On hypercentral units in integral group rings

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Abstract. For an arbitrary group $G$, and a $G$-adapted ring $R$ (for example, $R = \mathbb{Z}$), let $\mathcal{U}$ be the group of units of the group ring $RG$, and let $Z_{\infty}(\mathcal{U})$ denote the union of the terms of the upper central series of $\mathcal{U}$, the elements of which are called hypercentral units. It is shown that $Z_{\infty}(\mathcal{U}) \leq N_{\text{unip}}(G)$. As a consequence, hypercentral units commute with all unipotent elements, and if $G$ has non-normal finite subgroups with $R(G)$ denoting their intersection, then $[\mathcal{U}, Z_{\infty}(\mathcal{U})] \leq R(G)$. Further consequences are given as well as concrete examples.

1 Introduction

The study of the normalizer of a group $G$ in the units of its integral group ring $\mathbb{Z}G$ seems to have been initiated by Jackowski and Marciniak [18] and, later on, by Mazur [35]. There is an apparently ‘small’ quotient of the normalizer, naturally isomorphic to a certain subgroup of $\text{Out}(G)$, which measures the extend to which there are ‘non-obvious’ units normalizing $G$. This quotient accounts for a connection to the so-called isomorphism problem for integral group rings; cf. [11], [14], [19], [34]. Here we shall consider the normalizer in the context of hypercentral units.

For a group $H$, let $Z_n(H)$ be the $n$th term of the upper central series (so that $Z_1(H) = Z(H)$ is the center of $H$), and set $Z_{\infty}(H) = \bigcup_{n=1}^{\infty} Z_n(H)$. Elements of $Z_{\infty}(H)$ will be called hypercentral units. (If the upper central series $1 \leq Z_1(H) \leq Z_2(H) \leq \cdots$ of $H$ terminates, then $Z_{\infty}(H)$ coincides with the hypercenter of $H$, which is the terminal member of the transfinitely extended upper central series of $H$. See, for example, [39, p. 365].)

We will show that the hypercentral units of an integral group ring are contained in the normalizer, an already widely studied object. This turns out to be the key step for further investigations. Then, additional properties of hypercentral units will be used to limit the structure of the group generated by them.

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The normalizer, or rather the normalizer property, has become a prime topic of study in recent years, and is discussed in Sections 2 and 3. These sections have preliminary character for the present work, and many of the given results are taken from [22] and (unpublished parts of) the first author’s Habilitationsschrift [14], or are generalizations of results contained therein.

For a commutative ring \( R \), we write \( \text{Aut}_R(G) \) be the group of automorphisms of \( G \) which induce an inner automorphism of \( RG \). The quotient \( \text{Out}_R(G) = \text{Aut}_R(G)/\text{Inn}(G) \)—changing ‘A’ to ‘O’ will always have this fixed meaning—has two natural interpretations:

- \( \text{Out}_R(G) \) is the kernel of the natural map \( \text{Out}(G) \to \text{Out}(RG) \);
- \( \text{Out}_R(G) \cong N_{\mathcal{U}}(G)/\mathcal{U}G \), where \( \mathcal{U} = U(RG) \) is the group of units in \( RG \).

In particular, \( \text{Out}_R(G) = 1 \) means that the units in \( RG \) normalizing \( G \) are the ‘obvious’ ones. We remark that \( \text{Aut}_R(G) \) is always contained in \( \text{Aut}_c(G) \), the group of class-preserving automorphisms of \( G \) (see Proposition 2.6).

The group \( G \) is said to have the normalizer property if

\[
\text{Out}_Z(G) = 1, \quad \text{or, equivalently,} \quad N_{U(ZG)}(G) = Z(U(ZG))G.
\]

Sometimes this definition is broadened to allow \( G \)-adapted rings as coefficient rings. Thereby, an integral domain \( R \) of characteristic zero is called \( G \)-adapted (with respect to the given group \( G \)), if a rational prime \( p \) is not invertible in \( R \) whenever \( G \) has an element of order \( p \). Unless otherwise stated explicitly, \( R \) will always denote a \( G \)-adapted ring, and we set \( \mathcal{U} = U(RG) \).

Groups which do not have the normalizer property were constructed in [11], [12], [40]. Some classes of groups which do have the normalizer property are described in [10], [12]–[15], [18], [21], [22], [24], [33], [35], [37], [38]. Initially, only finite groups \( G \) were considered, but now there is in some sense a general reduction to the finite group case [22], as reviewed in Section 2.

The hypercenter of the unit group has been recently studied by several authors; cf. [1], [2], [14], [29]–[32]. Before describing the present work, we briefly comment on this development.

For a finite group \( G \), Arora, Hales and Passi [1] studied the multiplicative Jordan decomposition for elements of \( \mathcal{U} = U(ZG) \). They noted that \( [Z_2(\mathcal{U})/Z(\mathcal{U})] \leq Z(G) \), and that this implies that \( Z_{\infty}(\mathcal{U})/Z(\mathcal{U}) \) is a periodic group since \( Z(G) \) is finite. Thus \( Z_{\infty}(\mathcal{U}) \) consists of semisimple elements, and an inductive argument shows that elements of \( Z_{\infty}(\mathcal{U}) \) commute with all unipotent elements. These observations, together with Bovdi’s results from [4], [5], led them to the conclusion that \( Z_{\infty}(\mathcal{U}) = Z_2(\mathcal{U}) \).

