SYLOW NUMBERS FROM CHARACTER TABLES AND INTEGRAL GROUP RINGS

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Abstract. G. Navarro raised the question whether the ordinary character table $X(G)$ of a finite group G determines the Sylow numbers of G. In this note we show that this is the case when G is nilpotent-by-nilpotent, quasinilpotent, Frobenius group or a 2-Frobenius group. In particular Sylow numbers of supersoluble groups are determined by their ordinary character table. If G and H are finite groups with isomorphic integral group rings then it is well known that $X(G)$ and $X(H)$ coincide. In the last part of this note we show that the integral group ring $\mathbb{Z}G$ determines the number of Sylow q-subgroups provided G is q-contrained. It follows that $\mathbb{Z}G$ determines the Sylow numbers of G provided G is soluble.

Introduction

Groups naturally occur as symmetries of objects. The systematic study of finite groups was started by Ludwig Sylow in 1872. The Sylow Theorems give the existence and the number of maximal p-subgroups, i.e. the number of Sylow p-subgroups of a finite group G, denoted by $n_p(G)$. As usual $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups and $\mathrm{N}_G(P)$ denotes the normalizer of $P \in \mathrm{Syl}_p(G)$. By $\pi(G)$ we denote the set of all prime divisors p of $|G|$ and the set $\{\, n_p(G) \,|\, p \in \pi(G)\,\}$ is called Sylow numbers $\mathrm{sn}(G)$. The arithmetic composition of $n_p(G)$ was analysed by Marshall Hall in [5]. At the end of the 20th century the question arised what influence Sylow p-numbers have on properties of the given group.

In a series of articles finite groups with given Sylow numbers were studied. Florian Luca and Wenbin Guo gave evidence that for certain Sylow numbers the given group has to be soluble [22],[4], see also [23], [20]. Jiping Zhang and Naoki Chigira found an equivalent condition for p-nilpotency [30],[3, Main Theorem]. In 2006, Xianhua Li proved in [21, Theorem 3.3.1] the uniqueness of Sylow numbers for finite simple groups (excluding $B_n(q)$ and $C_n(q)$).

In 2003 Gabriel Navarro gave in [24] an overview of open problems on characters and Sylow subgroups. The question arised whether the ordinary character table $X(G)$ of a finite group G determines $|\mathrm{N}_G(P)|$ for $P \in \mathrm{Syl}_p(G)$ [24, Question 7]. It was shown that for cyclic Sylow p-subgroups the Sylow p-number is determined by the character table. I. M. Isaacs and G. Navarro proved that the primes dividing $|\mathrm{N}_G(P)|$ are known by $X(G)$ provided G is soluble, see [14].

In 2012 Alexander Moretó gave in [23, Theorem 2.1] a criterion for the existence of nilpotent Hall π-subgroups as a function of Sylow numbers. As nilpotent Hall π-subgroups are determined by the character table (see [19]), this again leads to the question whether Sylow numbers are determined by character tables.

MSC(2010): Primary: 16U60; Secondary: 20C10,16S34.
Keywords: Character tables, Integral group rings, Sylow numbers.

In the first section we analyse the situation under extensions. The use of nilpotent and central extensions enables us to prove in Section 2 that Sylow numbers of nilpotent-by-nilpotent groups are determined by their character table, see Theorem 2.4. In Section 3 we consider the question whether Sylow numbers are determined under the weaker assumption of class structures. If the groups G and H are in Jordan-Hölder-Sylow correspondence we show that $n_p(G)$ is determined provided Sylow p-subgroups are cyclic, cf. Theorem 3.6. Using results on cyclic Sylow subgroups we show at the end of Section 3 that $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$ in the case when G is a Frobenius or a 2-Frobenius group. In particular the Sylow numbers of soluble groups with disconnected prime graph are given by their character table.

In the last section we consider groups with isomorphic integral group rings and prove that $n_p(G)$ of a p-constrained group G is determined by its integral group ring. In particular this shows that the integral group ring of a finite soluble group G determines its Sylow numbers, cf. Theorem 4.4. This and the result on Frobenius groups in Section 3 supplements the Sylow-like theorems for integral group rings established in [18], [17, Theorem 8] respectively.
It is known that $\mathbb{Z}G$ determines $\mathrm{X}(G)$, see e.g. [13, 3.17], but the converse does not hold. The reason that we get a stronger result for integral group rings is the so-called F^*-theorem, cf. [26, Theorem 19], [27] and [9, Introduction]. This shows that a finite group G is determined by $\mathbb{Z}G$ up to isomorphism provided the generalized Fitting subgroup $F^*(G)$ is a p-group. This is certainly not the case for character tables. Note that in general $\mathbb{Z}G$ does not determine G up to isomorphism, the smallest counterexample known is a group of order $97^{28} \cdot 2^{21}$ [7]. Finally we establish for some classes of insoluble groups with abelian Sylow 2-subgroups or nilpotent extensions of simple groups that Sylow numbers are determined by its integral group ring.

