.3. *The induced morphism $(f, k)^*\colon \check{\mathrm{H}}^\bullet(Y; G) \to \check{\mathrm{H}}^\bullet(X; F)$ is a morphism of rings whenever k is a cohomomorphism of R–algebras. If G and F are presheaves of unital R–algebras and k is a cohomomorphism of unital R–algebras, then $(f, k)^*$ is a morphism of unital rings.*

Proof. If $\mathfrak{V}$ is a Čech cover of Y, then already on cochain level we have the commutative diagram

$$\begin{tikzcd}
\check{C}^p(\mathfrak{V};G) \times \check{C}^q(\mathfrak{V};G) \arrow[r, "\smile"] \arrow[d, "(f,k)^* \times (f,k)^*"'] & \check{C}^{p+q}(\mathfrak{V};G) \arrow[d, "(f,k)^*"] \\
\check{C}^p(f^{-1}\mathfrak{V};F) \times \check{C}^q(f^{-1}\mathfrak{V};F) \arrow[r, "\smile"] & \check{C}^{p+q}(f^{-1}\mathfrak{V};F)
\end{tikzcd}$$

Indeed, if $c \in \check{C}^p(\mathfrak{V};F)$ and $d \in \check{C}^q(\mathfrak{V};F)$ are arbitrary and σ is a $(p+q)$–simplex of $f^{-1}\mathfrak{V}$, we compute

$$\begin{aligned}
(f,k)^*(c \smile d)(\sigma) &= k_{\mathfrak{V}_\sigma}\big((c \smile d)(\sigma)\big) \\
&= k_{\mathfrak{V}_\sigma}\big(c(\sigma_0,\dots,\sigma_p)|_{\mathfrak{V}_\sigma} \cdot d(\sigma_p,\dots,\sigma_{p+q})|_{\mathfrak{V}_\sigma}\big) \\
&= k_{\mathfrak{V}_{(\sigma_0,\dots,\sigma_p)}}\big(c(\sigma_0,\dots,\sigma_p)\big)|_{\mathfrak{U}_\sigma} \cdot k_{\mathfrak{V}_{(\sigma_p,\dots,\sigma_{p+q})}}\big(d(\sigma_p,\dots,\sigma_{p+q})\big)|_{\mathfrak{U}_\sigma} \\
&= \big((f,k)^*(c) \smile (f,k)^*(d)\big)(\sigma),
\end{aligned}$$

because we are assuming that k is a cohomomorphism of algebras. If in addition F and G are unital algebras, and k is a cohomomorphism of unital algebras, then each of the algebras $F(f^{-1}\mathfrak{V}_\sigma)$ and $G(\mathfrak{V}_\sigma)$ possesses a unit, and these are mapped onto each other by $k_{\mathfrak{V}_\sigma}$. Hence, $(f,k)^*$ preserves the units of $\check{C}^\bullet(\mathfrak{V};G)$ and $\check{C}^\bullet(f^{-1}\mathfrak{V};F)$, which are the constant 0–simplicies $\sigma \mapsto 1$. ☐

A relative version of the previous proposition holds as well. More delicate, yet fundamental, is the following

Theorem 11.4. *If F is a presheaf of commutative R–algebras, i. .e. if $F(U)$ is a commutative R–algebra for all open subset U of X, then the cohomology cup product is graded–commutative: if $c \in \check{\mathrm{H}}^p(X)$ and $d \in \check{\mathrm{H}}^q(X)$, then $c \smile d = (-1)^{pq} \cdot d \smile c$.*

Proof. For the proof we cite the following generalization of lemma 8.2 (see [13, Section 20, Proposition 2]): if $\mathfrak{U}$ is a cover of X, then every cocycle in $\check{C}^r(\mathfrak{U})$ is cohomologous to an alternating cocycle. Here, a cochain $f \in \check{C}^r(\mathfrak{U})$ is said to be alternating if (i) $f(\sigma) = 0$ whenever two different vertices of σ are equal and (ii) $f(\sigma) = \mathrm{sgn}(\pi) \cdot f(\sigma_{\pi(0)},\dots,\sigma_{\pi(r)})$ for every permutation π on $\{0,\dots,r\}$. With this result at hand, the proof of the theorem is as follows. Consider for each $r \geq 0$ the map $\gamma\colon \check{C}^r(\mathfrak{U}) \to \check{C}^r(\mathfrak{U})$ given on an r–cochain c by

$$(\gamma c)(\sigma) = (-1)^{\frac{r(r+1)}{2}} \cdot c(\sigma_r,\dots,\sigma_0).$$

Note that if c is alternating, then $\gamma(c) = c$, because

$$\begin{aligned}
c(\sigma) &= (-1)^r \cdot c(\sigma_1,\dots,\sigma_r,\sigma_0) \\
&= (-1)^{r+(r-1)} \cdot c(\sigma_2,\dots,\sigma_r,\sigma_1,\sigma_r) \\
&= \dots \\
&= (-1)^{r+(r-1)+\dots+1} c(\sigma_r,\dots,\sigma_0) \\
&= (\gamma c)(\sigma).
\end{aligned}$$

Moreover, γ is a chain map, because

$$\begin{aligned}(\delta\gamma c)(\sigma) &= \sum_{j=0}^{r+1}(-1)^j(\gamma c)(\sigma|_{\{0,\dots,\hat{j},\dots,r+1\}}) \\ &= \sum_{j=0}^{r+1}(-1)^j(-1)^{\frac{r(r+1)}{2}}\cdot c(\sigma_{r+1},\dots,\hat{j},\dots,\sigma_0) \\ &= (-1)^{\frac{r(r+1)}{2}}\sum_{j=0}^{r+1}(-1)^{r+1-j}\cdot c(\sigma_{r+1},\dots,\widehat{\sigma_{r+1-j}},\dots,\sigma_0) \\ &= (-1)^{\frac{r(r+1)}{2}}\cdot(-1)^{r+1}\cdot(\delta c)(\sigma_{r+1},\dots,\sigma_0) \\ &= (-1)^{\frac{r(r+1)}{2}}\cdot(-1)^{r+1}\cdot(-1)^{\frac{(r+1)(r+2)}{2}}\cdot(\gamma\delta c)(\sigma) \\ &= (\gamma\delta c)(\sigma).\end{aligned}$$

Now let $c \in \check{C}^p(\mathfrak{U})$ and $d \in \check{C}^q(\mathfrak{U})$ be alternating cocycles. Then $c \smile d$ is cohomology to an alternating cocycle, that is, there exists an alternating cocycle $f \in \check{C}^{p+q}(\mathfrak{U})$ and a cochain $e \in \check{C}^{p+q-1}(\mathfrak{U})$ with $c \smile d = f + \delta e$. In particular, $\gamma(c \smile d) = f + \delta\gamma e$, and on the other hand, since F is a presheaf of commutative R–algebras and c, d are alternating,

$$\begin{aligned}(c \smile d)(\sigma_{p+q},\dots,\sigma_0) &= c(\sigma_{p+q},\dots,\sigma_q)|_{\mathfrak{U}_\sigma}\cdot d(\sigma_q,\dots,\sigma_0)|_{\mathfrak{U}_\sigma} \\ &= (-1)^{\frac{q(q+1)}{2}}\cdot(-1)^{\frac{p(p+1)}{2}}(\gamma d \smile \gamma c)(\sigma) \\ &= (-1)^{\frac{q(q+1)}{2}}\cdot(-1)^{\frac{p(p+1)}{2}}(d \smile c)(\sigma).\end{aligned}$$

It follows that,

$$\begin{aligned}\gamma(c \smile d) &= (-1)^{\frac{(p+q)(p+q+1)}{2}}\cdot(-1)^{\frac{q(q+1)}{2}}\cdot(-1)^{\frac{p(p+1)}{2}}\cdot(d \smile c)(\sigma) \\ &= (-1)^{pq}\cdot(d \smile c)(\sigma)\end{aligned}$$

and so

$$\begin{aligned}(-1)^{pq}\cdot[d \smile c] &= [\gamma(c \smile d)] \\ &= [f] \\ &= [c \smile d].\end{aligned}$$

Therefore, $\smile\colon \check{\mathrm{H}}^p(\mathfrak{U}) \times \check{\mathrm{H}}^q(\mathfrak{U}) \to \check{\mathrm{H}}^{p+q}(\mathfrak{U})$ is graded–commutative, and passing to the limit, the cup product on cohomology is graded–commutative too. □

Chapter III.

Spectral sequences

1. Abstract spectral sequences

Throughout this chapter, we fix a commutative ring R and denote by $\mathcal{W}$ either the category R–mod of R–modules or the category R–alg of R–algebras.

Definition 1.1. Let Γ be an Abelian group, M an R–module, and A an R–algebra.

(i) A collection of R–submodules $(M^\gamma)_{\gamma\in\Gamma}$ of M is a Γ*–grading* of M if $M = \bigoplus_{\gamma\in\Gamma} M^\gamma$. In this case M is said to be a Γ*–graded module.*

(ii) A Γ*–grading* of A *(as an algebra)* is a Γ–grading $(A^\gamma)_{\gamma\in\Gamma}$ of the R–module A which in addition satisfies $A^\gamma \cdot A^{\gamma'} \subseteq A^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. We call A a Γ*–graded algebra.*

(iii) A morphism $f\colon M \to N$ between Γ–graded objects M and N is just a morphism in $\mathcal{W}$. We say that f is *homogeneous of degree* γ, where $\gamma \in \Gamma$, if $f(M^{\gamma'}) \subseteq N^{\gamma+\gamma'}$ holds for all $\gamma' \in \Gamma$.

Example 1.2.

(i) Let $\mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$ be the group on two elements and consider the subgroups

$$Z_{\overline{0}} = \{2z \,|\, z \in \mathbb{Z}\} \text{ and } Z_{\overline{1}} = \{0, 2z+1 \,|\, z \in \mathbb{Z}\}$$

of $\mathbb{Z}$ (generated) by the sets of even and odd integers, respectively. Then $\mathbb{Z} = Z_{\overline{0}} \oplus Z_{\overline{1}}$ is a $\mathbb{Z}/2\mathbb{Z}$ grading of the $\mathbb{Z}$–module $\mathbb{Z}$. However, this is not a grading of algebras: if it was, then $Z_{\overline{0}} \cdot Z_{\overline{1}} \subseteq Z_{\overline{1}}$ would have to hold, but the product of an integer with an even integer is again an even integer.

(ii) For any topological space X and any presheaf of R–modules F on X, $\check{\mathrm{H}}^\bullet(X; F) = \bigoplus_{p\in\mathbb{Z}} \check{\mathrm{H}}^p(X; F)$ is a $\mathbb{Z}$–graded module, where we have set $\check{\mathrm{H}}^p(X; F) = 0$ for $p < 0$. If F is a presheaf of R–algebras, then, considering $\check{\mathrm{H}}^\bullet(X; F)$ as an R–algebra via the cup product, this grading is even a $\mathbb{Z}$–grading as an algebra.

Our objects of interest are not arbitrary graded objects, but complexes. A *(cochain) complex of modules* is a pair (C, d) consisting of a $\mathbb{Z}$–graded module C, say with grading $(C^p)_{p\in\mathbb{Z}}$, and a *differential* $d\colon C \to C$, i. e. a morphism of R–modules which is homogeneous of degree 1 with respect to the grading $(C^p)_{p\in\mathbb{Z}}$ and such that $d^2 = 0$. If C is an algebra, $(C^p)_{p\in\mathbb{Z}}$ is a grading of algebras, and if d in addition is an *anti–derivation*, i. e.

$$d(x \cdot y) = d(x) \cdot y + (-1)^n \cdot x \cdot d(y)$$

holds for every homogeneous element x of degree n, then we call (C, d) a *(cochain) complex of algebras.*

Given a complex (C, d), its *cohomology* is, as usual, $\mathrm{H}^\bullet(C) = \bigoplus_{p\in\mathbb{Z}} H^p(C)$, where

$$H^p(C) = \frac{\ker d \cap M^p}{\operatorname{im} d \cap M^p}.$$

A *morphism* or *chain map* $\alpha\colon (C, d) \to (C_0, d_0)$ between complexes C and C_0 is a morphism $\alpha\colon C \to C_0$ of the underlying objects, homogeneous of degree 0, with $\alpha \circ d = d_0 \circ \alpha$. Note that these conditions ensures that α induces a well–defined map $\alpha_*\colon \mathrm{H}^\bullet(C) \to \mathrm{H}^\bullet(C_0)$, homogeneous of degree 0.

Definition 1.3. Let a be an integer. A *spectral sequence* in $\mathcal{W}$ (starting at a) consists of the following data.

(i) For each integer $r \geq a$ a $(\mathbb{Z} \times \mathbb{Z})$–graded object $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$, called the *r–th page.* The object E_a is the *initial page* of the spectral sequence and elements of $E_r^{p,q}$ are said to be homogeneous of *bidegree* (p, q) and *total degree* $p + q$.

(ii) For each $r \geq a$ a morphism of R–modules $d_r\colon E_r \to E_r$, homogeneous of bidegree $(r, -r + 1)$, which is an anti–derivation on E_r with respect to total degree if $\mathcal{W}$ is the category of R–algebras: that is to say,

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^n \cdot x \cdot d_r(y)$$

must hold for all $x, y \in E_r$ if x is homogeneous of total degree n.

(iii) Isomorphisms $\Phi_r\colon \mathrm{H}^{\bullet,\bullet}(E_r) \to E_{r+1}$ for all $r \geq a$, homogeneous of bidegree $(0, 0)$. Here, $\mathrm{H}^{\bullet,\bullet}(E_r) = \bigoplus_{p,q\in\mathbb{Z}} \mathrm{H}^{p,q}(E_r)$ is the $(\mathbb{Z} \times \mathbb{Z})$–graded object with

$$\mathrm{H}^{p,q}(E_r) := \frac{\ker d \cap E_r^{p,q}}{\operatorname{im} d \cap E_r^{p,q}}.$$

Any $(\mathbb{Z}\times\mathbb{Z})$–graded object $C = \bigoplus_{p,q\in\mathbb{Z}} C^{p,q}$ can be regarded a $\mathbb{Z}$–graded object by collecting elements of the same total degree. Namely, consider the $\mathbb{Z}$–grading $(\mathrm{Tot}(C)^n)_{n\in\mathbb{Z}}$ on C with with $\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q}$. If we wish to indicate that we consider C as $\mathbb{Z}$–graded space with this gradation, we write $\mathrm{Tot}(C)$ in place of C, but note that, $\mathrm{Tot}(C) = C$ holds as ungraded modules. For example, if $(E_r)_{r\geq a}$ is a spectral sequence, then each map d_r is a differential on the total complex $\mathrm{Tot}(E_r)$ in the previously defined sense and E_{r+1} is obtained as the cohomology of this total complex.

A *morphism* between spectral sequences $(E_r)_{r\geq a}$ and $(E'_r)_{r\geq a}$ is a collection of homogeneous morphisms $f_r\colon E_r \to E'_r$ of bidegree $(0, 0)$, one for each integer $r \geq a$, such that $f_r \circ d_r = d'_r \circ f_r$ and such that

$$\begin{tikzcd}
\mathrm{H}^{\bullet,\bullet}(E_r) \arrow[r, "(f_r)_*"] \arrow[d, "\Phi_r"'] & \mathrm{H}^{\bullet,\bullet}(E'_r) \arrow[d, "\Phi'_r"] \\
E_{r+1} \arrow[r, "f_{r+1}"'] & E'_{r+1}
\end{tikzcd}$$

commutes for all r; here, Φ_r and Φ'_r are the isomorphisms provided with the respective spectral sequence and $(f_r)_*$ is the morphism induced by f_r on the level of cohomology.

Given a $\mathbb{Z}$–graded object $C = \bigoplus_{p\in\mathbb{Z}} C^p$ of $\mathcal{W}$, a *(decreasing) filtration* of C is a function $F_\bullet C$ assigning to every integer $p \in \mathbb{Z}$ a submodule $F_pC := (F_\bullet C)(p)$ of C which is a graded submodule in the sense that $(F_pC \cap C^k)_{k\in\mathbb{Z}}$ is a grading of F_pC and such that $F_{p+1}C \subseteq F_pC$ for all $p \in \mathbb{Z}$. If $\mathcal{W}$ is the category of R–algebras, then we additionally require $F_pC \cdot F_qC \subseteq F_{p+q}C$ to hold for all $p, q \in \mathbb{Z}$. The *associated graded object* of a filtration $F_\bullet C$ then is the $(\mathbb{Z} \times \mathbb{Z})$–graded object

$$\mathrm{gr}(F_\bullet C) = \bigoplus_{p,q\in\mathbb{Z}} \mathrm{gr}_p(F_\bullet C)^q \text{ with } \mathrm{gr}_p(F_\bullet C)^q = \frac{F_pC \cap C^{p+q}}{F_{p+1}C \cap C^{p+q}};$$

notice that $\mathrm{gr}(F_\bullet C)$ really is a graded algebra if C is an algebra, because the multiplication in C induces a well–defined "multiplication" $\mathrm{gr}_p(F_\bullet C)^q \times \mathrm{gr}_m(F_\bullet C)^n \to \mathrm{gr}_{p+m}(F_\bullet C)^{q+n}$, as $F_\bullet C$ is decreasing. With all these concepts at hand, we are ready to explain the notion of convergence of spectral sequences. Namely, a spectral sequence of modules $(E_r)_{r\geq a}$ *converges* or *abuts* to a $\mathbb{Z}$–graded module C if

(i) for every pair of integers p, q there is some r such that $E_r^{p,q} = E_{r+k}^{p,q}$ for all $k \geq 0$. We write $E_\infty^{p,q} := E_r^{p,q}$ and set $E_\infty := \bigoplus_{p,q\in\mathbb{Z}} E_\infty^{p,q}$.

(ii) There exists a filtration $F_\bullet C$ of C together with isomorphisms of graded modules $E_\infty \cong \mathrm{gr}(F_\bullet C)$.

If $(E_r)_{r\geq a}$ converges to C, we indicate this by writing $E_r^{p,q} \Longrightarrow C$. Of course, this notion also is defined in the category of R–algebras: if C is a $\mathbb{Z}$–graded algebra and $(E_r)_{r\geq a}$ is a spectral sequence of algebras, then we say that $(E_r)_{r\geq a}$ *converges* to C *(as a spectral sequence of algebras)* if $(E_r)_{r\geq a}$ abuts to the graded module C and if in addition the isomorphism $E_\infty \cong \mathrm{gr}(F_\bullet C)$ is an isomorphism of graded algebras; here, the ring structure on E_∞ is defined as follows. If we denote by $r(p,q)$ the smallest integer such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r \geq r(p,q)$, then the multiplication is defined so that

$$\begin{tikzcd}
E_r^{p,q} \times E_r^{m,n} \arrow[r] \arrow[d] & E_r^{p+m,q+n} \arrow[d] \\
E_\infty^{p,q} \times E_\infty^{m,n} \arrow[r] & E_\infty^{p+m,q+n},
\end{tikzcd}$$

commutes for all $r \geq \max\{r(p,q), r(m,n), r(p+m,q+n)\}$, where the upper horizontal map is the multiplication in E_r.

2. The spectral sequence of a filtered complex

In many situations a complex (C,d) in $\mathcal{W}$ comes equipped with a particular filtration from which we can build a spectral sequence converging to $\mathrm{H}^\bullet(C)$. In more detail, a *(decreasing) filtration* of the *complex* (C,d) is a filtration $F_\bullet C$ of C which in addition satisfies $d(F_pC) \subseteq F_pC$ for every integer p. Notice that in this case the differential d on C induces a differential on $\mathrm{gr}(F_\bullet C)$, homogeneous of (bi–)degree $(0,1)$. If $F_\bullet C$ is a filtration on (C,d), then we say that (C,d) is a *filtered complex.* A *morphism* between filtered complexes C and D with respective filtrations $F_\bullet C$ and $F_\bullet D$ is a chain map $\alpha\colon C \to D$ preserving the filtrations, that is, with $\alpha(F_pC) \subseteq F_pD$ for all $p \in \mathbb{Z}$.

We will mostly encounter *bounded* filtrations: these are filtrations $F_\bullet C$ such that for every integer n there exists integers t and s with

$$C^n = F_t \cap C^n \supseteq F_{t+1} \cap C^n \supseteq \ldots \supseteq F_{s-1} \cap C^n \supseteq F_s \cap C^n = 0.$$

A filtration is said to be *canonically bounded* if $s = n+1$ and $t = 0$.

Theorem 2.1. *For every filtration $F_\bullet C$ of C there exists a spectral sequence $(E_r)_{r\geq 1}$, called the* spectral sequence associated to $F_\bullet C$*, with the following properties.*

(i) *There exists an isomorphism of graded objects $E_1 \to \mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(F_\bullet C))$.*

(ii) *If $F_\bullet C$ is bounded, then the spectral sequence converges to $\mathrm{H}^\bullet(C)$ with respect to the filtration $F_\bullet H$ of $\mathrm{H}^\bullet(C)$ induced by $F_\bullet C$, i. e. the filtration in which F_pH is induced by the inclusion of complexes $F_pC \hookrightarrow C$.*

(iii) *If $\overline{C}$ is another filtered complex with filtration $F_\bullet\overline{C}$ and if $\alpha\colon C \to \overline{C}$ is a morphism of filtered complexes, then α induces a morphism of spectral sequences*

$$\big(\alpha_r\colon E_r \to \overline{E}_r\big)_{r\geq 1},$$

where $(\overline{E}_r)_{r\geq 1}$ is the spectral sequence associated to $F_\bullet\overline{C}$. Moreover, the diagram

$$\begin{tikzcd}
E_1 \arrow[r, "\alpha_1"] \arrow[d] & \overline{E}_1 \arrow[d] \\
\mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(F_\bullet C)) \arrow[r, "(\alpha)_*"] & \mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(F_\bullet \overline{C})),
\end{tikzcd}$$

with the isomorphisms coming from the first item, commutes, as does the diagram

$$\begin{tikzcd}
E_\infty^{p,q} \arrow[r, "\alpha_\infty"] \arrow[d, "\cong"'] & \overline{E}_\infty^{p,q} \arrow[d, "\cong"] \\
\mathrm{gr}_p(F_\bullet H)^q \arrow[r, "(\alpha)_*"] & \mathrm{gr}_p(F_\bullet \overline{H})^q
\end{tikzcd}$$

for all p, q, where $F_\bullet\overline{H}$ is the filtration of $\mathrm{H}^\bullet(\overline{C})$ induced by $F_\bullet\overline{C}$ and the isomorphisms are those with respect to which $(E_r)_{r\geq 1}$ and $(\overline{E}_r)_{r\geq 1}$ converge.

Proof. Let $Z_r^{p,q} = \{c \in F_pC \cap C^{p+q} \mid d(c) \in F_{p+r}C\}$ be those elements in F_pC which are homogeneous of bidegree (p,q) and cocycles modulo $F_{p+r}C$, and observe that

$$d(Z_r^{p,q}) \subseteq F_{p+r}C \cap C^{p+q+1} = F_{p+r}C \cap C^{(p+r)+(q-r+1)}$$

because d is homogeneous of degree 1. Since $d^2 = 0$, it follows that $d(Z_r^{p,q}) \subseteq Z_r^{p+r,q-r+1}$, and hence it makes sense to set, for all $r \geq 1$,

$$E_r^{p,q} := \frac{Z_r^{p,q}}{d(Z_{r-1}^{p-r+1,q+r-2}) + Z_{r-1}^{p+1,q-1}}.$$

With this choice, we can define d_r so as to make the diagram

$$\begin{tikzcd}
Z_r^{p,q} \arrow[r, "d"] \arrow[d, "\pi"'] & Z_r^{p+r,q-r+1} \arrow[d] \\
E_r^{p,q} \arrow[r, "d_r^{p,q}"] & E_r^{p+r,q-r+1}
\end{tikzcd}$$

commute, where the vertical maps are the canonical projections. Carefully observe that d_r becomes an anti–derivation with respect total degree if C is a complex of algebras. Now we notice that if $c \in Z_r^{p,q}$ is such that $d_r(\pi(c)) = 0$, then by construction

$$d(c) \in d(Z_{r-1}^{p+1,q-1}) + Z_{r-1}^{p+r+1,q-r},$$

or in other words, upon adding a suitable element $u \in Z_{r-1}^{p+1,q-1}$ we have $c + u \in Z_{r+1}^{p,q}$. But c and $c + u$ represent the same class in $E_r^{p,q}$, and since π is surjective, we see that $\ker d_r^{p,q}$ is contained in $\pi(Z_{r+1}^{p,q})$. Conversely, by the diagram above, every element in $\pi(Z_{r+1}^{p,q})$ is contained in the kernel of $d_r^{p,q}$, and therefore the canonical map

$$\frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{d(Z_{r-1}^{p-r+1,q+r-2}) + Z_{r-1}^{p+1,q-1}} \to \pi(Z_{r+1}^{p,q}) = \ker d_r^{p,q}$$

induced by π is an isomorphism. Next, observe that π induces

$$\frac{d(Z_r^{p-r,q+r-1}) + Z_{r-1}^{p+1,q-1}}{d(Z_{r-1}^{p-r+1,q+r-2}) + Z_{r-1}^{p+1,q-1}} \xrightarrow{\sim} \pi\left(d\left(Z_r^{p-r,q+r-1}\right)\right) = \operatorname{im} d_r^{p-r,q+r-1},$$

because $Z_{r-1}^{p-r+1,q+r-2}$ already is contained in $Z_r^{p-r,q+r-1}$. Therefore,

$$\frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{d(Z_r^{p-r,q+r-1}) + Z_{r-1}^{p+1,q-1}} \cong \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}},$$

and this isomorphism is again induced by π. Now consider the inclusion induced map

$$E_{r+1}^{p,q} = \frac{Z_{r+1}^{p,q}}{d(Z_r^{p-r,q+r-1}) + Z_r^{p+1,q-1}} \to \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{d(Z_r^{p-r,q+r-1}) + Z_{r-1}^{p+1,q-1}}.$$

This map is certainly surjective. It is injective, for if $c \in Z_{r+1}^{p,q}$ is such that $c = d(b) + z$ for some $b \in Z_r^{p-r,q+r-1}$ and some $z \in Z_{r-1}^{p+1,q-1}$, then $d(c) = d(z)$, so $z \in Z_r^{p+1,q-1}$, and hence c represents the zero class in $E_{r+1}^{p,q}$.