They also noted that the torsion elements of \( Z_{\infty}(\mathcal{U}) \) are contained in \( Z_2(G) \) since they form a periodic normal subgroup of \( \mathcal{U} \). Then, Arora and Passi [2] proved that \( Z_2(\mathcal{U}) = Z(\mathcal{U})T \), where \( T \) denotes the torsion subgroup of \( Z_2(\mathcal{U}) \). Their proof relies on Blackburn’s classification [3] of the finite groups in which the non-normal subgroups have non-trivial intersection. Moreover, they showed that \( G \) is a so-called \( Q^* \)-group provided that \( Z_2(\mathcal{U}) \neq Z(\mathcal{U}) \).
Let $G$ be a periodic group. Li [29] used a result of Krempa to show that $Z_2(\mathcal{U})/Z(\mathcal{U})$ has exponent 2, and consequently showed that the results from [1] carry over. That the result from [2] also carries over was established by Li and Parmenter [30]. However, they no longer use Blackburn’s classification [3]. Instead, they make use of certain ‘constructible’ units in integral group rings like Bass cyclic units. By contrast, Blackburn’s classification was still used in [14] to show that the (now definitive) results for periodic groups are still valid if one considers group rings of $G$ over $G$-adapted coefficient rings, but we would like to take the opportunity to mention that the proofs of [14, 22.4 Proposition, 23.8 Theorem] are incomplete, since Blackburn’s classification is misquoted (in particular, case (c) from [3, Theorem 2] is omitted in [14, 22.2 Theorem]). However, the gaps can be easily fixed along the given lines (see, e.g., [9]).

Li and Parmenter [31], [32] started to investigate the hypercentral units in $\mathcal{U}(\mathbb{Z}G)$ where $G$ is not necessarily torsion, concentrating on what happens if $Z_2(\mathcal{U})$ does not centralize all torsion elements of $G$.

Finally, we remark that for periodic $G$, the finite conjugacy center of $\mathcal{U}(\mathbb{Z}G)$ coincides with the second center of $\mathcal{U}(\mathbb{Z}G)$ (see [20], cf. [14, Chapter VII]).

Now we now briefly describe the contents of each section of this paper.

In Sections 2 and 3, we recall from [22] that the determination of the normalizer $N_{\mathcal{U}}(G)$ can essentially be reduced to the finite group case, which will be used to establish the normalizer property for various classes of infinite groups, extending known results (see the key Lemma 3.2, and Example 3.5, Corollary 3.6). Using the Ward–Coleman Lemma in a version appropriate for infinite groups (see Lemma 2.9), we generalize some results from [21], [22].

The main result from Section 4 is Proposition 4.1: each hypercentral unit of $\mathcal{U}$ lies in $N_{\mathcal{U}}(G)$ and commutes with all unipotent elements of $\mathcal{U}$, and $[\mathcal{U}, Z_\infty(\mathcal{U})] \leq G$. If $G$ has finite non-normal subgroups, then $R(G)$ is defined to be the intersection of all of these subgroups, and we shall call $G$ a Blackburn group if $R(G) \neq 1$. Existing results already indicate that Blackburn groups should play a decisive role in the determination of $Z_\infty(\mathcal{U})$. Our main result allows us to deduce that $[\mathcal{U}, Z_\infty(\mathcal{U})] \leq R(G)$ if $R(G)$ is defined (Corollary 4.7), again stressing the relevance of Blackburn groups in this context.

In Section 5, we investigate how hypercentral units can act on units of finite order. We prove that for $R = \mathbb{Z}$, a hypercentral unit of $\mathcal{U}$ normalizes each finite cyclic subgroup of $\mathcal{U}$ consisting of units of augmentation one, and acts trivially or by inversion (Lemma 5.5). Following the proof of [31, Theorem 3.5], we recover the definite result for a periodic group $G$: if $Z_\infty(G) \neq Z(\mathcal{U})$, then $G$ is a so-called Q*-group (Proposition 5.7).

In Section 6, we put some aspects of the action of a hypercentral unit in another context, namely that of units which do not satisfy an Engel condition.

In Section 7, examples are given where $Z_\infty(\mathcal{U})$ does not consist only of central units.

Finally, we remark that we do not know whether class-preserving automorphisms of infinite Blackburn groups are inner automorphisms. However, it is plausible that Blackburn groups in general have the normalizer property. If so, we would obtain $Z_\infty(\mathcal{U}) \leq Z(\mathcal{U})G$ for an arbitrary group $G$.
2 On the normalizer of the group

Recall that a $G$-adapted ring, for an arbitrary group $G$, is an integral domain of characteristic zero in which a rational prime $p$ is not invertible whenever $G$ has an element of order $p$.

It was shown in [7], [14], [20], [22], [23] that certain group ring problems can essentially be reduced to the finite group case, the most important being the determination of the normalizer $N_{U(RG)}(G)$ of $G$ in the group of units of $RG$ when $R$ is a $G$-adapted ring.

When studying $N_{U(RG)}(G)$, the first basic observation is that we can work in the group ring of the FC-center of $G$. Let us therefore recall the following definitions and elementary properties (see, for example, [36, Chapter 4, Section 1]). The set $D(G) = \{ g \in G \mid g^G \text{ is finite} \}$ is a characteristic subgroup of $G$, called the FC-center of $G$. If $G = D(G)$, then $G$ is said to be a finite conjugate group. The set $D^+(G)$ of torsion elements in $D(G)$ is a characteristic subgroup of $D(G)$. If $D(G)$ is finitely generated, then its center is of finite index in $D(G)$, and $D^+(G)$ is a finite group, with $D(G)/D^+(G)$ finitely generated torsion-free abelian.

