32. Kolloquium über Differentialgeometrie

Universität Stuttgart

4. Mai 2007

32. Differentialgeometrie-Kolloquium

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Vortragsauszüge

Gemäß einem langjährigen Zyklus wurde am 4. Mai 2007 an der Universität Stuttgart das „32. Kolloquium über Differentialgeometrie“ abgehalten. Die Vortragsauszüge sind auf den folgenden Seiten abgedruckt.

In diesem Jahr waren der Einladung Geometer aus Augsburg, Darmstadt, Karlsruhe, Konstanz und München gefolgt. Außerdem hatten wir zwei spezielle Gäste (sozusagen als „invited speaker“) aus Mannheim und Köln. Dies ist auch in Verbindung zu sehen mit dem neuen MAT Seminar zur Differentialgeometrie (3-Städte-Kolloquium Mannheim-Augsburg-Tübingen).

W.Kühnel

32. Kolloquium über Differentialgeometrie

Universität Stuttgart, 4. Mai 2007

Vortragsprogramm

10:00 – 10:30	Andreas Weber (Karlsruhe) Das Spektrum des Laplace-Beltrami-Operators auf Produkt-Mannigfaltigkeiten
10:40 – 11:20	Yong He (Darmstadt) Verzweigte Nodoide
	Teepause
11:50 – 12:40	Walter Freyn (Augsburg) Kac-Moody symmetrische Räume
	Mittagspause
15:00 – 16:00	Martin U. Schmidt (Mannheim) *cmc* tori in S^3
	Kaffeepause
16:30 – 17:30	Uwe Semmelmann (Köln) Nearly Kähler manifolds
ab ca. 18:00	gemeinsames Abendessen

Die Vorträge finden auf dem Campus Stuttgart-Vaihingen im Gebäude Pfaffenwaldring 57 statt, im 8. Stock, Raum 8.122. Das ist in der Nähe vom blauen Aufzug in dem Gebäude.
S-Bahn-Haltestelle "Universität" entlang den Linien S1, S2, S3 nach Flughafen, Filderstadt oder Böblingen, Herrenberg.

A general theory of affine Kac-Moody symmetric spaces

Walter Freyn

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1 Introduction

As is well known, in finite dimensions there is essentially an equivalence between symmetric spaces, polar representations and isoparametric submanifolds in Euclidean spaces. See for example [Dad85, Tho91].

Several papers by Ernst Heintze, Xiaobo Liu, Richard Palais, Chuu-Lian Terng, Gudlaugur Thorbergsson, [Ter89, Ter95, HL99, HPTT95] and others demonstrate, that similar results hold in infinite dimensions; however, the problem of constructing the associated infinite dimensional symmetric spaces was open.

In her article [Ter95] Chuu-Lian Terng showed that the symmetric spaces fitting into the theory should correspond to Kac-Moody groups, but remarks that the construction poses some problems. Bogdan Popescu [Pop05, Pop06] constructs Kac-Moody symmetric spaces by explicitly using loop groups. However, in his approach problems concerning the complexification — and so the dualization — arise. We will later describe more details of the work of Popescu, focusing on its problems.

For further details see also the article [Hei06] of Heintze which gives a good overview of the state of the art of the subject.

In this article we describe a complete theory of Kac-Moody symmetric spaces: we show, that from purely geometric and structural considerations, Kac-Moody symmetric spaces are the natural approach. Complete proofs and details will be contained in my forthcoming thesis and will be published elsewhere.

2 Why Kac-Moody symmetric spaces?

In this section we will review some important features of the theory of finite-dimensional symmetric spaces and explain how a generalization of those features to infinite dimensions leads to Kac-Moody symmetric spaces in a natural way. As a general reference for finite dimensional symmetric spaces, we recommend [Hel01].

We start with a compact simple Lie group G. It is a symmetric space with respect to its — up to scalar multiples — unique Ad-invariant metric $g(X,Y)$, which is constructed by left translates of the negative Cartan-Killing form $g(X,Y) := -B(X,Y) = -\mathrm{tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y))$. Following [Hel01] symmetric spaces constructed this way, are called „type II“-symmetric spaces. Taking an involution σ and denoting by K a subgroup of G such that $\mathrm{Fix}(\sigma)_0 \subset K \subset \mathrm{Fix}(\sigma)$, the quotient $X = G/K$ equipped with the quotient metric is a symmetric space, called „type I“-symmetric space. Symmetric spaces of type I and type II are compact and have sectional curvature $K \geq 0$. With respect to the $\{\pm 1\}$-eigenspaces of the derivative $d\sigma$ of the involution σ the Lie algebra $\mathfrak{g}$ splits into $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, so that $\mathfrak{k} = \mathrm{Lie}(K)$. Now one can dualize via complexification and obtains symmetric spaces of the form $X = G_{\mathbb{C}}/G$ and $X = G_d/K$, where G_d denotes the noncompact real form of G determined by the involution σ. These spaces are called symmetric spaces of „type IV“ and of „type III“ respectively. They are of noncompact type, have nonpositive sectional curvature $K \leq 0$ and are diffeomorphic to a vector space. This last fact is important, as it allows a nice completion at infinity: One can define a boundary, which has the structure of a spherical building; it is of great use for further developments of the theory as for example the Mostow rigidity theorem [Mos73].

Roughly speaking there is a correspondence between compact and noncompact symmetric spaces, namely

$$\begin{array}{ccc} \text{type I} & \Leftrightarrow & \text{type III,} \\ \text{type II} & \Leftrightarrow & \text{type IV.} \end{array}$$

For example $\mathbb{S}^n$ and $\mathbb{H}^n$ are dual symmetric spaces. If a Lie algebra of compact type splits as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, its associated dual Lie algebra of noncompact type splits as $\mathfrak{g} = \mathfrak{k} + i\mathfrak{p}$.

According to their construction (finite dimensional) symmetric spaces are classified by classifying semisimple Lie algebras and by classifying their involutions.

As the theory and the use of symmetric spaces relies heavily on the properties we described, a useful theory of infinite dimensional symmetric spaces should generalize those aspects. As we will see now, this leads to Kac-Moody symmetric spaces in a direct way.

1. A first approach for constructing infinite dimensional symmetric spaces via Lie groups is viewing Lie groups as matrix groups and generalizing them to groups of operators for example on Hilbert- or Banachspaces. This approach was studied by de la Harpe in [Har72]. It leads to a nice theory of Hilbert symmetric spaces — but these spaces do not generalize the finite dimensional structure theory, based on semisimple Lie algebras.

2. As a result, a more suitable adapted second approach has to focus on generalizing the structure theory of semisimple Lie groups. As semisimple Lie groups are locally determined by their (semisimple) Lie algebras, one can start by generalizing semisimple Lie algebras, trying to find appropriate groups associated to them, and defining a manifold structure on those groups. This is the approach we will use in this work.

In the classical theory of semisimple Lie algebras it is shown, that semisimple Lie algebras can be classified by their Cartan matrices [Hel01]. A Cartan matrix $A^{n\times n}$ is a square matrix with integer coefficients, sucht that

1. $a_{ii} = 2$ and $a_{i\neq j} \leq 0$.
2. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.
3. There is a vector $v > 0$ (componentwise), such that $Av > 0$.

Example 2.1. *(Rank 2 semisimple Lie algebras)*
There are - up to equivalence - four different Cartan matrices of rank 2:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

They describe the algebras: $A_1 \times A_1, A_2, B_2, G_2$.

There is a bijection between the set of complex semisimple Lie algebras and Cartan matrices: namely, for every Cartan matrix $A^{n\times n}$, one can construct a Lie algebra $\mathfrak{g}(A^{n\times n})$ associated to this matrix:

$$\mathfrak{g}(A^{n\times n}) = \langle e_i, f_i, h_i, i = 1, \ldots, n | \mathrm{R}_1, \ldots, \mathrm{R}_6 \rangle,$$

where

$$\begin{aligned}
\mathrm{R}_1: &\quad [h_i, h_j] = 0, \\
\mathrm{R}_2: &\quad [e_i, f_j] = h_i \delta_{ij}, \\
\mathrm{R}_3: &\quad [h_i, e_j] = a_{ji} e_j, \\
\mathrm{R}_4: &\quad [h_i, f_j] = -a_{ji} f_j, \\
\mathrm{R}_5: &\quad (\mathrm{ad} e_i)^{1-a_{ji}}(e_j) = 0 \ (i \neq j), \\
\mathrm{R}_6: &\quad (\mathrm{ad} f_i)^{1-a_{ji}}(f_j) = 0 \ (i \neq j).
\end{aligned}$$

Hence one has to generalize Cartan matrices in such a way that the realizations are infinite dimensional Lie algebras: The first condition ($a_{ii} = 2$ and $a_{i\neq j} \leq 0$) on a Cartan matrix A encodes geometric information, namely, the geometry of roots and reflections; since we want to preserve the geometry, we cannot change this condition [1]; as the second condition ($a_{ij} = 0 \Leftrightarrow a_{ji} = 0$) is necessary for the Jacobi identity to hold in the Lie algebra, one can in no case use a weaker form. In contrast, one can change the last condition, usign the following fact: For every (indecomposable) matrix A satisfying $a_{ii} = 2$, $a_{i\neq j} \leq 0$, and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, there is either a (componentwise) positive vector v such that $Av > 0$, or such that $Av = 0$, or such that $Av < 0$. In these three cases one can study the realization $\mathfrak{g}(A)$ (Cf. [Kac90]):

[1] Weakening this condition and allowing noninteger entries leads to Borcherds generalized Kac-Moody algebras, as they arise, for example, in connection to the Monster group and the moonshine conjectures, see [Wak99].

$$\begin{array}{lll} Av > 0 & \Leftrightarrow & \mathfrak{g}(A) \text{ is semisimple,} \\ Av = 0 & \Leftrightarrow & \mathfrak{g}(A) \text{ is affine Kac-Moody,} \\ Av < 0 & \Leftrightarrow & \mathfrak{g}(A) \text{ is indefinite Kac-Moody.} \end{array}$$

For completeness, we give the generalized Cartan matrices of rank $n = 2$, leading to algebras of affine Kac-Moody type:

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

It is an important feature of affine Kac-Moody algebras that they can be constructed explicitely as extensions of loop algebras. This is a good way of connecting them with geometry. Moreover, loop algebras or their associated loop groups can be seen as operator groups: Let G be a compact n-dimensional Lie group. Choose an embedding $i : G \hookrightarrow U(n)$, to get a matrix group. Let G act on a vector space via this embedding V^n. Now take the loop group $LG := \{f : \mathbb{S}^1 \longrightarrow G | \text{ f some regularity}\}$. Of course i induces an embedding $Li : LG \hookrightarrow LU(n) := \{f : \mathbb{S}^1 \longrightarrow LU(n)| \ f \text{ some regularity}\}$, which acts on the vectorspace $LV := \{f : \mathbb{S}^1 \longrightarrow V^n | \text{f some regularity}\}$. Thus an affine Kac-Moody algebra (or an associated affine Kac-Moody group) can also be seen as a group of operators on a vector space.

