Rigidity, stability and deformations in nearly parallel G_2-geometry

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Nearly parallel G_2-manifolds

Definition:
A nearly parallel G_2-structure on a manifold M^7 is given by a stable 3-form $\varphi \in \Omega^3_+(M^7)$ such that, w.r.t. to the metric and the orientation determined by φ, we have $d\varphi = \lambda * \varphi$ for some real number $\lambda \neq 0$. Here stable, in the sense of Hitchin, means that φ in any $p \in M$ lies in an open orbit of $\mathrm{GL}(7)$ acting on $\Lambda^3 T_p M$.

Properties and Motivation: Nearly parallel G_2-manifolds have the following interesting properties:

- The induced metric g_φ is Einstein with positive scalar curvature.
- The defining condition is equivalent to the existence of a Killing spinor Ψ, i.e. $\nabla_X \Psi = \frac{\lambda}{4} X \cdot \Psi$, $\forall X$.
- The metric cone $\hat{M} = M \times \mathbb{R}$ with metric $\hat{g} = r^2 g + dr^2$ has holonomy $\mathrm{Spin}(7)$.
- There exists a canonical metric connection with skew-symmetric and parallel torsion.
- All associated submanifolds (calibrated by φ) are minimal. There are no co-associatives (cf. part C).

Facts and examples:

1. There are three classes of nearly parallel G_2-manifolds:
   (i) 3-Sasakian manifolds $\widehat{=}$ 3-dim. space of Killing spinors
   (ii) Einstein-Sasaki manifolds $\widehat{=}$ 2-dim. space of Killing spinors
   (iii) proper $\widehat{=}$ 1-dim space of Killing spinors
2. We have classified homogeneous nearly parallel G_2-manifolds in [FKMS97]. Proper instances (iii) are:
   (i) The squashed sphere: S^7_{sq}
   (ii) The Aloff-Wallach spaces: $N(k,l) := \mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$, $(k,l) \neq (1,1)$
   (iii) The Berger space: $\mathrm{SO}(5)/\mathrm{SO}(3)_{max}$
3. The second Einstein metric in the canonical variation of a 3-Sasaki 7-manifold provides examples of proper nearly parallel G_2-structures, by our work in [FKMS97], e.g. on $N(1,1)$. Prototype non-homogeneous examples have a maximal isometric T^3-action and arbitrary second Betti number.

Remarks:
(1) Nearly parallel G_2-manifolds are in many respects very similar to 6-dimensional nearly Kähler manifolds, i.e. 6-dimensional manifolds with a Killing spinor. Many examples thereof, with T^3-symmetry, have been recently constructed from a second order ODE by Moroianu and Nagy.
(2) The focus of our project is on compact manifolds. However, the sine-cone construction over nearly Kähler manifolds provides a large class of non-complete nearly parallel G_2-metrics.
(3) A spin manifold (M^n, g) admits a Killing spinor iff it is the standard sphere or it has one of the following structures: Einstein-Sasaki ($n = 2k+1$), 3-Sasaki ($n = 4k+3$), nearly Kähler ($n = 6$) or nearly parallel G_2 ($n = 7$). In all cases the metrics are Einstein with positive scalar curvature.

A) Stability of Einstein metrics

Definition:
A closed Einstein manifold (M^n, g) is (linearly) stable if the second variation of the total scalar curvature functional is non-positive on tt-tensors h, i.e. tensors $h \in \Gamma(\mathrm{Sym}^2 TM)$ with $\mathrm{tr}(h) = 0 = \delta h$. Equivalently $\Delta_L \geq 2\mathrm{scal}/n$ on tt-tensors, where $\Delta_L = \nabla^* \nabla + q(R)$ is the Lichnerowicz Laplacian w.r.t. the Levi-Civita connection ∇. Here $q(R)$ is a symmetric endomorphism depending linearly on the Riemann curvature R.

Facts and examples:
Linear instability implies linear instability with respect to the ν-entropie and, for $\mathrm{scal} > 0$, also dynamic instability with respect to the Ricci flow [Kröncke]. The following classes of manifolds are known to be stable:

- Kähler-Einstein manifolds with $\mathrm{scal} \leq 0$
- Most symmetric spaces with $\mathrm{scal} > 0$, including the standard sphere [Koiso80]

The following manifolds are known to be unstable:

- Aloff-Wallach spaces $N(k,l)$ and many classes of non-homogeneous Einstein-Sasaki manifolds
- Nearly Kähler manifolds in dimension 6 with non-trival cohomology, by our recent work [SWW19].