For $u \in RG$, let $\text{supp}(u)$ denote the support of $u$, that is, if we write $u = \sum_{g \in G} r_g g$ (with all $r_g$ in $R$), then $\text{supp}(u)$ is the set $\{ g \in G \mid r_g \neq 0 \}$. The first observation (cf. [35, Corollary 1], [22, Lemma 1.1], and [14, 17.2 Lemma]) is as follows:

**Lemma 2.1.** Let $H \leq G$ and $u \in N_{U(RG)}(H)$. Then $H$ acts on $\text{supp}(u)$ via

$$x \mapsto h^{-1}xh^u \quad \text{for } x \in \text{supp}(u) \text{ and } h \in H,$$

and elements of an orbit under this operation have the same coefficient in $u$ (viewed as a linear combination of elements of $G$). Thus if $1 \in \text{supp}(u)$, then \(\{ h^{-1}h^u \mid h \in H \} \subseteq \text{supp}(u)\) and elements of $\text{supp}(u)$ have only finitely many $H$-conjugates.

Moreover, $\langle h^{-1}h^u \mid h \in H \rangle$ and $\langle \text{supp}(u) \rangle$ are $H$-invariant subgroups of $G$.

In particular, if $u \in N_{U(RG)}(G)$ and $1 \in \text{supp}(u)$, then $\langle \text{supp}(u) \rangle$ is a normal subgroup of $G$ contained in $\Delta(G)$.

The last statement forms the starting point for the proof of the results [22, Theorem 1.4] and [20, Theorem 1.4], here stated in a slightly more general form.

**Theorem 2.2.** Let $R$ be a $\Delta^+(G)$-adapted ring. Then for any $u \in N_{U(RG)}(G)$ with $1 \in \text{supp}(u)$, the support group $\langle \text{supp}(u) \rangle$ is a finite normal subgroup of $G$.

**Theorem 2.3.** Let $R$ be a $\Delta^+(G)$-adapted ring. Then for any $u \in \Delta(U(RG))$ with $1 \in \text{supp}(u)$, the support group $\langle \text{supp}(u) \rangle$ is a finite normal subgroup of $G$.

Our next result shows that the normal closure of the support group of a hypercentral unit also has a special structure.
Proposition 2.4. Let $G$ be a finitely generated group. Then for any $u \in \mathbb{Z}_n(\mathcal{U})$ the normal subgroup $\langle \text{supp}(u) \rangle^G$ is polycyclic-by-finite.

Proof. Let $u \in \mathbb{Z}_n(\mathcal{U})$ and let $S = \langle \text{supp}(u) \rangle^G$. We prove the statement by induction on $n$. If $n = 1$ then $u \in \mathbb{Z}_1(\mathcal{U})$ implies easily that $S$ is a finitely generated finite conjugacy group, and thus $S$ is polycyclic-by-finite (see [36, Chapter 4, Lemma 1.5]).

Suppose now that the result holds for $n$. Let $u \in \mathbb{Z}_{n+1}(\mathcal{U})$. Let $\{g_1, \ldots, g_m\}$ be a set of generators of $G$ that is closed under taking inverses and contains 1. For every $i$, write $g_i^{-1}ug_i = w_i$ with $w_i \in \mathbb{Z}_n(\mathcal{U})$. Put $H_i = \langle \text{supp}(w_i) \rangle^G$. It follows that

$$g_i^{-k}ug_i^k = uw_iw_i^g \ldots w_i^{g_{i-1}^{-1}} \quad \text{and} \quad w_iw_i^g \ldots w_i^{g_{i-1}^{-1}} \in RH_i.$$

Since $w_i \in \mathbb{Z}_n(\mathcal{U})$, the induction hypothesis yields that $H_i$ is polycyclic-by-finite. Hence $H = \langle H_i \mid 1 \leq i \leq m \rangle$ is a normal subgroup in $G$ that is polycyclic-by-finite. Choose $H_0 \subseteq H$ so that $H_0$ is an invariant subgroup, and hence is normal in $G$ (and poly-infinite cyclic) of finite index in $H$. Furthermore we also can take $H_0$ so that the natural homomorphism $G \to G/H_0$ is injective when restricted to the set $\text{supp}(u)$ (see [25, Corollary 2.3]). Let $G = G/H_0$ and $\overline{\text{supp}} = H/H_0$. Clearly $\overline{\text{supp}}$ is finite. Hence

$$\overline{\text{supp}}(u) = \overline{\text{supp}}(g_i^{k_i}u_i^{g_i}) = \overline{\text{supp}}(w_iw_i^g \ldots w_i^{g_{i-1}^{-1}}) \quad \text{and} \quad \overline{\text{supp}}(g_i^{-k}u_i^g) \subseteq \overline{\text{supp}}(u).$$