1. Sylow numbers

As mentioned above G always denotes a finite group. The following result of Marshall Hall yields a formula for Sylow p-numbers by Sylow numbers of certain subgroups and factor groups:

Theorem 1.1. [5, Theorem 2.1] *Let G have a normal subgroup M and assume $P \in \mathrm{Syl}_p(G)$. Then*

$$n_p(G) = n_p(M)n_p(G/M)n_p(\mathrm{N}_{PM}(P \cap M)/(P \cap M)).$$

The theorem contains the following basic observations for group extensions. We note that these observations may be easily established with direct arguments.

Proposition 1.2. *Let G have a normal q-subgroup Q.*

(i) *If $p \neq q$, then $n_p(G) = n_p(G/Q)n_p(PQ)$.*
(ii) *If $p = q$, then $n_p(G) = n_p(G/Q)$.*
(iii) *Let E be a finite central extension of G with kernel K. Then $n_p(E) = n_p(G)$ for each prime p.*

Proof: (i) and (ii) are immediate from 1.1. For (iii) we use induction. So we may assume that K is of prime order.
Let p be a prime and denote by P a Sylow p-subgroup of G. If $P \cap K = \{1\}$ then by (i) it follows that $n_p(G) = n_p(G/K)n_p(PK)$ and $n_p(PK) = 1$ because K is central. If $P \cap K \neq \{1\}$ then $P \cap K = K \trianglelefteq G$ and (ii) yields $n_p(G) = n_p(G/(P \cap K))$. □

In order to use induction (especially for soluble groups) with quotient modulo normal subgroups K where $p \notin \pi(K)$ the main problem is to determine the Sylow number of PK. A first basic observation is the following.

Proposition 1.3. [2, Proposition 1C] *Let $K \trianglelefteq G$, $P \in \mathrm{Syl}_p(G)$ and $K \cap P = \{1\}$. Then*

$$n_p(PK) = |K : \mathrm{C}_G(P) \cap K|$$

and

$$n_p(G) = n_p(G/K) \cdot |K : \mathrm{C}_G(P) \cap K|.$$

For the convenience of the reader we include a slightly more direct proof.

Proof: Let $P \in \mathrm{Syl}_p(G)$ and assume that $P^g \in \mathrm{Syl}_p(PK)$. Due to the trivial intersection of P and K and that K has order coprime to $|P|$ the Schur-Zassenhaus theorem yields $k \in K$ with $P^g = P^k$. Suppose for $k_1, k_2 \in K$, that $P^{k_1} = P^{k_2}$. Then $k_1 k_2^{-1} \in \mathrm{N}_G(P) \cap K$, i.e. $n_p(PK) = |K : \mathrm{N}_G(P) \cap K|$.
It remains to prove, that $\mathrm{N}_G(P) \cap K = \mathrm{C}_G(P) \cap K$. Let $k \in \mathrm{N}_G(P) \cap K$. It follows that $p^{-1}k^{-1}pk \in K \cap P = \{1\}$ for every $p \in P$. Consequently $pk = kp$ for every $k \in \mathrm{N}_G(P) \cap K$ and every $p \in P$. Thus, the result holds. □

Another obvious consequence of Proposition 1.3 is the following.

Corollary 1.4. *Let $P \in \mathrm{Syl}_p(G)$ and assume $K \trianglelefteq G$, where $p \notin \pi(K)$. Then the following assertions are equivalent:*

(i) $n_p(G) = n_p(G/K)$,
(ii) $K \subseteq \mathrm{N}_G(P)$,
(iii) $K \subseteq \mathrm{C}_G(P)$.

In some special cases it is possible to obtain the Sylow numbers explicitely.

Corollary 1.5. *Let G be a finite Frobenius group with Frobenius kernel K and Frobenius complement H. Let $\mathrm{sn}(H) = \{a_1, \ldots, a_n\}$. Then*

$$\mathrm{sn}(G) = \{1, |K| \cdot a_1, \ldots, |K| \cdot a_n\}.$$

In particular if H is nilpotent then $\mathrm{sn}(G) = \{1, |K|\}$.

Proof: If $p \in \pi(K)$ then $n_p(G) = 1$ because the Frobenius kernel is nilpotent. If $p \notin \pi(K)$ then we get by Propostion 1.3 that

$$n_p(G) = n_p(G/K) \cdot |K : \mathrm{C}_G(P) \cap K|.$$

Because H acts fixpointfreely on K we have always $\mathrm{C}_G(P) \cap K = 1$. This completes the proof. □

2. Character Tables

We first collect some of the known results on ordinary character tables concerning properties of the group G reflected by $\mathrm{X}(G)$.

2.1. *Let G be a finite group.*

(i) *The second orthogonality relations show that $\mathrm{X}(G)$ determines the length of the conjugacy classes of G.*
(ii) *The lattice of normal subgroups may be constructed out of $\mathrm{X}(G)$, see [13, p. 23]. The normal subgroups of G are given by intersections of the kernel of the irreducible characters. For each normal subgroup N of G the conjugacy classes in N are determined. Moreover the order of $|N|$ is determined.*

(iii) *Let C be a conjugacy class and $g \in C$. Then by a result of G. Higman* [13, Theorem (8.21)] *the prime divisors of the order of g may be calculated from* $\mathrm{X}(G)$.