Set $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$. What we have just shown is that the morphism

$$\Phi_r = \bigoplus_{p,q\in\mathbb{Z}} \Phi_r^{p,q} \colon \bigoplus_{p,q\in\mathbb{Z}} E_{r+1}^{p,q} \to \mathrm{H}^{\bullet,\bullet}(E_r)$$

with $\Phi_r^{p,q}([c]) = [c]$ is bijective, where on the left hand side $[c]$ denotes the equivalence class of $c \in Z_{r+1}^{p,q}$ in $E_{r+1}^{p,q}$ and on the right hand side the equivalence class of c in $\mathrm{H}^{\bullet,\bullet}(E_r)$.

Therefore, $(E_r)_{r\geq 1}$ is a spectral sequence. Its first page is

$$E_1^{p,q} = \frac{Z_1^{p,q}}{d(F_pC \cap C^{p+q-1}) + F_{p+1}C \cap C^{p+q}}$$

and an element $c \in F_pC \cap C^{p+q}$ is contained in $Z_1^{p,q}$ if and only if $d(c) \equiv 0$ modulo $F_{p+1}C$. This is exactly the condition for c to represent an element in the kernel of the induced differential $\mathrm{gr}_p(F_\bullet C)^q \to \mathrm{gr}_p(F_\bullet C)^{q+1}$. Thus, we have a commutative diagram

$$\begin{tikzcd}
Z_1^{p,q} \arrow[d] \arrow[dr] & \\
E_1^{p,q} \arrow[r] & \mathrm{H}^{p,q}(\mathrm{gr}(F_\bullet C))
\end{tikzcd}$$

where the vertical map is the quotient map and the diagonal map is induced by the quotient map $\eta\colon Z_1^{p,q} \to \mathrm{gr}_p(F_\bullet C)^q$. In particular, the horizontal map is surjective. Unwinding definitions we also see that is injective: indeed, an element $c \in Z_1^{p,q}$ maps to zero under the diagonal map if and only if $\eta(c)$ is contained in the image of the differential on $\mathrm{gr}(F_\bullet C)$. This in turn means that $c \equiv d(b)$ modulo $F_{p+1}C$ for some element $b \in F_pC$, and hence c represents 0 in $E_1^{p,q}$.

Now suppose that the filtration is bounded. Consider the filtration F_pH of $\mathrm{H}^\bullet(C)$ given by $F_pH := \mathrm{im}(\mathrm{H}^\bullet(F_pC) \to \mathrm{H}^\bullet(C))$, where the map is induced by the inclusion of complexes $F_pC \hookrightarrow C$. This filtration will verify that the spectral sequence just constructed converges to $\mathrm{H}^\bullet(C)$. To see this, fix integers p and q. If r is chosen so large that $F_{p+r}C \cap C^{p+q+1} = 0$ and $F_{p-r+1}C \cap C^{p+q-1} = C^{p+q-1}$, then

$$E_r^{p,q} = \frac{\ker d \cap F_pC \cap C^{p+q}}{d(C^{p+q-1}) \cap F_pC + \ker d \cap F_{p+1}C \cap C^{p+q}},$$

so $(E_r^{p,q})_{r\geq 1}$ stabilizes. On the other hand we have the chain of canonical isomorphisms

$$\begin{aligned}
\mathrm{gr}_p(F_\bullet H)^q &= \frac{F_pH \cap \mathrm{H}^{p+q}(C)}{F_{p+1}H \cap \mathrm{H}^{p+q}(C)} \\
&\xleftarrow{\sim} \frac{\ker d \cap F_pC \cap C^{p+q} + d(C^{p+q-1})}{\ker d \cap F_{p+1}C \cap C^{p+q} + d(C^{p+q-1})} \\
&\xleftarrow{\sim} \frac{\ker d \cap F_pC \cap C^{p+q} + d(C^{p+q-1}) \cap F_pC}{\ker d \cap F_{p+1}C \cap C^{p+q} + d(C^{p+q-1}) \cap F_pC} \\
&= E_\infty^{p,q},
\end{aligned}$$

where for the second isomorphism we once more use that $F_\bullet C$ is a decreasing filtration, so if $z + d(b) \in F_pC$ for some element $z \in F_{p+1}C$, then also $d(b) \in F_pC$.

Finally, let us discuss naturality of the spectral sequence. So assume that $\alpha\colon C \to \overline{C}$ is a morphism of filtered complexes, where $F_\bullet\overline{C}$ is the filtration on $\overline{C}$. Let $\overline{Z}_r^{p,q}$ be the elements in $F_p\overline{C}$ which are cocycles modulo $F_{p+r}\overline{C}$ and homogeneous of bidegree (p,q), and let $\overline{E}_r = \bigoplus_{p,q\in\mathbb{Z}} \overline{E}_r^{p,q}$ be the r–th page of the spectral sequence associated to $F_\bullet\overline{C}$. Since α preserves filtrations and commutes with the differentials, we can define α_r to be the unique morphism making commutative the diagram

$$\begin{tikzcd}
Z_r^{p,q} \arrow[r, "\alpha"] \arrow[d] & \overline{Z}_r^{p,q} \arrow[d] \\
E_r^{p,q} \arrow[r, "\alpha_r"] & \overline{E}_r^{p,q}
\end{tikzcd}$$

for all $r \geq 1$ and all $p, q \in \mathbb{Z}$. From this we immediately see that α_r commutes with the differentials on E_r and $\overline{E}_r$, and because the isomorphisms $E_{r+1} \to \mathrm{H}^{\bullet,\bullet}(E_r)$ and $\overline{E}_{r+1} \to \mathrm{H}^{\bullet,\bullet}(\overline{E}_r)$ are induced by inclusions, it follows that $(\alpha_r)_{r\geq 1}$ is a morphism of spectral sequences. Moreover, the commutativity of the diagram above shows that α_1 corresponds to α under the isomorphisms $E_1^{p,q} \to \mathrm{H}^{p+q}(\mathrm{gr}(F_\bullet C))$ and $\overline{E}_1^{p,q} \to \mathrm{H}^{p+q}(\mathrm{gr}(F_\bullet\overline{C}))$ constructed earlier. Analogously, since for fixed integers p, q we have $E_\infty^{p,q} = E_r^{p,q}$ and $\overline{E}_\infty^{p,q} = \overline{E}_r^{p,q}$ for sufficiently large r, and since the isomorphisms $E_\infty^{p,q} \to \mathrm{gr}_p(F_\bullet H)^q$ and $\overline{E}_\infty^{p,q} \to \mathrm{gr}_p(F_\bullet\overline{H})^q$ are induced by inclusions, we see that under these isomorphisms the map $\alpha_\infty = \alpha_r$ just corresponds to the map $\alpha_*\colon \mathrm{gr}_p(F_\bullet H)^q \to \mathrm{gr}_p(F_\bullet\overline{H})^q$. □

Remark 2.2. In categorical terms, we can rephrase theorem 2.1 as follows. Denote by $\mathcal{F}(\mathcal{W})$ the category of filtered differential complexes, in $\mathcal{W}$ i. e. pairs $(F_\bullet C, C)$, where C is a complex in $\mathcal{W}$ and $F_\bullet C$ is a filtration of this complex, and by $\mathcal{E}(\mathcal{W})$ the category of spectral sequences in $\mathcal{W}$ starting at a. The statement of theorem 2.1 then is that there exists a functor $E\colon \mathcal{F}(\mathcal{W}) \to \mathcal{E}(\mathcal{W})$ such that $E(F_\bullet C, C)$ is a spectral sequence whose first page is isomorphic to $\mathrm{H}^{\bullet,\bullet}(\mathrm{gr})$ and such that $E(F_\bullet C, C)$ is convergent if $F_\bullet C$ is bounded.

Note that we can actually replace the first page of $E(F_\bullet C, C)$ with $\mathrm{H}^{\bullet,\bullet}(\mathrm{gr})$ and still obtain a functorial assignment $\mathcal{F}(\mathcal{W}) \to \mathcal{E}(\mathcal{W})$. More generally, and more abstractly, this can be seen as follows. If $\mathcal{C}$ is any category and $E\colon \mathcal{C} \to \mathcal{E}(\mathcal{W})$ is a functor, let $\mathrm{Gr}(\mathcal{W})$ be the category of $\mathbb{Z}\times\mathbb{Z}$–graded objects and $p\colon \mathcal{C} \to \mathrm{Gr}(\mathcal{W})$ the functor which assigns to an object C of $\mathcal{C}$ the r–th page E_r of the spectral sequence $E(C) = ((E_r)_{r\geq a}, (d_r)_{r\geq a}, (\Phi_r)_{r\geq a})$. Suppose that $q\colon \mathcal{C} \to \mathrm{Gr}(\mathcal{W})$ is a functor and that $\eta\colon p \Rightarrow q$ is a natural isomorphism, i. e. a natural transformation such that η_C is an isomorphism for all objects C of $\mathcal{C}$. Then we can define a new functor $\overline{E}\colon \mathcal{C} \to \mathcal{E}(\mathcal{W})$ by letting $\overline{E}(C) = ((\overline{E}_r)_{r\geq a}, (\overline{d}_r)_{r\geq a}, (\overline{\Phi}_r)_{r\geq a})$ be the spectral sequence which equals $E(C)$ on all pages except the r–th, which we define to be

$$\overline{E}_r := q(C),\ \overline{d}_r := \eta_C \circ d_r \circ (\eta_C)^{-1},\ \text{and}\ \overline{\Phi}_r := \Phi_r \circ ((\eta_C)_*)^{-1}.$$

Here, $(\eta_C)_*\colon \mathrm{H}^{\bullet,\bullet}(E_r) \to \mathrm{H}^{\bullet,\bullet}(\overline{E}_r)$ is the map induced by η_C, which by construction commutes with the differentials d_r and $\overline{d}_r$ and is an isomorphism by assumption. Then $\overline{E}(C)$ is again a spectral sequence whose r–th page now equals $q(C)$.

This general reasoning applies to the functor $E\colon \mathcal{F}(\mathcal{W}) \to \mathcal{E}(\mathcal{W})$ of theorem 2.1, because the isomorphism between the first page of $E(F_\bullet C, C)$ and $\mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(F_\bullet C))$ is a natural by the second item of theorem 2.1.

3. Double complexes

Let $C = \bigoplus_{p,q\in\mathbb{Z}} C^{p,q}$ be a $(\mathbb{Z}\times\mathbb{Z})$–graded object and d_v, d_h two R–module morphisms on C. We call the triple (C, d_v, d_h) a *double complex* in $\mathcal{W}$ if the vertical differential d_v is homogeneous of bidegree $(0,1)$, the horizontal differential d_h is homogeneous of bidegree $(1,0)$, if $d_v d_h = -d_h d_v$, and if the total complex $\mathrm{Tot}(C)$ together with $d = d_v + d_h$ is a complex. A morphism of double complexes $C \to C'$ is just a morphism of the underlying graded objects commuting with the horizontal and vertical differentials.

Every double complex C admits two canonical filtrations of $\mathrm{Tot}(C)$, the *filtration by rows* and the *filtration by columns*, denoted by $V_\bullet C$ and $H_\bullet C$, respectively, and given by

$$V_{q_0}C = \bigoplus_{q\ge q_0, p\in\mathbb{Z}} C^{p,q} \text{ and } H_{p_0}C = \bigoplus_{p\ge p_0, q\in\mathbb{Z}} C^{p,q}.$$

To ensure that the spectral sequences associated to these filtrations converge against $\mathrm{H}^\bullet(\mathrm{Tot}(C))$, we assume that C is a *first quadrant* double complex, that is, $C^{p,q} = 0$ unless $p, q \ge 0$, or more generally that C is *bounded in total degree*, which means that $\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q}$ is a finite sum for each n. By theorem 2.1 there are two spectral sequences $(E_r)_{r\ge1}$ and $(I_r)_{r\ge1}$ associated to $V_\bullet C$ and $H_\bullet C$, respectively. By the same theorem, the first page of the spectral sequence associated to the filtration by columns is $E_1 \cong \mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(H_\bullet C))$. But

$$\mathrm{gr}_p(H_\bullet C)^q = \frac{H_pC \cap \mathrm{Tot}(C)^{p+q}}{H_{p+1}C \cap \mathrm{Tot}(C)^{p+q}} \xleftarrow{\sim} C^{p,q},$$

the isomorphism just being induced by the inclusion. Moreover, the differential on $\mathrm{gr}(H_\bullet C)$ is just the differential induced by d, and since $d_h(H_pC) \subseteq H_{p+1}C$, we see that under this identification the differential on $\mathrm{gr}(H_\bullet C)$ corresponds to d_v. Therefore, $E_1 \cong \mathrm{H}^{\bullet,\bullet}(C, d_v)$ as $(\mathbb{Z}\times\mathbb{Z})$–graded objects. An entirely analagous computation shows that $I_1 \cong \mathrm{H}^{\bullet,\bullet}(C^t, d_h)$, where $C^t = C$ as ungraded objects, but $(C^t)^{p,q} = C^{q,p}$.

Theorem 3.1. *For every double complex C there exists a spectral sequence $(E_r)_{r\ge2}$ having the following properties.*

- *(i) The initial page is $E_2 = \mathrm{H}^{\bullet,\bullet}\big(\mathrm{H}^{\bullet,\bullet}(C, d_v), d_h\big)$.*
- *(ii) If C is bounded in total degree, then the spectral sequence converges to $\mathrm{H}^\bullet(\mathrm{Tot}(C))$ with respect to the filtration induced by the filtration by columns.*

(iii) If (C', d'_v, d'_h) is another double complex and if $\alpha\colon C \to C'$ is a morphism of double complexes, then α induces a morphism $(\alpha_r)_{r\geq 2}$ between the spectral sequences $(E_r)_{r\geq 2}$ and $(E'_r)_{r\geq 2}$ which on the second page equals

$$(\alpha)_*\colon \mathrm{H}^{\bullet,\bullet}\big(\mathrm{H}^{\bullet,\bullet}(C, d_v), d_h\big) \to \mathrm{H}^{\bullet,\bullet}\big(\mathrm{H}^{\bullet,\bullet}(C, d'_v), d'_h\big).$$

Moreover, if C and C' are bounded in total degree, then the following diagram commute for all p, q:

$$\begin{tikzcd}
E_\infty^{p,q} \arrow[r, "\alpha_\infty"] \arrow[d, "\cong"'] & (E'_\infty)^{p,q} \arrow[d, "\cong"] \\
\mathrm{gr}_p(F_\bullet H)^q \arrow[r, "(\alpha)_*"] & \mathrm{gr}_p(F_\bullet H')^q
\end{tikzcd}$$

where $F_\bullet H$ and $F_\bullet H'$ are the filtrations of $\mathrm{H}^\bullet(\mathrm{Tot}(C))$ and $\mathrm{H}^\bullet(\mathrm{Tot}(C'))$ induced by the filtrations by columns and the isomorphisms are those with respect to which $(E_r)_{r\geq 2}$ and $(E'_r)_{r\geq 2}$ converge.

Proof. We will have to reexamine parts of the proof of theorem 2.1. Consider the horizontal filtration $H_\bullet C$ of the total complex $\mathrm{Tot}(C)$. Recall from the proof of theorem 2.1 that $E_1^{p,q}$, the (p,q)–th graded part of the first page of the spectral sequence $(E_r)_{r\geq 1}$ associated to $H_\bullet C$, is as a certain quotient of the set $Z_1^{p,q} \subseteq H_pC \cap \mathrm{Tot}(C)^{p+q}$ of homogeneous cocycles modulo $H_{p+1}C$. But $H_pC \cap \mathrm{Tot}(C)^{p+q} = C^{p,q}$ and an element $c \in C^{p,q}$ satisfies $d(c) \in H_{p+1}C$ if and only if $d_v(c) = 0$, because d_v is homogeneous of bidegree $(0,1)$, and thus $Z_1^{p,q} = \ker d_v \cap C^{p,q}$. Moreover, the differential d_1 on E_1 is defined so as to make the diagram

$$\begin{tikzcd}
Z_1^{p,q} \arrow[r, "d"] \arrow[d, "\pi"] & Z_1^{p+1,q} \arrow[d, "\pi"] \\
E_1^{p,q} \arrow[r, "d_1"'] & E_1^{p+1,q}
\end{tikzcd}$$

commute, where the vertical maps are quotient maps. Also recall that

$$\begin{tikzcd}
Z_1^{p,q} \arrow[d, "\pi"] \arrow[dr] & \\
E_1^{p,q} \arrow[r, "\cong"'] & \mathrm{H}^{p,q}(\mathrm{gr}(H_\bullet C))
\end{tikzcd}$$

is commutative, where the vertical map again is the quotient map and the diagonal map is induced by the quotient map $Z_1^{p,q} \to \mathrm{gr}_p(H_\bullet C)^q$. But we just checked in the paragraph preceeding this theorem that $\mathrm{H}^{p,q}(C, d_v) \xrightarrow{\sim} \mathrm{H}^{p,q}(\mathrm{gr}(H_\bullet C))$ and that this isomorphism is induced by the quotient map $C^{p,q} \to \mathrm{gr}_p(H_\bullet C)^q$. Thus, the kernel of the diagonal map above is precisely $\mathrm{im}\, d_v \cap C^{p,q}$ and so π induces an isomorphism $\overline{\pi}\colon \mathrm{H}^{p,q}(C, d_v) \to E_1^{p,q}$.

Since d_h descends to a well–defined map on $\mathrm{H}^{\bullet,\bullet}(C, d_v)$, we hence have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{p,q}(C, d_v) \arrow[r, "d_h"] \arrow[d, "\overline{\pi}"'] & \mathrm{H}^{p,q}(C, d_v) \arrow[d, "\overline{\pi}"] \\
E_1^{p,q} \arrow[r, "d_1"'] & E_1^{p+1,q}
\end{tikzcd}$$

in which both vertical maps are isomorphisms. Therefore, $\overline{\pi}$ induces an isomorphism $\mathrm{H}^{\bullet,\bullet}(\mathrm{H}^{\bullet,\bullet}(C, d_v), d_h) \to \mathrm{H}^{\bullet,\bullet}(E_1)$ of graded objects. Since $\mathrm{H}^{\bullet,\bullet}(E_1)$ in turn is isomorphic to E_2 as a $(\mathbb{Z} \times \mathbb{Z})$–graded object, we can replace E_2 with $\mathrm{H}^{\bullet,\bullet}(\mathrm{H}^{\bullet,\bullet}(C, d_v), d_h)$ in the spectral sequence $(E_r)_{r\geq 1}$, provided that this replacement defines is natural (cf. remark 2.2), and then all claims will be consequences of theorem 2.1.

Thus, let us now suppose that $\alpha\colon C \to C'$ is a morphism of double complexes. Then α commutes with d_h, d_v, and hence in particular with d. Also, α is homogeneous of bidegree $(0, 0)$, so α is a chain map between the $\mathbb{Z}$–graded complexes $(\mathrm{Tot}(C), d) \to (\mathrm{Tot}(C'), d')$ and preserves the horizontal filtrations on $\mathrm{Tot}(C)$ and $\mathrm{Tot}(C')$. That is to say, α is a morphism of filtered complexes and hence induces, by theorem 2.1, a morphism of spectral sequences $(\alpha_r\colon E_r \to E'_r)_{r\geq 1}$, where $(E'_r)_{r\geq 1}$ is the spectral sequence associated to the horizontal filtration $H_\bullet C'$. Now consider the diagram

$$\begin{tikzcd}
\mathrm{H}^{\bullet,\bullet}(C, d_v) \arrow[rrr, "(\alpha)_*"] \arrow[dd] \arrow[dr, "\overline{\pi}"'] & & & \mathrm{H}^{\bullet,\bullet}(C', d'_v) \arrow[dd] \arrow[dl, "\overline{\pi}'"] \\
& E_1 \arrow[r, "\alpha_1"] \arrow[dl] & E'_1 \arrow[dr] & \\
\mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(H_\bullet C)) \arrow[rrr, "(\alpha)_*"] & & & \mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(H_\bullet C')).
\end{tikzcd}$$

The outermost rectangle commutes, as do the left, right and lower triangles. Since $E'_1 \to \mathrm{H}^{\bullet,\bullet}(\mathrm{gr}(H_\bullet C'))$ is injective, it follows that the upper triangle commutes as well. But then so does the diagram

$$\begin{tikzcd}
\mathrm{H}^{\bullet,\bullet}\big(\mathrm{H}^{\bullet,\bullet}(C, d_v), d_h\big) \arrow[r, "(\alpha)_*"] \arrow[d, "\cong"'] & \mathrm{H}^{\bullet,\bullet}\big(\mathrm{H}^{\bullet,\bullet}(C', d'_v), d'_h\big) \arrow[d, "\cong"] \\
\mathrm{H}^{\bullet,\bullet}(E_1) \arrow[r, "(\alpha_1)_*"] \arrow[d, "\cong"'] & \mathrm{H}^{\bullet,\bullet}(E'_1) \arrow[d, "\cong"] \\
E_2 \arrow[r, "\alpha_2"] & E'_2
\end{tikzcd}$$

where the lower vertical maps are the isomorphisms coming from the respective spectral sequence. Thus, the spectral sequence associated to a double complex is functorial. □

Proposition 3.2. *Let $\alpha\colon C \to C'$ be a morphism between double complexes C and C' which are bounded in total degree, and let $(\alpha_r)_{r\geq 2}$ be the induced morphism between the spectral sequences $(E_r)_{r\geq 2}$ and $(E'_r)_{r\geq 2}$ of C and C', respectively. If α_r is an isomorphism for some r, then also $\alpha_*\colon \mathrm{H}^\bullet(\mathrm{Tot}(C)) \to \mathrm{H}^\bullet(\mathrm{Tot}(C'))$ is an isomorphism.*

Proof. If α_r is an isomorphism, then, since α_{r+1} corresponds under the isomorphisms $\mathrm{H}^{\bullet,\bullet}(E_r) \cong E_{r+1}$ and $\mathrm{H}^{\bullet,\bullet}(E_{r+1}) \cong E'_{r+1}$ to the map induced by α_r, it follows that α_{r+1} is an isomorphism. Inductively we see that each α_{r+k} is an isomorphism, and then α_∞ must be as well. Let then $F_\bullet H$ and $F_\bullet H'$ be the filrations of $\mathrm{H}^\bullet(\mathrm{Tot}(C))$ and $\mathrm{H}^\bullet(\mathrm{Tot}(C))$ with respect to which $(E_r)_{r\geq 2}$ and $(E'_r)_{r\geq 2}$ are convergent. Since for all p, q we have isomorphisms $E_\infty^{p,q} \cong \mathrm{gr}_p(F_\bullet H)^q$ and $(E'_\infty)^{p,q} \cong \mathrm{gr}_p(F_\bullet H')^q$ under which α_∞ corresponds to the map $\alpha_*\colon\ \mathrm{gr}(F_\bullet H) \to \mathrm{gr}(F_\bullet H')$ induced by α, it hence follows that the latter map is an isomorphism.