If $g \in G$ then we have $g = g_i^{k_i}g_{i+1}^{k_{i+1}} \ldots g_m^{k_m}$, with $1 \leq i \leq m$ and $k_i \geq 1$ for all $i$. Thus $\overline{\text{supp}}(g) \subseteq \overline{\text{supp}}(u)$ $\overline{\text{supp}}$. As $\overline{\text{supp}}$ is finite it follows that $\overline{\text{supp}}(u) \subseteq \Delta(G)$ and thus $\langle \text{supp}(u) \rangle^G \subseteq \Delta(G)$. Now by construction of $H_0$ we get that

$$\langle \text{supp}(u) \rangle^G = \overline{\text{supp}}(u).$$

Since $H_0$ is polycyclic-by-finite it follows that $S$ is polycyclic-by-finite, as desired. \hfill \Box

The next corollary (of Theorem 2.2) is already known in the case when $R = \mathbb{Z}$ (see [8], [21, Theorem 2.11], [35, Lemma 8] and Proposition 3.8 below), and so we omit the proof. It shows in particular that central units of finite order in $RG$ are trivial (cf. [27, Theorem 2.13]).

Corollary 2.5. Let $R$ be a $\Delta^+(G)$-adapted ring, and let $u$ a unit of augmentation one in $N_{U(RG)}(G)$ such that $u^k \in G$ for some $k \in \mathbb{N}$. Then $u \in G$.

The next result was proved for $G$-adapted coefficient rings $R$ in [22, Remark after Theorem 1.4], thus answering a question posed by Mazur in [35].

For $g \in G$, the partial augmentation $\varepsilon_{[g]}$ is the $R$-linear map $\varepsilon_{[g]} : RG \to R$ such that if $h \in G$ is conjugate to $g$ then $\varepsilon_{[g]}(h) = 1$, and $\varepsilon_{[g]}(h) = 0$ otherwise. Note that $\varepsilon_{[g]}(xy) = \varepsilon_{[g]}(x^{-1}xy)x = \varepsilon_{[g]}(yx)$ for all $x, y \in G$, and therefore $\varepsilon_{[g]}(ab) = \varepsilon_{[g]}(ba)$ for all $a, b \in RG$ by linearity.
Proposition 2.6. Let $G$ be a group and $R$ a commutative ring. Then $\text{Aut}_R(G) \subseteq \text{Aut}_C(G)$.

Proof. Let $\sigma \in \text{Aut}_R(G)$, and take any $g \in G$. There is $u \in N_{U(RG)}(G)$ such that $g\sigma = g^u$, and it follows that $e_{[g]}(g\sigma) = e_{[g]}(u^{-1}gu) = e_{[g]}(g) = 1$. As $g\sigma \in G$, this shows that $g\sigma$ is conjugate to $g$ within $G$, and the result follows. \qed

Next we prove the following key lemma.

Corollary 2.7. Let $R$ be a $\Delta^+(G)$-adapted ring, let $u \in \mathcal{U} = U(RG)$, $g \in G$ and suppose that $g^u \in N_\psi(G)$. Then $g^u \in G$.

Proof. By Theorem 2.2, there is a finite normal subgroup $N$ of $G$, and $h \in G$, such that $g^u \in (RN)h$. For $m \in RG$, let $m$ denote its image under the natural map $RG \to RG/N$. Then $g^u = h$. As in the proof of Proposition 2.6, it follows that $g$ and $h$ are conjugate in $G/N$. So $h \in Ng^x$ for some $x \in G$. It follows $g^{ux^{-1}} \in (RN)_g$, and certainly $g^{ux^{-1}} \in N_\psi(G)$. Since we have to show that $g^{ux^{-1}} \in G$, we can assume that $h = g$. Then $g^u \in (RN)_g$. Equivalently, $u^{\sigma^{-1}} \in U(RN)$. So, if we set $S = \text{supp}(u)$ then $S^{\sigma^{-1}} \subseteq SN$. It follows that $(SN)^{\sigma^{-1}} = SN$, and as $SN$ is a finite set, there is $k \in \mathbb{N}$ such that $g^k$ centralizes $SN$. Hence $[g^k, u] = 1$ and $(g^u)^k = (g^k)^u = g^k \in G$, and so $g^u \in G$, by Corollary 2.5. \qed

Recall that $\text{Out}_R(G) \cong N_\psi(G)/Z(\mathcal{U})G$, where $\mathcal{U} = U(RG)$. For another proof of the following corollary, see [35, Theorem 1].

Corollary 2.8. If $R$ is a $\Delta^+(G)$-adapted ring, then $\text{Out}_R(G)$ is a periodic group, the order of any element being divisible only by primes which also divide the order of an element of $\Delta^+(G)$.

Proof. Let $u \in N_{U(RG)}(G)$ with $1 \in \text{supp}(u)$. By Theorem 2.2, $N = \langle \text{supp}(u) \rangle$ is a finite normal subgroup of $G$. Thus conjugation by $u$ is an automorphism $\sigma$ of $G$ of finite order, and it follows that $\text{Out}_R(G)$ is a periodic group. Moreover, $\sigma$ induces a class-preserving automorphism $\sigma|_N$ of $N$ (see Proposition 2.6). Therefore primes which divide the order of $\sigma|_N$ also divide the order of $N$ (see [16, Kapitel I, §4, Aufgabe 14]). It easily follows that primes which divide the order of $\sigma$ also divide the order of $N$, and the proof is complete. \qed

We remark that the first statement remains true for an arbitrary commutative coefficient ring $R$, but not the second one (see [14, Corollary 17.9] and [14, Example 17.10]).

