Rational conjugacy of torsion units in integral group rings of non-solvable groups

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RTRA
Murcia, June 4th 2013
- $G$ a finite group.
- $RG$ group ring of $G$ over the ring $R$.
- $\mathbb{Q}_p$ the $p$-adic number field, $\mathbb{Z}_p$ ring of integers of $\mathbb{Q}_p$.
- $\varepsilon : RG \to R$ augmentation map, $\varepsilon_x : RG \to R$ partial augmentation on the conjugacy class $x^G$ for $x \in G$:
  \[ \varepsilon(\sum_{g \in G} u_g g) = \sum_{g \in G} u_g \quad \text{and} \quad \varepsilon_x(\sum_{g \in G} u_g g) = \sum_{g \in x^G} u_g \]
- $U(RG)$ units in $RG$, $V(RG)$ units of augmentation 1 aka normalized units ($\pm V(\mathbb{Z}G) = U(\mathbb{Z}G)$).
- $\exp(V(\mathbb{Z}G)) = \exp(G)$. (Cohn-Livingstone ’65)
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(First) Zassenhaus Conjecture (H.J. Zassenhaus, in the ’60s)

For \( u \in V(\mathbb{Z}G) \) of finite order there exist \( x \in U(\mathbb{Q}G) \) and \( g \in G \) s.t. \( x^{-1}ux = g \).

A weaker version of this conjecture is:

Prime Graph Question

Let \( p \) and \( q \) be different primes s.t. \( V(\mathbb{Z}G) \) has an element of order \( pq \). Does this imply that \( G \) has an element of that order? I.e.: Do \( G \) and \( V(\mathbb{Z}G) \) have the same prime graph?
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The Zassenhaus Conjecture is known for some series of solvable groups (e.g. nilpotent groups (Weiss ’91)) and for some non-solvable. E.g.:

- $A_5$ (Luthar-Passi ’89),
- $S_5$ (Luthar-Trama ’91),
- Central extensions of $S_5$ (Bovdi-Hertweck ’08),
- $\text{PSL}(2, p)$, $p$ a prime, $p \leq 17$. (Hertweck ’07, Gildea ’12, Kimmerle-Konovalov ’12),
- $\text{PSL}(2, 8)$ (Gildea ’12, Kimmerle-Konovalov ’12),
- $A_6 \cong \text{PSL}(2, 9)$ (Hertweck ’07).
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The Prime Graph Question is known for:

- Solvable Groups (Kimmerle ’06),
- $\text{PSL}(2, p)$, $p$ a prime (Hertweck ’07),
- Many sporadic simple groups (Bovdi, Konovalov et al. 07’ - ),
- Groups, whose order is divisible by at most three primes, if there are no units of order 6 in $V(\mathbb{Z}\text{PGL}(2, 9))$ and in $V(\mathbb{Z}M_{10})$. (Where $M_{10}$ denotes the Mathieu group of degree 10.) (Kimmerle-Konovalov ’12)
Our results

Theorem

The Zassenhaus Conjecture holds for $\text{PSL}(2, 19)$ and $\text{PSL}(2, 23)$.

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The connection between rational conjugacy and partial augmentations is:

**Lemma**

\[ u \in V(\mathbb{Z}G) \text{ is conjugate in } \mathbb{Q}G \text{ to an element of } G \text{ if and only if } \varepsilon_g(v) \geq 0 \text{ for every } v \in \langle u \rangle \text{ and every } g \in G. \]

General knowledge on partial augmentations of a torsion unit \( u \in V(\mathbb{Z}G) \):

- \( \varepsilon_1(u) = 0 \), if \( u \neq 1 \). (Berman-Higman ’39/’51)
- \( \varepsilon_x(u) \neq 0 \), then the order of \( x \) divides the order of \( u \). (Marciniak-Ritter-Sehgal-Weiss ’87, Hertweck ’07)
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The HeLP-method (Hertweck ’07, Luthar-Passi ’89) establishes a connection between partial augmentations and eigenvalues under representations:

Let \( u \in V(\mathbb{Z}G) \) be a torsion unit of order \( n \), \( K \) an algebraically closed field of characteristic not dividing \( n \) and \( D \) a \( K \)-representation of \( G \) with character \( \chi \). Let \( \zeta \) be a primitive \( n \)-th root of unity and \( \xi \) some \( n \)-th root of unity. Then the number of times \( \xi \) appears as an eigenvalue of \( D(u) \) is

\[
\frac{1}{n} \sum_{d \mid n, \ d \neq 1} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\xi^{-1}) + \frac{1}{n} \sum_{x \in G, \ p \mid o(x)} \epsilon_x(u) \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(x)\zeta^{-1}).
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\frac{1}{n} \sum_{d \mid n, \ d \neq 1} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\xi^{-1}) + \frac{1}{n} \sum_{x \in G, \ p \nmid o(x)} \varepsilon(x(u))\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(x)\zeta^{-1}).
$$
Part of the ordinary character table of $A_6$:

<table>
<thead>
<tr>
<th></th>
<th>1a</th>
<th>2a</th>
<th>3a</th>
<th>3b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi$</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

Let $D$ be a representation affording $\chi$ and $\zeta$ a prim. 3rd root of unity. Assume $u \in V(\mathbb{Z}A_6)$ is of order 6, s.t. $\varepsilon_{2a}(u) = -2$, $\varepsilon_{3a}(u) = 2$, $\varepsilon_{3b}(u) = 1$ and $u^4$ is rationally conjugate to an element in $3b$. Then $\chi(u^3) = 1$ and $\chi(u^4) = -1$, so