(iv) *For a given $N \trianglelefteq G$ the ordinary character table $\mathrm{X}(G/N)$ may be computed out of $\mathrm{X}(G)$ by deleting appropriate lines and columns* [13, p. 24].

Of course the question whether the character table determines the Sylow numbers has an affirmative answer when the character table determines the group up to isomorphism. This is for example the case when G is semisimple, i.e. the direct product of non-abelian simple groups [15, Satz 6.3], see also [19, Theorem 4 and 5].

Proposition 2.2. *Suppose that the finite group G is quasinilpotent. Then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof. Quasinilpotent groups are central extensions of semisimple groups. By Proposition 1.2 Sylow numbers remain unchanged under central extensions and by 2.1(iv) and [15] the result follows immediately. □

Lemma 2.3. *Let $K \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$ then it is possible to decide with $\mathrm{X}(G)$ whether the intersection $K \cap \mathrm{C}_G(P)$ is trivial or not.*

Especially, if K is cyclic of prime order $q \neq p$, then $n_p(G)$ may be calculated from $\mathrm{X}(G)$ provided $n_p(G/K)$ is known.

Proof: Regarding the character table we obtain using 2.1(iii) the conjugacy classes which are contained in K. By 2.1(i) we can compute $|\mathrm{C}_G(h)|$ for each $h \in K$. If $|P|$ divides $|\mathrm{C}_G(h)|$, then $h \in \mathrm{C}_G(P^g)$ for some $g \in G$. Consequently $h^{g^{-1}}$ centralizes P. Conversely, if a non-trivial element h of K centralizes P, then $|P|$ divides the order of its centralizer.
If $q = p$ then $n_p(G) = n_p(G/K)$. In the other case we use Proposition 1.3. As K has prime order the index $|K : K \cap \mathrm{C}_G(P)|$ equals q if and only if there is a non-trivial conjugacy class $k^G \in K$ such that $|P|$ divides $|\mathrm{C}_G(k)|$. □

Note that the previous result holds for arbitrary cyclic subgroups K. In general we cannot decide by means of the character table whether the whole conjugacy class g^G of g is contained in $\mathrm{C}_G(P)$. For nilpotent normal subgroups of a group we are able to compute the Sylow number of each factor seperately.

Theorem 2.4. *Suppose that the finite group G has a nilpotent normal subgroup N such that G/N is nilpotent then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof: By 2.1 we may assume that the Sylow numbers of all proper quotients of G are given. If $p \in \pi(N)$, then the intersection $P \cap N$, whereas $P \in \mathrm{Syl}_p(G)$, is normal in G. By Proposition 1.2 the Sylow p-number remains unchanged if we consider $n_p(G/(P \cap N))$ instead of $n_p(G)$.
So assume that $p \notin \pi(N)$. Then $P \cap N = 1$ and we obtain with Proposition 1.3

$$n_p(G) = n_p(PN).$$

A nilpotent group N is the direct product of its Sylow subgroups, i.e. $N = P_1 \times \ldots \times P_k$, $P_i \in \mathrm{Syl}_{p_i}(N)$ and $p_i \nmid p$ for all i. Consider $PN/\mathrm{Z}(P_1)$, where $\mathrm{Z}(P_1) \trianglelefteq PN$ denotes the center of P_1. By 1.1 we get

$$n_p(PN) = n_p(PN/\mathrm{Z}(P_1)) \cdot n_p(P\,\mathrm{Z}(P_1)).$$

Proposition 1.3 yields $n_p(P\,\mathrm{Z}(P_1)) = |\,\mathrm{Z}(P_1) : \mathrm{Z}(P_1) \cap \mathrm{C}_G(P)|$. Assume $a \in \mathrm{Z}(P_1) \cap \mathrm{C}_G(P)$. Consequently, as N is nilpotent, we obtain that $a \in \mathrm{Z}(N)$. By assumption G/N is nilpotent hence for each $\tilde{P} \in \mathrm{Syl}_p(G)$ there exists $n \in N$ such that $\tilde{P} = P^n$. It follows that $a \in \mathrm{C}_G(\tilde{P})$, as

$$\tilde{P}^a = (P^n)^a = (P^a)^n = P^n = \tilde{P}.$$

Let $a^g \in a^G$. Since $P^{a^g} = P$ we see that a^G is contained in $\mathrm{C}_G(P)$. Therefore $|n_p(P\,\mathrm{Z}(P_1))|$ may be computed by the character table. Note that the Sylow number $n_p(PN/\,\mathrm{Z}(P_1)) = n_p(G/\,\mathrm{Z}(P_1))$ is given by induction and thus $n_p(G)$ is determined. □

Proposition 2.5. *Suppose that G has a supersoluble normal subgroup N such that each chief factor of N is also a chief factor of G and $n_p(G/N)$ is known for $p \in \pi(G/N)$. Then $n_p(G)$ may be calculated from $\mathrm{X}(G)$.*

Proof. By assumption a maximal normal subgroup M within N is of prime index q in N. Now Lemma 2.3 implies that $n_p(G/M)$ is known. By induction the result follows. □

Corollary 2.6. *Suppose that G is supersoluble. Then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof. The corollary follows immediately from Proposition 2.5 where $N = G$. □

3. Class structures and cyclic Sylow subgroups

In general the character table without the headline does not determine the order of the representatives of all classes. By 2.1(iii) however one can decide which primes divide the order of a representative. Thus for each prime p the conjugacy classes of the p-elements are recognizable.