The Ward–Coleman Lemma states that for a finite group $G$ with a $p$-subgroup $P$, and any commutative ring $R$ with $pR \neq R$, we have

$$N_{U(RG)}(P) = N_G(P)C_{U(RG)}(P).$$

Coleman’s contribution [6] is well known, but the first version of the lemma appears in an article of Ward [43] as a contribution to a seminar run by Richard Brauer at
Harvard. In its present form, the lemma appears first in [41, Proposition 1.14]; see also [18, Theorem 2.6].

We shall need a version of the Ward–Coleman Lemma for infinite groups. The idea behind the proof is the same in both cases: for $P \leq G$ and $u \in N_{U(RG)}(P)$, the group $P$ acts on $\text{supp}(u)$ with elements of an orbit having the same coefficient in $u$ (see Lemma 2.1). If $G$ is finite, the consequences resulting from the action of $p$-subgroups $P$ of $G$ are important, whereas if $G$ is infinite, we shall consider the action of subgroups $P$ which act as $p$-groups.

Lemma 2.9. Let $R$ be a commutative ring with $pR \neq R$ for some rational prime $p$, and let $H \leq G$. Then for any $u \in N_{U(RG)}(H)$ there is $x \in \text{supp}(u)$ such that $ux^{-1}$ centralizes a subgroup of $H$ which is of finite $p^i$-index in $H$.

Proof. Let $u \in N_{U(RG)}(H)$, and let $H$ act as described above on $\text{supp}(u)$. Let $Q$ be the kernel of this operation (which is of finite index in $H$), and choose $Q \lhd P \leq H$ such that $P/Q$ is a Sylow $p$-subgroup of $H/Q$. Since the augmentation of $u$ is a unit in $R$, there is a fixed point $x \in \text{supp}(u)$ under the operation of $P$, that is, $ux^{-1}$ centralizes $P$.

We will later encounter unipotent elements in the context of the normalizer and hypercentral units. Theorem 2.2 can be used to show the following:

Lemma 2.10. Let $R$ be a $\Delta^+(G)$-adapted ring. Then $N_{U(RG)}(G)$ contains no non-trivial unipotent elements.

Proof. Let $u$ be a unipotent element in $N_{U(RG)}(G)$. Write $u = 1 + \theta$, so that $\theta$ is nilpotent. Since nilpotent elements in $RG$ have trace zero, i.e., have vanishing $1$-coefficient (see [36, Chapter 2, Lemma 3.3]), we have $1 \in \text{supp}(u)$. Hence $H = \langle \text{supp}(u) \rangle$ is a finite normal subgroup of $G$ by Theorem 2.2. Since $H$ is finite and $u \in N_{U(RG)}(G)$, there exists a positive integer $l$ such that $u^l \in Z(RH)$. We have $u^l = 1 + f(\theta)$ where $f$ is a polynomial with vanishing constant term. So $f(\theta)$ is a nilpotent central element in $RH$, which implies that it is zero. Consequently $u^l = 1$, and $u \in G$ by Corollary 2.5. As $1 \in \text{supp}(u)$, it follows that $u = 1$.

As will be seen later, $Z_{\infty}(U(RG)) \leq N_{U(RG)}(G)$, this gives an alternative proof of [32, Theorem 2.1], which states that $Z_{\infty}(\mathcal{U})$ contains no unipotent units.

Since it is known that $\Delta(U(ZH)) = Z_2(U(ZH)) \leq N_{U(ZH)}(H)$ for a finite group $H$ (see [20], and [14, Chapter VII]), using Theorem 2.3 the proof also shows that $\Delta(U(ZG))$ does not contain non-trivial unipotent elements.

3 Groups having the normalizer property

In [22], Theorem 2.2 was obtained and it was shown how it can be used to establish that various classes of groups have the normalizer property, extending (known) results for finite groups. In this section, we shall expand on this a little more.
First, however, we point out that to show that a group $G$ has the normalizer property, it suffices to prove the stronger statement that class-preserving automorphisms of $G$ are inner automorphisms (see Proposition 2.6). This is often an easier task. For example, the following observation implies a result of Li, Sehgal and Parmenter [33, Theorem 2], that they used to show by inspection that the finite Blackburn groups have the normalizer property (for the structure of these groups, see the discussion preceding Proposition 4.5).

Example 3.1. If the group $G$ has an abelian normal subgroup of index 2 then $\Out_c(G) = 1$.

Proof. Let $G = \langle A, g \rangle$, where $A$ is an abelian normal subgroup of index 2 in $G$ and $g \in G$, and take any $\sigma \in \Aut_c(G)$. We have to show that $\sigma \in \Inn(G)$, and we may assume without loss of generality that $g\sigma = g$. Note that for all $a \in A$, either $a\sigma = a$ or $a\sigma = a^g$. Clearly, we may assume that $\sigma$ is not the identity, so that we can choose $x \in A$ with $x \neq x\sigma = x^g$. Assume that there is $a \in A$ with $a\sigma \neq a^g$. Then $a\sigma = a$ and $x^g a = x^\sigma (a\sigma) = (xa)\sigma$, so that $x^g a$ is equal to $xa$ or $(xa)^g$. However, the first possibility contradicts $x^g \neq x$ and the second contradicts $a \neq a^g$. Hence $a\sigma = a^g$ for all $a \in A$, and $\sigma$ is conjugation by $g$. \qed

Our results will be achieved with the help of the following technical lemma, which is proved using the above version of the Ward–Coleman Lemma.

