Prime Graph Question for 4-primary groups II

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Brock International Conference on
Groups, Rings and Group Rings
August 1st 2014
As in Andreas talk we have:

- $G$ finite group
- $\mathbb{Z}G$ integral group ring over $G$
- Augmentation map: $\varepsilon : \mathbb{Z}G \rightarrow \mathbb{Z}$, $\varepsilon(\sum_{g \in G} z_g g) = \sum_{g \in G} z_g$
- $V(\mathbb{Z}G)$ group of units of augmentation 1, aka normalized units
- $\mathbb{Q}_p$ $p$-adic numbers, $\mathbb{Z}_p$ $p$-adic integers
Prime Graph Question (Kimmerle, 2006)

(PQ) Let $p$ and $q$ be different primes. If $V(\mathbb{Z}G)$ contains an element of order $pq$, does $G$?

Theorem (Kimmerle, Konovalov, 2012)

Suppose that (PQ) has an affirmative answer for each almost simple image of $G$, then it has also a positive answer for $G$. 
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Suppose that (PQ) has an affirmative answer for each almost simple image of $G$, then it has also a positive answer for $G.$
After applying HeLP to all almost simple 4-primary groups, we have the following cases left:

- Order 6 over $\text{PSL}(2, 2^f)$, for possibly infinitely many $f$
- Order 6 over $\text{PGL}(2, 3^f)$, for possibly infinitely many $f$
- Order 15 over $\text{PSL}(2, 81).\left(C_4 \times C_2\right)$
- Order 6 over $\text{PSL}(3, 4)$
- Order 15 over $\text{PSL}(3, 5).C_2$
- Order 21 over $\text{PSL}(3, 7).C_2$
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- Order 51 over $\text{PSL}(3, 17).C_2$
- Order 6 over $\text{PSU}(3, 4)$
- Order 21 over $\text{PSU}(3, 7).C_2$
- Order 10 over $\text{Sz}(32)$
- Order 21 over $G_2(3).C_2$
Let $u$ be a torsion unit of some given order in $V(\mathbb{Z}G)$. After applying HeLP we get restrictions on the partial augmentations of $u$ and thus on the eigenvalues of $D(u)$, where $D$ is a representation of $G$. Can we use this knowledge to get even more restrictions on the partial augmentations?

Original motivation:

Theorem (Kimmerle, Konovalov, 2012)

($PQ$) holds for all groups, whose order is divisible by exactly three different primes, if there are no units of order 6 in $V(\mathbb{Z}M_{10})$ and $V(\mathbb{Z}PGL(2,9))$.

Starting point: Hertweck’s handling of units of order 6 in $V(\mathbb{Z}A_6)$. 

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**Starting point:** Hertweck's handling of units of order 6 in $V(\mathbb{Z}A_6)$. 
Let \( R \) be a complete local ring with maximal ideal \( P \) containing the prime \( p \). Denote by \( K \) the field of fractions of \( R \), by \( k \) the residue class field of \( R \) and by \( \bar{\cdot} \) the reduction modulo \( P \).

### Proposition

Let \( G = \langle g \rangle \) be of order \( p^a m \) such that \( \gcd(p, m) = 1 \) and \( \xi \) a primitive \( m \)-th root of unity in \( R \). Let \( D \) be an \( R \)-representation of \( G \) and \( L \) an \( RG \)-lattice affording \( D \). Let \( A_i \) be sets with multiplicities s.t. \( \xi A_1 \cup \xi^2 A_2 \cup ... \cup \xi^m A_m \) are the complex eigenvalues of \( D(g) \).

Then \( L \cong L^{\xi} \oplus ... \oplus L^{\xi^m} \) and \( \bar{L} \cong \bar{L}^{\xi} \oplus ... \oplus \bar{L}^{\xi^m} \) s. t. \( \text{rank}_R(L^{\xi^i}) = \dim_k(L^{\xi^i}) = |A_i| \). Moreover if \( V_1, ..., V_m \) are \( KG \)-representations affording the representations \( E_1, ..., E_m \) s. t. \( E_i(g) \) has eigenvalues \( A_i \), then \( K \otimes_R L^{\xi^i} \cong V_i \) and \( L^{\xi^i} \) has only one composition factor up to isomorphism.
Main Proposition

Let $R$ be a complete local ring with maximal ideal $P$ containing the prime $p$. Denote by $K$ the field of fractions of $R$, by $k$ the residue class field of $R$ and by $\overline{-}$ the reduction modulo $P$.