$D(u^3) \sim \text{diag}(1, 1, 1, -1, -1), \quad D(u^4) \sim \text{diag}(1, \zeta, \zeta^2, \zeta, \zeta^2).$

The eigenvalues of $D(u)$ are products of the eigenvalues of $D(u^3)$ and $D(u^4)$ and

$\chi(u) = \varepsilon_{2a}(u)\chi(2a) + \varepsilon_{3a}(u)\chi(3a) + \varepsilon_{3b}(u)\chi(3b) = 1.$

This gives

$D(u) \sim \text{diag}(1, \zeta, \zeta^2, -\zeta, -\zeta^2).$
Basic idea

Natural question: What can we do with the eigenvalues? Let:

- $p$ a prime dividing the order of $u$,
- $D$ an ordinary representation of $G$,
- $K$ the $p$-adic completion of a number field affording $D$ with minimal ramification index over $\mathbb{Q}_p$,
- $R$ the ring of integers of $K$ with maximal ideal $P$, s.t. $p \in P$,
- $L$ an $RG$-lattice affording $D$,
- $k = R/P$ the quotient field of $R$,
- $\bar{\mathbb{F}}$ the reduction mod $P$.

Eigenvalues of $D(u) \longrightarrow$ Poss. iso’types of $\bar{L}$ as $k\langle \bar{u}\rangle$-module.

↓

$K\langle u\rangle$-composition factors of $L \longrightarrow$ Poss. iso’types of $L$ as $R\langle u\rangle$-lattice
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$K\langle u \rangle$-composition factors of $L \longrightarrow$ Poss. iso’types of $L$ as $R\langle u \rangle$-lattice
Notation as above, let \( \circ(u) = p^a m \) with \( p \nmid m \), let \( \zeta \) a prim. \( m \)-th root of unity s.t. \( \zeta \in R \). Let \( A_i \) be tuples of \( p^a \)-th roots of unity s.t. the eigenvalues of \( D(u) \) are \( \zeta A_1 \cup \zeta^2 A_2 \cup ... \cup \zeta^m A_m \). Then as \( R\langle u \rangle \)-lattice \( L \cong M_1 \oplus ... \oplus M_m \) s.t. \( \text{rang}_R(M_i) = |A_i| = \dim_k(\bar{M}_i) \) and \( \bar{M}_i \) has only one composition factor up to isomorphism.

Easiest case: \( K \) unramified over \( \mathbb{Q}_p \), \( \circ(u) = p \). There are three indecomposable \( R\langle u \rangle \)-lattices \( R \), \( I(RC_p) \), \( RC_p \) of rank 1, \( p - 1 \), \( p \) resp. with corresponding eigenvalues \( \{1\} \), \( \{\xi_p, ..., \xi_p^{p-1}\} \), \( \{1, \xi_p, ..., \xi_p^{p-1}\} \), where \( \xi_p \) is a primitive \( p \)-th root of unity. The reduction of any such lattice stays indecomposable.
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Let $G = \text{PSL}(2, 19)$. After applying HeLP the only critical case left is:

\[ \circ(u) = 10, (\varepsilon_{5a}(u), \varepsilon_{5b}(u), \varepsilon_{10a}(u)) = (1, -1, 1). \]

Let the setting be as above with $p = 5$ and let $\zeta$ be a primitive 5th root of unity.

There are ordinary representations $D_{18}$ and $D_{19}$ s.t. $D_{18}$ is a $\mathbb{Z}_5[\zeta + \zeta^{-1}]$-representation and $D_{19}$ a $\mathbb{Z}_5$-representation. If $L_{18}$ and $L_{19}$ are $RG$-lattices ($R$ is different) then $\bar{L}_{18} \leq \bar{L}_{19}$ and $\bar{L}_{19}/\bar{L}_{18} \cong \text{trivial } kG$-module.
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Using the given partial augmentations we compute:

\[ D_{18}(u) \sim \text{diag}(A, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4, -1, -\zeta, -\zeta^4, -\zeta, -\zeta^4), \]
\[ D_{19}(u) \sim \text{diag}(A', -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4), \]

where \( A \) and \( A' \) are some 5th roots of unity.

→ As \( k\langle \bar{u} \rangle \)-modules we have \( \bar{L}_{19} \cong M^1_{19} \oplus M^{-1}_{19} \) and \( \bar{L}_{18} \cong M^1_{18} \oplus M^{-1}_{18} \), where all compositions factors of \( M^*_1 \) are trivial and all composition factors of \( M^{-1}_* \) are non-trivial. Then \( M^{-1}_{19} \in \{ 2(k)_- \oplus 2I(kC_5)_-, (k)_- \oplus I(kC_5)_- \oplus (kC_5)_-, 2(kC_5)_- \} \).

As \( \bar{L}_{19}/\bar{L}_{18} \) is a trivial module, we have \( M^{-1}_{18} \cong M^{-1}_{19} \), but this is impossible by the composition factors of \( M^{-1}_{18} \) (by results of Jacobinski ’67 and Gudivok ’65).

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\[ \rightarrow \text{As } k\langle \bar{u} \rangle \text{-modules we have } \bar{L}_{19} \cong M_{19}^1 \oplus M_{19}^{-1} \text{ and } \bar{L}_{18} \cong M_{18}^1 \oplus M_{18}^{-1}, \]
where all compositions factors of \( M_* \) are trivial and all composition factors of \( M_*^{-1} \) are non-trivial. Then \( M_{19}^{-1} \in \{ 2(k)_- \oplus 2l(kC_5)_-, (k)_- \oplus l(kC_5)_- \oplus (kC_5)_-, 2(kC_5)_- \} \).

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Thank you for your attention!