Lemma 3.2. Suppose that

(i) $R$ is a commutative ring with $pR \neq R$, for some rational prime $p$;
(ii) $N$ is a finite normal subgroup of a group $G$ such that the center of a Sylow $p$-subgroup of $N$ is contained in $\text{O}_p(N)$;
(iii) $u \in N_{Z(U(RN))}(G)$ is such that $\sigma = \text{conj}(u) \in \Aut_R(G)$ is of $p$-power order.

Then there is $g \in Z(\text{O}_p(N))$ such that $ug \in Z(RG)$.

Proof. Clearly $[G, u] \leq N$, so that $[[G, u], u] = 1$ and $[G, u]^p = [G, u^p] = 1$. Moreover, $[[G, u], N] = 1$ by the Three Subgroup Lemma. In particular, it follows that $[G, u] \leq A = Z(\text{O}_p(N))$. By Lemma 2.9, there is a subgroup $P$ of $G$ of finite $p'$-index, and $x \in \text{supp}(u)$, such that $[P, ux^{-1}] = 1$. Since $u$ is of $p$-power order over the center, and $[u, x] = 1$, there is $y \in \langle x \rangle$ of $p$-power order with $[P, uy^{-1}] = 1$. Let $S$ be a Sylow $p$-subgroup of $N$. Then there is a fixed point under the multiplication action of $S$ on the set of left cosets of $P$ in $G$, say $gP$, and it follows that $[S^y, y^{-1}] = [S^y, uy^{-1}] = [P, uy^{-1}] = 1$. Hence $y \in A$ by the hypothesis on $N$. Now $\sigma = \text{conj}(uy^{-1})$ is an automorphism of $G$, which is of $p$-power order, induces the identity on both $A$ and $G/A$, and fixes $P$ element-wise. Using restriction-corestriction in 1-cohomology (see [16, (1 16.18)]), it follows that $\sigma = \text{conj}(a)$ for some $a \in A$. (Explicitly, $a = \prod_{i=1}^n g_i^{-1}(g_i^a)g_i$, where $g_1, \ldots, g_n$ is a system of right coset representatives of $P$ in $G$, and $m \in \mathbb{N}$ is such that $nm \equiv 1 \mod |A|$.) The proof is complete. \qed
Jackowski and Marciniak [18, Theorem 3.6] proved that $\text{Out}_\mathbb{Z}(G) = 1$ for a finite group $G$ with a normal Sylow 2-subgroup (in [15], it is shown that this is a special case of a more general result). This result was generalized to arbitrary periodic groups in [22, Theorem 2]. To state an even stronger generalization (see Example 3.5) for arbitrary groups we shall need the following lemma.

**Lemma 3.3.** Let $G$ be a finite group which has a normal Sylow $p$-subgroup, let $R$ be a commutative ring with $pR \neq R$, and let $\sigma \in \text{Aut}_R(G)$. If $\sigma$ is of $p$-power order, then $\sigma$ is an inner automorphism, given by conjugation by some element from $\text{Out}_R(G)$.

**Proof.** Suppose that $\sigma \in \text{Aut}_R(G)$ is of $p$-power order, and set $N = \text{O}_p(G)$. As $G/N$ is a $p'$-group, $\sigma$ induces the identity on $G/N$ (see [16, Kapitel 1, §4, Aufgabe 14]). By the Ward–Coleman Lemma, there is $g \in G$ such that $\sigma|_N = \text{conj}(g)|_N$, and clearly $g$ can be chosen to be a $p$-element. Then $g \in N$, so that $\text{conj}(g^{-1})\sigma$ fixes $G/N$ and $N$ element-wise, and is therefore an inner automorphism, given by conjugation by an element from $Z(N)$. □

**Corollary 3.4.** Let $p$ be a rational prime, and let $G$ be a group whose finite normal subgroups have a normal Sylow $p$-subgroup. Let $R$ be a $\Delta^+(G)$-adapted ring. Then $\text{Out}_R(G)$ has no $p$-torsion.

**Proof.** Let $u \in \text{N}_{U(RG)}(G)$ and set $\sigma = \text{conj}(u) \in \text{Aut}_R(G)$. By way of contradiction, assume that $\sigma \notin \text{Inn}(G)$, but that the image of $\sigma$ in $\text{Out}_R(G)$ has $p$-power order. Take $g \in \text{supp}(u)$; then $N = \langle \text{supp}(ug^{-1}) \rangle$ is a finite normal subgroup of $G$ by Theorem 2.2. Consequently, $\tau = \text{conj}(ug^{-1}) \in \text{Aut}(G)$ is of finite order, and there is $n \in N$, not divisible by $p$, such that $\tau^n$ has $p$-power order. By Corollary 2.8, $p$ divides the order of an element of $\Delta^+(G)$, and so Lemma 3.3 can be applied to give $h \in O_p(N)$ with $v = (ug^{-1})^n h \in Z(RN)$, and $\text{conj}(v) \in \text{Aut}(G)$ is of $p$-power order. Hence $\sigma^n \in \text{Inn}(G)$ by Lemma 3.2, a contradiction. □

By a result of J. Krempa (see [18, Theorem 3.2]), the group $\text{Out}_\mathbb{Z}(G)$ is, if non-trivial, of exponent 2, and Corollary 3.4 implies that the Jackowski–Marciniak result extends to infinite groups:

**Example 3.5.** Let $G$ be a group whose finite normal subgroups have a normal Sylow 2-subgroup. Then $\text{Out}_\mathbb{Z}(G) = 1$, i.e., $G$ has the normalizer property.