Lemma 3.1. *Let p^a be the exponent of a Sylow p-subgroup of G. Suppose that the intersection of different Sylow p-subgroups has smaller exponent and assume that the number m of the conjugacy classes of p-elements of order p^a and the number k of elements of order p^a of a Sylow p-subgroup is given by $\mathrm{X}(G)$. Then $n_p(G)$ may be computed from $\mathrm{X}(G)$.*

Proof. Let g be a p-element of order p^a. The length L of the conjugacy class of g may be calculated from $\mathrm{X}(G)$. By the assumptions we see that

$$m \cdot L = n_p(G) \cdot k.$$

This yields $n_p(G)$. □

Remarks 3.2.

(i) *The character table with the power map on the conjugacy classes (i.e with given headline) is usually called the spectral table* $\mathrm{Spec}(G)$ *of G. If the spectral table is given then the order of the class representatives may be calculated via the power map and the second assumption in the previous lemma is satisfied, i.e. the number m is known. Of course the same holds if the prime p divides $|G|$ only with the first power.*

(ii) *In* [24, Corollary 5] *M. Isaacs and G. Navarro have shown that if G has abelian Sylow p-subgroups then $\mathrm{X}(G)$ determines the order of the p-elements. If G has cyclic Sylow p-subgroups the other assumptions of Lemma 3.1 are satisfied. By* [24, Theorem 8] *$\mathrm{X}(G)$ determines whether G has cyclic Sylow p-subgroups. Consequently Lemma 3.1 shows that $n_p(G)$ is determined by $\mathrm{X}(G)$ provided G has cyclic Sylow p-subgroups.*

(iii) *Note that for an application of* [24, Theorem 8] *for p-elements $g \in G$ the field $F_g := \mathbb{Q}(\chi(g); \chi \in \mathrm{Irr}G)$ has to be computed. Thus one really needs to know the character values of the ordinary irreducible characters.*

Lemma 3.1 may also be applied in the situation when G has abelian T.I. Sylow subgroups.

Proposition 3.3. *Suppose that Sylow p-subgroups of G are T.I. and abelian. Then $\mathrm{X}(G)$ determines $n_p(G)$.*

Proof. Because Sylow p-subgroups are abelian we get by [24, Corollary 5] that the orders of the representatives of the conjugacy classes of p-elements are determined by $\mathrm{X}(G)$. Thus the number of conjugacy classes of p-elements of maximal order p^a is known. The T.I. property garantuees that the exponent of the intersection of two Sylow p-subgroup is 1. By [15, Satz 6.2] $\mathrm{X}(G)$ determines abelian Sylow p-subgroups of G up to isomorphism. Thus the number of elements of order p^a of a Sylow p-subgroup of G is given and by Lemma 3.1 $n_p(G)$ is determined. □

Next we show that even a JHS-class structure (an assumption which is weaker than the character table) permits the calculation of the Sylow numbers in the case that G has cyclic Sylow subgroups. For the convenience of the reader we recall the definitions of such class structures as introduced in [19].

Definition 3.4. Let G and H be finite groups.

- A class structure of G is a labelled poset given by its normal subsets such that the label contains at least the size of the corresponding subset.
- A class structure X is called of type JH if the labels give the information which elements of X form normal subgroups and which are of the form $N \cdot C$ for some conjugacy class C of G and $N \trianglelefteq G$.

  X is called of type JHS if it is of type JH and additionally the labels indicate for all primes p which conjugacy classes contain elements of order a power of p.

  X is called of type JHB if X is of type JH and the labels contain the power map on the conjugacy classes.
- A bijection $\tau : G \to H$ is called a class correspondence if it gives a bijection on the conjugacy classes of G and H.

  τ is called of type JH if it is a bijection on the normal subgroups of G and H and if $\tau(N \cdot C) = \tau(N) \cdot \tau(C)$ for each normal subgroup N and each conjugacy class of G.

  τ is called of type JHS if it is of type JH and $\tau(x)$ is a p-element if, and only if, x is a p-element for each prime p.

  τ is called of type JHB if it is of type JH and $\tau(C)^n = \tau(C^n)$ for each $n \in \mathrm{N}$ and all conjugacy classes C of G.