To conclude that $\alpha_*\colon \mathrm{H}^\bullet(\mathrm{Tot}(C)) \to \mathrm{H}^\bullet(\mathrm{Tot}(C'))$ is an isomorphism, choose n sufficiently large so that $F_n H \cap \mathrm{H}^n(\mathrm{Tot}(C))$ and $F_n H' \cap \mathrm{H}^n(\mathrm{Tot}(C'))$ are trivial. This is possible, because we are assuming that the double complexes are bounded in total degree, whence the filtrations $F_\bullet H$ and $F_\bullet H'$ are bounded as well. Then consider the restriction $\alpha_*\colon F_{n-k}H \cap \mathrm{H}^n(\mathrm{Tot}(C)) \to F_{n-k}H \cap \mathrm{H}^n(\mathrm{Tot}(C'))$. This map trivially is an isomorphism for $k = 0$, and proceeding inductively, suppose that this map is an isomorphism for some $k-1$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{n+1-k}H \cap \mathrm{H}^n(\mathrm{Tot}(C)) \arrow[r] \arrow[d, "\alpha_*"'] & F_{n-k}H \cap \mathrm{H}^n(\mathrm{Tot}(C)) \arrow[r] \arrow[d, "\alpha_*"'] & \mathrm{gr}_{n-k}(F_\bullet H)^k \arrow[r] \arrow[d, "\alpha_*"'] & 0 \\
0 \arrow[r] & F_{n+1-k}H' \cap \mathrm{H}^n(\mathrm{Tot}(C')) \arrow[r] & F_{n-k}H \ \cap \mathrm{H}^n(\mathrm{Tot}(C')) \arrow[r] & \mathrm{gr}_{n-k}(F_\bullet H')^k \arrow[r] & 0
\end{tikzcd}$$

We just observed that the right vertical map is an isomorphism, and we are assuming that that the left vertical map is an isomorphism. Because the rows of the above diagram are exact, it follows that the middle vertical map is an isomorphism for k, thus proving that the middle vertical map is an isomorphism for all choices of $k \geq 0$. Using once more that the filtrations $F_\bullet H$ and $F_\bullet H'$ are bounded, we conclude that $\alpha_*\colon \mathrm{H}^n(\mathrm{Tot}(C)) \to \mathrm{H}^n(\mathrm{Tot}(C'))$ is an isomorphism. ☐

As a first application of the spectral sequence of a double complex, let us prove the Universal Coefficent Theorem in cohomology. It asserts that for a space X and an R–module M the cohomology groups $\check{\mathrm{H}}^\bullet(X;\underline{M})$ can be expressed in terms of $\check{\mathrm{H}}^\bullet(X;\underline{R})$ and torsion products between $\check{\mathrm{H}}^\bullet(X;\underline{R})$ and M. To make this statement more precise, we have to introduce some notions from homological algebra.

First, recall that an R–module M is called *free*, if there exists an isomorphism of R–modules $\bigoplus_{i\in I} R \to M$ for some index set I. This is equivalent to demanding that there be an R–basis, that is, a collection of elements $(b_i)_{i\in I}$ of M such that every element $m \in M$ can be uniquely expressed as $m = \sum_{i\in I} r_i b_i$ for certain coefficients $r_i \in R$, finitely many of which are non–zero. Every R–module M can be *resolved* by a free R–module, which means that there exists a surjective map $F \to M$ from some free R–module M: simply consider the free R–module $F = \bigoplus_{m\in M} R$ and let $F \to M$ be the map which takes an element $f \in F$, say with components $f = (f_m)_{m \in M}$, to $\sum_{m \in M} f_m m$. Note that this expression is well–defined, because by definition of the direct sum only finitely many of the components of f are non–zero. Resolving the kernel of this map $F \to M$ and proceeding inductively, we see that every R–module M admits a *free resolution*, that is, there exists an exact sequence

$$\ldots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

with each F_i a free R–module. For an R–module N, the *n–th torsion group* $\operatorname{Tor}_n^R(M, N)$ then is defined to be

$$\operatorname{Tor}_n^R(N, M) = \frac{\ker(\mathrm{id}_N \otimes_R d_n)}{\operatorname{im}(\mathrm{id}_N \otimes_R d_{n+1})},$$

where $\mathrm{id}_N \otimes_R d_n \colon N \otimes_R F_n \to N \otimes_R F_{n-1}$ is the map induced by d_n. One can show that the isomorphism class of $\operatorname{Tor}_n^R(M, N)$ does not depend on the particular choice of free resolution, see [11, Theorem III.6.1].

Theorem 3.3 (Universal Coefficient Theorem)**.** *Let R be a principal ideal domain and M an R–module. Then for every p and all Čech covers $\mathfrak{U}$ of X there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \check{\mathrm{H}}^p(\mathfrak{U}; \underline{R}) \otimes_R M \arrow[r, "\mu_{\mathfrak{U}}"] \arrow[d] & \check{\mathrm{H}}^p(\mathfrak{U}; \underline{M}) \arrow[r] \arrow[d] & \operatorname{Tor}_1^R(\check{\mathrm{H}}^{p+1}(\mathfrak{U}; \underline{R}), M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \check{\mathrm{H}}^p(X; \underline{R}) \otimes_R M \arrow[r, "\mu"] & \check{\mathrm{H}}^p(X; \underline{M}) \arrow[r] & \operatorname{Tor}_1^R(\check{\mathrm{H}}^{p+1}(X; \underline{R}), M) \arrow[r] & 0
\end{tikzcd}$$

in which all vertical maps are induced by limit maps and where $\mu_{\mathfrak{U}}$ is induced by the map $\check{C}^p(\mathfrak{U}; \underline{R}) \otimes_R M \to \check{C}^p(\mathfrak{U}; \underline{M})$ sending a homogeneous tensor $c \otimes m$ to the cochain $c * m$ with $(c * m)(\sigma) = c(\sigma)m$ for all p–simplices σ.

Recall that a (commutative, associative) unital ring R is a *principal ideal domain* if R is an integral domain and every ideal $I \subseteq R$ is of the form $I = Rx$ for some $x \in R$

Proof. We shall use without proof that for a principal ideal domain R and every free R–module N also every submodule $Q \subseteq N$ is free. In particular, if we resolve M as $F \to M$ for some free R–module F, the kernel of this map automatically is free, and hence M admits a free resolution of the form

$$0 \to F_0 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} M \to 0.$$

By definition, we thus have $\operatorname{Tor}_1^R(N, M) = \ker \mathrm{id}_N \otimes_R \alpha$ for all R–modules N. Now let $\mathfrak{U}$ be a Čech cover of X and consider the double complex

$$C^{p,q} := \check{C}^p(\mathfrak{U}; \underline{R}) \otimes_R F_q,$$

where we have set $F_q = 0$ for $q \neq 0, 1$. The vertical differential d_v is just $\mathrm{id} \otimes_R \alpha$, trivially extended to all of C, and the horizontal differential d_h is determined by $d_h|_{C^{p,q}} =$

$(-1)^q \delta \otimes_R \mathrm{id}$, where $\delta\colon \check{C}^p(\mathfrak{U};\underline{R}) \to \check{C}^{p+1}(\mathfrak{U};\underline{R})$ is the boundary operator. Note that this double complex consists of two rows and lies in the first quadrant. To compute the cohomology of $\mathrm{Tot}(C)$ and to provide an explicit isomorphism, we introduce the complex

$$(C')^{p,q} := \begin{cases} \check{C}^p(\mathfrak{U};\underline{R}) \otimes_R M, & q = 1, \\ 0, & q \neq 1. \end{cases}$$

with trivial vertical differential and with horizontal differential $d'_h = -\delta \otimes_R \mathrm{id}$. Observe that we can consider $\mathrm{id} \otimes_R \beta$ as a morphism of double complexes $C \to C'$: indeed, since $\operatorname{im}\alpha = \ker\beta$, we see that $\mathrm{id} \otimes_R \beta$ vanishes on $C^{\bullet,0}$, and hence maps into C' and commutes with the differentials. We claim that $\mathrm{id} \otimes_R \beta$ induces an isomorphism between the cohomology of the total complexes, and to prove this assertion, we consider the spectral sequences $(E_r)_{r\geq 2}$ and $(E'_r)_{r\geq 2}$ of the double complexes C and C'. To determine E_2, let I be the index set on which $\mathfrak{U}$ is defined and consider first the commutative diagram

$$\begin{tikzcd}
\check{C}^p(\mathfrak{U};\underline{R}) \otimes_R F_1 \arrow[r] & \prod_{\sigma\in I^{p+1}} F_1 \\
\check{C}^p(\mathfrak{U};\underline{R}) \otimes_R F_0 \arrow[r] \arrow[u, "\mathrm{id}\otimes_R\alpha"] & \prod_{\sigma\in I^{p+1}} F_0 \arrow[u, "\prod_{\sigma\in I^{p+1}} \alpha"']
\end{tikzcd}$$

where the horizontal maps send a homogeneous tensor $c \otimes_R f$ with $f \in F_q$ to the element $(c(\sigma)f)_{\sigma\in I^{p+1}}$. These maps are injective: if $(b_k)_{k\in K}$ is a basis of F_q, then every element in $\check{C}^p(\mathfrak{U};\underline{R}) \otimes_R F_q$ is of the form $\sum_k c_k \otimes_R b_k$, with only finitely many c_k different from 0, and

$$\sum_{k\in K} c_k \otimes_R b_k \mapsto \left(\sum_k c_k(\sigma) b_k\right)_{\sigma\in I^{p+1}}.$$

If the right hand side is 0, then every component is 0, and thus $\sum_k c_k(\sigma) b_k = 0$ holds for all simplices σ. But since $(b_k)_{k\in K}$ is a basis, we must have $c_k(\sigma) = 0$ for all simplicies σ and all $k \in K$, and then $c_k = 0$ for all $k \in K$. It follows that in the diagram above the map on the left must be injective, because the map on the right is, α being injective. Since for any R–module N the sequence $N \otimes_R F_0 \to N \otimes_R F_1 \to N \otimes_R M \to 0$ is exact (see e. g. [11, Theorem V.5.1]), we thus see that

$$0 \to \check{C}^p(\mathfrak{U};\underline{R}) \otimes_R F_0 \xrightarrow{\mathrm{id}\otimes_R\alpha} \check{C}^p(\mathfrak{U};\underline{R}) \otimes_R F_1 \xrightarrow{\mathrm{id}\otimes_R\beta} \check{C}^p(\mathfrak{U};\underline{R}) \otimes_R M \to 0$$

is an exact sequence. This shows that

$$\mathrm{H}^{p,q}(C, d_v) \cong \begin{cases} \check{C}^p(\mathfrak{U};\underline{R}) \otimes_R M, & q = 1, \\ 0, & q \neq 1, \end{cases}$$

the isomorphism being induced by $\mathrm{id} \otimes_R \beta$. Then

$$\begin{aligned} E_2^{p,q} &= \mathrm{H}^{p,q}(\mathrm{H}^{\bullet,\bullet}(C, d_v), d_h) \\ &\cong \begin{cases} \mathrm{H}^p(\check{C}^\bullet(\mathfrak{U}; \underline{R}) \otimes_R M, -\delta \otimes_R \mathrm{id}), & q = 1, \\ 0, & q \neq 1, \end{cases} \\ &= (E_2')^{p,q} \end{aligned}$$

where the isomorphism again is induced by $\mathrm{id} \otimes_R \beta$. But this just says that the map of spectral sequences induced by $\mathrm{id} \otimes_R \beta$ is an isomorphism on the second page, and so, by proposition 3.2, $\mathrm{id} \otimes_R \beta$ induces an isomorphism

$$\mathrm{H}^n(\mathrm{Tot}(C)) \xrightarrow{\sim} \mathrm{H}^n(\mathrm{Tot}(C')) = \mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U}; \underline{R}) \otimes_R M).$$

Now we compute $\mathrm{H}^n(\mathrm{Tot}(C))$ in a different way. Consider the double complex $C^t = C$ with grading $(C^t)^{p,q} = C^{q,p}$, vertical differential $d_v^t = d_h$, and horizontal differential $d_h^t = d_v$. We have $\mathrm{Tot}(C^t) = \mathrm{Tot}(C)$ as $\mathbb{Z}$–graded modules, and so the spectral sequences $(E_r^t)_{r \geq 2}$ of C^t still converges against $\mathrm{H}^\bullet(\mathrm{Tot}(C))$. However, if F is a free R–module and $0 \to A \to B \to C \to 0$ is an exact sequence of R–modules, then also $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is exact, and this shows that

$$\mathrm{H}^{p,q}(C^t, d_v^t) = \frac{\ker(\delta \otimes_R \mathrm{id} \colon C^{q,p} \to C^{q+1,p})}{\mathrm{im}(\delta \otimes_R \mathrm{id} \colon C^{q-1,p} \to C^{q,p})} \xrightarrow{\sim} \check{\mathrm{H}}^q(\mathfrak{U}; \underline{R}) \otimes_R F_p,$$

the isomorphism being induced by the projection $\ker(\delta \otimes_R \mathrm{id}) = \ker(\delta) \otimes_R F_p \to \check{\mathrm{H}}^q(\mathfrak{U}; \underline{R}) \otimes_R F_p$. By definition we then have

$$(E_2^t)^{0,q} = \mathrm{H}^{0,q}(\mathrm{H}^\bullet(C^t, d_v^t), d_h^t) \cong \mathrm{Tor}_1^R(\check{\mathrm{H}}^q(\mathfrak{U}; \underline{R}), M).$$

On the other hand, E_2^t, if $F_\bullet H^t$ is the filtration of $\mathrm{H}^\bullet(\mathrm{Tot}(C))$ with respect to which $(E_r^t)_{r \geq 2}$ converges, then, because $F_0 H^t \cap \mathrm{H}^n(\mathrm{Tot}(C)) = \mathrm{H}^n(\mathrm{Tot}(C))$, we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F_1 H^t \cap \mathrm{H}^n(\mathrm{Tot}(C)) \arrow[r] & \mathrm{H}^n(\mathrm{Tot}(C)) \arrow[r] \arrow[d, "\mathrm{id} \otimes_R \beta"] & \mathrm{gr}_0(F_\bullet H^t)^n \arrow[r] \arrow[d, "\cong"] & 0 \\ & & \mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U}; \underline{R}) \otimes_R M) & \mathrm{Tor}_1^R(\check{\mathrm{H}}^n(\mathfrak{U}; \underline{R}), M) & \end{tikzcd}$$

and we just checked that $\mathrm{id} \otimes_R \beta$ is an isomorphism. It thus remains to determine the image of $\mathrm{id} \otimes_R \beta$ on $F_1 H^t \cap \mathrm{H}^n(\mathrm{Tot}(C))$. But $F_\bullet H^t$ is just the filration induced by the filtration $H_\bullet C^t$, the filtration of C^t by columns, and

$$H_1 C^t = \bigoplus_{p \geq 1, q \in \mathbb{Z}} (C^t)^{p,q} = \bigoplus_{q \in \mathbb{Z}} C^{q,1} = \bigoplus_{q \in \mathbb{Z}} \check{C}^q(\mathfrak{U}; \underline{R}) \otimes_R F_1.$$

Since the horizontal differential $d_h^t = d_v$ is zero on on $H_1 C^t$ and F_1 is a free R–module, we see that

$$\mathrm{H}^n(H_1 C^t) = \mathrm{H}^{1,n-1}(C^t, d_v^t) \xrightarrow{\sim} \check{\mathrm{H}}^{n-1}(\mathfrak{U}; \underline{R}) \otimes_R F_1,$$

as already observed earlier. It follows that

$$\begin{tikzcd}
F_1H^t \cap \mathrm{H}^n(\mathrm{Tot}(C)) \arrow[r] \arrow[d, "\mathrm{id}\otimes_R\beta"] & \mathrm{H}^n(\mathrm{Tot}(C)) \arrow[d, "\mathrm{id}\otimes_R\beta"] \\
\mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U};\underline{R}) \otimes_R M \arrow[r, "\mu"] & \mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U};\underline{R}) \otimes_R M)
\end{tikzcd}$$

commutes, where μ is determined by $\mu([c]\otimes m) = [c\otimes m]$ for all cocycles $c \in \check{C}^{n-1}(\mathfrak{U};\underline{R})$ and all $m \in M$. After identifying $\mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U};\underline{R}) \otimes_R M)$ with $\mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U};\underline{M})$ via the isomorphism induced by the canonical isomorphism $R \otimes_R M \to M$, we hence arrive at the exact sequence

$$0 \to \mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U};\underline{R}) \otimes_R M \xrightarrow{\mu} \mathrm{H}^{n-1}(\check{C}^\bullet(\mathfrak{U};\underline{M}) \to \mathrm{Tor}_1^R(\check{\mathrm{H}}^n(\mathfrak{U};\underline{R}), M) \to 0.$$

The remaining statement is purely algebraic and is a consequence of proposition 3.4 below. □

Proposition 3.4. *Let R be a principal ideal domain, M an R–module, and $A\colon I \to R\text{–}\mathsf{mod}$ a functor from a directed set I. Then the limit induced maps*

$$\varinjlim_{i\in I}(A_i \otimes_R M) \to (\varinjlim_{i\in I} A_i) \otimes_R M \text{ and } \varinjlim_{i\in I} \mathrm{Tor}_1^R(A_i, M) \to \mathrm{Tor}_1^R(\varinjlim_{i\in I} A_i, M)$$

are isomorphisms.

Proof. As already remarked in the proof of theorem 3.3, we can choose a free resolution of M of the form $0 \to F_1 \to F_0 \to M \to 0$. Since exact sequences are preserved under limits, we then obtain the commutative diagram

$$\begin{tikzcd}
\varinjlim A_i \otimes_R F_1 \arrow[r] \arrow[d] & \varinjlim A_i \otimes_R F_0 \arrow[r] \arrow[d] & \varinjlim A_i \otimes_R M \arrow[r] \arrow[d] & 0 \\
(\varinjlim A_i) \otimes_R F_1 \arrow[r] & (\varinjlim A_i) \otimes_R F_0 \arrow[r] & (\varinjlim A_i) \otimes_R M \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The vertical maps are surjective, because the limit maps $A_j \to \varinjlim A_i$ are exhaustive. Both left vertical maps are injective: to see this, just observe that for ever free R–module F and all $j \in I$ we have, by definition, a commutative diagram

$$\begin{tikzcd}
A_j \otimes_R F \arrow[r] \arrow[dr] & \varinjlim A_i \otimes_R F \arrow[d] \\
& (\varinjlim A_i) \otimes_R F
\end{tikzcd}$$

If an element x is in the kernel of the diagonal limit map, then, since $(-) \otimes_R F$ sends exact sequences to exact sequences, we may express x as a (finite) sum $x = \sum_k a_k \otimes_R f_k$ with each a_k in the kernel of the limit map $A_j \to \varinjlim A_i$. Hence, we can find $j_0 \geq j$ such that each a_k maps to zero under the map $A_j \to A_{j_0}$, and then x also lies in the kernel of the map $A_j \otimes_R F \to \varinjlim A_i \otimes_R F$, because the latter factors through $A_j \otimes_R F \to A_{j_0} \otimes_R F$. Thus, in the first diagram the left vertical maps are isomorphisms, and since the rows of this diagram are exact, also the right vertical map must be an isomorphism. This proves the first statement. In order to prove the second assertion, we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \operatorname{Tor}_1^R(A_i, M) \arrow[r] \arrow[d] & \varinjlim A_i \otimes_R F_1 \arrow[r] \arrow[d] & \varinjlim A_i \otimes_R F_0 \arrow[d] \\
0 \arrow[r] & \operatorname{Tor}_1^R(\varinjlim A_i, M) \arrow[r] & (\varinjlim A_i) \otimes_R F_1 \arrow[r] & (\varinjlim A_i) \otimes_R F_0
\end{tikzcd}$$

where the left vertical map is the limit map induced by the maps $\operatorname{Tor}_1^R(A_j, M) \to A_j \otimes_R F_1 \to (\varinjlim A_i) \otimes_R F_1$ for varying j. The rows of this commutative diagram are again exact, and we just saw that the middle and right vertical maps are isomorphisms. Then the left vertical map must be an isomorphism too. ☐

4. Collapsing of spectral sequences

We say that a spectral sequence $(E_r)_{r\geq a}$ *collapses on the b–th page*, if the differentials d_r are trivial for all $r \geq b$.

Example 4.1. Suppose that $(E_r)_{r\geq a}$ is a spectral sequence consisting of a single row, that is, we assume there exists an integer q_0 such that $E_a^{p,q} = 0$ unless $q = q_0$. Then the spectral sequence collapses. The same is true if the spectral sequence just consists of a single column.

Example 4.2. Let us assume that we are given a spectral sequence $(E_r)_{r\geq a}$ which converges against a $\mathbb{Z}$–graded module $C = \bigoplus_{p\in\mathbb{Z}} C^p$ with respect to a canonically bounded filtration $F_\bullet C$, and that for each n, $\operatorname{Tot}(E_a)^n = \bigoplus_{p+q=n} E_a^{p,q}$ consists of at most one non–trivial summand. Fix n and let $E_a^{p,q}$ be this non–trivial summand. Then $E_r^{p,q}$ is the only possibly non–trivial summand in $\operatorname{Tot}(E_r)^n$ for all subsequent pages, and in particular $E_\infty^{r,s} = 0$ for all pairs (r, s) with $r + s = n$, unless $(r, s) = (p, q)$. Hence, since the spectral sequence is convergent, also $\operatorname{gr}_r(F_\bullet C)^s$ is trivial unless $(r, s) = (p, q)$. It follows that $F_r C \cap C^n = F_{r+1} C \cap C^n$ for all $r \neq p$, and since the filtration is canonically bounded, that is, $F_0 C \cap C^n = C^n$ and $F_{n+1} C \cap C^n = 0$, we thus have

$$F_r C \cap C^n = \begin{cases} C^n, & r \leq p, \\ 0, & r > p. \end{cases}$$

Therefore, $\operatorname{Tot}(\operatorname{gr}_\bullet(F_\bullet C)) = C$ as $\mathbb{Z}$–graded modules, and hence also $\operatorname{Tot}(E_a)$ is isomorphic to C as a $\mathbb{Z}$–graded module.

If C is a $\mathbb{Z}$–graded algebra and the spectral sequence is a spectral sequence of algebras, then the isomorphism $\operatorname{Tot}(E_a) \cong C$ is, in general, not an isomorphism of $\mathbb{Z}$–graded algebras: for example, it might happen that the multiplication map

$$\operatorname{gr}_p(F_\bullet C)^q \times \operatorname{gr}_r(F_\bullet C)^s \to \operatorname{gr}_{p+r}(F_\bullet C)^{q+s}$$

is trivial, simply because the target (or equivalently $E_\infty^{p+r,q+s}$) is trivial, even though the corresponding multiplication in C is not (we will encounter such a phenomenon in the computation of $B\mathbb{Z}_p$ for p a prime, see remark 5.2). However, if none of $E_\infty^{p,q}$, $E_\infty^{r,s}$, and $E_\infty^{p+r,q+s}$ are trivial, then, because of our assumption that only one summand in $\mathrm{Tot}(E_a)^n$ is non–trivial, we have a commutative diagram

$$\begin{tikzcd}
E_\infty^{p,q} \times E_\infty^{r,s} \arrow[r] \arrow[d] & E_\infty^{p+r,q+s} \arrow[d] \\
\mathrm{gr}_p(F_\bullet C)^q \times \mathrm{gr}_r(F_\bullet C)^s \arrow[r] \arrow[d, equal] & \mathrm{gr}_{p+r}(F_\bullet C)^{q+s} \arrow[d, equal] \\
C^{p+q} \times C^{r+s} \arrow[r] & C^{p+q+r+s}
\end{tikzcd}$$

where all horizontal maps are the respective multiplications. This show for example that the previously constructed isomorphism $\mathrm{Tot}(E_a) \cong C$ is an isomorphism of $\mathbb{Z}$–graded algebras if the spectral sequence just consists of the single row $E_a^{\bullet,0}$ or column $E_a^{0,\bullet}$.

Example 4.3. As a further application of the spectral sequence of a double complex, let use prove the de Rham Theorem. It states that for a manifold M Čech cohomology $\check{\mathrm{H}}^\bullet(M;\underline{\mathbb{R}})$ coincides with the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}^\bullet(M)$, which is the cohomology of the complex $(\Omega^\bullet(M), d)$ of alternating forms on M, equipped with the exterior derivative d of forms. For the proof, we follow [2, Example 14.16].

Consider for each q the sheaf Ω^q on M which assigns to an open subset $U \subseteq M$ the $\mathbb{R}$–algebra $(\Omega^q)(U) := \Omega^q(U)$ of alternating q–forms on U. Our first claim is that $\check{\mathrm{H}}^p(\mathfrak{U};\Omega^q)$ is trivial for all $p \geq 1$, where for the moment $\mathfrak{U}$ is an arbitrary Čech cover of M. In fact, if I is the index set on which $\mathfrak{U}$ is defined, let $(\xi_i)_{i\in I}$ be a partition of unity subordinate to the cover $\{\mathfrak{U}_i \,|\, i \in I\}$ of M. Now assume that c is a cocycle in $\check{C}^p(\mathfrak{U};\Omega^q)$ and define $f \in \check{C}^{p-1}(\mathfrak{U};\Omega^q)$ on a $(p-1)$–simplex τ by

$$f(\tau) = \sum_{i\in I} \xi_i \cdot c(i, \tau_0, \dots, \tau_{p-1});$$

this expression is well–defined, because the right hand side is a finite sum at each point of M. Observe that for every p–simplex σ and every index $i \in I$

$$0 = \delta c(i, \sigma_0, \dots, \sigma_p) = c(\sigma)|_{\mathfrak{U}_i\cap\mathfrak{U}_\sigma} - \sum_{j=0}^{p}(-1)^j c(i, \sigma_0, \dots, \widehat{\sigma_j}, \dots, \sigma_p)|_{\mathfrak{U}_i\cap\mathfrak{U}_\sigma}.$$

Since $(\xi_i)_{i\in I}$ is a partition of unity and subordinated to $\{\mathfrak{U}_i \,|\, i \in I\}$, it thus follows that

$$\begin{aligned} c(\sigma) &= \sum_{i\in I}\sum_{j=0}^{p}(-1)^j \xi_i \cdot c(i, \sigma_0, \dots, \widehat{\sigma_j}, \dots, \sigma_p) \\ &= (\delta f)(\sigma), \end{aligned}$$

and hence $\check{\mathrm{H}}^p(\mathfrak{U};\Omega^q)=0$.