As described, the class of affine Kac-Moody algebras generalizes the most important basic features of finite dimensional semisimple Lie groups to infinite dimensions. We will see that the associated infinite dimensional symmetric spaces do the same.

3 Kac-Moody algebras as tame Fréchet loop algebras

In this section we will describe an explicite realization of Kac-Moody Lie algebras as tame Fréchet Lie algebras of holomorphic mappings. We start with a

Definition 3.1. *(Loop algebra)*

Let $\mathfrak{g}_{\mathbb{C}}$ be a finite dimensional semisimple complex Lie algebra. The loop algebra $M\mathfrak{g}$ is the Lie algebra

$$M\mathfrak{g} := \{f : \mathbb{C}^* \longrightarrow \mathfrak{g}_{\mathbb{C}} | \ f \textit{ is holomorphic}\}$$

equipped with the natural Lie bracket:

$$[f, g]_{M\mathfrak{g}}(z) := [f, g]_0(z) := [f(z), g(z)]_{\mathfrak{g}}.$$

We define an extension:

Definition 3.2. *(Kac-Moody algebra)*

Let $\widehat{M\mathfrak{g}} := M\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with the Lie bracked defined by

$$\begin{aligned} &[d, f] := f'; [c, c] = [c, d] = [c, f] = [d, d] = 0; \\ &[f, g] := [f, g]_0 + \omega(f, g)c \end{aligned}$$

where ω is an antisymmetric 2-form on $M\mathfrak{g}$; we choose

$$\omega(f, g) := \frac{1}{2\pi} Res(\langle f, g' \rangle).$$

Every Kac-Moody algebra equipped with the family of norms $\|f\|_n := \sup_{z \in A_n} |f_z|$ with $A_n := \{e^{-n} \leq |z| \leq e^n\}$ on the loop algebra part and the euclidean metric on the part generated by c and d is a tame Fréchet Lie algebra (see: A.8).

As holomorphic functions on $\mathbb{C}^*$ can be expanded into Laurent series, one can represent every element of a loop algebra by a series:

$$f(z) := \sum_n g_n z^n$$

with $g_n \in \mathfrak{g}$. So any element of a Kac-Moody algebra can be represented by a tripel $((g_{n,n\in\mathbb{N}}) \subset \mathfrak{g}, r_c, r_d)$.

A real form of compact type of a Kac-Moody algebra $\widehat{M\mathfrak{g}}$ is defined to be an algebra:

$$\widehat{M\mathfrak{g}}_{\mathbb{R}} := M\mathfrak{g}_{\mathbb{R}} \oplus \mathbb{R}c \oplus \mathbb{R}d$$

where $\mathfrak{g}_{\mathbb{R}}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$ and

$$M\mathfrak{g}_{\mathbb{R}} := \{f \in M\mathfrak{g}_{\mathbb{C}} | f(\mathbb{S}^1) \subset \mathfrak{g}_{\mathbb{R}}\}.$$

From an elementary point of view this definition is justified by the embedding: $M\mathfrak{g}_{\mathbb{R}} \subset L\mathfrak{g} := \{f : \mathbb{S}^1 \longrightarrow \mathfrak{g} | f \text{ satisfies some regularity condition}\}$.

The Kac-Moody groups (Kac-Moody symmetric spaces), associated to these real forms correspond to the simple finite dimensional symmetric spaces of „type II“.

To construct Kac-Moody symmetric spaces corresponding to the finite dimensional symmetric spaces of „type I“ and „type III“ one has to define a splitting $\widehat{M\mathfrak{g}} := \mathfrak{k} + \mathfrak{p}$; to do this, one can use two further involutions on $M\mathfrak{g}$:

$$M\sigma_*(f)(z) := \overline{-f(\overline{z})}^t$$
$$M\sigma_0 f(z) := \overline{f\left(\frac{1}{\overline{z}}\right)}$$

Then $M\mathfrak{g}_{\mathbb{R}} = Fix(M\sigma_0)$ and the loop algebras $(M\mathfrak{g}, M\sigma_*)$ and $(Fix(M\sigma_*), M\sigma_0)$ are dual.

After embedding $M\mathfrak{g}_{\mathbb{R}} \subset L\mathfrak{g}$ the involutions $M\sigma_0$ and $M\sigma_*$ coincide with the involutions

$$L\sigma_*(f)(t) := \sigma_*(f(-t))$$
$$L\sigma_0 f(t) := \overline{f(t)}$$

as should be the case.

4 Groups of holomorphic maps

Up to now, we only have defined Kac-Moody algebras, but not studied the associated Kac-Moody groups. This will be the topic of this section. We start with some definitions:

Definition 4.1. *(Kac-Moody groups)*

Let $MG_{\mathbb{C}} := \{f : \mathbb{C}^* \longrightarrow G_{\mathbb{C}} | \ f \text{ is holomorphic}\}$ *and let* $MG_{\mathbb{R}} := \{f \subset MG_{\mathbb{C}} | f(\mathbb{S}^1) \subset G_{\mathbb{R}}\}$. *Then the Kac-Moody group* $\widehat{MG}_{\mathbb{R}}$ *is the semidirect product of* $\mathbb{S}^1$ *with a* $\mathbb{S}^1$*-bundle* $\widetilde{MG}_{\mathbb{R}} := \mathbb{S}^1/MG_{\mathbb{R}}$.

The complex Kac Moody group $\widehat{MG}_{\mathbb{C}}$ *is its complexification: a semidirect product of* $\mathbb{C}^*$ *with* $\widetilde{MG}_{\mathbb{C}}$*, which is a* $\mathbb{C}^*$*-bundle over* MG.

A detailed construction of the group $\widetilde{MG}_{\mathbb{R}}$ is described in the book [PS86]. The action of the semidirect product factor is in both cases given by change of the argument:

$\mathbb{C}^* \ni w : MG \to MG : f(z) \mapsto f(z \cdot w)$.

Clearly, there is an exponential function:

$M\exp : M\mathfrak{g} \longrightarrow MG$ is defined via composition with the group exponential function: $\exp : \mathfrak{g} \to G$:

$$(M\exp(f))(z) := \exp(f(z)).$$

This exponential can be uniquely extended to the complete Kac-Moody algebra $\widehat{M\mathfrak{g}}$. It maps $\mathbb{C}c$ into the fiber of the $\mathbb{C}^*$-bundle and the $\mathbb{C}d$-term into the semidirect $\mathbb{C}^*$-factor.

The exponential map allows a definition of the adjoint action of the Lie group; it satisfies the usual identity: $Ad \circ \exp = e^{\mathrm{ad}}$. We remark that the adjoint action of λd is translation of the argument of the loop by λ.

To clarify the relationship between $\widehat{M\mathfrak{g}}$ and $\widehat{MG}$ we give two theorems:

Theorem 4.1. *(Tangential space)*
Let $\mathfrak{g}$ be the Lie algebra of G. Then

$$\widehat{M\mathfrak{g}} := T_e(\widehat{MG}).$$

Moreover it is the algebra of left-invariant vector fields on $\widehat{MG}$. On the other hand, we find:

Theorem 4.2. *The exponential map is not a local diffeomorphism.*

The image of the exponential map consists of loops whose image is completely contained in the image of the Lie group exponential map. We will give further comments on this situation in section 7, where we study the behaviour of geodesics.

At first glance, these properties look contradictory; however it is due to a change of limits arising because of the noncompactness of $\mathbb{C}^*$: theorem 4.1 describes locally uniform convergence on each compact subset in $\mathbb{C}^*$ while theorem 4.2 shows, this needs no longer to be true globally. One of the main problems arising is that one can no longer choose exponential charts, to proof that the loop group (Kac-Moody group) is actually a manifold.

Moreover this means that 1-parameter subgroups do not form a neighborhood of the identity.

One can clarify the situation by using an inverse limit Banach structure: let $L\mathfrak{g}^{(n)} := \{f : A_n \longrightarrow \mathfrak{g} | f \text{ is holomorphic}\}$ be made into a Banach Lie algebra, using the topology of uniform convergence and $LG^{(n)} := \{f : A_n \longrightarrow G | f \text{ is holomorphic}\}$ the associated Banach Lie group. Then the exponential map $L\exp^{(n)} : L\mathfrak{g}^{(n)} \longrightarrow LG^{(n)}$ is defined by composition with the group exponential function $\exp : \mathfrak{g} \to G$. By the inverse function theorem for Banach spaces, it is a local diffeomorphism. With these definitions at hand, one has:

$$M\mathfrak{g} := \varprojlim L\mathfrak{g}^{(n)} \qquad \text{and} \qquad MG := \varprojlim LG^{(n)}$$

.

For every $n \in \mathbb{N}$ the map $L\exp^{(n)}$ is a local diffeomorphism $L\exp^{(n)} : U^{(n)} \longrightarrow V^{(n)}$. Of course in the limit, we only get that $M\exp$ is a diffeomorphism of $\bigcap U^{(n)}$ onto $\bigcap V^{(n)}$, which are not necessarily open; to remedy the situation one would have to introduce the assumption:

There are open neighborhoods $U^\infty \subset \bigcap U^{(n)}$ and $V^\infty \subset \bigcap V^{(n)}$ such that $L\exp^{(n)}$ is invertible in U^∞ for each $n \in \mathbb{N}$.