Remarks. JH is an abbreviation for Jordan-Hölder respectively JHS for Jordan-Hölder-Sylow and JHB for Jordan-Hölder-Brauer. For the following relationship with spectral and character tables we refer to [19, §1]:

- If $\mathrm{Spec}(G) = \mathrm{Spec}(H)$, then G and H are in class correspondence of type JHB.
- If $\mathrm{X}(G) = \mathrm{X}(H)$, then G and H are in class correspondence of type JHS.
- If G and H are in class correspondence of type JHB, then G and H are in class correspondence of type JHS.

In terms of class structures (cf. [19]) Lemma 3.1 may be strengthened. Obviously a class structure on G of type JHB determines $n_p(G)$ provided G has cyclic Sylow p-subgroups. But in this case the situation is even better.

Proposition 3.5. *Let G be a finite group and suppose that G has cyclic Sylow p-subgroups. If H is a group which is in class correspondence of type JH to G then H has also cyclic Sylow p-subgroups.*

Proof. Suppose that G and H are in class correspondence of type JH. Suppose that G is p-soluble. Using [19, Lemma 1.8] we may assume that $O_{p'}(G) = O_{p'}(H) = 1$. The normal subgroup correspondence given by a class correspondence of type

JH shows that H has a normal Sylow p-subgroup P^* which has for each divisor d of its order precisely one normal subgroup of order d.

We claim that this implies that P^* is cyclic. For $M := P^*/\Phi(P^*)$ is a semisimple $F_p(G/P^*)$-module because $|G/P^*|$ and $|P^*|$ are coprime. By the normal subgroup structure of P^* we see that M has to be of order p. So $\Phi(P^*)$ has index p in P^* and it follows that P^* is cyclic. This proves part (ii) in the p-soluble case.

Suppose now that G is not p-soluble. We still can assume that $O_{p'}(G) = 1$. As G is not p-soluble it follows from a theorem of R. Brauer [2, Theorem 3C] that for a normal subgroup N of G either $P \subseteq N$ or $P \subseteq G/N$ for some $P \in \mathrm{Syl}_p(G)$. As $O_{p'}(G) = 1$ we obtain $P < N$. Thus G has precisely one minimal normal subgroup N. Since G is not p-soluble and p divides precisely one chief factor by Brauer's theorem we get that the generalized Fitting subgroup $F^*(G)$ is a simple nonabelian group S and it follows that G is an almost simple group of type S [1]. Moreover p does not divide $|G/S|$. A class structure of type JH determines the chief factors [19, Theorem 5]. Thus S is determined up to isomorphism and a group H in class correspondence of type JH to G must be an almost simple group of type S such that p does not divide $|H/S|$. Consequently H has cyclic Sylow p-subgroups as well. Hence the second part of the claim is established. □

If G has cyclic Sylow p-subgroups then the order of the centralizer of a Sylow p-subgroup P coincides with the smallest order of the centralizer of a p-element. Thus $\mathrm{X}(G)$ or even a class structure of type JHS of G determines $|\mathrm{C}_G(P)|$. The inertia group of the principal p-block of $\mathrm{C}_G(P)$ in $\mathrm{N}_G(P)$ is $\mathrm{N}_G(P)$. Let B_0 be the principal p-block of G. Then the Brauer-Dade theory of cyclic blocks shows that $|\mathrm{N}_G(P) : \mathrm{C}_G(P)|$ coincides with the number of non-exceptional characters in B_0 [6]. Consequently $\mathrm{X}(G)$ determines $n_p(G)$. We show that even a class structure of type JHS determines $n_p(G)$ provided G has cyclic Sylow p-subgroups.

Theorem 3.6. *Suppose that G has a cyclic Sylow p-subgroup and let H be a group such that G and H are in class correspondence of type JHS. Then* $n_p(G) = n_p(H)$. *Moreover* $n_p(G)$ *may be calculated from data given by a JHS-class structure.*

Proof. By Proposition 3.5(ii) we know that H has as well a cyclic Sylow p-subgroup. The result is clear when the Sylow p-subgroup is central. Thus we assume that $|\mathrm{C}_G(P)| < |G|$. Let h be a generator of a cyclic Sylow p-subgroup P. Certainly for each subgroup U of P we have that

$$\mathrm{C}_G(P) \subseteq \mathrm{C}_G(U) \text{ and } \mathrm{N}_G(P) \subseteq \mathrm{N}_G(U).$$

Let $U = \langle h^k \rangle$. Moreover $\mathrm{N}_G(P) < \mathrm{N}_G(U)$ if, and only if, $\mathrm{C}_G(h) < \mathrm{C}_G(h^k)$. Consequently subgroups generated by p-elements with minimal centralizer order have normalizers of the same order. Let M be the number of conjugacy classes of p-elements g such that $|\mathrm{C}_G(g)|$ is minimal. Note that a class correspondence of type JHS determines M. Because Sylow p-subgroups are cyclic we see that there is a certain $n \in \mathbb{N}_0$ such that all p-elements of order bigger than p^n have centralizers of minimal length. The number of such p-elements inside of P is $p^a - p^n$ if $|P| = p^a$. Thus we get

$$p^a - p^n = M \cdot \mathrm{N}_G(P)/\mathrm{C}_G(P).$$

The number of all such p-elements in G is $M \cdot |G/\mathrm{C}_G(P)|$. Note that a p-element with centralizer of minimal order cannot be contained in different Sylow p-subgroups.