Next, consider the double complex $C=\bigoplus_{p,q\in\mathbb{Z}}C^{p,q}$ with

$$C^{p,q}=\check{C}^p(\mathfrak{U};\Omega^q),$$

with horizontal differential $d_h=\delta$ induced by the boundary of the Čech complex and with vertical differential $d_v|_{C^{p,q}}=(-1)^p d$ induced by the exerior derivative. We assume that $\mathfrak{U}$ is chosen such that $\mathfrak{U}_\sigma$ is smoothly homotopy equivalent to a point for all simplices σ. Covers of this form (named "good covers" in [2]) are cofinal: for example, if we choose an auxiliary Riemannian metric on M, then every point has a neighborhood basis consisting of open and (geodesically) convex subsets, and the intersection of such subsets is again convex, hence smoothly contractible, see [15, Ex. 32(f)]. Then, since whenever $0\to A\to B\to C\to 0$ is an exact sequence of presheaves on M also $0\to\check{C}^\bullet(\mathfrak{U};A)\to\check{C}^\bullet(\mathfrak{U};B)\to\check{C}^\bullet(\mathfrak{U};C)\to 0$ is an exact sequence of complexes, we see that

$$\begin{aligned}\mathrm{H}^{p,q}(C,d_v)&=\frac{\ker\bigl(d\colon\check{C}^p(\mathfrak{U};\Omega^q)\to\check{C}^p(\mathfrak{U};\Omega^{q+1})\bigr)}{\operatorname{im}\bigl(d\colon\check{C}^p(\mathfrak{U};\Omega^{q-1})\to\check{C}^p(\mathfrak{U};\Omega^q)\bigr)}\\&\cong\frac{\check{C}^p(\mathfrak{U};\ker d|_{\Omega^q})}{\check{C}^p(\mathfrak{U};\operatorname{im}(d|_{\Omega^{q-1}}))}\\&\cong\check{C}^p(\mathfrak{U};\mathrm{H}^q_{\mathrm{dR}}),\end{aligned}$$

where $\mathrm{H}^q_{\mathrm{dR}}$ is the presheaf with $(\mathrm{H}^q_{\mathrm{dR}})(U)=\mathrm{H}^q_{\mathrm{dR}}(U)$ for all open subsets $U\subseteq M$. However, since $\mathfrak{U}_\sigma$ is contractible for all simplices σ, also $\check{C}^p(\mathfrak{U};\mathrm{H}^q_{\mathrm{dR}})=0$ unless $q=0$. On the other hand, the canonical map $\mathbb{R}\to\mathrm{H}^0_{\mathrm{dR}}$ induces an isomorphism $\check{C}^p(\mathfrak{U};\mathbb{R})\to\check{C}^p(\mathfrak{U};\mathrm{H}^0_{\mathrm{dR}})$, and therefore

$$\mathrm{H}^{p,q}(\mathrm{H}^{\bullet,\bullet}(C,d_v),d_h)\cong\begin{cases}0, & q>0,\\ \check{\mathrm{H}}^p(\mathfrak{U};\mathbb{R}), & q=0.\end{cases}$$

In particular, the spectral sequence $(E_r)_{r\geq 2}$ of the double complex C consists of a single row; hence, it collapses, so $E_2\cong E_\infty$, and $\mathrm{Tot}(E_\infty)\cong\mathrm{Tot}(C)$ as graded $\mathbb{R}$–algebras by example 4.2 and example 4.1 above. On the other hand, we can also apply theorem 3.1 to the double complex $C^t=C$ with grading $(C^t)^{p,q}=C^{q,p}$. The resulting spectral sequence $(E^t_r)_{r\geq 2}$ has as second page

$$E^t_2=\mathrm{H}^{\bullet,\bullet}(\mathrm{H}^{\bullet,\bullet}(C^t,d_h),d_v)$$

and we already observed that $\mathrm{H}^{q,p}(C^t,d_h)=\check{\mathrm{H}}^p(\mathfrak{U};\Omega^q)$ is non–trivial only if $p=0$. Moreover, under the canonical isomorphisms $\Omega^q(M)\to\check{\mathrm{H}}^0(\mathfrak{U};\Omega^q)$ the vertical differential $d_v=d$ just corresponds to the exterior derivative $d\colon\Omega^q(M)\to\Omega^{q+1}(M)$, and so

$$(E^t_2)^{p,q}\cong\begin{cases}0, & q>0,\\ \mathrm{H}^p_{\mathrm{dR}}(M), & q=0.\end{cases}$$

Again we see that $E^t_2=E^t_\infty$ and that $\mathrm{Tot}(E^t_\infty)\cong\mathrm{Tot}(C^t)$ as graded $\mathbb{R}$–algebras. But $\mathrm{Tot}(C^t)=\mathrm{Tot}(C)$, and so $\mathrm{Tot}(E^t_2)\cong\mathrm{Tot}(E_2)$ as graded $\mathbb{R}$–algebras. Since $\mathrm{Tot}(E^t_2)^n\cong\mathrm{H}^n_{\mathrm{dR}}(M)$ and $\mathrm{Tot}(E_2)^n\cong\check{\mathrm{H}}^n(\mathfrak{U};\mathbb{R})$, the de Rham Theorem follows.

5. Edge maps

If $(E_r)_{r\geq a}$ is a first quadrant spectral sequence (i. e. $E_r^{p,q} \neq 0$ only if $p, q \geq 0$) which converges against a $\mathbb{Z}$–graded object C with respect to a canonically bounded filtration $F_\bullet C$, then there are two canonically defined *edge maps*: since the spectral sequence lies in the first quadrant, we see that no differential enters $E_r^{0,\bullet}$, so $\mathrm{H}^{0,\bullet}(E_r)$ is a submodule of $E_r^{0,\bullet}$. Hence, using the given identifications $\mathrm{H}^{\bullet,\bullet}(E_r) \cong E_{r+1}$ of the spectral sequence, we have for each n and all sufficiently large k a chain of injections

$$E_a^{0,n} \hookleftarrow E_{a+1}^{0,n} \hookleftarrow \ldots \hookleftarrow E_{n+1}^{0,n} \cong E_{n+2}^{0,n} \cong \ldots \cong E_{n+k}^{0,n} = E_\infty^{0,n}.$$

On the other hand, since the spectral sequence converges against C, there is given an isomorphism $E_\infty^{0,n} \cong \mathrm{gr}_0(F_\bullet C)^n$. Since C^n surjects onto $\mathrm{gr}_0(F_\bullet C)^n$, we thus obtain a map $e_F \colon C^n \to E_a^{0,n}$. This is the first edge map.

For the second edge map, observe that the differentials on $E_r^{\bullet,0}$ are all trivial, so $\mathrm{H}^{\bullet,0}(E_r)$ is a quotient of $E_r^{\bullet,0}$ and, using once more the identifications provided by the spectral sequence, we have a chain of surjections

$$E_a^{n,0} \to E_{a+1}^{n,0} \cdots \to E_{n+1}^{n,0} \cong E_\infty^{n,0}.$$

Then $E_\infty^{n,0} \cong \mathrm{gr}_n(F_\bullet C)^0 = F_n C \cap C^n$, and the resulting map $e_B \colon E_a^{n,0} \to F_n C \cap C^n \subseteq C^n$ again is what we call an edge map.

Proposition 5.1. *Let $(E_r)_{r\geq a}$ be a first quadrant spectral sequence convering against a $\mathbb{Z}$–graded object C with respect to a canonically bounded filtration $F_\bullet C$. If $(E_r)_{r\geq a}$ collapses on the a–th page, the edge map $e_B \colon E_a^{n,0} \to C^n$ is injective and the edge map $e_F \colon C^n \to E_a^{0,n}$ is surjective for all n.*

Proof. If the spectral sequence collapses on the initial page, then each of the maps $E_{a+\ell}^{n,0} \to E_{a+\ell+1}^{n,0}$ in the definition of the edge map is an isomorphism, and hence the edge map $e_B \colon E_a^{n,0} \to F_n C \cap C^n$ is an isomorphism too. Similarly, all maps $E_{a+\ell+1}^{0,n} \hookrightarrow E_{a+\ell}^{0,n}$ appearing in the definition of the edge map e_F must be isomorphisms, and then the edge map $e_F \colon C^n \to E_a^{0,n}$ still is surjective. ☐

If $A = \bigoplus_{p\in\mathbb{Z}} A^p$ and $B = \bigoplus_{q\in\mathbb{Z}} B^q$ are $\mathbb{Z}$–graded R–algebras, then the *(graded–commutative) tensor product* of A and B is the R–module $A \otimes_R B = \bigoplus_{p,q\in\mathbb{Z}} A^p \otimes_R B^q$ with multiplication given by

$$a \otimes b \cdot a' \otimes b' = (-1)^{qp'} \cdot (aa') \otimes (bb')$$

whenever $a \in A^p$, $a' \in A^{p'}$, $b \in B^q$, and $b' \in B^{q'}$. Under certain situations, we have a converse to proposition 5.1:

Proposition 5.2. *In the situation of proposition 5.1, additionally suppose that*

(i) $(E_r)_{r\geq a}$ is a spectral sequence of R–algebras,

(ii) *that $E_r = A \otimes_R B$ is the graded–commutative tensor product of $\mathbb{Z}$–graded R–algebras A and B,*

(iii) *and that A and B are unital R–algebras.*

Then, if the edge map $e_F\colon C^n \to E_a^{n,0}$ is injective for all n, the spectral sequence collapses on the initial page.

Proof. If the edge map is surjective, then all the maps $E^{n,0}_{a+\ell+1} \to E^{n,0}_{a+\ell}$ appearing in the definition of the edge map must be isomorphisms, because they are already injective. Therefore, each of the differentials $d_r|_{E_r^{0,q}}$ in the spectral sequence must be trivial, and since the spectral sequence lies in the first quadrant, also the differentials $d_r|_{E_r^{p,0}}$ must be trivial. But $(E_r)_{r\geq a}$ is a spectral sequence of algebras, so the differentials d_r are anti–derivations with respect to total degree. In particular, for $x \in A^p$, $y \in B^q$ we have

$$d_a(x \otimes y) = \pm d_a(x \otimes 1_A \cdot 1_B \otimes y) = \pm d_a(x) \otimes 1_A \pm 1_B \otimes d_a(y) = 0,$$

because necessarily $1_A \in A^0$ and $1_B \in B^0$, as A and B are graded algebras. Thus, d_a is trivial and $E_{a+1} \cong A \otimes_R B$ as $(\mathbb{Z} \times \mathbb{Z})$–graded algebras. Proceeding inductively, we see that all differentials d_r are trivial. □

6. The sheaf associated to a presheaf

To any presheaf F (of R–modules) on a topological space X one can associate its *etale space* or *sheaf space* F^+. This is the space defined by

$$F^+ := \overline{F}/_\sim, \text{ with } \overline{F} := \coprod_{\substack{U \subseteq X \\ U \text{ open}}} U \times F(U),$$

where for each open subset $U \subseteq X$ we consider $F(U)$ as a discrete space and two elements (u,s) and (v,t) with $u \in U$ and $v \in V$ are equivalent if $u = v$ and $s|_u = t|_v$. The canonical projection $\overline{F} \to X$ induces a well–defined continuous map $\pi\colon F^+ \to X$, which is a local homeomorphism. Indeed, since the projection $\overline{F} \to X$ is an open map and F^+ is endowed with the quotient topology, also π must be an open map. Moreover, if $V \subseteq X$ is any open subset and $A \subseteq F(V)$ is arbitrary, then the under the quotient map $q\colon \overline{F} \to F^+$ the set $V \times A$ maps to an open subset of F^+ by definition of the quotient topology, because for any open subset U and any $s \in F(U)$ with $s|_x = t|_x$ for some $t \in A$ there must be a whole neighborhood $W \subseteq U \cap V$ of x with $s|_W = t|_W$ by remark 3.3. This shows that $q\colon \overline{F} \to F^+$ is an open map, so in particular $V \times \{t\} \subseteq F^+$ is open for every $t \in F(V)$. Now we certainly have $\pi(V \times \{t\}) = V$, and if $[u,s]$, $[w,r]$ are two elements in $V \times \{t\}$ with $\pi([u,s]) = \pi([w,r])$, then $u = w$ and $s|_u = t|_u = r|_t$, whence $[u,s] = [w,r]$ and $\pi|_{V\times\{t\}}$ is a homeomorphism onto V.

One then defines a presheaf on X, again denoted by F^+ and called the *sheaf associated* to F (also: *sheaf of sections* or *sheafification* of F), by declaring $F^+(U)$ to be the set of all sections over U, that is, continuous morphisms $s\colon U \to F^+$ such that $\pi \circ s = \mathrm{id}_U$.

Note that F^+ actually is a sheaf and that for every open subset we have a canonical map $F(U) \to F^+(U)$, which associates to $s \in F(U)$ the section $U \to F^+$, $x \mapsto [x, s]$. That is to say, there exists a canonical morphism of presheaves $F \to F^+$. In particular, for every $x \in X$ we also have a map $F_x \to (F^+)_x$.

Proposition 6.1. *Let X be a topological space, F a presheaf on X, F^+ its sheafification, and $\Phi\colon F \to F^+$ the canonical morphism of presheaves.*

(i) The induced map on stalks $\Phi_x\colon F_x \to (F^+)_x$ is an isomorphism.

(ii) Φ is an isomorphism if and only if F is a sheaf.

Proof.

(i) Any element in the stalk F_x is represented by some element $s \in F(U)$ for some open neighborhood U of x. If $\Phi_x(s|_x) = 0$, then $\Phi_U(s)|_x = 0$, and hence shrinking U if necessary, we can assume that $\Phi_U(s) = 0$. In particular $(\Phi_U(s))(x) = [x, s]$ is trivial. Hence, on some even smaller neighborhood W of x we have $s|_W = 0$, whence $s|_x = 0$. It follows that Φ_x is injective. To see hat Φ_x is surjective, represent any given element of $(F^+)_x$ by $f \in F^+(U)$ for some open neighborhood U. By definition, $f(x) = [x, s]$ for some element $s \in F(W)$ and some open neighborhood $W \subseteq U$ of x, and shrinking W if necessary, we may assume that $\pi|_{\pi^{-1}(W)}$ is a homoeomorphism onto W. In particular, $y = \pi([y, s])$ for all $y \in W$, which shows that $f(y) = [y, s]$ on W. Hence $\Phi_W(s) = f$ and Φ_x is surjective.

(ii) First, assume that F is a sheaf and let $U \subseteq X$ be an open subset. We wish to show that $\Phi_U\colon F(U) \to F^+(U)$ is injective, so suppose that $\Phi_U(s) = \Phi_U(t)$ holds. By definition, this means that for every $x \in U$ we have

$$[x, s] = (\Phi_U(s))(x) = (\Phi_U(t))(x) = [x, t]$$

and thus $s|_x = t|_x$. Then there must exist an open neighborhood $U_x \subseteq U$ of x with $s|_{U_x} = t|_{U_x}$, and then $s = t$, because F is a sheaf and $(U_x)_{x \in U}$ covers U. Thus, Φ is injective. To show that Φ is surjective, let $f \in F^+(U)$ be given, $U \subseteq X$ open. For every point $x \in U$ we find an open neighborhood U_x of x and an element $s_x \in F(U_x)$ with $f(y) = [y, s_x]$ for all $y \in U_x$. Note that this just asserts that $\Phi_{U_x}(s_x) = f|_{U_x}$. In particular,

$$\Phi_{U_x \cap U_y}(s_x|_{U_x \cap U_y}) = f|_{U_x \cap U_y} = \Phi_{U_x \cap U_y}(s_y|_{U_x \cap U_y}),$$

and since we just showed that Φ is injective, we see that $s_x|_{U_x \cap U_y} = s_y|_{U_x \cap U_y}$. As F is a sheaf, there thus exists $s \in F(U)$ with $s|_{U_x} = s_x$ for all $x \in U$, and so

$$\Phi_U(s)|_{U_x} = \Phi_{U_x}(s_x) = f|_{U_x}$$

and hence $\Phi_U(s) = f$, because F^+ is a sheaf.

Now assume that Φ is an isomorphism, let $s \in F(U)$, and let $U = \bigcup_{i \in I} U_i$ be a cover such that $s|_{U_i} = 0$. Then in particular $s|_x = 0$ for all $x \in U$, and by definition of F^+ we have $\Phi_U(s)(x) = [x, s] = 0$ for all $x \in X$. Since Φ is assumed to be an isomorphism, $s = 0$. On the other hand, suppose that $U = \bigcup_{i \in I} U_i$ is a cover and that we are given $s_i \in F(U_i)$ for each i in such a way that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ holds for all i, j. This says that the element $f \in F^+(U)$ with $f(x) = [x, s_i]$ whenever $x \in U_i$ is well–defined and continuous. But Φ_U is surjective, so there must be an element $s \in F(U)$ with $\Phi_U(s) = f$. Hence,

$$\Phi_{U_i}(s|_{U_i}) = \Phi_U(s)|_{U_i} = f|_{U_i} = \Phi_{U_i}(s_i),$$

the last equality being satisfied by definition of Φ_{U_i} and f, and so, by injectivity of Φ, we have $s|_{U_i} = s_i$. ☐

Proposition 6.2. *Let $f\colon X \to Y$ be a morphism of topological spaces and G a presheaf on X. Then there is a canonical morphism of presheaves $\alpha\colon f_*(G^+) \to (f_*G)^+$ making the diagram*

$$\begin{tikzcd}
(f_*G)(V) \arrow[d] \arrow[dr] & \\
(f_*G)^+(V) \arrow[r, "\alpha_V"'] & f_*(G^+)(V)
\end{tikzcd}$$

commute for all open subsets $V \subseteq Y$.

Proof. On an open subset $V \subseteq Y$, the map α_V is defined as follows. If $s \in (f_*G)^+(V)$ is any section and $x \in f^{-1}(V)$, then $s(f(x)) = [f(x), t]$ for some element $t \in (f_*G)(W) = G(f^{-1}(W))$, where W is an open neighborhood of $f(x)$. Thus, $f^{-1}(W)$ is an open neighborhood of x and it makes sense to set

$$(\alpha_V s)(x) = [x, t].$$

Note that this definition is independent of the particular representative t of $s(f(x))$: if also $s(f(x)) = [f(x), t_0]$, then $t|_{f(x)} = (t_0)|_{f(x)}$, by construction of the equivalence relation on $(f_*G)^+$. That is to say, there exists an open neighborhood $Q \subseteq Y$ of $f(x)$ such that $t|_Q = (t_0)|_Q$, where the restriction maps are those of the presheaf $(f_*G)(Q)$. But by definition of f_*G, these are precisely $t|_Q = t|_{f^{-1}(Q)}$ and $(t_0)|_Q = (t_0)|_{f^{-1}(Q)}$, where the right hand side is the restriction map of the presheaf G. Hence, $t|_x = (t_0)|_x$ in G_x and so $[x, t] = [x, t_0]$ in G^+. This also shows that $\alpha_V(s)$ is a continuous map: given $x \in f^{-1}(V)$ we can always choose the neighborhood W so small that $s(f(y)) = [f(y), t]$ holds for all $y \in f^{-1}(W)$, and then $(\alpha_V s)(y) = [y, t]$ is continuous. ☐

Remark 6.3. In general, this morphism is not an isomorphism.

7. The Leray spectral sequence

Let X be a topological space and F a presheaf on X. The *q–th Leray presheaf* on X is the presheaf $H^q(-; F)$ on X given by

$$H^q(-; F)(U) = \check{\mathrm{H}}^q(U; F|_U).$$

Note that if $(f,k)\colon (X,F) \to (X_0,F_0)$ is a morphism, where F_0 is a presheaf on the topological space X_0, then (f,k) induces a natural transformation $(f,k)\colon H^q(-;F_0) \Rightarrow f_*H^q(-;F)$. In fact, if $V \subseteq X_0$ is an open subset, then consider $f|_{f^{-1}(V)}$ as a morphism $f|_{f^{-1}(V)}\colon f^{-1}(V) \to V$. We can restrict the f–cohomomorphism $k\colon F_0 \Rightarrow f_*F$ to

$$k|_V\colon (F_0)|_V \Rightarrow (f|_{f^{-1}(V)})_*(F|_{f^{-1}(V)})$$

and hence obtain a morphism $(f|_{f^{-1}(V)}, k|_V)\colon (f^{-1}(V), F|_{f^{-1}(V)}) \to (V,(F_0)|_V)$. Thus, we have an induced map $(f|_{f^{-1}(V)}, k|_V)^*\colon \check{\mathrm{H}}^q(V,(F_0)|_V) \to \check{\mathrm{H}}^q(f^{-1}(V); F|_{f^{-1}(V)})$ which is compatible with restrictions and hence provides the desired natural transformation.

Theorem 7.1. *Let X and Y be paracompact Hausdorff spaces, F a presheaf on X and $\pi\colon X \to Y$ a continuous map. There exists a first quadrant spectral sequence*

$$E_2^{p,q} = \check{\mathrm{H}}^p(Y;\pi_*\check{\mathrm{H}}^q(-;F)) \Longrightarrow \check{\mathrm{H}}^{p+q}(X;F),$$

called the Leray spectral sequence *of π, with the following properties.*

(i) The spectral sequence converges with respect to a canonically bounded filtration $F_\bullet H$.

(ii) The spectral sequence is natural with respect to commutative diagrams

$$\begin{tikzcd}
(X,F) \arrow[r, "{(f,k)}"] \arrow[d, "\pi"'] & (\overline{X},\overline{F}) \arrow[d, "\overline{\pi}"] \\
Y \arrow[r, "g"] & \overline{Y}.
\end{tikzcd}$$

That is to say, if $(\overline{E}_r)_{r\geq 2}$ is the Leray spectral sequence of $\overline{\pi}$, then there is an induced morphism of spectral sequences $(E_r \to \overline{E}_r)_{r\geq 2}$ which on the second page equals the morphism

$$(g,(f,k))^*\colon \check{\mathrm{H}}^p(\overline{Y};\overline{\pi}_*\check{\mathrm{H}}^q(-;\overline{F})) \to \check{\mathrm{H}}^p(Y;\pi_*\check{\mathrm{H}}^q(-;F)).$$

Moreover, if $F_\bullet\overline{H}$ is the filtration with respect to $(\overline{E}_r)_{r\geq 2}$ converges, then the morphism $(f,k)^*\colon \check{\mathrm{H}}^\bullet(\overline{X};\overline{F}) \to \check{\mathrm{H}}^\bullet(X;F)$ is filtration preserving and makes the following diagram commute for all p, q:

$$\begin{tikzcd}
E_\infty^{p,q} \arrow[r] \arrow[d, "\cong"'] & \overline{E}_\infty^{p,q} \arrow[d, "\cong"'] \\
\mathrm{gr}_p(F_\bullet H)^q \arrow[r, "{(f,k)^*}"] & \mathrm{gr}_p(F_\bullet\overline{H})^q
\end{tikzcd}$$

(iii) If F is a presheaf of R–algebras, then the spectral sequence converges as a spectral sequence of R–algebras. The multiplication on $E_2^{\bullet,\bullet}$ is given by $c\cdot c' = (-1)^{qp'}\cdot c \smile c'$ whenever $c \in E_2^{p,q}$ and $c' \in E_2^{p',q'}$.