This is exactly the additional condition arrising in the Nash-Moser inverse function theorem (C.f. [Ham82]).

Conversely, for every $\gamma \in MG$ sufficently close to the identity, there exists $n \in \mathbb{N}$ such that $\gamma|_{A_n} \in \Im(L\exp^{(n)})$. Suppose there exists a maximal number n with this property, and let $\dot\gamma$ be such that $\gamma|_{A_n} = L\exp^n \dot\gamma$ then we call the curve $(\gamma(t))\,|_{A_n} := L\exp^{(n)}(\dot\gamma)$ a n-quasigeodesic.

Quasigeodesics can be used as a tool to investigate the local structure of loop groups.

To remedy the bad behaviour of the exponential function, one could imagine to take loops $\mathbb{S}^1 \longrightarrow G$. In this case it is easy to see, that the exponential map is always a diffeomorphism as a neighborhood of the identity element of such a loop group is given by loops whose image lies in a small neighborhood V of the identity of the subjacend Lie group; this neighborhood can be chosen in a way, such that the group exponential is a diffeomorphism from an open neighborhood U in the Lie algebra onto it.

To see the problems arising in this setting we will review part of the theory of [Pop05]: Let us start with the definitions:

Definition 4.2. *Let* $L\mathfrak{g} := \{f : \mathbb{S}^1 \longrightarrow G_{\mathbb{R}} | f \text{ is } H^1\}$ *be a loop algebra and* $LG := \{f : \mathbb{S}^1 \longrightarrow G_{\mathbb{R}} | f \text{ is } H^1\}$ *the associated Lie group.*

The $L\mathfrak{g}$ is the Lie algebra of LG. Let $U \subset \mathfrak{g}$ and $V \subset G$ be such that, $\exp : U \longrightarrow V$ is a diffeomorphism.

Then $L\exp : LU \longrightarrow LV$ is a diffeomorphism; so one can proof that LG is a Banach-manifold of continuous loops. We want to put a metric on it. Unfortunately as Terng noticed [Ter95] the H^1-metric is not Ad-invariant. So one has to choose the H^0-metric. But with respect to this metric the loop group is only a Pre-Hilbert-manifold. Nevertheless Popescu shows in [Pop05] that one can define nice pre-Hilbert symmetric spaces corresponding to all four types: One can define dual symmetric spaces and quotients G/K, where K is the Fixed point group of an involution. Of course it will be not possible, to define the extension corresponding to d, as $d : H^1 \longrightarrow H^0$. So one cannot get a theory of Kac-Moody groups. Let us remark that the term „Kac-Moody group“ is sometimes used differently, denoting loop groups or the central extension of the loop group [Mic89].

So one tries a new setting:

Definition 4.3. *Let* $L\mathfrak{g} := \{f : \mathbb{S}^1 \longrightarrow G_{\mathbb{R}} |\, f \text{ is } C^\infty\}$ *be a loop algebra and* $LG := \{f : \mathbb{S}^1 \longrightarrow G_{\mathbb{R}} |\, f \text{ is } C^\infty\}$ *the associated Lie group.*

Popescu shows, that this gives tame Fréchet Loop groups. Moreover he defines an extension, corresponding to $\mathbb{R}c \oplus \mathbb{R}d$ which consists of a $\mathbb{S}^1$-bundle and a semidirect product with $\mathbb{S}^1$. The exponential function is again a local diffeomorphism. Unfortunately a complexification — and thus a dualization — is not possible: The argument centers in studying d. As we described, the adjoint action of d corresponds to a change of variables in the loop group. We calculate the adjoint action of an element $\exp(i\lambda d)$ in the Kac-Moody group on a loop $f(t)$ in the Kac-Moody algebra:

$$\begin{aligned} Ad\left(\exp(i\lambda d)\right)(f(t)) &= \left(e^{\mathrm{ad}(i\lambda d)}\right)(f(t)) = \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{ad}(i\lambda d)^n f(t)}{n!} = \\ &= \sum_{n=0}^{\infty} \frac{i^n \lambda^n}{n!} f^{(n)}(t) = \\ &= f(t+i\lambda) \notin \mathbb{R}. \end{aligned}$$

Using exponential notation: $t = e^{i\phi} \in \mathbb{S}^1$, we find:

$$Ad\left(\exp i\lambda d\right)(f(e^{i\phi})) = f(e^{i\phi}e^{-\lambda})$$

which of course no longer is in $\mathbb{S}^1$. Take $f(z) := \frac{1}{z-2}$; this is holomorphic — and thus C^∞ — on $\mathbb{S}^1$ but has a pole at $z = 2$. So the series diverges for $\lambda = -\ln 2$.

So also in this case, one does not get a complete theory. At the end, one sees, that one has to use loops of $\mathbb{C}^*$ inspite of their complicated behaviour.

One has now to see, that $\widehat{MG}$ is a tame Fréchet manifold:

Theorem 4.3. *$\widehat{MG}_{\mathbb{C}}$ and $\widehat{MG}_{\mathbb{R}}$ are tame Fréchet manifolds.*

The part, that $\widehat{MG}_{\mathbb{R}/\mathbb{C}}$ are locally convex topological manifolds is contained in a recent preprint of Neeb [NW07]. He proves his result by using the logarithmic derivative, strongly relying on Glöckners inverse function theorem [Glö03]. If one is — as we are — only interested in the case of Kac-Moody groups, one can simplify the proof. In either case, the tame Fréchet nature of the Kac-Moody groups has to be verified.

Having prooved, that $\widehat{MG}_{\mathbb{C}/\mathbb{R}}$ are tame Fréchet manifolds, one can easily show, that the same is true for the quotients $\widehat{MG}_{\mathbb{R}}/Fix(\sigma)$, and $\widehat{MG}_{\mathbb{C}}/\widehat{MG}_{\mathbb{R}}$.

5 Symmetric spaces of the compact type

In this chapter we want to equip the Kac-Moody groups of compact type with an Ad-invariant metric to get symmetric spaces. Of course the spaces are — as they are infinite dimensional — not compact. So we want to call a Kac-Moody symmetric space „of compact type“, if it is a (compact type) real form of a Kac-Moody group or a quotient of such a real form or equivalently, if $\langle K(X,Y)X, Y\rangle \geq 0$.

We start by defining an Ad-invariant scalar product on $M\mathfrak{g}$:

Definition 5.1. *Let $\langle , \rangle$ be an* Ad*-invariant scalar product on $\mathfrak{g}$. Then an Ad-invariant scalarproduct of index one on $M\mathfrak{g}$ is defined by: $\langle f, g\rangle_{M\mathfrak{g}} := \frac{1}{2\pi}\int_{\mathbb{S}^1}\langle f(z), g(z)\rangle dz$ for $f, g \in M\mathfrak{g}$, $\langle c, d\rangle = -1$; $\langle c, c\rangle = \langle d, d\rangle = \langle u, c\rangle = \langle u, d\rangle = 0$.*

This is a Lorentz-skalarproduct. As in the finite dimensional case, we call a vector v

1. spacelike iff $|v| > 0$,
2. lightlike iff $|v| = 0$,
3. timelike iff $|v| < 0$.

c and d are lightlike vectors, $c+d$ is timelike, elements of the loop algebra $M\mathfrak{g}$ are spacelike.

It can be verified by elementary calculation, that this scalar product is Ad-invariant. By left translation, it defines a left invariant metric on $\widehat{MG}$.

By a general theorem of Popescu [Pop05], there exists a Levi-Civita connection for tame Fréchet Lie groups. Adapted to our setting, he proved:

Theorem 5.1. *(Levi-Civita-Connection)*

$(\widehat{MG}, g)$ *admits a unique Levi-Civita connection* ∇. *For* X, Y *left invariant vector fields,* $\nabla_X Y = \frac{1}{2}[X, Y]$.

For the proof, see [Pop05].

One can show, that this metric and connection induce metric and connections on the quotient spaces $X = \widehat{MG}_{\mathbb{R}}/\mathrm{Fix}(\sigma)$.

Further, elementary calculation shows, that $\langle K(X,Y)X, Y\rangle \geq 0$.

So, the resulting symmetric spaces have „compact type behaviour".

6 Symmetric spaces of the noncompact type

As described before, one can construct dual symmetric spaces in the canoncial way, known from finite dimensional symmetric spaces.

The continuation of the Ad-invariant Lorentz metric to $\widehat{MG}_{\mathbb{C}}$ induces a unique Lorentz metric on the quotientspace $\widehat{MG}_{\mathbb{C}}/(\widehat{MG}_{\mathbb{R}}$.

Elementary calculation shows, that $\langle K(X,Y)X, Y\rangle \geq 0$.

One can proof the following theorem:

Theorem 6.1. *Kac-Moody symmetric spaces of the noncompact type are diffeomorphic to a vector space.*

This last fact suggests the existence of a nice boundary at infinity of the Kac-Moody symmetric spaces; the existence of rigidity of quotient spaces and a Mostow-type behaviour seem quite probable.

7 Some remarks concerning the geometry

We will concentrate on three points, which show, that the geometry of Kac-Moody symmetric spaces differs from the classical well-known finite dimensional Riemannian geometry:

1. Geodesics
2. Curvature
3. Hermitian symmetric spaces

The most important fact concerning the structure of geodesics is that geodesic completeness fails even locally. The introduction of quasi-geodesics in section 4 helps to remedy this problem, but quasi-geodesics cannot completely replace geodesics. There is no chance to get a Hopf-Rinow-type theorem. This is a typical feature of Lorentz-geometry. We will first give an easy example of a finite dimensional Lorentz-manifolds which is complete in the sense that every geodesic is infinitely long but which is not geodesically complete in the sense, that every pair of two points can be connected by a geodesic:

Example 7.1. ($SL(2,\mathbb{R})$)

Take $SL(2,\mathbb{R})$ with its natural ad-invariant-metric defined by the left translate of Cartan-Killing form. There is a unique Levi-Civita-connection associated to this metric. As this group is non-compact, with a compact subgroup $\mathbb{S}^1$, it is a Lorentz-manifold. As it is homogenous and locally symmetric it is a Lorentz symmetric space. Geodesics through e are exactly 1-parameter subgroups. This means, that e can only be connected with a geodesic to a point p, if p lies on a 1-parameter subgroup, so in the image of $\exp\mathfrak{sl}(2,\mathbb{R})$. *But* $\exp\mathfrak{sl}(2,\mathbb{R})$ *is not surjectiv. For example, points conjugate to* $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ *are not in the image of* $\exp\mathfrak{sl}(2,\mathbb{R})$.