[1]If S is a finite non-abelian simple group then we call a group $A(S)$ an almost simple group of type S if

$$\mathrm{Inn}(S) \leq A(S) \leq \mathrm{Aut}(S).$$

Consequently

$$n_p(G) = M \cdot \frac{L}{p^a - p^n},$$

where $L = |G|/|\mathrm{C}_G(g)|$ and g is a p-element with minimal centralizer order. M, L and p^a are given by a class structure of type JHS. By Sylow's theorem we have $n_p(G) \equiv 1 \mod p$. Thus there is precisely one $n \in \mathbb{N}_0$ such that this congruence holds and $n_p(G)$ is determined. □

Looking for possible candidates of soluble groups whose Sylow numbers are not determined by their character table the following result shows that they must have at least with respect to two different primes non-cyclic Sylow subgroups.

Proposition 3.7. *Let G be a group of order $q^a \cdot p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$, where p_i are pairwise different primes and q is a prime different from all p_i. Assume further that all Sylow p_i-groups of G are cyclic. Then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof. Assume first that G is soluble. Let G be a counterexample of minimal order. Note that the result holds when G is a p-group. If N is a normal subgroup of G then G/N suffices the hypothesis of the theorem as well. So we may assume that the result holds for G/N. Assume that N is a minimal normal subgroup of G and that G is not simple. If N is not a q-group then N is cyclic and $|N| = p_i^k$ for some i and $k \leq a_i$. By Lemma 2.3 we get that $\mathrm{sn}(G)$ is determined by $\mathrm{X}(G)$. If N is a q-group then $n_q(G) = n_q(G/N)$ and $n_{p_i}(G)$ is determined by Theorem 3.6.

Assume now that G is insoluble. By the Feit-Thompson theorem 2 divides $|G|$ and it follows by [11, IV, Satz 2.8] that $q = 2$. As in the soluble case it follows that a minimal counterexample does not have minimal normal subgroups which are cyclic or of order 2^m.

Thus the generalized Fitting subgroup $F^*(G)$ only consists of the layer $E(G)$. As $\mathrm{C}(G) \leq \mathrm{C}_G(F^*(G)) \leq F(G) = 1$ we see that $E(G)$ is a simple non-abelian group S. Then G is isomorphic to an almost simple group of type S. By [1], see also [12, p.190], the only simple groups all of whose Sylow subgroups of odd order are cyclic are $\mathrm{PSL}(2, 2^f)$, $f > 1$, $\mathrm{PSL}(2, p)$, $p > 3$, $\mathrm{Sz}(2^{2n+1})$ and the Janko group J_1 of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$.
All these simple groups have a cyclic outer automorphism group. Thus for each divisor m of $|\mathrm{Out}(S)|$ there is precisely one almost simple group of type S of order $m \cdot |S|$. By 2.1(ii), [16, Theorem 6.1] and [15, Propositionen 6.5 und 6.6] $\mathrm{X}(G)$ determines G up to isomorphism and thus in particular $\mathrm{sn}(G)$. □

Especially for induction the following result is helpful.

Proposition 3.8. *Let G be a finite group with non-trivial normal subgroups M and N of coprime order. Suppose that all Sylow numbers of $n_p(G/M)$, $n_p(G/N)$ and $n_p(G/MN)$ are known. Then $n_p(G)$ is known for each p.*

Proof. Without loss of generality assume that $p \notin \pi(G)$. By Proposition 1.3 we get

$$n_p(G) = n_p(G/M) \cdot n_p(M) \cdot |M : M \cap \mathrm{C}_G(P)|,$$

where $P \in \mathrm{Syl}_p(G)$ and similarly, again because $p \notin \pi(M)$

$$n_p(G/N) = n_p(G/MN) \cdot n_p(M) \cdot |MN/N : \mathrm{C}_{G/N}(PN/N) \cap MN/N|.$$

Consider the restriction $\kappa|_{\mathrm{C}_G(P) \cap M} : \mathrm{C}_G(P) \cap M \to \mathrm{C}_{G/N}(PN/N) \cap MN/N$ of the reduction map $\kappa : G \to G/N$. This map is injective. Suppose for $x \in M$ that $xN \in \mathrm{C}_{G/N}(PN/N)$, i.e. for every $y \in P$ we obtain $y \cdot x = x \cdot y \cdot n$ for some $n \in N$. But as $n = [x, y] \in M$ and $M \cap N = 1$ the map is also surjective and therefore

$$|M : M \cap \mathrm{C}_G(P)| = |MN/N : \mathrm{C}_{G/N}(PN/N) \cap MN/N|.$$

As $n_p(G/M)$, $n_p(G/N)$ and $n_p(G/MN)$ are known, it follows that $n_p(G)$ is determined provided p does not divide $|M|$. □

Proposition 3.8 may be used to give a different proof of Theorem 2.4.