We will mostly be interested in applying theorem 7.1 for coefficients in the constant presheaf $\underline{M}$ associated to some R–module M and when π is a locally trivial fiber bundle with typical fiber F, because we can say more about the Leray presheaf in this situation: observe that any local trivialization $\Phi = (\Phi_1, \Phi_2)\colon X|_U \to U \times F$ of π over some open subset $U \subseteq Y$ provides a morphism of R–modules

$$(\Phi_2|_{\pi^{-1}(V)})^*\colon \check{\mathrm{H}}^q(F; \underline{M}) \to \check{\mathrm{H}}^q(\pi^{-1}(V); \underline{M})$$

for all open subsets $V \subseteq U$ and that the right hand side of this map is just the value of the presheaf $\pi_*\check{\mathrm{H}}^q(-; \underline{M})$ on V. Considering the left hand side as a constant presheaf, we thus obtain an induced morphism

$$f_\Phi\colon \underline{\check{\mathrm{H}}^q(F; \underline{M})} \to (\pi_*\check{\mathrm{H}}^q(-; \underline{M}))|_U$$

between the constant presheaf on U associated to $\check{\mathrm{H}}^q(F; \underline{M})$ and the restricted Leray presheaf $(\pi_*\check{\mathrm{H}}^q(-; \underline{M}))|_U$. Since the stalks of a presheaf and its restriction are canonically isomorphic, we thus obtain a commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^q(F; \underline{M}) \arrow[r] \arrow[d, "\cong"'] & (\pi_*\check{\mathrm{H}}^q(-; \underline{M}))_y \arrow[d, "\cong"'] \\
(\underline{\check{\mathrm{H}}^q(F; \underline{M})})_y \arrow[r, "(f_\Phi)_y"] & ((\pi_*\check{\mathrm{H}}^q(-; \underline{M}))|_U)_y
\end{tikzcd}$$

and we denote the upper horizontal map again by $(f_\Phi)_y$. Note that for fixed $y \in U$, we also have a map going in the opposite direction: namely, we can define $h_{\Phi,y}$ to be the unique map making the diagram

$$\begin{tikzcd}
(\pi_*\check{\mathrm{H}}^q(W; \underline{M})) \arrow[r] \arrow[d] & \check{\mathrm{H}}^q(F; \underline{M}) \\
(\pi_*\check{\mathrm{H}}^q(-; \underline{M}))_y \arrow[ur, "h_{\Phi,y}"] &
\end{tikzcd}$$

commute for all open neighborhoods $W \subseteq U$ of y, where the upper horizontal map is induced by the map $F \to X_y \subseteq W$, $p \mapsto \Phi^{-1}(y, p)$. We certainly have $h_{\Phi,y} \circ (f_\Phi)_y = \mathrm{id}$, but in general $h_{\Phi,y}$ is not injective.

Proposition 7.2. *The map $(f_\Phi)_y$ is an isomorphism with inverse $h_{\Phi,y}$ for all $y \in U$ whenever at least one of the following conditions is satisfied:*

(i) Y is locally contractible or

(ii) F is homotopy equivalent to a compact Hausdorff space.

Proof. Note that this is a purely local question: since Y is paracompact Hausdorff, hence normal, the closed neighborhoods are cofinal in the set of all neighborhoods (cf. the argument given in example 7.3), so we can shrink U and assume that $U \subseteq N$ for some closed neighborhood $N \subseteq Y$ of y and that $\Phi\colon X|_N \to N \times F$ still is a homeomorphism. Then N and $X|_N$ are paracompact Hausdorff, as they are closed subsets of the paracompact Hausdorff spaces Y and X. Moreover, for each open neighborhood W of y we have a commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^q(F) \arrow[r, equal] \arrow[d] \arrow[dd, bend right=60, "(f_{\mathrm{id}})_y"'] & \check{\mathrm{H}}^q(F) \arrow[d] \arrow[dd, bend left=60, "(f_{\Phi})_y"] \\
\check{\mathrm{H}}^q(W \times F) \arrow[r, "\Phi^*"] \arrow[d] & \check{\mathrm{H}}^q(W|_U) \arrow[d] \\
((p_1)*\check{\mathrm{H}}^q(-))_y \arrow[r] & (\pi_*\check{\mathrm{H}}^q(-))_y
\end{tikzcd}$$

in which all horizontal maps are isomorphisms, induced by Φ, and p_1, p_2 are the canonical projections from $N \times F$ onto N, F, respectively; here and in what follows, cohomology is taken with coefficients in the constant presheaf associated to M. Thus, we may assume that $X = Y \times F$ is trivial, so $\pi = p_1$ and $\Phi = \mathrm{id}$, and in particular $\Phi_2 = p_2$.

Then if Y is locally contractible, the set of all contractible open neighborhoods V of y is cofinal in the set of all open neighborhoods of y, so in the formation of the limit $((p_1)_*\check{\mathrm{H}}^q(-))_y$ we may restrict to such neighborhoods. However, for any such neighborhood the map $p_2\colon V \times F \to F$ induces an isomorphism, and hence $(f_{\mathrm{id}})_y$ is an isomorphism.

Now let Y be completely arbitrary and assume that F is homotopy equivalent to a compact Hausdorff space. Passing to a trivial bundle $Y \times F_0$ with F_0 compact Hausdorff and homotopy equivalent to F if necessary, we can assume that F is compact Hausdorff. Then consider the commutative diagram

$$\begin{tikzcd}
 & \check{\mathrm{H}}^q(F) \arrow[dl] \arrow[d] \arrow[dr] & \\
((p_1)_*\check{\mathrm{H}}^q(-))_y \arrow[r] & \varinjlim_{Z \ni y} \check{\mathrm{H}}^q(Z \times F) & \varinjlim_{N \ni y} \check{\mathrm{H}}^q(N \times F) \arrow[l]
\end{tikzcd}$$

where the rightmost limit ranges over all closed (in Y) neighborhoods N of y and the centered limit is taken over all (arbitrary) neighborhoods Z of y. The horizontal maps, which are the limit maps, are isomorphisms, because the set of all open and closed neighborhoods of y are cofinal in the set of all neighborhoods of y: for the open neighborhoods this is immediate, and for the closed neighborhoods this is a consequence of normality of Y, as already observed earlier. Next, we note that

$$\begin{tikzcd}
 & \check{\mathrm{H}}^q(F) \arrow[d] \arrow[dr] & \\
 & \varinjlim_{N \ni y} \check{\mathrm{H}}^q(N \times F) \arrow[r] \arrow[d] & \check{\mathrm{H}}^q(\{y\} \times F) \\
 & \varinjlim_{A \supseteq \{y\} \times F} \check{\mathrm{H}}^q(A \times F) \arrow[ur] &
\end{tikzcd}$$

commutes, where now the lower limit ranges over all closed neighborhoods $A \subseteq Y \times F$ of $\{y\} \times F$. Certainly, the upper diagonal map is an isomorphism. As F is compact Hausdorff, $Y \times F$ is paracompact Hausdorff and $\{y\} \times F$ is closed in $Y \times F$. Therefore, the lower diagonal map is an isomorphism by the tautness property of Čech cohomology (theorem 9.5), and we wish to show that the lower vertical map is an isomorphism. To do so, it will suffice to show that the set of all neighborhoods of the form $N \times F$ for N a closed neighborhood of y is cofinal in the set of all closed neighborhoods $A \subseteq X \times F$ of $\{y\} \times F$. Thus, let A be a closed neighborhood of $\{y\} \times F$. Because F is compact, there must be an open neighborhood W of y such that $W \times F \subseteq A \times F$. Then if we let N be a closed neighborhood of y with $N \subseteq W$, we have $N \times F \subseteq A \times F$ as required. Therefore, $(f_\Phi)_y \colon \check{\mathrm{H}}^q(F) \to ((p_1)_*\check{\mathrm{H}}^q(-))_y$ is an isomorphism, and since we already observed that $h_{\Phi,y} \circ (f_\Phi)_y = \mathrm{id}$, its inverse must be given by $h_{\Phi,y}$. □

For a locally trivial fiber bundle $\pi \colon X \to Y$ with typical fiber F, let us say that the associated Leray presheaf with coefficients in some R–module M is *simple* if there is a cover $(U_i)_{i \in I}$ of Y by open subsets and for each $i \in I$ a local trivialization $\Phi_i \colon X|_{U_i} \to U_i \times F$ over U_i with the following properties.

(i) For each $i \in I$, all $y \in U$, and every q the induced map $(f_{\Phi_i})_y \colon \check{\mathrm{H}}^q(F; \underline{M}) \to (\pi_*\check{\mathrm{H}}(-; \underline{M}))_y$ is an isomorphism.

(ii) For all $i, j \in I$, all $y \in U_i \cap U_j$, and all q the following diagram commutes:

$$\begin{tikzcd}
\check{\mathrm{H}}^q(F; \underline{M}) \arrow[rr, "\mathrm{id}"] \arrow[dr, "(f_{\Phi_i})_y"'] & & \check{\mathrm{H}}^q(F; \underline{M}) \arrow[dl, "(f_{\Phi_j})_y"] \\
 & (\pi_*\check{\mathrm{H}}(-; \underline{M}))_y &
\end{tikzcd}$$

Proposition 7.3. *Let $\pi \colon X \to Y$ be a locally trivial fiber bundle with typical fiber F and M an R–module. Suppose that the associated Leray presheaf $\pi_*\check{\mathrm{H}}^\bullet(-; \underline{M})$ of π is simple. Then in the Leray spectral sequence $(E_r)_{r \geq 2}$ of π we may take*

$$E_2^{p,q} = \check{\mathrm{H}}^p\big(Y; \underline{\check{\mathrm{H}}^q(F; \underline{M})}\big).$$

If M is an R–algebra, then this equality holds as bigraded R–algebras (with multiplication on E_2 defined as in theorem 7.1).

Proof. For the sake of simplicity, we assume that M is an R–algebra and suppress coefficients in the notation. We collect the various Leray presheaves into one presheaf $\pi_*\check{\mathrm{H}}^\bullet(-) := \bigoplus_{q\geq 0}\pi_*\check{\mathrm{H}}^q(-)$, which then becomes a presheaf of R–algebras via the cup product. Let then $\mathcal{H}^q$ be the sheaf of R–modules associated to the q–th Leray presheaf $\pi_*\check{\mathrm{H}}^q(-)$ and $\mathcal{H}^\bullet$ the sheaf of R–algebras associated to $\pi_*\check{\mathrm{H}}^\bullet(-)$. Let further $(\Phi_i)_{i\in I}$ be a collection of local trivializations of X as in the definition of a simple Leray presheaf. For a non–empty open subset $V \subseteq Y$ we define a morphism $\Psi_V = \Psi_V^q \colon \check{\mathrm{H}}^q(F) \to \mathcal{H}^q$ as follows. Given $c \in \check{\mathrm{H}}^q(F)$, the section $\Psi_V(c)\colon V \to (\pi_*\check{\mathrm{H}}^q(-))^+$ is defined by

$$\Psi_V(c)(y) := [y, ((\Phi_i)_2)^*(c)]$$

whenever $y \in U_i$. This definition is independent of the particular trivialization chosen, since $((\Phi_i)_2)^*(c)|_y = ((\Phi_j)_2)^*(c)|_y$ holds by definition of simplicity. This also shows that $\Psi_V(c)$ is continuous and that Ψ_V is a morphism of R–modules, so the various Ψ_V piece together to give a morphism of preshaves of R–modules $\Psi\colon \underline{\check{\mathrm{H}}^q(F)} \to \mathcal{H}^q$. Note that if $y \in V$, then the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^q(F) \arrow[r] \arrow[d, "\cong"'] & (\pi_*\check{\mathrm{H}}^q(-))_y \arrow[d, "\cong"] \\
\underline{\check{\mathrm{H}}^q(F)}_y \arrow[r, "\Psi_y"] & (\mathcal{H}^q)_y
\end{tikzcd}$$

commutes and that the upper horizontal map is an isomorphism, because we are assuming the Leray presheaf to be simple. Hence, the lower map is an isomorphism as well. Now consider the canonical morphism $\pi_*\check{\mathrm{H}}^q(-)) \to \mathcal{H}^q$ of proposition 6.1. It induces stalkwise isomorphisms and therefore also an isomorphism in cohomology

$$E_2^{p,q} \to \check{\mathrm{H}}^p(Y;\mathcal{H}^q) =: A^{p,q}$$

for all p, q, because Y is paracompact Hausdorff. These isomorphisms assemble to an isomorphism of bigraded R–modules $E_2 \to A^{\bullet,\bullet} = \bigoplus_{p,q} A^{p,q}$. Declare a product on A by

$$\begin{tikzcd}
\check{\mathrm{H}}^p(Y;\mathcal{H}^q) \times \check{\mathrm{H}}^{p'}(Y;\mathcal{H}^{q'}) \arrow[r] \arrow[d] & \check{\mathrm{H}}^{p+p'}(Y;\mathcal{H}^{q+q'}) \\
\check{\mathrm{H}}^p(Y;\mathcal{H}^\bullet) \times \check{\mathrm{H}}^{p'}(Y;\mathcal{H}^\bullet) \arrow[r, "(-1)^{qp'}\smile"] & \check{\mathrm{H}}^{p+p'}(Y;\mathcal{H}^\bullet) \arrow[u]
\end{tikzcd}$$

where the left vertical map is induced by the canonical morphisms of presheaves $\mathcal{H}^q \to \mathcal{H}^\bullet$ and $\mathcal{H}^{q'} \to \mathcal{H}^\bullet$, and the right vertical map is induced by the canonical morphism of presheaves $\mathcal{H}^\bullet \to \mathcal{H}^{q+q'}$. Then the isomorphism $E_2 \to A^{\bullet,\bullet}$ is actually an isomorphism of $(\mathbb{Z}\times\mathbb{Z})$–graded R–algebras, because by definition of the multiplicative structure on

E_2 we have a commutative square

$$\begin{tikzcd}
 & A^{p,q} \times A^{p',q'} \arrow[rr] \arrow[dd] & & A^{p+p',q+q'} \\
E_2^{p,q} \times E_2^{p',q'} \arrow[ur] \arrow[rr] \arrow[dd] & & E_2^{p+p',q+q'} \arrow[ur] & \\
 & \check{\mathrm{H}}^p(Y;\mathcal{H}^\bullet) \times \check{\mathrm{H}}^{p'}(Y;\mathcal{H}^\bullet) \arrow[rr, "(-1)^{qp'}\smile"] & & \check{\mathrm{H}}^{p+p'}(Y;\mathcal{H}^\bullet) \arrow[uu] \\
\check{\mathrm{H}}^p(Y;\pi_*\check{\mathrm{H}}^\bullet(-)) \times \check{\mathrm{H}}^{p'}(Y;\pi_*\check{\mathrm{H}}^\bullet(-)) \arrow[ur] \arrow[rr, "(-1)^{qp'}\smile"] & & \check{\mathrm{H}}^{p+p'}(Y;\pi_*\check{\mathrm{H}}^\bullet(-)) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

in which all diagonal maps are isomorphisms, as they are induced by maps from a presheaf into its associated sheaf. Similarly, we can make $B^{\bullet,\bullet} = \bigoplus_{p,q} B^{p,q}$ with $B^{p,q} = \check{\mathrm{H}}^p(Y;\underline{\check{\mathrm{H}}^q(F)})$ a $(\mathbb{Z}\times\mathbb{Z})$–graded R–algebra by

$$\begin{tikzcd}
\check{\mathrm{H}}^p(Y;\underline{\check{\mathrm{H}}^q(F)}) \times \check{\mathrm{H}}^{p'}(Y;\underline{\check{\mathrm{H}}^{q'}(F)}) \arrow[r] \arrow[d] & \check{\mathrm{H}}^{p+p'}(Y;\check{\mathrm{H}}^{q+q'}(F)) \\
\check{\mathrm{H}}^p(Y;\underline{\check{\mathrm{H}}^\bullet(F)}) \times \check{\mathrm{H}}^{p'}(Y;\underline{\check{\mathrm{H}}^\bullet(F)}) \arrow[r, "(-1)^{qp'}\smile"] & \check{\mathrm{H}}^{p+p'}(Y;\underline{\check{\mathrm{H}}^\bullet(F)}) \arrow[u]
\end{tikzcd}$$

This R–algebra structure is exactly the algebra structure claimed, and Ψ induces an isomorphism of $(\mathbb{Z}\times\mathbb{Z})$–graded R–algebras $B^{\bullet,\bullet}\to A^{\bullet,\bullet}$. As these isomorphisms are natural, the claim follows from remark 2.2. □

Corollary 7.4. *In the situation of proposition 7.3, suppose that M is a free R–module. Then we even may take*

$$E_2^{p,q} = \check{\mathrm{H}}^q(Y;\underline{R}) \otimes_R \check{\mathrm{H}}^p(F;\underline{M}),$$

as $(\mathbb{Z}\times\mathbb{Z})$–graded R–modules. If M is an R–algebra which is free as an R–module, this equality also holds as graded $(\mathbb{Z}\times\mathbb{Z})$–graded R–algebras, where the right hand side is to be understood as (the (p,q)–th graded component of) the graded–commutative tensor product of the R–algebras $\check{\mathrm{H}}^\bullet(Y;\underline{R})$ and $\check{\mathrm{H}}^\bullet(F;\underline{M})$.

Proof. If M is a free R–module, this is an immediate consequence of the Universal Coefficient Theorem (theorem 3.3) and remark 2.2, since under these assumptions we have $\mathrm{Tor}_1^R(N,M)=0$ for all R–modules N. If M is an R–algebra which is free as an

R–module, we in addition observe that

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{V};\underline{R}) \otimes_R \check{\mathrm{H}}^q(F;\underline{M}) \times \check{\mathrm{H}}^r(\mathfrak{V};\underline{R}) \otimes_R \check{\mathrm{H}}^s(F;\underline{M}) \arrow[r] \arrow[d, "\mu_{\mathfrak{V}} \times \mu_{\mathfrak{V}}"'] & \check{\mathrm{H}}^{p+r}(\mathfrak{V};\underline{R}) \otimes_R \check{\mathrm{H}}^{q+s}(F;\underline{M}) \arrow[d, "\mu_{\mathfrak{V}}"] \\
\check{\mathrm{H}}^p(\mathfrak{V};\underline{\check{\mathrm{H}}^q(F;\underline{M})}) \times \check{\mathrm{H}}^r(\mathfrak{V};\underline{\check{\mathrm{H}}^s(F;\underline{M})}) \arrow[r, "(-1)^{qr}\smile"] & \check{\mathrm{H}}^{p+r}(\mathfrak{V};\underline{\check{\mathrm{H}}^{q+s}(F;\underline{M})})
\end{tikzcd}$$

commutes, where the undecorated horizontal map is multiplication. This is because for all cocycles $c \in \check{C}^p(\mathfrak{V};\underline{R})$ and $c' \in \check{C}^r(\mathfrak{V};\underline{R})$, all elements $d \in \check{\mathrm{H}}^q(F;\underline{M})$ and $d' \in \check{\mathrm{H}}^s(F;\underline{M})$, and all $(p+r)$–simplices σ we have, in the notation of theorem 3.3:

$$\begin{aligned}
((c * d) \smile (c' * d'))(\sigma) &= (c(\sigma_0,\dots,\sigma_p)\cdot d) \smile (c'(\sigma_p,\dots,\sigma_{p+r})\cdot d') \\
&= c(\sigma_0,\dots,\sigma_p)c(\sigma_p,\dots,\sigma_{p+r})\cdot(d \smile d') \\
&= ((c \smile c') * (d \smile d'))(\sigma),
\end{aligned}$$

since the cup product is R–bilinear. ☐

Example 7.5. Let $\pi\colon E \to B$ be a (locally trivial) fiber bundle, where E, B are paracompact Hausdorff and B is locally contractible or the typical fiber F is homotopy equivalent to a compact Hausdorff space. Suppose there exists a cover $(U_i)_{i\in I}$ of B by open subsets and trivializations $\Phi_i\colon E|_{U_i} \to U_i \times F$ such that for all i, j for which $U_i \cap U_j$ is non–empty and all $x \in U_i \cap U_j$ the pointwise transition functions

$$t_{ji,x}\colon F = \{x\} \times F \xrightarrow{\Phi_j \circ \Phi_i^{-1}|_{U_i \cap U_j}} \{x\} \times F = F$$

induce the identity map on cohomology. Then the Leray presheaf of π with coefficients in any R–module M is simple: in fact, $t_{ji,x}$ makes the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^q(F) \arrow[r, "(t_{ji,x})^*"] & \check{\mathrm{H}}^q(F) \\
\check{\mathrm{H}}^q(E_x) \arrow[u, "((\Phi_j)^{-1})^*"] & \check{\mathrm{H}}^q(E_x) \arrow[u, "((\Phi_i)^{-1})^*"'] \\
\check{\mathrm{H}}^q(E|_W) \arrow[u] \arrow[r, equal] & \check{\mathrm{H}}^q(E|_W) \arrow[u]
\end{tikzcd}$$

commute for all open neighborhoods $W \subseteq U_i \cap U_j$ of y. Hence, also the diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^q(F) \arrow[r, "(t_{ji,x})^*"] & \check{\mathrm{H}}^q(F) \\
(\pi_*\check{\mathrm{H}}^q(-))_x \arrow[u, "h_{\Phi_j,x}"] \arrow[r, equal] & (\pi_*\check{\mathrm{H}}^q(-))_x \arrow[u, "h_{\Phi_i,x}"']
\end{tikzcd}$$

is commutative. Since $(t_{ji,x})^* = \mathrm{id}$ by assumption and $(h_{\Phi_i,x})^{-1} = (f_{\Phi_i})_x$ as well as $(h_{\Phi_j,x})^{-1} = (f_{\Phi_j})_x$ by proposition 7.2, it follows that the Leray presheaf of π is simple.

Example 7.6. Assume that $E = E_0 \times_G F$ is associated to a principal G–bundle $\tau\colon E_0 \to B$ and that G acts trivially on $\check{\mathrm{H}}^\bullet(F)$. That is to say, for any $g \in G$, let $T_g\colon F \to F$ be the map $T_g(p) = gp$ and assume that $(T_g)^* = \mathrm{id}$. Then, if E, B are paracompact Hausdorff and B is locally contractible or F is homotopy equivalent to a compact space, the previous example applies and shows that the Leray presheaf of $\pi\colon E \to B$ is simple. To see this, recall that if we choose a trivialization $\Phi = (\Phi_1, \Phi_2)$ of $E_0 \to B$ over U, then by proposition 2.10 we obtain an induced trivialization of $E \to B$ over U given by

$$E|_U \to U \times F,\ [x,p] \mapsto (\tau(x), \Phi_2(x)p).$$

Thus, if Ψ is another trivialization of $E_0 \to B$ over some open subset V, then the transition function $\Phi \circ \Psi^{-1}|_{U \cap V}$ is given by

$$(U \cap V) \times F \to (U \cap V) \times F,\ (x,p) \mapsto (x, \Phi_2(\Psi^{-1}(x,1))p),$$

so the pointwise transition function $F \to F$ over x is just the map T_g for $g = \Phi_2(\Psi^{-1}(x,1))$.

Note that G always acts trivially on F if G is connected, for any path γ from $\gamma(0) = g$ to $\gamma(1) = 1$ provides a homotopy $F \times [0,1] \to F$, $p \mapsto \gamma(t)p$ between T_g and id_F.

8. Edge maps in the Leray spectral sequence

Proposition 8.1. *Let $\pi\colon X \to Y$ be a locally trivial fiber bundle between paracompact Hausdorff spaces and that the Leray presheaf of π is simple.*

(i) If the typical fiber $F = \pi^{-1}(*)$ is connected, then in the Leray spectral sequence $(E_r)_{r \geq 2}$ of π the edge map

$$e_B\colon E_2^{n,0} \to \check{\mathrm{H}}^n(X; \underline{R})$$

is, up to isomorphism on $E_2^{n,0} \cong \check{\mathrm{H}}^n(Y; \underline{R})$, the projection $\pi^*\colon \check{\mathrm{H}}^n(Y; \underline{R}) \to \check{\mathrm{H}}^n(X; \underline{R})$.

(ii) If Y is connected, then, up to isomorphism on $E_2^{0,n} \cong \check{\mathrm{H}}^n(F; \underline{R})$ the edge map

$$e_F\colon \check{\mathrm{H}}^n(X; \underline{R}) \to E_2^{0,n}$$

equals the fiber inclusion $\check{\mathrm{H}}^n(X; \underline{R}) \to \check{\mathrm{H}}^n(F; \underline{R})$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[d, "\pi"'] & Y \arrow[d, "\mathrm{id}_Y"] \\
Y \arrow[r, "\mathrm{id}_Y"] & Y
\end{tikzcd}$$

We have an induced map $(\alpha_r\colon \overline{E}_r \to E_r)_{r \geq 2}$, where E_r is the spectral sequence of π and $\overline{E}_r$ is the spectral sequence of id_Y, the latter considered as a locally trivial fiber bundle with fiber $*$ a point with $\pi^{-1}(*) = F$. Since $(\alpha_r)_{r\geq 2}$ is a map of spectral sequences, we have a commutative diagram

$$\begin{tikzcd}
\overline{E}_2^{n,0} \arrow[r] \arrow[d, "\alpha_2"] & \overline{E}_3^{n,0} \arrow[r] \arrow[d, "\alpha_3"'] & \dots \arrow[r] & \overline{E}_{n+1}^{n,0} \arrow[r, "\cong"] \arrow[d, "\alpha_{n+1}"'] & \mathrm{gr}_n(F_\bullet \overline{H})^0 \arrow[r] & \check{\mathrm{H}}^n(Y) \arrow[d, "\pi^*"] \\
E_2^{n,0} \arrow[r] & E_3^{n,0} \arrow[r] & \dots \arrow[r] & E_{n+1}^{n,0} \arrow[r, "\cong"] & \mathrm{gr}_n(F_\bullet H)^0 \arrow[r] & \check{\mathrm{H}}^n(X)
\end{tikzcd}$$

in which $F_\bullet H$ and $F_\bullet \overline{H}$ are the filtrations with respect to which the spectral sequences $(E_r)_{r\geq 2}$ and $(\overline{E}_r)_{r\geq 2}$ converge, respectively; the composition of the upper and lower vertical maps gives the edge maps of the corresponding spectral sequence. But since $Y \to Y$ has fiber $*$, the spectral sequence $(\overline{E}_r)_{r\geq 2}$ is a single row, concentrated in bidegrees $(\bullet, 0)$. In particular, it collapses on the second page, $F_n\overline{H} \cap \check{\mathrm{H}}^n(Y) = \check{\mathrm{H}}^n(Y)$, and its edge map $\overline{e}_B$ is an isomorphism by proposition 5.1. Moreover,

$$\begin{tikzcd}
\check{\mathrm{H}}^n(Y; \underline{\check{\mathrm{H}}^0(*)}) \arrow[r, equal] \arrow[d, "(\mathrm{id}_Y, (\pi|_F)^*)^*"'] & \overline{E}_2^{n,0} \arrow[d, "\alpha_2"] \\
\check{\mathrm{H}}^n(Y; \underline{\check{\mathrm{H}}^0(F)}) \arrow[r, equal] & E_2^{n,0}
\end{tikzcd}$$

commutes, where $(\pi|_F)^*$, the map on cohomology induced by $\pi|_F \colon F \to *$, is an isomorphism, because F is connected. Thus, α_2 is an isomorphism and the edge map $e_B \colon E_2^{n,0} \to \check{\mathrm{H}}^n(X)$ is as desired.