So Hopf-Rinow-type behaviour is not typical for Lorentz manifolds.

There are hard theorems showing the impossibility of Lorentz manifolds to be geodesically complete. For details see [BE81].

The fact, that Lorentz-manifolds are not globally geodesically complete leads directly to the fact that Kac-Moody symmetric spaces are not locally geodesically complete:

We start again with an example, this time taking the loop group $LSL(2,\mathbb{C})$: We want to show that there exists a sequence $f_n \in LSL(2,\mathbb{C})$ which converges to the identity map in the CompactOpen-(tame Fréchet) topology but such that the image of every map containes points outside the image of $\exp(\mathfrak{sl}(2,\mathbb{C}))$. We take

$$f_n(z) = \begin{pmatrix} e^{\pi z/n} & -iz/n \\ 0 & e^{-\pi z/n} \end{pmatrix}.$$

$$\text{Then } f_n(in) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

So f_n is not contained in $\Im(M\exp(\mathfrak{sl}(2,\mathbb{C}))$. On the other hand, for all $z_0 \in \mathbb{C}^*$ fixed

$$\lim_{n\to\infty} f_n(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Id}.$$

So in the Compact open topology for every neighborhood U_k of the identity, there exists $n_k \in \mathbb{N}$ such that $\forall n \geq n_k : f_n \in U_k$.

One has some theorems about the behaviour of geodesics:

Theorem 7.1. *(1-parameter subgroups)*

A curve in MG through e is a geodesic iff it is a 1-parameter subgroup.

A very important point is the close connection to geodesics in the subjacend Lie groups:

Theorem 7.2. *$\gamma(t) \subset MG$ is a geodesic iff $\gamma_z(t) \subset G$ is a geodesic for every $z \in \mathbb{C}^*$.*

This gives rise to the following theorem for $\widehat{MG}$:

Theorem 7.3. *(Geodesic through e)*

Let $\pi : \widehat{MG} \longrightarrow MG$ be the bundle projection. Let $\dot{\gamma} := (f, r_c, r_d) \in M\mathfrak{g}$ and $\gamma(t)$ the associated geodesic. Then $(\pi\gamma)_{\phi(z,r_d)}(t)$ is a geodesic in G.

Concerning curvature, an important result in finite dimensional Lorentz geometry is the theorem of Kulkarni. It states that for pseudo Riemannian manifolds bounded curvature implies constant curvature. For Kac-Moody symmetric spaces, a direct generalization holds:

Theorem 7.4. *(Kulkarni)*

Let M be a tame Fréchet manifold of Kac-Moody type. Let $p \in M$. Then the following conditions are equivalent:

1. *K is constant.*
2. *$a \leq K$ or $K \leq b$ for all $a, b \in \mathbb{R}$.*
3. *$a \leq K \leq b$ on all definite planes for all $a \leq b \in \mathbb{R}$.*
4. *$a \leq K \leq b$ on all indefinite planes for all $a \leq b \in \mathbb{R}$.*

For the finite dimensional proof of this theorem — and more generally finite dimensional Lorentz geometry — see [O'N83].

As Kac-Moody symmetric spaces — by their construction — have flats of dimension at least 3, their sectional curvature has to be unbounded.

The last point to mention is the question of existence of hermitian Kac-Moody symmetric spaces. One can show that — as in finite dimensions — the existence of a complex structure leads to a direct $\mathbb{S}^1$-factor in the isometry group. Unfortunately — as with finite dimensional Lorentz symmetric spaces — it is not possible to construct Hermitian Kac-Moody symmetric spaces:

Theorem 7.5. *There are no hermitian Kac-Moody symmetric spaces.*

To proof this result, one has to study the behaviour of the timelike vector $v = c + d$ with respect to the complex structure J. The question of the existence of para-hermitian Kac-Moody symmetric spaces is open.

8 Conclusion and Outlook

As we have shown, there exists a nice theory of infinite dimensional symmetric spaces, which generalizes many features of the well-known finite dimensional theory. Most important points are a classification similar to the finite dimensional case by classifying Kac-Moody algebras and their involutions and the duality between spaces of compact type and of noncompact type.

A further technical tool, we only mentioned in this article is the introduction of Inverse-Limit-Banach (ILB)-structures and Inverse-limit-Hilbert (ILH)-structures — as defined by [Omo74] — on the Kac-Moody groups. Those concepts help to clarify the relationship between tame Fréchet-Kac-Moody groups and the classical theory of Hilbert- and Banach Lie and loop groups. Details about the construction and some applications will be presented in a subsequent paper.

The possibility, to generalize further topics known from finite dimensional symmetric spaces — symplectic structures, arithmetic groups and rigidity, embeddings of spaces of the noncompact type into spaces of the compact type, factorization theorems, paracomplex structures, twistor theory... — seem quite good.

A Appendix

A.1 Fréchet spaces

Fréchet spaces form a class of very general topological vectorspaces. Fréchet manifolds are manifolds, modelled locally on Fréchet spaces. We follow closely [Ham82].

Definition A.1. *(Fréchet vector space)*

A Fréchet vector space is a complete, locally convex, Hausdorff topological vector space, whose topology is generated by a countable family of seminorms.

We give some examples:

Example A.1.

1. *Every Banach space is a Fréchet space.*
2. *Let* $Hol(\mathbb{C}, \mathbb{C})$ *denote the holomorphic functions* $f : \mathbb{C} \longrightarrow \mathbb{C}$. *Denote by* $B_n := \{z|\ |z| \leq n\}$. *Let* $\|f\|_n := \sup_{z \in B_n} |f(z)|$.
3. *Let Let* $Hol(\mathbb{C}^*, \mathbb{C})$ *denote the holomorphic functions* $f : \mathbb{C}^* \longrightarrow \mathbb{C}$. *Let* $A_n := \{e^{-n} \leq |z| \leq e^n\}$. *Let* $\|f\|_n := \sup_{z \in A_n} |f(z)|$. *Then* $(Hol(\mathbb{C}^*, \mathbb{C}), \| \|_n)$ *is a Fréchet space.*
4. $Hol(\mathbb{C}^*, \mathbb{C}^n)$ *is a Fréchet space.*

A.2 Tame Fréchet spaces

The failure of inverse function theorems for Fréchet spaces is a serious problem for the differential geometry of Fréchet spaces. For example, this could lead to manifolds whose geodesics have switches. Fortunately there is a class of Fréchet spaces, which allow inverse function theorems, namely the subclass of tame Fréchet spaces; the applications we have in mind are tame Fréchet.

The definition of tame Fréchet spaces involves a further structure, namely gradings:

Definition A.2. *(Grading)*

Let F *be a Fréchet space. A grading on* F *is a collection of seminorms* $\{\| \|_n), n \in \mathbb{N}_0\}$ *that define the topology and satisfy:*

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \|f\|_3 \leq \dots .$$

Following Hamilton, we call a graded Fréchet space a Fréchet space with a grading.

Lemma A.1. *(Constructions of graded Fréchet spaces)*

1. *A closed subspace of a graded Fréchet space is a graded Fréchet space.*
2. *Direct sums of graded Fréchet spaces are graded Fréchet spaces.*

Definition A.3. *(Tame equivalence of gradings)*

Let F *be a graded Fréchet space. The two gradings* $\{\| \|_n\}$ *and* $\{\widetilde{\| \|}\}$ *are called* $(r, b, C(n))$*-equivalent iff*

$$\| \|_n \leq C(n) \widetilde{\| \|}_{n+r} \quad and \quad \widetilde{\|f\|}_n \leq C(n) \|f\|_{n+r}.$$

They are called tame equivalent iff they are $(r, b, C(n))$*-equivalent for some* $(r, b, C(n))$.

The following example is basic:

Example A.2. *Let* B *be a Banach space with norm* $\| \|_B$. *Denote by* $\Sigma(B)$ *the space of all exponentially decreasing sequences* $\{c_k\}_{k \in \mathbb{N}_0}$ *of elements of* B. *On this space, we can define different gradings:*

$$\|f\|_{l_1^n} := \sum_{k=0}^{\infty} e^{nk} \|c_k\|_B$$

$$\|f\|_{l_\infty^n} := \sup_{k \in \mathbb{N}_0} e^{nk} \|f\|_B$$

Lemma A.2. *On the space $\Sigma(B)$ the two gradings $\|f\|_{l_1^n}$ and $\|f\|_{l_\infty^n}$ are tamely equivalent.*

Corollary A.1. *The spaces of exponentially decreasing sequences of elements in $B = \mathbb{C}$, together with the euclidean norm and in $B = \mathbb{C}^2$ together with the norm $\|(c_1, c_1')\|_B := sup(|c_1|, |c_1'|)$ are tame Fréchet spaces.*

From now on let F and G denote graded Fréchet spaces.

Definition A.4. *(Tame linear map)*
A linear map $\phi : F \longrightarrow G$ is called $(r, b, C(n))$-tame if it satisfies:

$$\|\phi(f)\|_n \leq C(n)\|f\|_{n+r}$$

ϕ is called tame iff it is $(r, b, C(n))$-tame for some $(r, b, C(n))$.

Definition A.5. *(Tame isomorphism)*
$\phi : F \longrightarrow G$ is called a tame isomorphism, iff it is a linear isomorphism and ϕ and ϕ^{-1} satisfies tame estimates.

Definition A.6. *(Tame direct summand)*
F is a tame direct summand of G iff there exist tame linear maps $\phi : F \longrightarrow G$ and $\psi : G \longrightarrow F$ such that $\psi \circ \phi : F \longrightarrow F$ is the identity.

Definition A.7. *(Tame Frechet space)*
A graded Fréchet space is tame iff it is a tame direct summand for a space $\Sigma(B)$ for some Banach space B.

One can show that all examples in A.1 are tame Fréchet.