Corollary 3.9.

(i) *Let G be a Frobenius group. Then* $\mathrm{X}(G)$ *determines* $\mathrm{sn}(G)$*.*
(ii) *Let G be a 2-Frobenius group. Then* $\mathrm{X}(G)$ *determines* $\mathrm{sn}(G)$*.*

Proof. (i) By Corollary 1.5 it suffices to show that the Sylow numbers of a Frobenius complement H are determined. Denote by K the Frobenius kernel then $\mathrm{X}(G)$ determines $\mathrm{X}(G/K) = \mathrm{X}(H)$. But Sylow subgroups of odd order of H are cyclic. Thus H satisfies the hypothesis of Proposition 3.7 and the result follows.
(ii) Because G is a 2-Frobenius group it has a normal series

$$1 < M < T < G$$

such that T is a Frobenius group with kernel M and G/M is a Frobenius group with kernel T/M. By the structure theorems of Frobenius groups it follows that M is nilpotent and T/M is cyclic of odd order. Moreover because automorphism groups of cyclic groups are abelian we see that G/T has to be cyclic again.
Let G be a counterexample to (ii) of minimal order. If M is not a p-group then we get by Proposition 3.8 that $n_p(G)$ is determined by $\mathrm{X}(G)$ for each prime p which does not divide $|M|$. By Proposition 1.2 it follows that $n_p(G) = n_p(G/P)$, where P is a normal Sylow p-subgroup of M. Using 2.1(iv) we see that G cannot be a counterexample. Thus M is a p-group. By the previous for $q \neq p$ all Sylow q-subgroups of G are cyclic. But by Proposition 3.7 we see that G is also in this case not a counterexample. This completes the proof. □

4. Integral Group Rings

First we collect known results on properties of G determined by $\mathbb{Z}G$.

4.1. *Let G be a finite group.*

(i) *It is a result due to G. Glauberman that $\mathbb{Z}G \cong \mathbb{Z}H$ implicates that* $\mathrm{X}(G) = \mathrm{X}(H)$ [13, 3.17]*. Thus G and H share all properties given in 2.1.*
(ii) *If $\mathbb{Z}G \cong \mathbb{Z}H$ then there is a bijection $\sigma : G \to H$ such that the conjugacy classes of $\sigma(g)$ and g have the same length and representatives of the same order. The power map on the classes is determined.*
(iii) *Let $N \trianglelefteq G$ and $\mathbb{Z}G \cong \mathbb{Z}H$. Then exists $M \trianglelefteq H$ such that $\mathbb{Z}G/N \cong \mathbb{Z}H/M$. If $g \in N$ then $\sigma(g) \in M$, where σ is the bijection in (ii).*
(iv) **F*-Theorem**. *Assume that the generalized Fitting subgroup $F^*(G)$ is a p-group then $\mathbb{Z}G \cong \mathbb{Z}H$ implies that $G \cong H$.*

Remarks 4.2.

(i) Note that G is p-constrained provided $\bar{G} := G/O_{p'}(G)$ has a normal p-subgroup $\bar{P}$ such that $C_{\bar{G}}(\bar{P}) \subset \bar{P}$. If G is p-constrained then by the F^*-theorem $\mathbb{Z}G$ determines $G/O_{p'}(G)$. In the case that G is soluble the F^*-Theorem only indicates that $\mathbb{Z}G$ determines G provided $O_{p'}(G) = 1$.
(ii) The F^*-Theorem has been discovered by K. W. Roggenkamp and L. L. Scott, see [26, Theorem 19] and [27]. However they did not publish a complete proof. A proof of it may be collected via [10], [8] and [9].
(iii) If $\mathbb{Z}G \cong \mathbb{Z}H$ then G is p-constrained if, and only if, H is p-constrained.

Proposition 4.3. *Suppose that $O_p(G) \neq 1$ and that G is p-constrained. Then $\mathbb{Z}G$ determines $n_q(G)$ for each prime q where $n_q(G/O_p(G))$ is known.*

Proof. Because G is p-constrained the F^*-theorem shows that $\mathbb{Z}G$ determines $G/O_{p'}(G)$ up to isomorphism. If $O_{p'}(G) = 1$, then G is already determined. So assume that $O_{p'}(G) \neq 1$. Now apply Proposition 3.8 with $M = O_{p'}(G)$ and $N = O_p(G)$. □

Corollary 4.4. *Let G be finite soluble. Then $\mathbb{Z}G$ determines* $\mathrm{sn}(G)$.

Proof. The result holds when G is a p-group. Soluble groups are p-constrained for each prime p. By induction on the order we may assume that the integral group ring of each proper quotient of G determines the Sylow numbers of the quotient. Thus we may apply Proposition 4.3 for each prime q. □

Remarks 4.5. A different proof of Corollary 4.4 may be given using a recent result of G. Navarro and N. Rizo [25]. They showed that the spectral table of a p-soluble group determines $n_p(G)$ and $\mathrm{sn}(G)$ for soluble groups. Because $\mathbb{Z}G \cong \mathbb{Z}H$ implies that the finite groups G and H have the same spectral table (see 4.1) this establishes Corollary 4.4.