To identify the other edge map, we once more exploit naturality of spectral sequences and consider the commutative diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\pi|_F"'] & X \arrow[d, "\pi"] \\
* \arrow[r] & Y
\end{tikzcd}$$

Let $(\overline{E}_r)_{r\geq 2}$ be the spectral sequence of $\pi|_F$ and $(\beta_r \colon E_r \to \overline{E}_r)_{r\geq 2}$ the map of spectral sequences arising from the commutative diagram above. We have a commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^n(X) \arrow[r] \arrow[d] & \mathrm{gr}_0(F_\bullet H)^n \arrow[r, "\cong"] \arrow[d] & E_{n+1}^{0,n} \arrow[r] \arrow[d, "\beta_{n+1}"'] & \dots \arrow[r] & E_3^{0,n} \arrow[r] \arrow[d, "\beta_3"] & E_2^{0,n} \arrow[d, "\beta_2"] \\
\check{\mathrm{H}}^n(F) \arrow[r] & \mathrm{gr}_0(F_\bullet \overline{H})^n \arrow[r, "\cong"] & \overline{E}_{n+1}^{0,n} \arrow[r] & \dots \arrow[r] & \overline{E}_3^{0,n} \arrow[r] & \overline{E}_2^{0,n}
\end{tikzcd}$$

and the spectral sequence $(\overline{E}_r)_{r\geq 2}$ is a single column, concentrated in bidegrees $(0, \bullet)$. Hence, it collapses, $\mathrm{gr}_0(F_\bullet \overline{H})^n = \check{\mathrm{H}}^n(F)$, and its edge map $\overline{e}_F$ is an isomorphism. Then since Y is connected, the vertical maps in the commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^0(Y;\underline{\check{\mathrm{H}}^n(F)}) \arrow[r, equal] \arrow[d, "(\pi|_F{,}\mathrm{id}_F)^*"'] & E_2^{0,n} \arrow[d, "\beta_2"] \\
\check{\mathrm{H}}^0(*;\underline{\check{\mathrm{H}}^n(F)} \arrow[r, equal] & \overline{E}_2^{0,n}
\end{tikzcd}$$

are isomorphisms, and we see that $e_F\colon H^n(X) \to E_2^{0,n}$ has the required properties. $\square$

Corollary 8.2. *(Vietoris–Begle Mapping Theorem) Let* $\pi\colon X \to Y$ *be a locally trivial fiber bundle between paracompact Hausdorff spaces and suppose that* Y *is locally contractible or that the typical fiber* F *of* π *is homotopy equivalent to a compact Hausdorff space. Then if* $\check{\mathrm{H}}^q(F;\underline{M}) = \check{\mathrm{H}}^q(*;\underline{M})$ *for all* $q \leq N$ *and some* R*–module* M*, also*

$$\pi^*\colon \check{\mathrm{H}}^q(Y;\underline{M}) \to \check{\mathrm{H}}^q(X;\underline{M})$$

is an isomorphism for all $q \leq N$*.*

Proof. Under these assumptions, the stalk $(\pi_*\check{\mathrm{H}}^q(-))_y$ of any point $y \in Y$ is isomorphic to $\check{\mathrm{H}}^q(F;\underline{M})$. But by assumption this group is trivial unless $q = 0$, in which case $\check{\mathrm{H}}^0(F;\underline{M}) \cong M$. Therefore, since Y is paracompact Hausdorff, corollary 9.3 implies that the Leray spectral sequence $(E_r)_{r\geq 2}$ of π has as initial page

$$E_2^{p,q} \cong \begin{cases} 0, & 0 < q \leq N, \\ \check{\mathrm{H}}^p(Y;\underline{M}), & q = 0, \end{cases}$$

for all $q \leq N$. This means that also $E_r^{p,q} = 0$ for all r and all $0 < q \leq N$, and in particular all differentials $d_r^{p,q}$ with $0 < q \leq N$ must be trivial. However, since $(E_r)_{r\geq 2}$ is a first quadrant spectral sequence, the only possible differentials $d_r^{s,t}$ such that $\mathrm{im}(d_r^{s,t}) \subseteq E_r^{p,0}$ for some p are those with $t < p$, which is why we must have $E_2^{p,0} = E_\infty^{p,0}$ for all $p \leq N$. In particular, the edge maps $e_B\colon E_2^{p,0} \to \check{\mathrm{H}}^p(X;\underline{M})$ are injective and hence isomorhisms for all $p \leq N$, and these maps are, by the same reasoning as in the proof of proposition 8.1, up to isomorphism equal to $\pi^*\colon \check{\mathrm{H}}^p(Y;\underline{M}) \to \check{\mathrm{H}}^p(X;\underline{M})$ for all $p \leq N$. $\square$

Example 8.3. Suppose X and Y are connected, paracompact Hausdorff spaces and that Y is locally contractible or that X is homotopy equivalent to a compact Hausdorff space. The canonical projection $\pi\colon Y \times X \to Y$ is a locally trivial fiber bundle with typical fiber X, and the associated Leray presheaf is simple: the identity map $Y \times X \to Y \times X$ is a trivialization and induces a stalkwise identification $\check{\mathrm{H}}^q(X;\underline{\mathbb{K}}) \to \pi_*\check{\mathrm{H}}^q(-;\underline{\mathbb{K}})$ by proposition 7.2. Hence, by corollary 7.4 we have

$$E_2^{p,q} = \check{\mathrm{H}}^p(Y;\underline{\check{\mathrm{H}}^q(F;\underline{\mathbb{K}})}) \cong \check{\mathrm{H}}^p(Y;\underline{\mathbb{K}}) \otimes_{\mathbb{K}} \check{\mathrm{H}}^q(X;\underline{\mathbb{K}}) =: \overline{E}_2^{p,q}$$

as graded $\mathbb{K}$–algebras. We claim that the spectral sequence $(E_2)_{r\geq 2}$ and hence also $(\overline{E}_2)_{r\geq 2}$ collapses on the second page. To see this, it suffices by proposition 5.2 to show that the edge map $e_F\colon \check{\mathrm{H}}^n(X) \to \check{\mathrm{H}}^n(Y \times X)$ is surjective. But this edge map is up to isomorphism just the fiber inclusion ι^*, and we have the projection $\pi\colon Y \hookrightarrow X$, which satisfies $\pi \circ \iota = \mathrm{id}_X$. Thus, $\iota^* \circ \pi^* = \mathrm{id}$ and ι^* is surjective. It follows that

$$\check{\mathrm{H}}^n(Y \times X; \underline{\mathbb{K}}) \cong \bigoplus_{p+q=n} \check{\mathrm{H}}^p(Y; \underline{\mathbb{K}}) \otimes_{\mathbb{K}} \check{\mathrm{H}}^q(X; \underline{\mathbb{K}})$$

as $\mathbb{K}$–vector spaces. This is a weak form of the Künneth Theorem, which states that this abstract isomorphism of $\mathbb{K}$–vector spaces can in fact be realized by an isomorphism of $\mathbb{K}$–algebras: namely, by the maps $y \otimes x \mapsto (p_1)^*(y) \smile (p_2)^*(x)$ for all $y \in \check{\mathrm{H}}^p(Y; \underline{\mathbb{K}})$ and $x \in \check{\mathrm{H}}^q(X; \underline{\mathbb{K}})$, where p_1 and p_2 are the canonical projections onto the factors of $Y \times X$.

Chapter IV.

Equivariant cohomology

Within this chapter, unless stated otherwise, cohomology will always be taken in a constant presheaf $\underline{\mathbb{K}}$ for some field $\mathbb{K}$.

1. Definition and first computations

Definition 1.1. Let X be a topological space and $\rho\colon X \times G \to X$ a (left–) action. The *equivariant cohomology* of ρ (or G, if there is no ambiguity) is $\mathrm{H}^\bullet_G(X) := \check{\mathrm{H}}^\bullet(EG \times_G X)$, where $EG \times_G X$ is the total space of the fiber bundle with fiber X associated to the universal principal G–bundle $EG \to BG$ arising from the Milnor construction. We set $\mathrm{H}^\bullet_G := \check{\mathrm{H}}^\bullet(BG)$.

Remark 1.2. If G and X are compact and X is Hausdorff, we can compute $\mathrm{H}^n_G(X)$ using any principal G–bundle $E \to B$ such that E is compact Hausdorff and such that $\check{\mathrm{H}}^q(E) = \check{\mathrm{H}}^q(*)$ for all $q \leq n$. To see this, first of all note that $EG = \bigcup_{n \geq 1} E_n G$ is a countable union of compact subspaces and hence paracompact, cf [12, p. 66]. Also, EG is Hausdorff, whence EG and BG are paracompact Hausdorff. Now view $E \times EG$ as a G–space via the product action and consider

$$\begin{tikzcd} & (E \times EG) \times_G X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E \times_G X & & EG \times_G X \end{tikzcd}$$

where p_1 and p_2 are induced by the canonical projections from $E \times EG \times X$ onto $E \times X$ and $EG \times X$, respectively. Note that p_1 is a locally trivial fiber bundle with typical fiber EG, which is contractible, and that $(E \times EG) \times_G X$ is paracompact Hausdorff, because E and X are compact Hausdorff. Hence, by the Vietoris–Begle Theorem (corollary 8.2) $(p_1)^*$ induces an isomorphism in cohomology. Similarly, since E has the homotopy type of a point in degrees less than n, $(p_2)^*$ induces an isomorphism in degree n and so $((p_1)^*)^{-1} \circ (p_2)^*$ provides an isomorphism between $\mathrm{H}^n_G(X)$ and $\check{\mathrm{H}}^n(E \times_G X)$.

Principal G–bundles $E \to B$ such that $\check{\mathrm{H}}^q(E) = \check{\mathrm{H}}^q(*)$ for all $q \leq n$ are called n–*universal*, and n–universal bundles with compact Hausdorff total space always exist for compact Lie groups. In fact, one can simply take $E_N G \to B_N G$ for N large enough, because one can show (but we will not) that $\check{\mathrm{H}}^q(E_N G) = 0$ for all $0 < q \leq N - 2$. In the case of most interest to us, namely, if G is a subgroup of an r–torus $T = (S^1)^r$, we can be explicit, however. In this case, we can just take $E = (S^{2k-1})^r$ with the standard S^1–action and $B = E/G$ for any $k > n$, since by the Künneth formula $\check{\mathrm{H}}^q(E)$ is trivial in degrees $0 < q < k$ (see also example 1.3 below).

Example 1.3. Let us use the previous reasoning to show that as $\mathbb{K}$–algebras

$$\check{\mathrm{H}}^{\bullet}(BT) = \mathbb{K}[t_1, \dots, t_r]$$

for $T = (S^1)^r$ an r–torus and $r \geq 1$, where $t_1, \dots, t_r \in \check{\mathrm{H}}^2(BT)$. To this end, fix $k \geq 1$ and consider $S^{2k+1} \subseteq \mathbb{C}^{k+1}$ as an S^1–space via the action of $S^1 \subseteq \mathbb{C}$ given on S^{2k+1} by multiplication in each factor. Then $E = (S^{2k+1})^r$ is a T–space with $B = E/S^1 = (\mathbb{CP}^k)^r$. In example 8.7 we already computed that $\mathbb{CP}^k = \mathbb{K}[x]/(x^{k+1})$ for some element $x \in \check{\mathrm{H}}^2(\mathbb{CP}^k)$ in the image of the inclusion induced map $\check{\mathrm{H}}^2(\mathbb{CP}^k) \to \check{\mathrm{H}}^2(\mathbb{CP}^2)$, and so by the Künneth formula we have

$$\check{\mathrm{H}}^q(B) \cong \bigoplus_{p_1+\ldots+p_r=q} \check{\mathrm{H}}^{p_1}(\mathbb{CP}^k) \otimes_{\mathbb{K}} \ldots \otimes_{\mathbb{K}} \check{\mathrm{H}}^{p_r}(\mathbb{CP}^k) \cong \mathbb{K}^q[x_1, \dots, x_r],$$

as $\mathbb{K}$–algebras in degrees $q \leq 2k$. The reasoning of remark 1.2 thus shows that also

$$\check{\mathrm{H}}^q(BT) \cong \check{\mathrm{H}}^q_T(*) \cong \check{\mathrm{H}}^q(B) \cong \mathbb{K}^q[x_1, \dots, x_r]$$

holds as $\mathbb{K}$–algebras in degrees $q \leq 2k$. Since k was arbitrary, we see that $\check{\mathrm{H}}^{\bullet}(BT)$ is as claimed.

Remark 1.4. Note that the existence of n–universal bundles for compact Lie groups G allows us to generalize remark 1.2. Namely, if X is a compact Hausdorff G–space, $E \to B$ is an n–universal bundle, and E is paracompact Hausdorff, then we have maps

$$E \times_G X \leftarrow (E \times E_N G) \times_G X \to E_N G \times_G X \leftarrow (E_N G \times EG) \times_G X \to EG \times_G X$$

each of which induce isomorphisms in cohomology in degrees $p \leq n$, provided that N is sufficiently large.

Theorem 1.5. *Let G be a compact Lie group which acts smoothly and freely on a compact manifold M. Then the projection $\pi\colon EG \times_G M \to M/G$ induces an isomorphism*

$$\check{\mathrm{H}}^{\bullet}(M/G) \to \mathrm{H}^{\bullet}_G(M).$$

Proof. Since G acts freely on M, the projection $M \to M/G$ is a principal G–bundle, and so π is a locally trivial fiber bundle with typical fiber the contractible space EG. Since M is compact, so that $EG \times_G M$ is paracompact Hausdorff, the Vietoris–Begle Theorem applies and shows that π^* is an isomorphism. □

Example 1.6. Consider the S^1 action on $S^2 \subseteq \mathbb{C} \times \mathbb{R}$ given by multiplication in the first factor, i. e. the action by rotation around the $\{0\} \times \mathbb{R}$ axis. Let $N = (0, 1)$ and $S = (0, -1)$ be the north and south poles of S^2 and consider the open neighborhoods $U = S^2 - \{N\}$ and $V = S^2 - \{S\}$. Note that these are S^1–invariant subsets and that $ES^1 \times_{S^1} S^2 = ES^1 \times_{S^1} U \cup ES^1 \times_{S^1} V$ is a union by open subsets. Furthermore, both of the sets on the right hand side are homotopy equivalent to a point, because there is an S^1–equivariant deformation retract of U and V onto S and N, respectively. Hence, we can apply the Mayer–Vietoris sequence (theorem 10.1) and obtain an exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0_{S^1}(S^2) \to \mathrm{H}^0_{S^1}(U) \oplus \mathrm{H}^0_{S^1}(V) \to \mathrm{H}^0_{S^1}(U \cap V) \to \mathrm{H}^1_{S^1}(S^2) \to \dots \\ \dots \to \mathrm{H}^p_{S^1}(S^2) \to \mathrm{H}^p_{S^1}(U) \oplus \mathrm{H}^p_{S^1}(V) \to \mathrm{H}^p_{S^1}(U \cap V) \to \dots \end{aligned}$$

Note that $U \cap V$ deformation retracts in an S^1–equivariantly fashion onto the equator $S^1 \times \{0\} \subseteq S^2$ and that the restricted action of S^1 on this equator is just the action by left multiplication. Since this action is free, theorem 1.5 shows that $\mathrm{H}^p_{S^1}(U \cap V) \cong \check{\mathrm{H}}^p(*)$. Therefore, in degrees $p \geq 2$ the sequence reduces to

$$0 \to \mathrm{H}^p_{S^1}(S^2) \to \mathrm{H}^p_{S^1}(S) \oplus \mathrm{H}^p_{S^1}(N) \to 0$$

where the map is induced by the inclusions $\{S\} \hookrightarrow S^2$ and $\{N\} \hookrightarrow S^2$. In degree 0 we have the exact sequence

$$\begin{tikzcd} & \mathbb{K} \arrow[d, equal] & \mathbb{K} \oplus \mathbb{K} \arrow[d, equal] & \mathbb{K} \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{H}^0_{S^1}(S^2) \arrow[r] & \mathrm{H}^0_{S^1}(U) \oplus \mathrm{H}^0_{S^1}(V) \arrow[r] & \mathrm{H}^0_{S^1}(U \cap V), \end{tikzcd}$$

so for dimensional reasons the last map must be surjective, and also $\mathrm{H}^1_{S^1}(S^2) \cong \mathrm{H}^1_{S^1}(S) \oplus \mathrm{H}^1_{S^1}(N)$. In total, we see that the inclusion induced map $\varphi\colon \mathrm{H}^p_{S^1}(S^2) \to \mathrm{H}^p_{S^1}(S) \oplus \mathrm{H}^p_{S^1}(N)$ is an injection. If we choose elements $u \in \mathrm{H}^2_{S^1}(S) \cong \check{\mathrm{H}}^2(BS^1)$ and $v \in \mathrm{H}_{S^1}(N)$ such that $\mathrm{H}^\bullet_{S^1}(S) = \mathbb{K}[u]$ and $\mathrm{H}^\bullet_{S^1}(N) = \mathbb{K}[v]$, then

$$\operatorname{im} \varphi \cong \frac{\mathbb{K}[u] \oplus \mathbb{K}[v]}{\{(f, g) \in \mathbb{K}[u] \oplus \mathbb{K}[v] \mid f(0) = g(0)\}}.$$

2. The module structure

The equivariant cohomology $\mathrm{H}^\bullet_G(X)$ of a G–space X is not only a $\mathbb{K}$–algebra, but even a $\mathrm{H}^\bullet_G$–algebra: the projection $\pi\colon EG \times_G X \to BG$ of the locally trivial fiber bundle $EG \times_G X \to BG$ induces a map $\pi^*\colon \mathrm{H}^\bullet_G \to \mathrm{H}^\bullet_G(X)$, and since $\mathrm{H}^\bullet_G(X)$ is a ring via the cup product, we hence obtain a module structure $\mathrm{H}^\bullet_G \times \mathrm{H}^\bullet_G(X) \to \mathrm{H}^\bullet_G(X)$, $(f, s) \mapsto \pi^*(f) \smile s$.

Example 2.1. Let G be a Lie group, $K \subseteq G$ a Lie subgroup and X a G–space. Restricting the G–action to K, we also can consider X as a K–space. By functoriality of the Milnor construction, the inclusion $K \hookrightarrow G$ induces maps $EK \to EG$ and $BK \to BG$, and these induce the upper horizontal map in the commutative diagram

$$\begin{tikzcd} EK \times_K X \arrow[r] \arrow[d] & EG \times_G X \arrow[d] \\ BK \arrow[r] & BG \end{tikzcd}$$

Taking cohomology of this diagram, we see that the map $\mathrm{H}^\bullet_G(X) \to \mathrm{H}^\bullet_K(X)$ is a morphism of $\mathrm{H}^\bullet_G$–modules, where the $\mathrm{H}^\bullet_G$–module structure on $\mathrm{H}^\bullet_K(X)$ is given by "restricting" elements $\mathrm{H}^\bullet_G$ to $\mathrm{H}^\bullet_K$ via the map induced by $BK \to BG$ and then using the $\mathrm{H}^\bullet_K$–module structure.

Proposition 2.2. *Let T be a and $S \subsetneq T$ a closed subgroup. Then the canonical map $\mathrm{H}^\bullet_T \to \mathrm{H}^\bullet_S$ has a non–trivial kernel if S is connected or $\mathbb{K}$ is a field of characteristic* 0*. In particular, in such cases, for any compact T–space X the S–equivariant cohomology $\mathrm{H}^\bullet_S(X)$ is a $\mathrm{H}^\bullet_T$*–torsion module*: that is to say, for every element $x \in \mathrm{H}^\bullet_S(X)$ there exists a non–zero element $f \in \mathrm{H}^\bullet_T$ such that $fx = 0$.*

Proof. Let us first assume that S is connected and hence a torus. Since $S \subsetneq T$, we see that $\dim S < \dim T$, and because $\dim_{\mathbb{K}} \check{\mathrm{H}}^1(S) = \dim S$ and $\dim_{\mathbb{K}} \check{\mathrm{H}}^1(T) = \dim T$, the map $\iota^* \colon \check{\mathrm{H}}^1(T) \to \check{\mathrm{H}}^1(S)$ induced by the inclusion $\iota\colon S \to T$ must have a non–trivial kernel. Now consider the spectral sequences $(E_r)_{r\geq 2}$ and $(\overline{E}_r)_{r\geq 2}$ of $ET \to BT$ and $ES \to BS$, respectively. By functoriality, ι induces maps $E\iota\colon ES \to ET$ and $B\iota\colon BS \to BT$, which in turn induce a morphism of spectral sequences $(\alpha_r \colon E_r \to \overline{E}_r)_{r\geq 2}$. Since S and T are connected, the Leray presheaf is simple and so on the second page α_2 is just the map

$$(B\iota)^* \otimes_{\mathbb{K}} (\iota^*) \colon \mathrm{H}^\bullet_T \otimes_{\mathbb{K}} \check{\mathrm{H}}^\bullet(T) \to \mathrm{H}^\bullet_S \otimes_{\mathbb{K}} \check{\mathrm{H}}(S).$$

On the other hand, since the spectral sequence $(E_r)_{r\geq 2}$ converges against $\check{\mathrm{H}}^\bullet(ET)$ and ET is contractible, the differential d_2 is an isomorphism on $E_2^{0,1}$. Hence, if we let $x \in \check{\mathrm{H}}^1(T)$ be a non–trivial element in the kernel of ι^*, then $y := d_2(1 \otimes_{\mathbb{K}} x)$ is a non–zero element and

$$\alpha_2(y) = \overline{d}_2(\alpha_2(1 \otimes_{\mathbb{K}} x)) = \overline{d}_2(1 \otimes_{\mathbb{K}} \iota^*(x)) = 0.$$

But $\check{\mathrm{H}}^0(T) = \mathbb{K} \cdot 1$, so if we write $y \in \mathrm{H}^2_T \otimes_{\mathbb{K}} \check{\mathrm{H}}^0(T)$ as $y = \sum_i y_i \otimes_{\mathbb{K}} \lambda_i$ for elements $\lambda_i \in \mathbb{K}$, then $\alpha_2(y) = \sum_i (B\iota)^*(y_i) \otimes_{\mathbb{K}} \lambda_i$, because ι^* is a morphism of rings, and so $\sum_i \lambda_i y_i$ is a non–zero element in the kernel of $(B\iota)^*$.