Proposition

Let $G = \langle g \rangle$ be of order $p^a m$ such that $\gcd(p, m) = 1$ and $\xi$ a primitive $m$-th root of unity in $R$. Let $D$ be an $R$-representation of $G$ and $L$ an $RG$-lattice affording $D$. Let $A_i$ be sets with multiplicities s.t. $\xi A_1 \cup \xi^2 A_2 \cup ... \cup \xi^m A_m$ are the complex eigenvalues of $D(g)$.

Then $L \cong L^\xi \oplus ... \oplus L^{\xi^m}$ and $\overline{L} \cong \overline{L^\xi} \oplus ... \oplus \overline{L^{\xi^m}}$ s. t. $\text{rank}_R(L^\xi) = \dim_k(L^\xi) = |A_i|$. Moreover if $V_1, ..., V_m$ are $KG$-representations affording the representations $E_1, ..., E_m$ s.t. $E_i(g)$ has eigenvalues $A_i$, then $K \otimes_R L^\xi \cong V_i$ and $L^\xi$ has only one composition factor up to isomorphism.
Let $k$ be a field of characteristic $p$ and $G = \langle g \rangle$ of order $p^a m$. Denote by $\xi$ an primitive $m$-th r.o.u.

**Lemma**

Up to isomorphism there are $m$ simple $kG$-modules. They are all 1-dim. and $g^m$ acts trivially on them while $g^{p^a}$ acts as $\xi, \ldots, \xi^m$. The projective indecomposable $kG$-modules have dimension $p^a$, are uniserial and have only one composition factor up to isomorphism. An indecomposable $kG$-module is a submodule of an projective, indecomposable $kG$-module.
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The easiest case

Next we need to establish a connection between $A_i$ and the iso’type of $L^{\xi^i}$. This depends on the $p$-part of $\langle g \rangle$ and the ramification index of $R$ over $\mathbb{Z}_p$ and is a wild problem in general. The easiest case is the following:

**Proposition**

Assume $R$ is unramified over $\mathbb{Z}_p$ and let $\zeta$ be a primitive $p$-th root of unity. Up to isomorphism there are exactly 3 indecomposable $RC_p$-lattices $R, I(RC_p)$ and $RC_p$. All of them stay indecomposable when reduced modulo $P$. The corresponding eigenvalues are 1 and $\zeta, \zeta^2, ..., \zeta^{p-1}$ and $1, \zeta, ..., \zeta^{p-1}$ respectively.
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First applications for the Lattice-Method:

**Theorem**
There are no units of order 6 in $V(\mathbb{Z}M_{10})$ and $V(\mathbb{Z} \text{PGL}(2, 9))$, thus (PQ) holds for groups, whose order is divisible by exactly three different primes.

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The Zassenhaus-Conjecture holds for $\text{PSL}(2, 19)$.

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Let $G = \text{PSL}(2, 2^f)$ with $3 \nmid f$. Then (PQ) holds for $G$.
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After applying HeLP to all almost simple 4-primary groups, we have the following cases left:

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After applying HeLP to all almost simple 4-primary groups, we have the following cases left

- **Order 6 over PSL(2, 2^f)**, for possibly infinitely many \( f \)
- **Order 6 over PGL(2, 3^f)**, for possibly infinitely many \( f \) (Decomposition)
- **Order 15 over PSL(2, 81).\((C_4 \times C_2)\)** (Data)
- **Order 6 over PSL(3, 4)**
- **Order 15 over PSL(3, 5).C_2**
- **Order 21 over PSL(3, 7).C_2**
- **Order 6 over PSL(3, 8)** (Representation type)
- **Order 51 over PSL(3, 17).C_2** (Data)
- **Order 6 over PSU(3, 4)**
- **Order 21 over PSU(3, 7).C_2**
- **Order 10 over Sz(32)** (Representation type)
- **Order 21 over G_2(3).C_2**
Sometimes the Lattice-Method is not successful when applied to $G$, but proves (PQ) for $G$ when applied to some group $A$ containing $G$.

For example $A = \text{Aut}(G)$ can help using Cliffords Theorem. Or $M_{22}$ for $\text{PSL}(3,4)$.

→ Can we construct such “helping groups” to solve the open cases?
Thank you for your attention!