Except for a few symmetric spaces there are no known examples of stable manifolds with $\mathrm{scal} > 0$. It seems to be very likely that manifolds admitting Killing spinors (not isometric to the standard sphere) are unstable. This is already checked in many cases. An important remaining class are the nearly parallel G_2-manifolds.

Project goal: Prove that nearly parallel G_2-manifolds (with non-trivial cohomology) are unstable.

Related to this topic is the following problem. By the work of Koiso the question of stability is decided for all symmetric spaces except for: $\mathbb{HP}^2$, $\mathrm{Sp}(n+m)/\mathrm{Sp}(n) \times \mathrm{Sp}(m), n \geq m \geq 2$ and $\mathrm{F}_4/\mathrm{Spin}_9$. Here stability is an open and challenging question, which we plan to consider.

Methods:
In our proof of the instability of 6-dimensional nearly Kähler manifolds with non-trivial cohomology in [SWW19] we used an invariant identification, mapping harmonic forms to eigenforms of the Lichnerowicz Laplacian Δ_L corresponding to subcritical eigenvalues. An important tool was the canonical Hermitian connection with skew-symmetric and parallel torsion having holonomy $\mathrm{SU}(3)$.

- Generalisation to nearly parallel G_2 manifolds M with $b_2(M) \neq 0$ or $b_3(M) \neq 0$ will explore the effect of mapping harmonic forms via the algebraic identification: $\Lambda^3_{27} \cong \mathrm{Sym}^2_0\mathrm{T}$ on small eigenvalues of Δ_L. The inclusion $\mathrm{Sym}^2_0 T \subset \mathrm{Sym}^2(\Lambda^2_{14})$ leads to a notion of square for harmonic forms in Λ^2_{14} on which we will attempt to calculate the action of the Lichnerowiz Laplacian Δ_L.
- For the Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)_{max}$, which is a homology sphere, we plan to use Killing tensors. Killing tensors are symmetric tensors h with $(\nabla_X h)(X, \dots, X) = 0$ for all tangent vectors X. If divergence free they satisfy the equation $\Delta_L h = 2q(R)h$. Key to this approach is that $q(R)$ is determined by the Casimir operator on normal homogeneous spaces. To compute the space of Killing tensors methods of harmonic analysis will be used.

B) Rigidity of nearly parallel G_2-structures

Definition:
An infinitesimal deformation of a nearly parallel G_2-structure (M^7, φ) is a tangent vector $v \in \Omega^3(M)$ to a curve φ_t of nearly parallel G_2-structures with $\varphi_0 = \varphi$. A G_2-structure φ is called rigid if there exists (up to diffeomorphims) no curve of G_2-structures through φ, i.e. φ defines an isolated point in the moduli space.

Recall that any nearly parallel G_2-structure defines an Einstein metric. So G_2-deformations are a subclass of Einstein deformations.

Facts: The following statements for proper G_2-structures we proved in [AS12]:

- The space $\mathcal{E}$ of infinitesimal deformations is a subspace of a certain Δ-eigenspace on 3-forms. More precisely, an infinitesimal deformation $v \in \Omega^3(M)$ has to satisfy: $*dv = -\lambda v$ and $v \in \Omega^3_{27}(M)$.
- The squashed 7-sphere and the Berger-space are rigid, since they do not have infinitesimal deformations.
- The second Einstein metric on the 3-Sasaki Aloff-Wallach space $N(1,1) = \mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$ has an 8-dimensional space of infinitesimal deformations.

Other important results: Pedersen and Poon showed that 3-Sasaki metrics are rigid. However, in the toric case they can be deformed through smooth families of Einstein-Sasaki metrics, as showed by van Coevering.

Project goals: Develop (reformulate) the full deformation theory for proper nearly parallel G_2-manifolds. Describe the obstruction to deformations. We plan to test the new approach on the following open problems:

- Determine the space of infinitesimal deformations for the Aloff-Wallach spaces $N(k,l)$ with $(k,l) \neq (1,1)$ and for the second Einstein metric on 3-Sasaki manifolds in dimension 7. Moreover give explicitly obstructions to deformations.
- Prove rigidity for the second Einstein metric on the 3-Sasaki Aloff-Wallach space $N(1,1)$.

Note the somewhat mysterious role of $\mathrm{SU}(3)$: the nearly Kähler $\mathrm{SU}(3)/T^2$ has infinitesimal deformations as well as the nearly parallel G_2-manifold $\mathrm{SU}(3)/T^1$. The integrability of infinitesimal Einstein deformations of the symmetric metric on $\mathrm{SU}(3)$, shown to exist by Koiso, is a longstanding open problem. In this respect it could be interesting to use that $\mathrm{SU}(3)$ carries a harmonic $\mathrm{PSU}(3)$-structure defined by a stable 3-form ψ.