Now suppose that S is an arbitrary proper subgroup of T, not necessarily connected, but that $\mathbb{K}$ is a field of characteristic 0. Let S_0 be the identity component of S and consider the commutative diagram

$$\begin{tikzcd}
ES_0/S_0 \arrow[r] \arrow[dr] & ES/S_0 \arrow[r] \arrow[d] & ES/S \arrow[dl] \\
& BT &
\end{tikzcd}$$

in which all maps are induced by inclusions. The left horizontal map induces an isomorphism $\check{\mathrm{H}}^\bullet(ES/S_0) \to \mathrm{H}^\bullet_{S_0}$, because $ES \to ES/S_0$ is a classifying bundle for S_0 by corollary 3.11 and because the base spaces of any two universal bundles of a fixed Lie group are homotopy equivalent. The map $ES/S_0 \to ES/S = BS$ is a covering map with finite fiber S/S_0, since $ES \to BS$ is a principal S–bundle. Hence, it induces a monomorphism $\mathrm{H}^\bullet_S \to \check{\mathrm{H}}^\bullet(ES/S_0)$ by proposition 2.3 below. Therefore, as $\mathrm{H}^\bullet_T \to \mathrm{H}^\bullet_{S_0}$ has non–trivial kernel, also $\mathrm{H}^\bullet_T \to \mathrm{H}^\bullet_S$ must have a non–trivial kernel. □

Proposition 2.3. *Let $f\colon X \to Y$ be a k–sheeted covering map. Then there exists a morphism $\alpha\colon \check{\mathrm{H}}^\bullet(X) \to \check{\mathrm{H}}^\bullet(Y)$ such that $\alpha \circ f^* = k\mathrm{id}$. In particular, if k is invertible in $\mathbb{K}$, then f^* is injective.*

Proof. Consider pairs $(\mathfrak{U}, \mathfrak{V})$ consisting of Čech covers $\mathfrak{U}$ of X and $\mathfrak{V}$ of Y with the following properties:

(i) $\mathfrak{V}$ is defined on the index set I and $\mathfrak{U}$ is defined on the index set $I \times \{1, \dots, k\}$.

(ii) for fixed i and all $j = 1, \dots, k$ the map $f|_{\mathfrak{U}_{i,j}}\colon \mathfrak{U}_{i,j} \to \mathfrak{V}_i$ is a homeomorphism.

We claim that the assignments sending such a pair $(\mathfrak{U}, \mathfrak{V})$ to $\mathfrak{U}$ and $\mathfrak{V}$ are cofinal in the category of all Čech covers of X and Y, respectively. For Y this is immediate, because f is a covering map. To see that the assignment $(\mathfrak{U}, \mathfrak{V}) \mapsto \mathfrak{U}$ is cofinal, let $\mathfrak{W}$ be an arbitrary Čech cover of X. Let $y \in Y$ be arbitrary and choose an open neighborhood $V \subseteq Y$ such that $f^{-1}(V) = U_1 \cup \ldots \cup U_k$ is a disjoint union of open subsets such that $f|_{U_i}\colon U_i \to V$ is a homeomorphism. Then $f^{-1}(y) = \{x_1, \dots, x_k\}$ for certain elements $x_i \in U_i$. Let $j_1, \dots, j_k$ be such that $x_i \in \mathfrak{W}_{j_i}$ and consider

$$\mathfrak{V}_y := f(\mathfrak{W}_{j_1} \cap U_1) \cap \ldots \cap f(\mathfrak{W}_{j_k} \cap U_k) \subseteq V.$$

This is an open neighborhood of y and f restrict to a homeomorphism from $\mathfrak{U}_{y,i} = f^{-1}(\mathfrak{V}_y) \cap U_i$ onto $\mathfrak{V}_y$: it is immediate that $f|_{\mathfrak{U}_{y,i}}$ is injective. To see that this map surjects onto $\mathfrak{V}_y$, let $z \in \mathfrak{V}_y$. By definition, there exists an element $x \in \mathfrak{W}_{j_i} \cap U_i$ with $f(x) = z$, and so $x \in \mathfrak{U}_{y,i}$.

Now for such a pair of covers $(\mathfrak{U}, \mathfrak{V})$ consider the map $\alpha_{(\mathfrak{U},\mathfrak{V})}\colon \check{C}^p(\mathfrak{U}) \to \check{C}^p(\mathfrak{V})$ given by

$$\alpha_{(\mathfrak{U},\mathfrak{V})}(c)(\sigma) = \sum_{j=1}^{k} c((\sigma_0, j), \dots, (\sigma_p, j))$$

for all p–simplices $\sigma = (\sigma_0, \dots, \sigma_p)$. This map is certainly a chain map and hence induces, by passage to the limit, a map $\alpha\colon \check{\mathrm{H}}^p(X) \to \check{\mathrm{H}}^p(Y)$. To see that α has the claimed properties, let $(\mathfrak{U}, \mathfrak{V})$ again be a pair of covers as before. Observe that $\mathfrak{U}$ is a refinement of $f^{-1}\mathfrak{V}$, because $f^{-1}(\mathfrak{V}_i) = \mathfrak{U}_{i,1} \cup \ldots \cup \mathfrak{U}_{i,k}$ by choice of $\mathfrak{U}$. An explicit refinement projection is given by $p(i,j) = i$. Hence we have a commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^p(\mathfrak{V}) \arrow[r, "f^*"] \arrow[dd] & \check{\mathrm{H}}^p(f^{-1}\mathfrak{V}) \arrow[d, "p_*"] & \\
& \check{\mathrm{H}}^p(\mathfrak{U}) \arrow[r, "(\alpha_{(\mathfrak{U},\mathfrak{V})})^*"] \arrow[d] & \check{\mathrm{H}}^p(\mathfrak{V}) \arrow[d] \\
\check{\mathrm{H}}^p(Y) \arrow[r, "f^*"'] & \check{\mathrm{H}}^p(X) \arrow[r, "\alpha"] & \check{\mathrm{H}}^p(Y)
\end{tikzcd}$$

But if $c \in \check{C}^p(\mathfrak{V})$ is a p–cocycle, then

$$\big(((\alpha_{(\mathfrak{U},\mathfrak{V})})^* \circ p_* \circ f^*)c\big)(\sigma) = \sum_{j=1}^{k} (p_* f^* c)((\sigma_0, j), \ldots, (\sigma_k, j)) = kc(\sigma),$$

so $\alpha \circ f^* = k\mathrm{id}$. □

Remark 2.4. Proposition 2.3 is a result of its own interest: it shows for example that if $\mathbb{K}$ is a field of characteristic 0 or different from p, then $\check{\mathrm{H}}^\bullet(B\mathbb{Z}_p; \mathbb{K}) = 0$, because we have the p–sheeted covering $E\mathbb{Z}_p \to B\mathbb{Z}_p$ and $E\mathbb{Z}_p$ is contratible.

Example 2.5. Let T be a torus, X a T–space, and fix a point $p \in X$. We want to compute the T–equivariant cohomology of the orbit Tp as an $\mathrm{H}_T^\bullet$–module. To this end, consider the continuous map

$$f\colon ET/T_p \to ET \times_T (Tp),\ xT_p \mapsto [x, p].$$

It is well–defined, because $[xt, p] = [x, tp] = [x, p]$ holds for all $t \in T_p$; it is surjective, because $[x, tp]$ equals $[xt, p]$ in $ET \times_T (Tp)$; and it is injective, for if $[x, p] = [y, p]$, there must exist an element $t \in T$ such that $(x, p) = (yt, t^{-1}p)$, and this means that $t \in T_p$ and hence $xT_p = yT_p$. Thus, by the Vietoris–Begle Theorem, f induces an isomorphism on cohomology. On the other hand, we the commutative diagram

$$\begin{tikzcd}
BT_p = ET_p/T_p \arrow[d] & \\
ET/T_p \arrow[r, "f"] \arrow[d] & ET \times_T (Tp) \arrow[dl, "\pi"] \\
BT &
\end{tikzcd}$$

in which the upper vertical map is induced by $ET_p \to ET$, which in turn is the map induced by the inclusion $T_p \hookrightarrow T$, and the lower vertical map is the canonical map $ET/T_p \to BT = ET/T$ induced by the identity; the map π is the bundle projection. Note that $BT_p \to ET/T_p$ induces an isomorphism on cohomology, because both ET/T_p and BT_p are classifying spaces for T_p. Hence, by commutativity of the diagram above, $\mathrm{H}_{T_p}^\bullet \cong \mathrm{H}_T^\bullet(Tp)$ as $\mathrm{H}_T^\bullet$–modules, where the $\mathrm{H}_T^\bullet$–module structure on $\mathrm{H}_{T_p}^\bullet$ is induced by the map $BT_p \to BT$. In particular, if p is not a T–fixed point and if T_p is connected or $\mathbb{K}$ is a field of characteristic 0, then $\mathrm{H}_T^\bullet \to \mathrm{H}_T^\bullet(Tp)$ has a non–trivial kernel by proposition 2.2.

3. Localization

If a torus acts on compact Hausdorff space, then there is a strong connection between the equivariant cohomology of the space and the equivariant cohomology of the fixed point set of the action, and this connection can be concisely stated using the notion of localization. We shall quickly review this concept in what follows, referring the reader to [7, Section I.2] for more details.

Let R be a commutative (associative), unital ring. A subset $S \subseteq R$ is *multiplicatively closed*, if $1 \in S$ and if whenever $s, t \in S$ also $st \in S$ holds. For an R–module M we then define the *localization* of M at S to be the set $M[S^{-1}] := (M \times S)/\sim$, where $(m, s) \sim (n, t)$ holds if and only if there exists an element $u \in S$ such that $utm = usn$. We denote the equivalence class of (m, s) by m/s and make $M[S^{-1}]$ a group by

$$\frac{m}{s} + \frac{n}{t} := \frac{tm + sn}{st}.$$

Note that we can in particular localize R at S and that $R[S^{-1}]$ is in fact a ring, with multiplication defined by

$$\frac{r}{s} \cdot \frac{r'}{s'} := \frac{rr'}{ss'}.$$

Also, if we localize an R–module M at S, then $M[S^{-1}]$ is an $R[S^{-1}]$–module with scalar multiplication given by

$$\frac{r}{s} \cdot \frac{m}{t} := \frac{rm}{st}$$

for all $r \in R$, $m \in M$, and $s, t \in S$.

We can consider localization as a functor R–$\mathsf{mod} \to R[S^{-1}]$–$\mathsf{mod}$: all that remains to be observed is that if $f\colon M \to N$ is a morphism of R–modules, then we have an induced morphism of $R[S^{-1}]$–modules

$$\frac{f}{1}\colon M[S^{-1}] \to N[S^{-1}],\ \frac{m}{s} \mapsto \frac{f(m)}{s}.$$

For us, the most important property of localization is that it preserves exactness:

Proposition 3.1. *Localization at a multiplicatively closed subset S of a ring R is an exact functor. That is, if*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence of R–modules, then so is the sequence

$$0 \to A[S^{-1}] \xrightarrow{f/1} B[S^{-1}] \xrightarrow{g/1} C[S^{-1}] \to 0.$$

Proof. It is immediate that $g/1$ is surjective and that $\mathrm{im}(f/1) \subseteq \ker(g/1)$. To show that the latter inclusion is an equality, let b/s be an element with $(g/1)(b/s) = 0$. Since $(g/1)$ is $R[S^{-1}]$–linear and $s/1$ is invertible, we see that $g(b)/1 = 0$, which by definition of localization means that there exists $t \in S$ with $tg(b) = 0$. But g is R–linear, so $tb \in \ker g$, and hence by exactness $tb = f(a)$ for some element $a \in A$. Then

$$\frac{b}{s} = \frac{tb}{ts} = \frac{f(a)}{ts}$$

and $\operatorname{im}(f/1) = \ker(g/1)$. Similarly, if $a/s \in \ker(f/1)$, then there exists $t \in S$ with $f(ta) = 0$. Since f is injective, we have $ta = 0$, and since $t/1$ is invertible in $A[S^{-1}]$, it follows that $a/s = ta/ts = 0$. Therefore, $f/1$ is injective. $\square$

Proposition 3.2. *Let T be a torus, X a compact Hausdorff T–space, and $A \subseteq X$ a T–invariant closed subspace. We have*

$$\varinjlim_{N \supseteq A} \mathrm{H}^\bullet_T(N) \xrightarrow{\sim} \mathrm{H}^\bullet_T(A),$$

where the limit ranges over all closed, T–invariant neighborhoods $N \subseteq X$ of A and the map is induced by the inclusions $A \hookrightarrow N$ for all such neighborhoods N.

Proof. Fix an integer $p \geq 0$ and recall from remark 1.2 that we can find a compact princpial T–bundle $E \to B$ such that each of the maps

$$E \times_T Y \leftarrow (E \times ET) \times_T Y \to ET \times_T Y$$

induces isomorphisms in cohomology in degree p for all T–spaces Y. Since these maps are natural with respect to T–equivariant morphisms, it thus suffices to show that

$$\varinjlim_{N \supseteq A} \check{\mathrm{H}}^p(E \times_T N) \to \check{\mathrm{H}}^p(E \times_T A),$$

is an isomorphism. This will be a consequence of the tautness property of Čech cohomology (theorem 9.5) if we can show that the sets of the form $E \times_T N$, with $N \subseteq X$ a closed invariant neighborhood of A, are cofinal in the set of all closed neighborhoods of $E \times_T A \subseteq E \times_T X$. So, let $W \subseteq E \times_T X$ be a closed neighborhood of $E \times_T A$ and $\pi \colon E \times X \to E \times_T X$ the canonical projection. Write $W_0 := \pi^{-1}(W)$ and note that this is a T–invariant neighborhood of $E \times A$ in $E \times X$. Hence, by compactness of E, we can find for each point $a \in A$ a closed neighborhood N_a of a such that $E \times N_a \subseteq W_0$. As T acts diagonally on $E \times X$, we then have

$$W_0 \supseteq T \cdot (E \times N_a) = E \times (T \cdot N_a),$$

and $T \cdot N_a$ is a compact, hence closed, T–invariant neighborhood of a in X. As A is compact, we therefore can find finitely many closed T–invariant neighborhoods $N_1, \ldots, N_k$ covering A and such that $E \times N_i \subseteq W_0$ for all $i = 1, \ldots, k$. Then $N := N_1 \cup \ldots \cup N_k$ is a T–invariant closed neighborhood of A with $E \times N \subseteq W_0$. Since $E \times_T X$ is Hausdorff, $E \times_T N \subseteq W$ is a compact and thus closed neighborhood of $E \times_T A$. $\square$

Theorem 3.3 (Borel Localization)**.** *Let X be a compact Hausdorff T–space, where T is a torus. Assume that $\mathbb{K}$ is a field of characteristic 0 and denote by $X^T = \{p \in M \mid T_p = T\}$ the set of T–fixed points in X. Then the inclusion $X^T \hookrightarrow X$ induces an isomorphism*

$$\mathrm{H}^\bullet_T(X)[(\mathrm{H}^\bullet_T - 0)^{-1}] \to \mathrm{H}^\bullet_T(X^T)[(\mathrm{H}^\bullet_T - 0)^{-1}].$$

Proof. Let us first suppose that X^T is empty. Then the statement is that $\mathrm{H}_T^\bullet(X)[(\mathrm{H}_T^\bullet - 0)^{-1}]$ is trivial, or equivalently, that $\mathrm{H}_T^\bullet(X)$ is a $\mathrm{H}_T^\bullet$–torsion module. To see this, let $x \in X$ be an arbitrary element. By tautness of equivariant cohomology (proposition 3.2) we can find a closed, T–invariant neighborhood N of x such that the inclusion induced map $\mathrm{H}_T^p(N) \to \mathrm{H}_T^p(Tx)$ is an isomorphism. Because x is not a fixed point, $\mathrm{H}_T^\bullet \to \mathrm{H}_T^\bullet(N)$ has a non–trivial kernel by example 2.5. As X is compact, we hence can find finitely many such neighborhoods $N_1, \ldots, N_k$ covering X and non–trivial elements $f_i \in \mathrm{H}_T^\bullet$ such that $f_i|N_i$ is zero, where we the latter denotes the image of f_i in $\mathrm{H}_T^\bullet(N_i)$. Then $f := f_1 \smile \ldots \smile f_k$ is a non–zero element (because $\mathrm{H}_T^\bullet$ is a polynomial ring over $\mathbb{K}$ in $\dim T$ variables) which restricts to zero in each $\mathrm{H}_T^\bullet(N_i)$, and we inductively show that f^i restricts to zero on $N_1 \cup \ldots \cup N_i$ for all $i = 1, \ldots, k$. We just argued why this statement is true for $i = 1$, so assume that the claim holds up to some integer $i < k$. Write $A := N_1 \cup \ldots \cup N_i$, $B := N_{i+1}$, and consider

$$\mathrm{H}_T^\bullet(A \cap B) \xrightarrow{\partial} \mathrm{H}_T^\bullet(A \cup B) \to \mathrm{H}_T^\bullet(A) \oplus \mathrm{H}_T^\bullet(B),$$

which is a part of the Mayer–Vietoris sequence. Since $f^i|_A = 0$ and $f^i|_B = 0$, we see that $f^i|_{A \cup B} = \partial(x)$ for some $x \in \mathrm{H}_T^\bullet(A \cap B)$. But since already $f|_B = 0$, we also have $f|_{A \cap B} = 0$, and because ∂ is $\mathrm{H}_T^\bullet$–linear, we see that

$$f^{i+1}|_{A \cup B} = f|_{A \cup B} \cdot \partial(x) = \partial(f|_{A \cap B} \cdot x) = 0.$$

Hence $\mathrm{H}_T^\bullet \to \mathrm{H}_T^\bullet(X)$ has non–trivial kernel and $\mathrm{H}_T^\bullet(X)$ is an $\mathrm{H}_T^\bullet$–torsion module.

Now let X^T be arbitrary. Since X^T is a closed T–invariant subspace, we can find a closed T–invariant neighborhood A of X^T such that $\mathrm{H}_T^\bullet(A) \to \mathrm{H}_T^\bullet(X^T)$ is an isomorphism. Note that $\mathring{A}$ is an open T–invariant neighborhood of X^T, so $B = X - \mathring{A}$ is a closed T–invariant neighborhood such that $A \cup B = X$. We again consider the exact sequence

$$\mathrm{H}_T^\bullet(A \cap B) \xrightarrow{\partial} \mathrm{H}_T^\bullet(X) \to \mathrm{H}_T^\bullet(A) \oplus \mathrm{H}_T^\bullet(B),$$

coming from the Mayer–Vietoris sequence. Because localization is an exact functor (proposition 3.1), we still have an exact sequence after localizing each term in the sequence above. However, $A \cap B$ and B are compact Hausdorff spaces without T–fixed points and hence are $\mathrm{H}_T^\bullet$–torsion by what we have just shown. Therefore, the localized sequence reduces to an isomorphism $\mathrm{H}_T^\bullet(X)[(\mathrm{H}_T^\bullet - 0)^{-1}] \to \mathrm{H}_T^\bullet(A)[(\mathrm{H}_T^\bullet - 0)^{-1}]$, and since $\mathrm{H}_T^\bullet(X) \to \mathrm{H}_T^\bullet(X^T)$ factors through the isomorphism $\mathrm{H}_T^\bullet(A) \to \mathrm{H}_T^\bullet(X^T)$, the claim follows. ☐

4. More on the module structure

Let C be a $\mathbb{Z}$–graded $\mathbb{K}$–algebra and $(E_r)_{r \geq 2}$ a first quadrant spectral sequence of algebras convering to C with respect to a canonically bounded filtration $F_\bullet C$. Suppose further that $E_2 = A \otimes_{\mathbb{K}} B$ is the graded commutative tensor product of $\mathbb{Z}$–graded $\mathbb{K}$–algebras A and B. If B is unital, then we can consider C and the associated graded

$\mathbb{K}$–algebra $\mathrm{gr}(F_\bullet C)$ as (ungraded) A–algebras via the module structure induced by the edge map $e_B\colon E_2^{\bullet,0} \to C$: indeed, since $(E_r)_{r\geq 2}$ converges against C, we have preferred isomorphisms $E_\infty^{\bullet,0} \cong \mathrm{gr}_\bullet(F_\bullet C)^0 \subseteq C$, and by definition the edge map is the composition of this map $E_\infty^{\bullet,0} \hookrightarrow C$ with a map $\overline{e}_B\colon E_2^{\bullet,0} \to E_\infty^{\bullet,0}$. Since the edge map is a morphism of rings, so is

$$A \xrightarrow{\sim} A \otimes_{\mathbb{K}} 1 \subseteq A \otimes_{\mathbb{K}} B \xrightarrow{\overline{e}_B} E_\infty^{\bullet,0} \cong \mathrm{gr}_\bullet(F_\bullet C)^0,$$

and hence we can use this map to turn $\mathrm{gr}(F_\bullet C)$ into an A–algebra.

Theorem 4.1. *In this situation, suppose further that A is an integral domain and write $Q = A[(A-0)^{-1}]$ for the quotient field of A. Then*

(i) $\mathrm{gr}(F_\bullet C)[(A-0)^{-1}]$ admits a basis $(x_i/1)_{i\in I}$ with each $x_i \in \mathrm{gr}(F_\bullet C)$ homogeneous of bidegree (p_i, q_i), and

(ii) for each such basis, the map of Q–vector spaces

$$\mathrm{gr}(F_\bullet C)[(A-0)^{-1}] \to C[(A-0)^{-1}],\ \frac{x_i}{1} \mapsto \frac{\overline{x}_i}{1},$$

is an isomorphism, where $\overline{x}_i \in F_{p_i}C \cap C^{p_i+q_i}$ is a lift of x_i with respect to the canonical projection $F_{p_i}C \cap C^{p_i+q_i} \to \mathrm{gr}_{p_i}(F_\bullet C)^{q_i}$.

Proof. We write $G := \mathrm{gr}(F_\bullet C)$ and $G_{(0)} := G[(A-0)^{-1}]$ as well as $C_{(0)} := C[(A-0)^{-1}]$. To see the first item, we just need to adjust the well–known argument that every vector space has a basis. In more detail, assume that $G_{(0)}$ is non–trivial and consider the set

$$\mathcal{B} = \{B \subseteq G_{(0)} \,|\, B \text{ linearly independent and } \forall b \in B \exists p, q \in \mathbb{Z} \exists \overline{b} \in G_p^q : b = \overline{b}/1\}.$$

It is non–empty, because $G_{(0)} = Q \cdot G$ and $G_{(0)}$ is non–empty, and partially ordered by inclusion. Moreover, if $\mathcal{C} \subseteq \mathcal{B}$ is a chain, i. e. a totally ordered subset, then $C := \bigcup_{B\in\mathcal{C}} B$ is linearly independet and an upper bound for each element $B \in \mathcal{C}$. Hence, by Zorn's Lemma, $\mathcal{B}$ has a maximal element B, and this is the desired homogeneous basis.

Now fix a basis $(x_i/1)_{i\in I}$ of $G_{(0)}$ with each $x_i \in G$ homogeneous of some bidegree (p_i, q_i), and let $\overline{x}_i \in F_{p_i}C \cap C^{p_i+q_i}$ be a lift. The statement in the second item is equivalent to the statement that $(\overline{x}_i/1)_{i\in I}$ is a basis of $C_{(0)}$, and this is what we will show. So, suppose that

$$\sum_{i\in I} \lambda_i \frac{\overline{x}_i}{1} = 0$$

holds, where only finitely many $\lambda_i \in Q$ are non–zero. Clearing denominator of each λ_i and using the definition of $C_{(0)}$, we see that this holds if and only if there exist elements $a_i \in A$, finitely many of which are non–zero, such that $\sum_i a_i\overline{x_i} = 0$ holds in C. Let $(a_i)_k$ be the k–th graded component of a_i, so $a_i = \sum_k (a_i)_k$ and $(a_i)_k \in A^k$, and let $n = \max\{k \,|\, (a_i)_k \neq 0\}$ be the maximal non–zero component occuring among all a_i.

Likewise, let $\ell = \max\{r \mid a_i \neq 0, p_i + q_i = r\}$ be the maximal total degree of an element $\overline{x}_i$ whose coefficient a_i is non–zero, and denote by c_k the k–th component of $c \in C$. Then by definition of the module structure in C, we have

$$0 = \left(\sum_{i\in I} a_i \overline{x}_i\right)_{n+\ell} = \sum_{j\in J} (a_j)_n \overline{x}_j,$$

where now $J \subseteq I$ denotes the set of all indices $i \in I$ such that $p_i + q_i = \ell$. Hence, if we can show that $(a_j)_n = 0$ for all $j \in J$ (and thus also $(a_i)_n = 0$ for all $i \in I$), then $a_i = 0$ follows by induction. Thus, we are left with showing that if we are given lifts $\overline{z}_k \in F_{v_k}C \cap C^\ell$ of elements $z_1, \dots, z_r \in G$ which are homogeneous of bidegree (v_k, w_k), total degree $v_k + w_k = \ell$, and such that $z_1/1, \dots, z_r/1 \in G_{(0)}$ are linearly independet, then $\sum_k a_k \overline{z}_k = 0$ can only hold for elements $a_1, \dots, a_r \in A^n$ if $a_1 = \dots = a_r = 0$. Assume that $v_1 \geq \dots \geq v_r$ and note that the diagram

$$\begin{tikzcd}
A^n \times (F_{v_k} \cap C^\ell) \arrow[r] \arrow[d] & (F_{v_k+n} \cap C^{n+\ell}) \arrow[d] \\
A^n \times G^{w_k}_{v_k} \arrow[r] & G^{w_k}_{v_k+n}
\end{tikzcd}$$

commutes for all k, where the horizontal maps are the scalar multiplication maps of the A–modules C and G, and the vertical maps are induced by the identity map $A^n \to A^n$ and the canonical quotient projections. Hence, if $v_1 = \dots = v_k > v_{k+1}$, then since the filtration $F_\bullet C$ is decreasing also

$$a_1\overline{z}_1 + \dots + a_k\overline{z}_k \in F_{v_1+n+1}C \cap C^{n+\ell}$$

and thus $a_1 z_1 + \dots + a_k z_k = 0$. But $z_1/1, \dots, z_k/1$ are linearly independet, and so $a_1/1 = \dots = a_k/1 = 0$, and then $a_1 = \dots = a_k = 0$, because A is an integral domain. Repeating this argument, we conclude that $a_1 = \dots = a_r = 0$.