Definition A.8. *(Tame Fréchet Lie algebra)*
A Fréchet Lie algebra is tame, iff it is a tame Fréchet vector space and $ad(X)$ is tame for every X.

We now give some definitions for non-linear tame Fréchet objects:

Definition A.9. *(Tame maps)*
Let F and G be graded spaces and $P : U \subset F \longrightarrow G$ a nonlinear map.
P is tame if there exist (r, b, C) such that

$$\|P(f)\|_n \leq C(1 + \|f\|_{n+r}).$$

Definition A.10. *(Tame manifold)*
A tame Fréchet-manifold is a Fréchet-manifold with charts in a tame space such that the chart transition functions are tame.

The reason for introducing tame maps is the (Nash-Moser) inverse function theorem for:

Theorem A.1. *(Nash-Moser inverse function theorem)*
Let F and G be tame frechet spaces. $P : U \subset F \longrightarrow G$ a smooth tame map with a unique invertible derivative for all $z \in U$ and a tame, smooth dependence of z. Then P is locally invertible and the inverse is a smooth tame map.

B Thanks

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DIE VON NODOIDEN VERZWEIGTEN FLÄCHEN

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Es gibt zwei-parametrige Familien von H-Flächen, d.h. Flächen konstanter mittlerer Krümmung, die von Nodoidenfamilien verzweigen. Wir werden die Konstruktionsverfahren dieser Flächen vorstellen.

1. Delaunay-Flächen

1841 bestimmte Delaunay alle Rotationsflächen in $\mathbb{R}^3$, diese Flächen sind einfach periodische, vollständige H-Flächen. Sie werden *Delaunay-Flächen* genannt.
Der Meridian dieser Flächen ist die Spur der Brennpunkte von Kegelschnitten beim Rollen auf der Euklidischen Ebene.

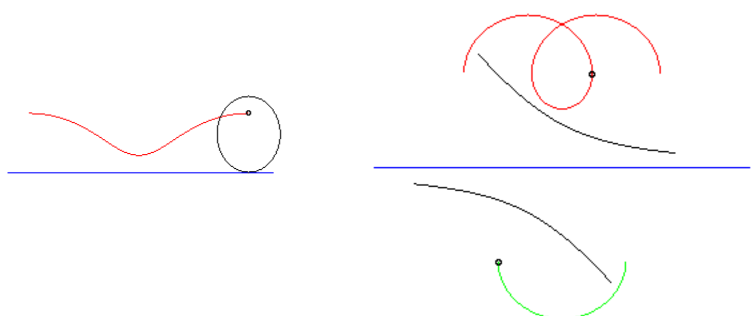

Figure 1.1. Meridian der Delaunay-Flächen

Links im Bild ist eine eingebettete ebene Kurve, die von einer Ellipse erzeugt wird. Rechts im Bild ist eine immersierte ebene Kurve, die von einer Hyperbel erzeugt wird. Entsprechend heißen die eingebetteten Delaunay-Flächen *Unduloide* und die immersierten *Nodoiden*. Wir bezeichenen mit r_{min}(bzw. r_{max}) den minimalen (bzw. maximalen) Abstand der Fläche zu der Rotationsachse. So bilden alle Delaunay-Flächen eine 1-parametrige Familie $\{D_a\}$ bzgl. des Parameters $|a| = \frac{r_{min}}{r_{max}}$. Es gilt

- $a \in [-1, 1]$.
- Für $a = 1$ ist D_a ein Zylinder.
- Für $0 < a < 1$ ist D_a ein Unduloid.
- Für $a = 0$ ist D_a eine Kette von Sphären.

- Für $-1 < a < 0$ ist D_a ein Nodoid.
- Für $a = -1$ ist D_a ein Zylinder.

Delaunay-Flächen sind die konjugierten Flächen von sphärischen Helicoiden. Für Details siehe [1] (537-539).

2. Konstruktionsverfahren

Wir konstruieren neue H-Flächen, für die wir hinreichend hohe Symmetrie voraussetzen, nämlich ebene Spiegelsymmetrie. Wir gehen wie folgt schematisch vor:

1) Das Fundamentalstück von Nodoiden ist bekannt, das liegt in der konvexen Hülle der Symmetrie-Ebenen. Die Randkurven des Fundamentalstücks bilden ein Viereck, welches am Rand dieser konvexen Hülle liegt.
2) Wegen Lawson-Korrespondenz existiert eine Minimalfläche (welche wir auch Cousine nennen) in $\mathbb{S}^3$, die zu dem Fundamentalstück aus (1) konjugiert ist.
3) Die Randkurven des sphärischen Viereckes sind Integralkurven der Hopf-Vektorfelder (Hopffaserung). Sie sind explizit anhand Quaternionen anzugeben. Somit haben wir eine angenehme Situation - festes Randwertproblem in $\mathbb{S}^3$.
4) Wir deformieren das sphärische Viereck durch Verlängern einer Kante und gleichzeitiges Verkürzen der gegebenüber liegenden Kante. Somit erhalten wir ein neues sphärisches Viereck.
5) Wir lösen das Plateau-Problem zu dem neuen Vierecke und erhalten eine Minimalfläche in $\mathbb{S}^3$. Mit Lawson-Korrespondenz erhalten wir eine H-Fläche in $\mathbb{R}^3$.
6) Durch Schwarz-Spiegelungen (unendlich oft) erhalten wir eine neue vollständige H-Fläche in $\mathbb{R}^3$.

2.1. **Lawson-Korrespondenz.** In der Arbeit [2] (363-367) beschreibt Lawson eine 1-1 Korrespondenz zwischen H-Flächen in $\mathbb{R}^3$(bzw. in H^3) und minimalflächen in $\mathbb{S}^3$ (bzw. in $\mathbb{R}^3$). Wir benutzen die Identifikationen $T_1\mathbb{S}^3 = \mathrm{Im}\mathbb{H} = \mathbb{R}^3$ und $\mathbb{R} = \mathrm{Re}\mathbb{H}$.

Theorem 2.1. *Sei $\Omega \subset \mathbb{R}^2$ ein einfachzusammenhängendes Gebiet und $f \colon \Omega \to \mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ eine Immersion konstanter mittlerer Krümmung. Dann hat die Gleichung*

$$d\tilde{f} = \tilde{f} df \circ J$$

eine Lösung $\tilde{f} \colon \Omega \to \mathbb{S}^3 \subset \mathbb{H}$. Wobei J die $90°$- Drehung in $T\Omega$ ist. Die Flächen f und $\tilde{f}$ sind isometrisch und $\tilde{f}$ ist eine minimale Immersion.

2.2. **Symmetrie der konjugierten Minimalflächen.** Sei $\gamma \subset \Omega \subset \mathbb{R}^2$ eine nach Bogenlänge parametrisierte Kurve und E_u eine Ebene in $\mathbb{R}^3$ mit der Normalen $u \in \mathbb{S}^2$. We nennen $f \circ \gamma$ eine *Kurve der ebenen Spiegelung mit der Symmetrie-Ebene E_u*, wenn $f \circ \gamma$ konstante Konormale $u \in \mathbb{S}^2$ hat.

Corollary 2.2. *$f \circ \gamma$ ist eine Kurve der ebenen Spiegelung bzgl. der Ebene E_u genau dann wenn die konjugierte Kurve $\tilde{f} \circ \gamma$ Teil des Grosskreises $\cos t + u \sin t$ ist.*

3. Bifurkationsflächen

3.1. **Nodoide.** Da Nodoiden periodisch und rotationssymmetrisch sind, betrachten wir ein einfachzusammenhängendes Fundamentalstück, z.B ein φ-Segment mit $\varphi = \frac{\pi}{j}$, $j \in \mathbb{N} \setminus 1$ um eine halbe Periode. Mit Γ bezeichnen wir das Randpolygon des Flächenstücks, mit p_i und γ_i die Ecken und Kanten von Γ.

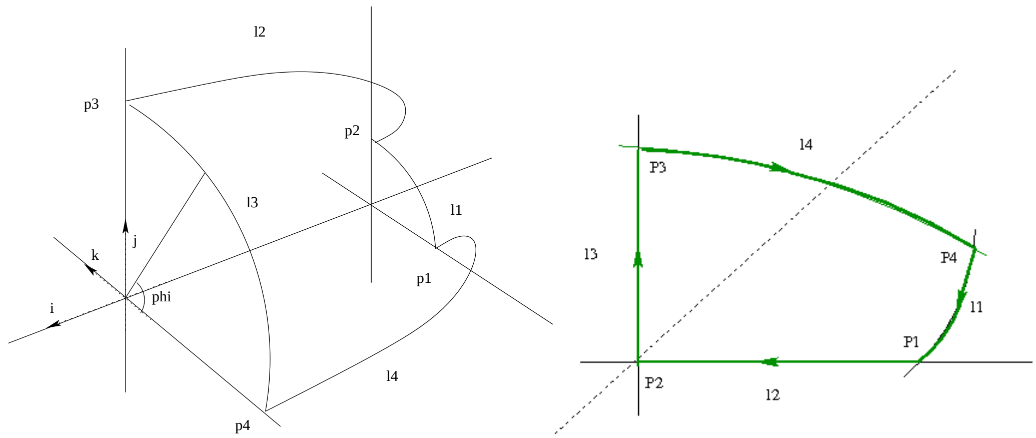

FIGURE 3.1. Nodoid

Die Länge und die Hopfvektorfelder zu Γ werden in folgender Tabelle aufgelistet:

	γ_1	γ_2	γ_3	γ_4
Länge	l	$\frac{\pi}{2}$	$\varphi + l$	$\frac{\pi}{2}$
Hopffeld	i	$j\cos\varphi + k\sin\varphi$	i	$-j$

Die Eckpunkte des spharischen Polygons Γ genügen

$$p_1 = 1 \in S^3$$

$$p_2 = p_1 e^{\mathbf{i}l} = \cos l + \mathbf{i}\sin l$$

$$\begin{aligned} p_3 &= p_2 e^{(\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)\frac{\pi}{2}} \\ &= (\cos l + \mathbf{i}\sin l)(\cos\frac{\pi}{2} + (\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)\sin\frac{\pi}{2}) \\ &= (\cos l + \mathbf{i}\sin l)(\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi) \\ &= \mathbf{j}\cos l\cos\varphi + \mathbf{k}\sin l\cos\varphi + \mathbf{k}\cos l\sin\varphi - \mathbf{j}\sin l\sin\varphi \\ &= \mathbf{k}\sin(l+\varphi) + \mathbf{j}\cos(l+\varphi) \end{aligned}$$

$$\begin{aligned} p_4 &= p_3 e^{\mathbf{i}(\varphi + l)} \\ &= (\mathbf{j}\cos(l+\varphi) + \mathbf{k}\sin(l+\varphi))(\cos(l+\varphi) + \mathbf{i}\sin(l+\varphi)) \\ &= \mathbf{j}\cos^2(l+\varphi) + \mathbf{k}\sin(l+\varphi)\cos(l+\varphi) - \mathbf{k}\cos(l+\varphi)\sin(l+\varphi) + \mathbf{j}\sin^2(l+\varphi) = \mathbf{j} \end{aligned}$$

$$p_5 = p_4 e^{-\mathbf{j}\frac{\pi}{2}} = 1 = p_1$$

3.2. **Neue CMC-Flächen.** Sei $t \in [0, \frac{\pi}{2})$. Wir verlängern den Grosskreisbogen γ_2 entlang $j\cos\varphi + k\sin\varphi$ um t und gleichzeitig verkürzen wir den Grosskreisbogen γ_4 um t. Somit erhalten wir ein neues sphärisches Polygon $\Gamma_{l,t}$. Das Plateau-Problem zu $\Gamma_{l,t}$ ist lösbar.