Combining these results we obtain for $\mathbb{Z}G$ an even stronger result:

Theorem 4.6. *Assume that G is a q-constrained group. Then $\mathbb{Z}G$ determines $n_p(G)$ for each p not dividing $|O_{q'}(G)|$. In particular $\mathbb{Z}G$ determines $n_q(G)$.*

Proof. Let $P \in \mathrm{Syl}_p(G)$. By Theorem 1.1 we obtain that

$$n_p(G) = n_p(O_{q'}(G))n_p(G/O_{q'}(G))n_p(\mathrm{N}_{PO_{q'}(G)}(P \cap O_{q'}(G))/(P \cap O_{q'}(G))).$$

As $p \nmid |O_{q'}(G)|$, we get $n_p(G) = n_p(G/O_{q'}(G))n_p(PO_{q'}(G))$. The F^*-theorem determines $G/O_{q'}(G)$ up to isomorphism. By [25, Theorem 1.1] we get that the spectral table (and thus $\mathbb{Z}G$) determines $|O_{q'}(G) \cap C_G(P)|$. Thus we get $n_p(G)$ from Proposition 1.3. □

Finally we consider special classes of insoluble groups.

Proposition 4.7. *$\mathbb{Z}G$ determines $n_2(G)$ provided G has abelian Sylow 2-subgroups.*

Proof. If G is a finite group with abelian Sylow 2-subgroup then G has a normal series

$$1 < M < N < G$$

such that $M = O_{2'}(G)$, G/N has odd order and N/M is a direct product of simple groups with abelian Sylow 2-subgroups and an abelian 2-group [29]. Again we consider a counterexample of minimal order.
Suppose that G has a minimal normal subgroup V which is not soluble. Clearly $n_2(G) = n_2(N)$. Then V is a normal subgroup of N with $\mathrm{C}_N(V) \cap V = 1$. Let $g \in N$. Then conjugation with g induces an inner automorphism of V.

Thus $N = \mathrm{C}_N(V) \cdot V$. It follows that N is a direct product of the form $V \times \mathrm{C}_N(V)$ and $n_2(N) = n_2(V) \times n_2(\mathrm{C}_N(V))$. Moreover $n_2(\mathrm{C}_N(V)) = n_2(G/V)$. Using 4.1 V is determined up to isomorphism by $\mathbb{Z}G$ and G/V is not a counterexample. Thus we may assume that a minimal counterexample does not have an insoluble minimal normal subgroup.

If $M = 1$ then N has to be a 2-subgroup and G has a normal Sylow 2-subgroup.

Assume that $M \neq 1$ and that G contains two minimal normal soluble subgroups of coprime order then by Proposition 3.8 we see that G is not a minimal counterexample.

Finally, if G has only minimal normal subgroups which are q-groups for some prime q then $\mathrm{soc}(G)$ must be contained in M if $M \neq 1$. Moreover the Fitting subgroup $F(G)$ is a q-group.

Consider the layer $E(G)$. The components of $E(G)$ involve simple groups with abelian Sylow 2-subgroups. All these simple groups do not have a Schur multiplier involving odd primes. So G has an insoluble minimal normal subgroup if $E(G) \neq 1$.

Thus $E(G) = 1$ and $F(G)$ coincides with $F^*(G)$. It follows that $\mathrm{C}_G(F(G)) \subset F(G)$. Beacuse $F(G)$ is a q-group we get by the F^*-Theorem 4.1 that $\mathbb{Z}G$ determines G up to isomorphism. Thus G is not a counterexample. □

Proposition 4.8. *Suppose that G has a nilpotent normal subgroup N such that G/N is simple. Then $\mathbb{Z}G$ determines* $\mathrm{sn}(G)$.

Proof. If G/N is simple abelian the result follows from Theorem 2.4. If G is non-abelian simple then $\mathrm{X}(G)$ determines G up to isomorphism, see the remarks after 2.1.
Let G be a counterexample of minimal order. If N is not a p-group then by Proposition 3.8 G cannot be a counterexample of minimal order. Thus we may assume that N is a p - group and that G/N is non-abelian simple. Suppose that $\mathrm{C}_G(N) \not\subset N$. Then the structure of G shows that $N \cdot \mathrm{C}_G(N) = G$. Thus $N \cap \mathrm{C}_G(N) = 1$ or $N \cap \mathrm{C}_G(N)$ is a non-trivial central subgroup of G. In the first case $n_q(G) = 1$ if q does not divide $|G/N|$ and $n_q(G) = n_q(G/N)$ if q divides $|G/N|$. In the second case we see by Proposition 1.2(iii) that $\mathrm{sn}(G) = \mathrm{sn}(G/(N \cap \mathrm{C}_G(N))$. Thus the case that $\mathrm{C}_G(N) \subset N$ remains. But then the F^*-theorem shows that $\mathbb{Z}G$ determines G even up to isomorphism. □

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