Methods: Use Hitchin's duality map for stable forms $T \mapsto \hat{T}$ and the preliminary observation that the deformations theory is governed by a Maxwell type operator $D(T) := d\hat{T} - \lambda T$. Determine explicitly the cubic polynomial Q on $\mathcal{E} = \ker D$ obstructing the deformations at second order.

- We plan to use elliptic theory for the operator D in order to describe the set of G_2-structures near φ as a level set of a naturally defined functional on stable 4-forms. The main tool for establishing existence here is the implicit function theorem on Banach spaces.
- From our previous work the obstruction polynomial Q in the case of $N(1,1)$ can be interpreted as a $\mathrm{SU}(3)$-invariant polynomial on the Lie algebra of $\mathrm{SU}(3)$. The plan is to use harmonic analysis and representation theory to show that it does not vanish identically, thus establishing rigidity. A similar program was carried out by Foscolo in his proof of the rigidity of the nearly Kähler $\mathrm{SU}(3)/T^2$.
- For the second Einstein metric on quasi-regular 3-Sasaki spaces we will determine $\mathcal{E}$ in terms of objects defined on the 4-dimensional orbifold associated to the 3-Sasaki structure. We plan to treat the more general case of Einstein deformations in a similar way, partly following ideas of van Coevering.

C) Deformation of associative submanifolds

Definition:
An associative submanifold in a general G_2-manifold (M^7, φ) is a 3-dimensional submanifold $i : N \hookrightarrow M$, calibrated w.r.t. the stable form φ, i.e. such that $i^*\varphi = \mathrm{vol}_N$. The nearly parallel G_2-manifolds are characterised among the non torsion-free G_2-manifolds by the fact that all associated submanifolds are minimal.

Facts and examples: Up to now associatives in nearly parallel G_2 manifolds were mainly studied in the case of the 7-sphere. Associatives can be constructed from other geometries, e.g. from Legendrian submanifolds in the Einstein-Sasaki case or Lagrangians in nearly Kähler 6-manifolds via the sine-cone construction.

- Homogeneous associative submanifolds in S^7 were classified by Lotay (2012) and in the squashed 7-sphere S^7_{sq} by Kawai (2015). Only one instance, $A_3 \subset S^7$, does not arise from other geometries. In addition, constant curvature and totally geodesic associatives in S^7 are well understood.
- Associatives ruled by geodesic circles correspond to pseudo-holomorphic curves in the Grassmannian $\mathrm{Gr}^+_2(\mathbb{R}^8)$. Moreover any minimal surface in S^6 induces a family of associatives via its pseudoholomorphic lift to $\mathrm{Gr}^+_2(\mathbb{R}^8)$. A key role in this construction is played by the twistor fibration $\mathbb{CP}^3 \hookrightarrow \mathrm{Gr}^+_2(\mathbb{R}^8) \to S^6$.
- The general deformation theory of associative submanifolds has been developed by McLean (1998) in the torsion free case and by Akbulut and Salur (2008) for general G_2 structures. Infinitesimal deformations are given as an eigenspace of a twisted Dirac operator. This has been calculated on the homogeneous examples. Based on this, Kawai showed that A_3 is unobstructed to second order.

Project goals: New examples of associatives, classification under geometric conditions, study deformations.

- Find examples of associative submanifolds in the Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)$. In particular, classify homogeneous or curvature pinched associatives together with their isometric deformations.
- In general, study classes of associative submanifolds in nearly parallel G_2 manifolds, e.g. totally geodesic or with index of nullity one, i.e. where the second fundamental form vanishes on a line subbundle.
- Study obstructions to deformations of associative submanifolds and prove in particular that formally obstructed is the same as smoothly obstructed, in analogy to the nearly Kähler case. Test this for A_3.

As a related problem we plan to answer the question whether the volume is constant under deformations of associatives in nearly parallel G_2 manifolds. This was proved by Verbitsky (2013) for the similar situation of deformations of Lagrangians in 6-dimensional nearly Kähler manifolds.

Methods: We propose to search for a geometric construction of a space allowing a correspondence between pseudoholomorphic curves in it and associative submanifolds in e.g. the Berger space. We believe that the realisation of the Berger space as a S^3-fibration over S^4 (cf. Goette et al. (2004)) will play an impotrant role. If the second fundamental form is additionally constrained we will use Cartan-Kähler theory to capture the local geometry. For the obstruction theory we plan to use ideas from the nearly Kähler case and the work.

References

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