Theorem 3.1. *Für jedes $j \in \mathbb{N}$ mit $j \geq 2$ gibt es eine Familie von periodischen H-Flächen $M_{j,t}$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ mit folgenden Eigenschaften:*

[1.] Jede Fläche $M_{j,t}$ *repräsentiert eine einfach periodische Immersion vom topologischen Typ* $S^2 \setminus \{p_1, p_2\}$*, wobei* $p_1, p_2 \in S^2$ *mit* $p_1 \neq p_2$*.*

[2.] Es ist $M_{j,t} = M_{j,-t}$ *bis auf eine* $\frac{\pi}{j}$*-Drehung. Sonst sind alle* $M_{j,t}$ *verschieden.*

[3.] Bifurkation: Die Fläche $M_{j,0}$ *ist ein Nodoid mit Kragenweite* $N_r = \frac{j-1}{2}$*.*

[4.] Symmetrie: Jede Fläche $M_{j,t}$ *besitzt eine* $\frac{\pi}{j}$*-Rotationssymmetrie.*

[5.] Degenerierter Fall: Bei $t \to \frac{\pi}{2}$ *ist die Hauptkrümmung von* $M_{j,t}$ *unbeschränkt. Die Fläche* $M_{j,t}$ *konvergiert gegen eine Kette von Sphären.*

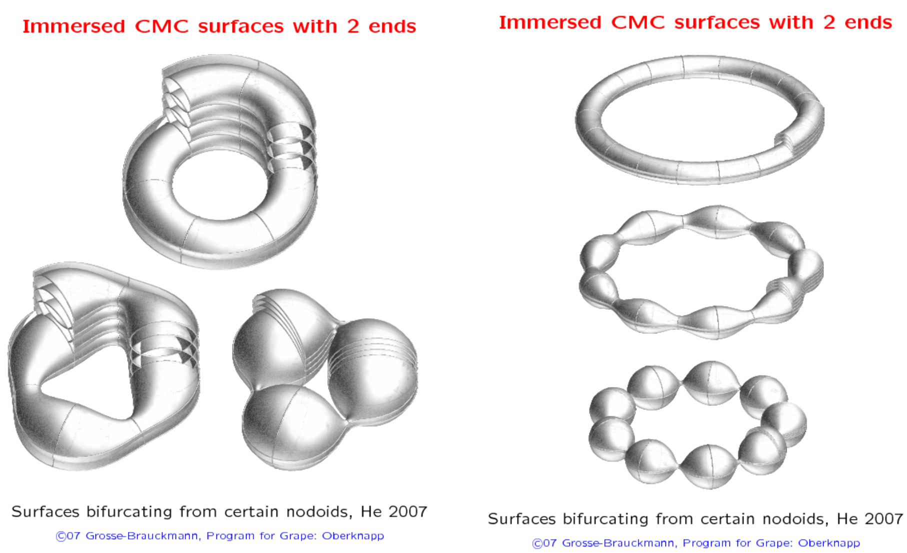

FIGURE 3.2. Nodoide

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Constant mean curvature tori in S^3

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Introduction

1958 Alexandrov: Alexandrov emb. cmc surfaces in $\mathbb{R}^n$, $\mathbb{S}^n_+$ and $\mathbb{H}^n$ are round spheres.

1969 Hsiang/Lawson: Minimal surfaces in $\mathbb{S}^3$. Conjecture: unique emb. minimal torus.

1986 Wente: cmc tori in $\mathbb{R}^3$

1988 Meeks & Korevaar/Kusner/Salomon: properly embedded cmc cylinders in $\mathbb{R}^3$.

1989 Pinkall/Sterling: cmc tori in $\mathbb{R}^3$.

1990 Hitchin: minimal tori in $\mathbb{S}^3$.

1991 Bobenko: cmc tori in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$

Spectral curves of cmc tori in $\mathbb{S}^3$

- X compact hyperrelliptic Riemann surface with two marked points x^+ and x^-.

- hyperelliptic involution σ and anti–holomorphic involution η without fixed points.

- meromorphic function λ with second order pole at x^+ and second order zero at x^-

$$\sigma^*\lambda = \lambda \qquad\qquad \eta^*\lambda = \bar{\lambda}^{-1}$$

- non–zero holomorphic functions μ_1 and μ_2 on $X \setminus \{x^+, x^-\}$ with

$$\sigma^*\mu_i = \mu_i^{-1} \qquad\qquad \eta^*\mu_i = \bar{\mu}_i$$

$d\ln(\mu_i)$ second order poles at $x^\pm$.

- 4 points with $\mu_1(x_j) = \mu_2(x_j) = \pm 1$ $x_1, x_2 = \sigma x_1 = \eta x_1, x_3, x_4 = \sigma x_3 = \eta x_3$.

1–sided Alexandrov embedded tori

Alexandrov embedded:
Immersion $f : \mathbb{T}^2 \to \mathbb{S}^3$ extends to
Immersion $f : M^3 \to \mathbb{S}^3$ with $\partial M^3 = \mathbb{T}^2$.

1–sided Alexandrov embedded:
Mean curvature vector H does not vanish and points outward M^3.

Theorem *Let $f : (0,1) \times \mathbb{T}^2 \to \mathbb{S}^3$ be a family of cmc–immersions,*

- *depends contiuously on $t \in (0,1)$ (up to second derivatives)*
- *mean curvature $H(t) > 0$ for all $t \in (0,1)$.*
- *for $t_0 \in (0,1)$ 1–sided Alexandrov emb.*

$\Rightarrow$ *1–sided Alexandrov emb. for all $t \in (0,1)$.*

ODE of spectral data

Spectral curves: $\kappa = \imath\frac{\lambda+1}{\lambda-1}$, $\sigma^*\kappa = \kappa$, $\eta^*\kappa = \bar{\kappa}$

$$\nu^2 = \frac{a(\kappa)}{\kappa^2+1} \qquad \deg(a) = 2g$$

$$d\ln\mu_i = \frac{b(\kappa)}{(\kappa^2+1)^2}d\kappa \qquad \deg(b_i) = g+1$$

Families of spectral curves with parameter t.
a, μ_1, μ_2, b_1, b_2 functions of (κ, t).

Ansatz: $$\frac{\partial \ln\mu_i}{\partial t} = \frac{c_i(\kappa)}{\kappa^2+1} \quad \deg(c_i) = g+1.$$

Closing condition: $\frac{\partial \ln\mu_i(x_j)}{\partial t} = 0$. $\kappa_j = \kappa(x_j)$.

$$\frac{\partial \ln\mu_2}{\partial t}d\ln\mu_1 - \frac{\partial \ln\mu_1}{\partial t}d\ln\mu_2 = C\frac{(\kappa-\kappa_1)(\kappa-\kappa_3)}{(\kappa^2+1)^2}d\kappa.$$

$$c_1b_2 - c_2b_1 = C(\kappa-\kappa_0)(\kappa-\kappa_1)a$$
$$2a\dot{b}_i - \dot{a}b_i = (\kappa^2+1)(2ac_i' - a'c_i) - 2\kappa ac_i$$

ODE: $a, b_1, b_2, \kappa_1, \kappa_3 \implies c_1, c_2, \dot{a}, \dot{b}_1, \dot{b}_2, \dot{\kappa}_1, \dot{\kappa}_3$.

Moduli space

Theorem *Spectral data of cmc torus at* t_0*. Solution of ODE yields spectral data for all* t*.*

Moduli space of spectral data of cmc tori is a 1–dim manifold with bifurcation points.

Bifurcation points: spectral data with
double points: $\mu_i(x) = \mu_i(\sigma x) = \pm 1$.
Discontinuouties of the genus.

Problem: determine connected components of spectral data of 1–sided A.e. cmc tori in $\mathbb{S}^3$.

Flat tori in $\mathbb{S}^3$

Flat tori are invariant under a 2–dimensional subgroup of the isometries $SO(4)$.

Spectral curve $\simeq \mathbb{P}^1$

Classification of embedded flat tori:
For every $H \geq 0$ there exists one embedded flat torus in $\mathbb{S}^3$ up to isometry.
For $H = 0$ this is the Clifford torus.
Conformal classes are rectangular.

Classification of flat tori: All flat tori in $\mathbb{S}^3$ are isogenic to an embedded flat torus.

1–sided Alexandrov embedded flat tori

Embedded flat tori are boundaries of two solid tori $\simeq \mathbb{S}^1 \times \mathbb{D}$.

H points outward $\Longrightarrow$ unique solid torus.

1–sided A.e. immersion $f : M^3 \to \mathbb{S}^3$ with $f|_{\partial M = \mathbb{T}^2}$ isogeny onto flat torus.