Other classes of groups to which Lemma 3.2 applies include the class of locally nilpotent groups; cf. [22].

**Corollary 3.6.** Let $R$ be a $\Delta^+(G)$-adapted ring, and let $u \in \text{N}_{U(RG)}(G)$ with $1 \in \text{supp}(u)$ be such that $N = \langle \text{supp}(u) \rangle$ is nilpotent. Then there is $g \in N$ such that $ug \in Z(RG)$.

**Proof.** By Theorem 2.2, $N = \langle \text{supp}(u) \rangle$ is a finite normal subgroup of $G$, and by the Ward–Coleman Lemma, there is $x \in N$ such that $v = ux \in Z(RN)$. The auto-
morphism \( \sigma = \text{conj}(v) \in \text{Aut}(G) \) is of finite order. For some prime \( p \) dividing the order of \( \sigma \), let \( n \in \mathbb{N} \) be such that \( \sigma^n \) is of \( p \)-power order. Obviously, it suffices to show that there is \( h \in \mathbb{N} \) such that \( v^nh \in \mathbb{Z}(RG) \). But this follows from Lemma 3.2 since \( p \) divides the order of \( N \).

**Example 3.7.** Let \( G \) be a group whose finite normal subgroups are nilpotent. Then \( \text{Out}_R(G) = 1 \) for any \( \Delta^\times(G) \)-adapted ring \( R \).

It was noted in [22] that Theorem 2.2 can be used to reduce the proof of a result of Farkas and Linnell [8] to the (known) finite group case. Similarly, one can prove the following slightly stronger result:

**Proposition 3.8.** Let \( U \) be a subgroup of \( \mathcal{U} \) of augmentation one units which contains a subgroup \( B \) of \( G \) which is of finite index in both \( U \) and \( G \). Then for all \( u \in U \setminus \{1\} \), we have \( 1 \neq \text{supp}(u) \).

**Proof.** Replacing \( B \) by \( \bigcap_{u \in U} B^u \), if necessary, we can assume in addition that \( B \) is normal in \( U \).

Let \( u \in U \) with \( 1 \in \text{supp}(u) \) and set \( S = \langle \text{supp}(u) \rangle \). As \( |U : B| < \infty \), there is \( l \in \mathbb{N} \) such that \( u^l \in B \). By Lemma 2.1, \( S \) is an \( B \)-invariant subgroup of \( \Delta(G) \), so that the normal subgroup \( H = S^G = \langle S^g \mid g \in G \rangle \) of \( G \) is a finitely generated FC-group. By [25, Corollary 2.3], we may choose a characteristic subgroup \( K \) of \( H \) of finite index such that the homomorphism \( \pi : RG \to RG = RG/K \) is injective on the set \( \text{supp}(u) \). Now \( \tilde{u} \in RH \) where \( H \) is a finite subgroup of \( \tilde{G} \), and \( \tilde{u}^l \in B \leq \tilde{G} \). Hence \( \tilde{u} \) is of finite order. Further \( 1 \in \text{supp}(\tilde{n}) \) as \( \pi \) is injective on \( \text{supp}(u) \). It is well known that this implies \( \tilde{u} = 1 \) (see, for example, [26, Theorem 2.1]), and so \( u = 1 \) by injectivity of \( \pi \) on \( \text{supp}(u) \). \( \square \)

### 4 The hypercenter normalizes the group

Throughout this section, \( R \) always denotes a \( G \)-adapted ring, for a given arbitrary group \( G \), and we set \( \mathcal{U} = U(RG) \). We show that a hypercentral unit of \( \mathcal{U} \) lies in \( N_{\mathcal{U}}(G) \) and commutes with all unipotent elements of \( \mathcal{U} \). In the case when \( R = \mathbb{Z} \), the commutation result was first proved by Li and Parmenter [32, Lemma 2.2] (we proceed differently), and for \( G \) torsion, Li [29, Lemma 1] proved the containment of the hypercenter in the normalizer.

**Proposition 4.1.** The following hold:

1. \( Z_{\infty}(\mathcal{U}) \subseteq N_{\mathcal{U}}(G) \);
2. \( [\mathcal{U}, Z_{n+1}(\mathcal{U})] \subseteq Z_n(G) \) for each \( n \in \mathbb{N} \);
3. each element of \( Z_{\infty}(\mathcal{U}) \) commutes with all unipotent element of \( \mathcal{U} \).

**Proof.** As to (1), we shall prove inductively that \( Z_n(\mathcal{U}) \subseteq N_{\mathcal{U}}(G) \) for all \( n \in \mathbb{N} \). The case \( n = 1 \) being trivial, let \( n > 1 \), and take any \( u \in Z_n(\mathcal{U}) \) and \( g \in G \). Then
Suppose thus that \( n \geq 1 \) and \( \mathcal{Z}_n(\mathcal{U}) \subseteq \mathcal{Z}_{n-1}(G) \). Let \( u \in \mathcal{Z}_{n+1}(\mathcal{U}) \) and \( v \in \mathcal{U} \). Using the identity \([x, yz] = [x, z][x, y][xy, z]\) we get inductively that \([v, u^k] = [v, u]^k h_k\) with \( h_k \in \mathcal{Z}_n(G) \cap \mathcal{Z}_n(\mathcal{U})\). Because of Proposition 2.4 the subgroup