We still need to show that $(\overline{x}_i/1)_{i\in I}$ generates $C_{(0)}$, and for this it suffices to prove that each $F_pC \cap C^q \subseteq C_{(0)}$ is generated by the $\overline{x}_i/1$. We shall prove by induction on $k \geq 0$ that $F_{n+1-k} \cap C^n$ is generated by the $\overline{x}_i/1$, for all n. Since the filtration is canonically bounded, $F_{n+1}C \cap C^n = 0$, so the induction base $k = 0$ is etablished. Hence, suppose that the induction hypothesis holds up to some k and let n be arbitrary. We need to show that in this case also $F_{n-k}C \cap C^n$ is generated by the $\overline{x}_i/1$. Hence, let $\overline{c} \in F_{n-k}C \cap C^n$ and consider its image $c \in G^k_{n-k}$. Since $(x_i/1)_{i\in I}$ is a basis of $G_{(0)}$, we can find elements $\lambda_i \in Q$, finitely many of which are non–zero, with $c/1 = \sum_i \lambda_i x_i/1$. Again, we clear denominators and use the definition of $G_{(0)}$ to conclude that then

$$sc = \sum_i a_i x_i$$

must hold in G for certain elements $a_i, s \in A$. First, assume that s is homogeneous of degree r. Then, as c and all x_i are homogeneous, we may assume that each a_i is homogeneous, and so $sc - \sum_i a_i x_i = 0$ in G^k_{n-k+r} implies that

$$s\overline{c} - \sum_i a_i \overline{x}_i \in F_{n-k+r+1}C \cap C^{n+r}.$$

By induction hypothesis, we thus have

$$\frac{s\overline{c}}{1} - \sum_i \frac{a_i}{1} \cdot \frac{\overline{x}_i}{1} = \sum_i \mu_i \frac{\overline{x}_i}{1}$$

for certain elements $\mu_i \in Q$. Since s is invertible in $C_{(0)}$, we see that $\overline{c}/1$ lies in the span of the $\overline{x}_i/1$ if s is homogeneous. If s is arbitrary, let s_r be the highest non–vanishing component of s. Then, denoting by $t^q_p \in G^q_p$ the (p,q)–component of $t \in G$,

$$s_r c = (sc)^k_{n-k+r} = \sum_i (a_i x_i)^k_{n-k+r}$$

and because each x_i is homogeneous, the right hand side equals $\sum_i (a_i)^{s_i}_{t_i} x_i$, where t_i, s_i are such that $t_i + p_i = n-k+r$ and $s_i + q_i = k$. By the previously dealt with case, $\overline{c}/1$ again lies in the span of the $\overline{x}_i/1$. □

As a corollary, we obtain the following

Theorem 4.2. *Suppose a torus T acts on a compact Hausdorff space X. Then*

(i) $\mathrm{H}^\bullet_T(X)[(\mathrm{H}^\bullet_T - 0)^{-1}]$ *is a finite–dimensional vector space over* $Q = \mathrm{H}^\bullet_T[(\mathrm{H}^\bullet_T - 0)^{-1}]$,

(ii) $\dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{even}}(M^T) \le \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{even}}(M)$ *and* $\dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{odd}}(M^T) \le \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{odd}}(M)$,

(iii) $\dim_{\mathbb{K}} \check{\mathrm{H}}^\bullet(M^T) = \dim_{\mathbb{K}} \check{\mathrm{H}}^\bullet(M)$ *holds if and only if the Leray spectral sequence of* $ET \times_T X \to BT$ *collapses on the second page.*

(iv) $\chi_{\mathbb{K}}(M) = \chi_{\mathbb{K}}(M^T)$,

Proof. Since X is compact and T is connected, the Leray presheaf of $\pi\colon ET \times_T X \to X$ is simple by example 7.6. Hence the Leray spectral sequence of π has as second page

$$E_2 = \mathrm{H}^\bullet_T \otimes_{\mathbb{K}} \check{\mathrm{H}}^\bullet(X),$$

which is a graded–commutative tensor product. Now endow each page E_r with the $\mathrm{H}^\bullet_T$–module structure induced by the maps

$$E_2^{\bullet,0} \to \mathrm{H}^{\bullet,0}(E_2) \cong E_3^{\bullet,0} \to \mathrm{H}^{\bullet,0}(E_3) \cong E_4 \to \ldots \to E_r^{\bullet,0}$$

Since $\mathrm{H}^\bullet_T$ is an evenly–graded space, the subspaces $\mathrm{Tot}^{\mathrm{even}}(E_r)$ and $\mathrm{Tot}^{\mathrm{odd}}(E_r)$ of even– and odd–degree elements in $\mathrm{Tot}(E_r)$ (which as an ungraded $\mathrm{H}^\bullet_T$–module is just E_r) are $\mathrm{H}^\bullet_T$–submodules. Also note that since $\dim_{\mathbb{K}} \check{\mathrm{H}}^\bullet(X)$ is finite (because X is compact), we have $E_n = E_\infty$ for some n and that, by definition, the edge map e_B is just the composition of the above map for $r = n$ with the map $E_\infty^{\bullet,0} \hookrightarrow \check{\mathrm{H}}^\bullet(BT) = \mathrm{H}^\bullet_T$. Moreover, the differential d_r on E_r is $\mathrm{H}_T^\bullet$–linear, since E_r lies in the first quadrant and d_r is an anti–derivation with respect to total degree. Thus, since localization is an exact functor (proposition 3.2) and the isomorphisms $\mathrm{H}^{\bullet,\bullet}(E_r) \cong E_{r+1}$ restrict to isomorphisms $\mathrm{H}^\bullet(\mathrm{Tot}^{\mathrm{even}}(E_r)) \cong \mathrm{Tot}^{\mathrm{even}}(E_{r+1})$ of $\mathrm{H}_T^\bullet$–modules, we see that

$$\begin{aligned}\dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{even}}(X) &= \dim_Q \left(\mathrm{Tot}^{\mathrm{even}}(E_2)\right)_{(0)}\\ &\geq \dim_Q \left(\mathrm{Tot}^{\mathrm{even}}(E_3)\right)_{(0)}\\ &\dots\\ &\geq \dim_Q \left(\mathrm{Tot}^{\mathrm{even}}(E_\infty)\right)_{(0)}\\ &= \dim_Q \left(\mathrm{H}_T^{\mathrm{even}}(X)\right)_{(0)},\end{aligned}$$

where the first equality holds, because the module structure defined on E_2 is just the module structure induced by multiplication with elements of $E_2^{\bullet,0} \cong \mathrm{H}_T^\bullet$, and the last equality holds by theorem 4.1 if we consider $\mathrm{H}_T(X)$ as a module over $E_2^{\bullet,0} \cong \mathrm{H}_T^\bullet$ via the edge map e_B. However, by proposition 8.1, the edge map is, up to isomorphism on $\mathrm{H}_T^\bullet$, just the map π^* induced by $ET \times_T X \to BT$, and hence the module structures on $\mathrm{H}_T(X)$ coming from e_B and π^* coincide up to isomorphism of $\mathrm{H}_T^\bullet$–modules. In particular, $(\mathrm{H}_T^\bullet(X))_{(0)}$ is a finite–dimensional Q–vector space with respect to the module structure coming from π^*. This proves the first part. Moreover, since T acts trivially on X^T, combining the previous inequalities with the Borel Localization Theorem, we obtain:

$$\dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{even}}(X) \geq \dim_Q (\mathrm{H}_T^{\mathrm{even}}(X))_{(0)} = \dim_Q (\mathrm{H}_T^{\mathrm{even}}(X^T))_{(0)} = \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{even}}(X^T).$$

The same reasoning shows that $\dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{odd}}(X) \geq \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{odd}}(X^T)$, and thus the second item follows. If both of these inequalities are equalities, then

$$\dim_Q (E_2)_{(0)} = \dim_Q (E_r)_{(0)} = \ldots = \dim_Q (E_\infty)_{(0)}$$

holds as Q–vector spaces, and so $\dim_Q (E_2)_{(0)} = \dim_Q (\ker d_2)_{(0)}$, meaning that d_2 induces the trivial map on $(E_2)_{(0)}$. But E_2 is a free $\mathrm{H}_T^\bullet$–module, in particular an integral domain, so this is only possible if d_2 is trivial. Moreover, the isomorphism $\mathrm{H}^{\bullet,\bullet}(E_2) \cong E_3$ is an isomorphism of (ungraded) $\mathrm{H}_T^\bullet$–modules, so E_3 also is a free $\mathrm{H}_T^\bullet$–module. Inductively we see that each E_r must be a free $\mathrm{H}_T^\bullet$–module and that all differentials are trivial. Hence, the spectral sequence collapses.

Finally, to prove the statement about the Euler characteristics, let (V, d) be a finite–dimensional complex over Q which is $\mathbb{Z}_2$–graded; that is, $V = V_0 \oplus V_1$ is a finite–dimensional $\mathbb{Z}_2$–graded vector space and d is homogeneous of degree 1 in the sense that that $d(V_0) \subseteq V_1$ and $d(V_1) \subseteq V_0$. Then also $\mathrm{H}(V, d) = \mathrm{H}_0(V, d) \oplus \mathrm{H}_1(V, d)$ is $\mathbb{Z}_2$–graded, where

$$\mathrm{H}_0(V, d) = \frac{\ker d \cap V_0}{\operatorname{im} d \cap V_1} \text{ and } \mathrm{H}_1(V, d) = \frac{\ker d \cap V_1}{\operatorname{im} d \cap V_0}.$$

In particular, if we set $\chi(W) = \dim W_0 - \dim W_1$ for every $\mathbb{Z}_2$–graded Q–vector space, then by the dimension formula,

$$\begin{aligned}\chi(\mathrm{H}(V)) &= \dim \mathrm{H}_0(V) - \dim \mathrm{H}_1(V) \\ &= \dim(\ker d \cap V_0) - \dim(\operatorname{im} d \cap V_1) - \dim(\ker d \cap V_1) + \dim(\operatorname{im} d \cap V_0) \\ &= \dim V_0 - \dim V_1 \\ &= \chi(V).\end{aligned}$$

We apply this reasoning to the Q–vector spaces $(\mathrm{Tot}(E_r))_{(0)}$: $\mathrm{Tot}(E_r) = \mathrm{Tot}^{\mathrm{even}}(E_r) \oplus \mathrm{Tot}^{\mathrm{odd}}(E_r)$ is a $\mathbb{Z}_2$–grading of $\mathrm{H}_T^\bullet$–modules, and thus induces a $\mathbb{Z}_2$–grading on $(\mathrm{Tot}(E_r))_{(0)}$ with respect to which the map induced by the differential d_r is homogeneous of degree 1. Therefore,

$$\begin{aligned}\chi_{\mathbb{K}}(X) &= \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{even}}(M) - \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{odd}}(M) \\ &= \dim_Q (\mathrm{Tot}^{\mathrm{even}}(E_2))_{(0)} - \dim_Q (\mathrm{Tot}^{\mathrm{odd}}(E_2))_{(0)} \\ &= \dim_Q (\mathrm{Tot}^{\mathrm{even}}(E_\infty))_{(0)} - \dim_Q (\mathrm{Tot}^{\mathrm{odd}}(E_\infty))_{(0)} \\ &= \dim_Q (\mathrm{H}_T^{\mathrm{even}}(X))_{(0)} - \dim_Q (\mathrm{H}_T^{\mathrm{odd}}(X))_{(0)},\end{aligned}$$

where the last equality again holds by theorem 4.1. But by the Borel Localization Theorem, the latter is equal to

$$\begin{aligned}&\dim_Q (\mathrm{H}_T^{\mathrm{even}}(X^T))_{(0)} - \dim_Q (\mathrm{H}_T^{\mathrm{odd}}(X^T))_{(0)} \\ &= \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{even}}(X^T) - \dim_{\mathbb{K}} \check{\mathrm{H}}^{\mathrm{odd}}(X^T) \\ &= \chi_{\mathbb{K}}(X^T).\end{aligned}$$

□

5. Finite groups acting on spheres

Theorem 5.1. *If $\mathbb{K} = \mathbb{Z}_p$ is the field with p elements, p prime, then $\mathrm{H}_{\mathbb{Z}_p}^\bullet \cong \mathbb{Z}_p[s]$ as graded $\mathbb{K}$–vector spaces, with s of degree 1.*

Proof. Recall from remark 1.2 that $\check{\mathrm{H}}^n(B\mathbb{Z}_p)$ can be computed using any n–universal bundle $E \to B$ with compact Hausdorff total space E. Consider then the principal $\mathbb{Z}_p$–bundles $S^{2n+1} \to S^{2n+1}/\mathbb{Z}_p$, where we regard S^{2n+1} as a subset of $\mathbb{C}^{n+1}$ and the $\mathbb{Z}_p$–action is the restriction of the standard free action of $S^1 \subseteq \mathbb{C}$ on $\mathbb{C}^{n+1}$ (i. e. the diagonal action). Let us first observe that $\check{\mathrm{H}}^1(S^{2n+1}/\mathbb{Z}_p)$ is non–trivial: we already saw in example 2.7 that there is a bijection between $\check{\mathrm{H}}^1(X; C_G(X))$ and the set of isomorphism classes of principal G–bundles over X, where $C_G(X)$ is the sheaf of G–valued functions on X. In particular, if $G = \mathbb{Z}_p$ and X is connected, $C_G(X)$ is the constant presheaf associated to $\mathbb{Z}_p$, and hence also $\check{\mathrm{H}}^1(X)$ bijectively corresponds to the set of isomorphism classes of principal $\mathbb{Z}_p$–bundles. Thus, to show that $\check{\mathrm{H}}^1(S^{2n+1}/\mathbb{Z}_p)$ is non–trivial, it suffices to construct a non–trivial bundle over $S^{n+1}/\mathbb{Z}_p$. But $S^{2n+1} \to S^{2n+1}/\mathbb{Z}_p$ is one such bundle: indeed, if this bundle was trivial, then S^{2n+1} and $(S^{2n+1}/\mathbb{Z}_p) \times \mathbb{Z}_p$ would be homeomorphic, which is impossible, because S^{2n+1} is connected, while $(S^{2n+1}/\mathbb{Z}_p) \times \mathbb{Z}_p$ is not.

Next, consider the compact, connected, Abelian Lie group $S := S^1/\mathbb{Z}_p$ and the principal S–bundle $S^{2n+1}/\mathbb{Z}_p \to S^{2n+1}/S^1 = \mathbb{CP}^n$ induced by the identity map. Its Leray presheaf is simple, because S is connected (example 7.6), and hence the second page of the Leray spectral sequence $(E_r)_{r\geq 2}$ of this bundle reads

$$E_2^{p,q} = \check{\mathrm{H}}^p(\mathbb{CP}^n) \otimes_{\mathbb{K}} \check{\mathrm{H}}^q(S).$$

In particular, since $S \cong S^1$ as Lie groups, it consists of the two rows $E_2^{\bullet,0} \cong \check{\mathrm{H}}^\bullet(\mathbb{CP}^n)$ and $E_2^{\bullet,1} \cong \check{\mathrm{H}}^\bullet(\mathbb{CP}^n)$, and $E_3 = E_\infty$ is the final page. The differential $d_2 \colon E_2^{0,1} \to E_2^{2,0} \cong \check{\mathrm{H}}^2(\mathbb{CP}^n)$ must be trivial: as $E_2^{0,1} \cong \mathbb{K}$, the latter statement will be true if d_2 has a non–trivial kernel on $E_2^{0,1}$, and this is the case, because $E_\infty^{1,0} = 0$, so that $E_\infty^{0,1} \cong \check{\mathrm{H}}^1(S^{2n+1}/\mathbb{Z}_p)$ and hence $E_3^{0,1} = E_\infty^{0,1}$ must be non–trivial. As d_2 is an anti–derivation, it follows that $d_2 = 0$, the spectral sequence collapses, and

$$\begin{aligned}\check{\mathrm{H}}^k(S^{2n+1}/\mathbb{Z}_p) &\cong \begin{cases}\check{\mathrm{H}}^{k-1}(\mathbb{CP}^n), & k \text{ odd},\\ \check{\mathrm{H}}^k(\mathbb{CP}^n), & k \text{ even}\end{cases}\\ &\cong \mathbb{K}^k[s]/(s^{2n+2})\end{aligned}$$

as $\mathbb{Z}$–graded $\mathbb{K}$–vector spaces with s of degree 1. □

Remark 5.2. Consider the Leray spectral sequence $(E_r)_{r\geq 2}$ of the principal $S = S^1/\mathbb{Z}_p$–bundle $B\mathbb{Z}_p \to \mathbb{CP}^\infty$, where we consider $\mathbb{Z}_p$ as a subgroup of S^1 and $B\mathbb{Z}_p$ as $B\mathbb{Z}_p = S^\infty/\mathbb{Z}_p$. Because S is connected, the second page is

$$E_2^{p,q} = \check{\mathrm{H}}^p(\mathbb{CP}^\infty) \otimes_{\mathbb{K}} \check{\mathrm{H}}^q(S),$$

and as in the proof of theorem 5.1 it follows that the spectral sequence collapses on the second page. Thus, as algebras over $\mathbb{K} = \mathbb{Z}_p$, we have $E_\infty = \mathbb{K}[s]/(s^2) \otimes_{\mathbb{K}} \mathbb{K}[t]$, where s is an element of degree 1, corresponding to the generator of $H^1(S)$, and t is an element of degree 2, corresponding to the generator of $\check{\mathrm{H}}^2(\mathbb{CP}^\infty)$. However, it is known (e. g. [3, p. 373]) that $\check{\mathrm{H}}^\bullet(B\mathbb{Z}_2) = \mathbb{K}[s]$ as $\mathbb{K}$–algebras, so even though $\mathrm{Tot}(E_\infty)$ consists of one non–trivial summand, $\check{\mathrm{H}}^\bullet(B\mathbb{Z}_p)$ and $\mathrm{Tot}(E_\infty)$ need not be isomorphic as algebras.

Corollary 5.3. *No group $\mathbb{Z}_m \oplus \mathbb{Z}_m$ with $m \geq 2$ can act freely on a sphere S^n.*

Proof. Suppose that $G = \mathbb{Z}_m \oplus \mathbb{Z}_m$ would act freely on S^n, $n \geq 1$, for a contradiction. Since every cyclic group $\mathbb{Z}_m$ contains a subgroup $\mathbb{Z}_p$ with p–prime, we may assume that $m = p$ is prime. Now consider the Leray presheaf $\pi_*\check{\mathrm{H}}^q(-;\mathbb{Z}_m) = \pi_*\check{\mathrm{H}}^q(-)$ of $\pi\colon EG \times_G S^n \to BG$ with coefficients in $\mathbb{Z}_p$. We claim that this Leray presheaf is simple. In fact, since π is associated to a principal G–bundle, we only need to show that every element of G acts trivially on $\check{\mathrm{H}}^\bullet(S^n)$, cf. example 7.6. For $p = 2$ this is automatic, because there is only one non–trivial morphism of groups on $\mathbb{Z}_2$. Thus, let us assume $p > 2$ and let $\rho\colon G \times S^n \to S^n$, $(g, x) \mapsto \rho_g(x)$, be the action map, and $g \in G$ a generator of $\mathbb{Z}_p \times \{0\} \subseteq G$. Consider first the action of $(\rho_g)^*$ on $\check{\mathrm{H}}^n(S^n;\underline{\mathbb{Z}})$. Since $\check{\mathrm{H}}^n(S^n;\underline{\mathbb{Z}}) \cong \mathbb{Z}$ and ρ_g is a homeomorphism, $(\rho_g)^* = \pm\mathrm{id}$. However, $k := g^2$ still is a generator of $\mathbb{Z}_p$, because p is prime and $p > 2$, and in addition $\rho_k = (\rho_g)^2 = \mathrm{id}$. Therefore, $\rho_g = \mathrm{id}$. Now consider the commutative square

$$\begin{tikzcd}
\check{\mathrm{H}}^n(S^n; \underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[r] \arrow[d, "(\rho_g)^* \otimes_{\mathbb{Z}} \mathrm{id}"'] & \check{\mathrm{H}}^n(S^n; \underline{\mathbb{Z}_p}) \arrow[d, "(\rho_g)^*"] \\
\check{\mathrm{H}}^n(S^n; \underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[r] & \check{\mathrm{H}}^n(S^n; \underline{\mathbb{Z}_p})
\end{tikzcd}$$

The horizontal maps are the isomorphisms of the Universal Coefficient Theorem (theorem 3.3), and we just checked that the left vertical map is the identity. Hence also the right vertical map must be the identity. Applying the same argument to the other $\mathbb{Z}_p$ factor of G, we see that the Leray presheaf is simple.

The Leray spectral sequence of π thus has as second page

$$E_2^{p,q} = \check{\mathrm{H}}^p(BG; \underline{\mathbb{Z}_p}) \otimes_{\mathbb{K}} \check{\mathrm{H}}^q(S^n; \underline{\mathbb{Z}_p})$$

which just consists of the two rows $E_2^{\bullet,0} \cong \check{\mathrm{H}}^\bullet(BG)$ and $E_2^{\bullet,n} \cong \check{\mathrm{H}}^\bullet(BG)$. In particular, $E_2 \cong E_3 \cong \ldots \cong E_{n+1}$ and d_{n+1} is the only possibly non–trivial differential. On the other hand, since G acts freely on S^n, we know from theorem 1.5 that π^* induces an isomorphism $\mathrm{H}_G^\bullet(S^n) \cong \check{\mathrm{H}}^\bullet(S^n/G)$. As S^n/G is an n–dimensional smooth manifold, $\check{\mathrm{H}}^{n+k}(S^n/G)$ is trivial for all $k > 0$, and hence the same must be true for abutment $\mathrm{H}_G^\bullet(S^n)$ of the spectral sequence $(E_r)_{r \geq 2}$. Therefore, the differentials

$$\begin{tikzcd}
d_{n+1} \colon E_{n+1}^{k,n} \arrow[r] \arrow[d, "\cong"'] & E_{n+1}^{n+k+1,0} \arrow[d, "\cong"] \\
\check{\mathrm{H}}^k(BG; \underline{\mathbb{Z}_p}) \arrow[r] & \check{\mathrm{H}}^{n+k+1}(BG; \underline{\mathbb{Z}_p})
\end{tikzcd}$$

are isomorphisms for all $k > 0$. That is to say, the $\mathbb{Z}_p$–cohomology of $\check{\mathrm{H}}^\bullet(BG)$ is periodic. But $B\mathbb{Z}_p \times B\mathbb{Z}_p$ also is a classifying space for G, and hence also

$$\dim_{\mathbb{Z}_p} \check{\mathrm{H}}^k(BG) = \sum_{r+s=k} \dim_{\mathbb{Z}_p} \check{\mathrm{H}}^r(B\mathbb{Z}_p) \cdot \dim_{\mathbb{Z}_p} \check{\mathrm{H}}^s(B\mathbb{Z}_p) = k+1,$$

by the Künneth Formula (example 8.3) and theorem 5.1 above. This is impossible. ☐

As a further corollary we obtain a classical result by P. Smith [14].

Corollary 5.4. *If a finite group G acts freely on a sphere S^n, then all of its Abelian subgroups are cyclic.*

Remark 5.5. Finite groups with all Abelian subgroups cyclic have been classified by Suzuki–Zassenhaus, see [1, Theorem 6.15].

Proof. Let us first show that $\mathbb{Z}_p \oplus \mathbb{Z}_q$ can only act freely on S^n if p and q are coprime. If this was not the case, then there would exist an integer $1 < k < p, q$ dividing both p and q, that is, there would exist integers $r, s \geq 1$ such that $kr = p$ and $ks = q$. But then $\mathbb{Z}_k \oplus \mathbb{Z}_k \to \mathbb{Z}_p \oplus \mathbb{Z}_q$, $(i, j) \mapsto (ir, js)$ would be an injection and $\mathbb{Z}_k \times \mathbb{Z}_k$ would act freely on S^n, which we know is not the case. Hence, p and q are coprime.

Now let G be a finite Abelian group acting freely on S^n. Then G is of the form $G \cong \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_r}$, where each $q_i = (p_i)^{n_i}$ is a non–trivial power of a prime p_i. We just observed that $p_i \neq p_j$ for all $i \neq j$, and if p and q are coprime, then $\mathbb{Z}_{pq} \to \mathbb{Z}_p \oplus \mathbb{Z}_q$, $k \mapsto (k, k)$, is an isomorphism. Hence, $G \cong \mathbb{Z}_m$ with $m = q_1 \cdots q_r$ is cyclic. □

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