$\Longrightarrow$ f is finite–sheeted unbranched covering of solid torus $\mathbb{S}^1 \times \mathbb{D}$.

uniquely determined by cofinite subgroup of $\pi_1(\mathbb{S}^1 \times \mathbb{D}) \simeq \mathbb{Z}$. $\quad \longleftrightarrow \quad L\mathbb{Z}$ with $L \in \mathbb{N}$.

Classification of flat 1–sided A.e. tori:
For every $L \in \mathbb{N}$ there exists a family of 1–sided *A.e.* flat tori parameterized by $H > 0$.
Conformal classes are rectangular.

Bifurcation points of flat tori

The spectral curve X is the limit of spectral curves of higher genus.

$$\Updownarrow$$

X has double points, i.e. $\mu_i(x) = \mu_i(\sigma x) = \pm 1$.

$X \simeq \mathbb{P}^1$ and x double point $\Longrightarrow \rho x = \sigma\eta x = x$.

a discrete infinite subset of every family of flat tori are limits of spectral curves of genus one.

the family of embedded flat tori has for every $K \in \mathbb{N} \setminus \{1\}$ one doublepoint.

Families of $g = 1$ spectral curves

Deformation equation

Every $g = 1$ spectral curve belongs to a one–dimensional family

Every $g = 1$ family has as a limit the spectral curve of a flat torus.

families of rectangular classes have sequences of branchpoints with $\lambda \to \infty$ and $\lambda \to 0$

1–sided A.e. cmc tori with $g = 1$

The L–wrapped family of 1–sided A.e. flat tori have for every $K \in \mathbb{N}$ with $2L^2 < K^2$ a bifurcation point to a 1–sided A.e. $g = 1$ family.

rectangular conformal clasess.

$\sqrt{2}L < K < 2L$ ends in minimal $g = 1$ cmc–torus.

$K = 2L$ ends in minimal chains of spheres.

$2L < K$ ends in non–minimal chains of spheres.

Known connected components of spectral curves of 1–sided A.e. tori

$g = 1$ and x double point $\Longrightarrow \rho x = \sigma\eta x = x$.

Imaginary part of $\ln \mu_1$ and $\ln \mu_2$

$g = 1$ families of spectral curves of 1–sided A.e. cmc tori have no double points.

Known connected components of spectral curves of 1–sided A.e. cmc tori in $\mathbb{S}^3$:
For all $L \in \mathbb{N}$ one connceted component contains one minimal flat cmc torus.

contains for all $\sqrt{2}L < K$ a $g = 1$ family.
the $g = 1$ families with $\sqrt{2}L < K < 2L$ end in a $g = 1$ minimal cmc torus

they contain no spectral curves with $g > 1$.

they contain all $g \leq 1$ spectral curves of A.e. embedded cmc tori in $\mathbb{S}^3$.

Chains of spheres

rectangular $g = 1$ families ends in spectral curves of chains of spheres

brachpoints with $\lambda \to 0$ and $\lambda \to \infty$.

K isometric round spheres in $\mathbb{S}^3$ along a geodesic

for $K = 2$ minimal.

families continuos beyond the chains of spheres with a $g = 1$ family of cmc tori.

beyond the chains of spheres A.e.
but not 1–sided A.e.

connect different connected components of the moduli space.
Conjectrue: connect all components of the moduli space.

Spectral curves of cmc tori in $\mathbb{R}^3$

- X compact hyperrelliptic Riemann surface with two marked points x^+ and x^-.

- hyperelliptic involution σ and anti–holomorphic involution η without fixed points.

- meromorphic function λ with second order pole at x^+ and second order zero at x^-
$$\sigma^*\lambda = \lambda \qquad\qquad \eta^*\lambda = \bar{\lambda}^{-1}$$

- non–zero holomorphic functions μ_1 and μ_2 on $X \setminus \{x^+, x^-\}$ with
$$\sigma^*\mu_i = \mu_i^{-1} \qquad\qquad \eta^*\mu_i = \bar{\mu}_i$$
$d\ln(\mu_i)$ second order poles at $x^\pm$.

- 2 points $x_1, x_2 = \sigma x_1 = \eta x_1$ with $\mu_1(x_j) = \mu_2(x_j) = \pm 1$ and $d\mu_i(x_j) = 0$.

Cmc tori in $\mathbb{R}^3$

Spectral curves are limits $H \to \infty$ of spectral curves of cmc tori in $\mathbb{S}^3$.

cmc tori in $\mathbb{S}^3$ shrink to a point.

blow up yields cmc torus in $\mathbb{R}^3$.

all A.e. cmc tori in $\mathbb{S}^3_+$ are round spheres.
$\Longrightarrow$ Spectral curves are no limits of families of spectral curves of 1–sided A.e. cmc tori.

$g \geq 2$

Wente tori with $g = 2$ for all $K \in \mathbb{N} \setminus \{1, 2\}$.

Limits of spectral curves

A cmc cylinder in $\mathbb{R}^3$: $H \to \infty$

conformal class $\tau \to \infty$.

branchpoints bounded.

limits of $g \leq 1$ families.

B chains of spheres: H bounded.

$\tau \to \infty$.

branchpoints with $\lambda \to 0$ and $\lambda \to \infty$.

limits of $g = 1$ families.

C cmc tori in $\mathbb{R}^3$: $H \to \infty$.

τ bounded.

branchpoints bounded.

limits of $g \geq 2$ families.

Theorem *Let $X(t)$ be continuous family of spectral curves of cmc tori in $\mathbb{S}^3$ with $H(t) \geq$*

$H_0 > 0$. *Then a sequence* $X(t_n)$ *has a subsequence*

A or B *with* $g \leq 1$ *or*

C *not 1–sided A.e. or*

D *spectral curve* ($g' \leq g$) *of cmc torus in* $\mathbb{S}^3$

Deformation of Nearly Kähler manifolds

Uwe Semmelmann

Definition 0.1 *A nearly Kähler manifold is an almost Hermitian manifold* (M, g, J) *such that* $(\nabla_X J)(X) = 0$ *for all vector fields* X*. For a strict nearly Kähler manifold one has in addition that* $\nabla_X J \neq 0$*.*

Let ω be the Kähler form, i.e. $\omega(X, Y) = g(JX, Y)$. Then the nearly Kähler condition is equivalent to $\nabla\omega$ being a 3-form. In particular ω is coclosed. Such forms are called Killing 2-forms.

There are two sources of nearly Kähler manifolds: 3-symmetric spaces and twistor spaces of quaternion Kähler manifolds. A 3-symmetric space can be written as G/H, where G admits an automorphism s of order 3, such that H is the set of elements in G fixed by s. In this situation one has a canonical almost complex structure defined by

$$s_* \;=\; -\frac{1}{2}\,\mathrm{id}_{TM} \;+\; \frac{\sqrt{3}}{2}\,J \;,$$

i.e. the summands $T^{10}M$ and $T^{01}M$ of the complexified tangent bundle are given by the two eigenspaces of J for the non-real eigenvalues. If the group G is compact and simple, and if the metric g is induced by the Killing form of G, then the 3-symmetric space $(G/K, g, J)$ is a nearly Kähler manifold.

A quaternion Kähler manifold is a Riemannian manifold M with holonomy a subgroup of $Sp(n) \cdot Sp(1)$. Equivalently M has locally three almost complex structures, satisfying the quaternion relations and spanning a rank 3 subbundle Q of the endomorphism bundle of TM, which is preserved by the Levi Civita connection. Such manifolds are automatically Einstein. The twistor space Z is the total space of the sphere bundle in Q. It is a $\mathbb{C}P^1$-fibration over M and, if the scalar curvature of M is positive, it possesses a Kähler-Einstein structure. However there is also an non-integrable almost complex structure and a second Einstein metric, which together define a nearly Kähler structure on Z.

The only known compact examples in dimension 6 are:

$$S^6 = G_2/SU(3), \qquad S^3 \times S^3 = SU(2) \times SU(2) \times SU(2)/\Delta SU(2),$$

$$\mathbb{C}P^3 = Sp(2)/SU(2) \cdot U(1), \qquad F(1,2) = SU(3)/T^2$$

These are all 3-symmetric spaces, $\mathbb{C}P^3$ and $F(1,2)$ are the twistor spaces of S^4 resp. $\mathbb{C}P^2$. J.-B. Butruille showed that these examples are the only homogenous nearly Kähler manifolds in dimension 6

In the following we will restrict ourself to 6-dimensional strict nearly Kähler manifolds. This is the most interesting case due to the Reduction Theorem of P. Nagy, which states that any strict nearly Kähler manifold is locally a product of naturally reductive 3-symmetric spaces, twistor spaces of quaternion Kähler manifolds or 6-dimensional nearly Kähler manifolds.

In dimension 6 there are several very interesting additional properties, e.g. 6-dimensional nearly Kähler manifolds are precisely spin manifolds with Killing spinors, i.e. sections ψ of the spinor bundle satisfying the equation $\nabla_X \psi = \lambda X \cdot \psi$, for all vector fields X and some constant λ. Killing spinors are eigenspinors of the Dirac operator for the smallest possible eigenvalue. Their existence forces the manifold to be Einstein. Moreover the metric cone over such a manifold has holonomy G_2.

A 6-dimensional nearly Kähler manifold can also be characterized as a manifold with a special $SU(3)$-structure, i.e. one has a tuple (g, J, ω, Ψ), where g, J, ω is as above and $\Psi = \Psi^+ + i\Psi^-$ is a non-vanishing complex volume form, which satisfies the equations

$$d\omega = 3\,\Psi^+ \qquad d\Psi^- = -\,2\,\omega \wedge \omega$$

Here the scalar curvature is normalized to $scal = 30$. Moreover one can show that the Kähler form ω is a so-called special Killing 2-form and in particular an eigenform of the Laplace operator for the eigenvalue 12.

Let $\bar{\nabla}$ be the canonical connection defined by $\bar{\nabla}_X = \nabla_X - \frac{1}{2}\, J\, \nabla_X J$. For this connection the metric g and the almost complex structure J are parallel. Hence its holonomy is a subgroup of $SU(3)$. R. Cleyton and A. Swann showed that 6-dimensional nearly Kähler manifolds can also be characterized by the property that $\bar{\nabla}$ has skew-symmetric and $\bar{\nabla}$-parallel torsion.

There are two other interesting classification results in dimension 6: F. Belgun and A. Moroianu showed that the holonomy of $\bar{\nabla}$ is $\mathbb{C}$-reducible if and only if the nearly Kähler manifold is isometric to one of the two twistor spaces. If the holonomy is only $\mathbb{R}$-reducible than P. Nagy showed that the nearly Kähler manifold has to be isometric to $S^3 \times S^3$. Due to a result of A. Moroianu, P. Nagy and U. Semmelmann the nearly Kähler manifold $S^3 \times S^3$ is also characterized by the existence of a unit length Killing vector field.

Finally we want to state a result on the infinitesimal deformations of nearly Kähler structures. These deformations are only interesting if we deform the metric and the almost complex structure at the same time. Indeed Th. Friedrich showed that if the manifold (M^6, g) is not isometric to S^6 then there exists at most one nearly Kähler J compatible with g. One the other side M. Verbitsky showed that on an almost complex manifold (M^6, J) there exists (up to scaling) at most one nearly Kähler g compatible with J. Moreover any nearly Kähler deformation is automatically an Einstein deformation. Hence no such deformation exists on the standard S^6, since it is well known that there is no Einstein deformation.

Let $(g_t, J_t, \omega_t, \Psi_t)$ be a curve of nearly Kähler structures. Then we call its tangent vector an infinitesimal deformation.

Theorem 0.2 (A. Moroianu, P. Nagy, U. Semmelmann) *The space of infinitesimal deformations of a strict nearly Kähler structure (modulo the action of the diffeomorphism group) is given by the space*

$$E(12) = \{\phi \in \Omega^{11}_0(M) \mid d^*\phi = 0, \quad \Delta\phi = 12\phi\}$$

Here the scalar curvature is normalized to scal = 30.

Explicit calculations on the examples show that only on the flag manifold exist infinitesimal deformations. However up to now it is not clear whether they correspond to real deformations of the nearly Kähler structure.

Theorem 0.3 (A. Moroianu, U. Semmelmann) *Let (M^6, g, J) be a strict nearly Kähler manifold of scalar curvature 30. Then the space of infinitesimal Einstein deformations decomposes as:*

$$E(2) \oplus E(6) \oplus E(12)$$

Again explicit calculations for the known examples show that only the nearly Kähler metric on $F(1,2)$ admits infinitesimal Einstein deformations.

Das L^p-Spektrum von Produktmannigfaltigkeiten

Andreas Weber*

Zusammenfassung

Wir zeigen, dass das L^p-Spektrum einer Riemann'schen Produktmannigfaltigkeit $M_1 \times M_2$ mit der mengentheoretischen Summe der L^p-Spektren von M_1 und M_2 übereinstimmt.

1 Einleitung

Es ist bekannt, dass für kompakte Riemann'sche Mannigfaltigkeiten M_1 und M_2 das L^2-Spektrum des Produkts $M_1 \times M_2$ mit der mengentheoretischen Summe der L^2-Spektren von M_1 und M_2 übereinstimmt (vgl. [1] oder Abschnitt 2). Wir verallgemeinern dieses Resultat indem wir einerseits nicht-kompakte Riemann'sche Mannigfaltigkeiten M_1 und M_2 zulassen und andererseits deren L^p-Spektren, $p \in [1, \infty)$, betrachten. Für Details verweisen wir auf [5].

Im Folgenden sei M eine vollständige Riemann'sche Mannigfaltigkeit. Der Laplace-Beltrami-Operator $\Delta_M := -\mathrm{div}(\mathrm{grad})$, definiert auf der Menge $C_c^\infty(M) \subset L^2(M)$ der differenzierbaren Funktionen mit kompaktem Träger, ist wesentlich selbst-adjungiert. Somit ist sein Abschluss $\Delta_{M,2}$ ein selbst-adjungierter Operator auf dem Hilbert-Raum $L^2(M)$. Da $\Delta_{M,2}$ positiv ist, erzeugt $-\Delta_{M,2}$ eine beschränkt analytische Halbgruppe $T_2(t) = \mathrm{e}^{-t\Delta_{M,2}}$. Diese Halbgruppe ist eine Submarkov-Halbgruppe, und man erhält deshalb durch Extrapolation eine konsistente Familie von C_0-Halbgruppen $T_p(t)$ auf $L^p(M)$, $p \in [1, \infty)$, vgl. [2]. Hierbei heißen die Halbgruppen $T_p(t)$ konsistent, wenn die Einschränkungen von $T_p(t)$ und $T_q(t)$ auf $L^p(M) \cap L^q(M)$ übereinstimmen.
Den Erzeuger von $T_p(t)$ nennen wir im Folgenden $-\Delta_{M,p}$ und das Spektrum $\sigma(\Delta_{M,p})$ von $\Delta_{M,p}$ ist das L^p-Spektrum von M.

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2 Hauptresultat

Seien (M_1, g_1) und (M_2, g_2) Riemann'sche Mannigfaltigkeiten und $(M_1 \times M_2, g = g_1 \oplus g_2)$ deren Produkt. Dann gilt für alle $f \in C_c^\infty(M_1 \times M_2)$ die Beziehung

$$\Delta_{M_1 \times M_2} f(x_1, x_2) = \Delta_{M_1} f(\cdot, x_2)|_{x_1} + \Delta_{M_2} f(x_1, \cdot)|_{x_2}. \tag{2.1}$$

Dies folgt leicht aus der Formel

$$\Delta_M = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial}{\partial x^j} \right).$$

Theorem 2.1. *Seien $(M_1, g_1), (M_2, g_2)$ vollständige Riemann'sche Mannigfaltigkeiten und $(M = M_1 \times M_2, g = g_1 \oplus g_2)$ deren Produkt. Dann gilt*

$$\sigma(\Delta_{M,p}) = \sigma(\Delta_{M_1,p}) + \sigma(\Delta_{M_2,p}) \tag{2.2}$$

für alle $p \in (1, \infty)$.

Der Beweis von Theorem 2.1 ist einfach, falls M_1 und M_2 kompakt sind: In diesem Fall sind die L^2-Spektren $\sigma(\Delta_{M_1,2})$ und $\sigma(\Delta_{M_2,2})$ diskret und bestehen ausschließlich aus Eigenwerten $\lambda_i, i \in \mathbb{N}$ (bzw. $\mu_j, j \in \mathbb{N}$). Desweiteren gibt es vollständige Orthonormalsysteme $\{f_i : i \in N\}$ (bzw. $\{g_j : j \in \mathbb{N}\}$) bestehend aus Eigenfunktionen für $\Delta_{M_1,2}$ (bzw. $\Delta_{M_2,2}$). Die Menge $\{f_i g_j : i, j \in \mathbb{N}\}$ ist dann ein vollständiges Orthonormalsystem bestehend aus Eigenfunktionen für $\Delta_{M,2}$ mit $\Delta_{M,2} f_i g_j = (\lambda_i + \mu_j) f_i g_j$, d.h. (2.2) gilt für $p = 2$. Für mehr Details vgl. [1, S. 144].
Da das L^p-Spektrum von kompakten Mannigfaltigkeiten nicht von p abhängt (vgl. z.B. [4]), folgt das Resultat für alle p.

Im nicht-kompakten Fall funktioniert dieser Beweis aus zwei Gründen nicht: Einerseits braucht das L^2-Spektrum nicht mehr aussschließlich aus Eigenwerten zu bestehen und andererseits kann das L^p-Spektrum mit p variieren. Ob das Spektrum einer Riemann'schen Mannigfaltigkeit M mit p variiert, hängt von der Geeometrie von M ab [4]. Ein Beispiel für eine Riemann'sche Mannigfaltigkeit deren L^p-Spektrum mit p variiert ist $\mathbb{H}^n$, der n-dimensionale hyperbolische Raum [3].

Im Beweis von Theorem 2.1 verwenden wir die Theorie der Tensor Produkte. Für weitere Details verweisen wir auf [5].

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32. Kolloquium über Differentialgeometrie
Universität Stuttgart, 4. Mai 2007

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Die bisherigen Kolloquien über Differentialgeometrie

1.	Kolloquium am	18.06.1976	in	München
2.	Kolloquium am	10.06.1977	in	Darmstadt
3.	Kolloquium am	26.05.1978	in	München
4.	Kolloquium am	15.06.1979	in	Würzburg
5.	Kolloquium am	06.06.1980	in	Karlsruhe
6.	Kolloquium am	19.06.1981	in	Darmstadt
7.	Kolloquium am	11.06.1982	in	Stuttgart
8.	Kolloquium am	03.06.1983	in	München
9.	Kolloquium am	22.06.1984	in	Würzburg
10.	Kolloquium am	07.06.1985	in	Karlsruhe
11.	Kolloquium am	30.05.1986	in	Darmstadt
12.	Kolloquium am	19.06.1987	in	Stuttgart
13.	Kolloquium am	03.06.1988	in	München
14.	Kolloquium am	26.05.1989	in	Würzburg
15.	Kolloquium am	15.06.1990	in	Karlsruhe
16.	Kolloquium am	31.05.1991	in	Thessaloniki
17.	Kolloquium am	29.05.1992	in	Dresden
18.	Kolloquium am	11.06.1993	in	Darmstadt
19.	Kolloquium am	03.06.1994	in	Stuttgart
20.	Kolloquium am	19.05.1995	in	München
21.	Kolloquium am	07.06.1996	in	Würzburg
22.	Kolloquium am	30.05.1997	in	Karlsruhe
23.	Kolloquium am	22.05.1998	in	Dresden
24.	Kolloquium am	14.05.1999	in	Darmstadt
25.	Kolloquium am	02.06.2000	in	Wien
26.	Kolloquium am	25.05.2001	in	Stuttgart
27.	Kolloquium am	10.05.2002	in	Würzburg
28.	Kolloquium am	26.06.2003	in	München
29.	Kolloquium am	11.06.2004	in	Karlsruhe
30.	Kolloquium am	27.05.2005	in	Dresden
31.	Kolloquium am	16.06.2006	in	Darmstadt
32.	Kolloquium am	04.05.2007	in	Stuttgart