\[
\mathcal{H} := \langle \text{supp}(u), \text{supp}(v) \rangle^G
\]

is polycyclic-by-finite and so we may choose \( H_0 \triangleleft G, H_0 \subseteq H \) and of finite index in \( H \) such that the natural homomorphism \( \pi : RG \to RG/H_0 \) is injective when restricted to \( \text{supp}(u) \cup \text{supp}(u) \) (see [25, Corollary 2.3]). So \( \pi \in U(RG) \) and \( H \) is finite. Because of Proposition 2.4, \( \pi \in \mathbb{N}_G(G) \). By (1) and Corollary 2.8, we can choose \( k_0 \in \mathbb{N} \) such that \( \pi^{k_0} \in U(RG) \) (see also the proof of [22, Theorem 1.4]). From the above we get that \([v, u^{k_0}] = [v, u]^{k_0} h_{k_0}\) for some \( h_{k_0} \in \mathcal{Z}_n(G) \cap \mathcal{Z}_n(\mathcal{U}) \), and thus

\[
1 = \pi([v, u^{k_0}]) = \pi([v, u]^{k_0}) \pi(h_{k_0}).
\]

Therefore \( \pi([v, u]^{k_0}) = \pi(h_{k_0}^{-1}) \in \mathcal{G} \). Because of Proposition 2.4 we have \([v, u] \in \mathbb{N}_G(G)\), and thus \( \pi([v, u]) \in \mathbb{N}_G(G) \). By Corollary 2.5 it follows that \( \pi([v, u]) \in G \). The injectiveness of \( \pi \) yields then that \([v, u] \in G \). Now, for any \( g \in G \),

\[
[[v, u], g] \in \mathcal{Z}_{n-1}(\mathcal{U}) \cap G \subseteq \mathcal{Z}_{n-1}(G).
\]

Hence \([v, u] \in \mathcal{Z}_n(G)\) and therefore \([\mathcal{Z}_{n+1}(\mathcal{U}), \mathcal{U}] \subseteq \mathcal{Z}_n(G)\), as desired.

As to (3), let \( x \in \mathcal{U} \) be a unipotent element and set \( \theta = x - 1 \), so that \( \theta^n = 0 \) for some \( n \in \mathbb{N} \). We prove inductively that \([x, \mathcal{Z}_n(\mathcal{U})] = 1\), the case \( n = 1 \) being trivial. Let \( n > 1 \), and take \( u \in \mathcal{Z}_n(\mathcal{U}) \). Then

\[
1 + \theta^n = x^n = x v = v + \theta v
\]

with \( v = [x, u] \in \mathcal{Z}_{n-1}(\mathcal{U})\), and we may assume inductively that \([x, v] = 1\). Thus \( v, \theta \) and \( \theta^n \) are pairwise commuting elements. Note that \( v \in G \) by (2). We now invoke the fact that nilpotent elements have trace zero, i.e., have vanishing 1-coefficient (see [36, Chapter 2, Lemma 3.3]). Then, as \( \theta v \) is nilpotent, taking traces on both sides of (4.1) yields \( v = 1 \), so that \( x \) commutes with \( u \). \( \square \)

An alternative proof of (1), using Proposition 2.4, goes as follows. The case \( n = 1 \) is obvious and so we suppose that it holds for \( n - 1 \) and that \( G \) is finitely generated. For \( g \in G, u \in \mathcal{Z}_n(\mathcal{U}) \) we put \( G_0 = \langle \langle \text{supp}(u) \rangle^G, g \rangle \). By Proposition 2.4, the group \( G_0 \) is polycyclic-by-finite and so we may choose an invariant finite index subgroup \( H \) of \( G_0 \) for which the natural map \( \pi : G \to \mathcal{G} = G/H \) is injective when restricted to the set \( \text{supp}(u^{-1} g u) \) (see [25, Corollary 2.3]). As \( u^{-1} g u = g [g, u] \) and \([g, u] \in \mathcal{Z}_{n-1}(\mathcal{U})\), the induction hypothesis implies that \([g, u] \in \mathbb{N}_G(G)\). Clearly \( \pi(u^{-1} g u) \in \mathbb{N}_G(G)\).
As \( \pi(u^{-1}gu) \) is of finite order it follows, by Corollary 2.5, that \( \pi(u^{-1}gu) \in G \). Hence \( |\text{supp}(u^{-1}gu)| = |\text{supp}(\pi(u^{-1}gu))| = 1 \). So \( u^{-1}gu \in G \).

**Corollary 4.2.** We have \( G \cap Z\pi(\mathcal{U}) \leq \mathcal{U} \).

*Proof.* Let \( H = G \cap Z\pi(\mathcal{U}) \). By Proposition 4.1(2), \( [\mathcal{U}, H] \leq G \cap Z\pi(\mathcal{U}) = H \), and so \( H \leq \mathcal{U} \). \( \square \)

It is a long-standing question whether \( \mathcal{U} = R^* \times G \) for \( G \) torsion-free (here \( R^* \) denotes the group of units of \( R \)). By [21] (Theorem 2.2) and [35, Corollary 7], we know that \( N_\mathcal{U}(G) = R^* \times G \), and so Proposition 4.1(1) gives the following:

**Corollary 4.3.** Suppose that \( G \) is torsion-free. Then \( Z\pi(\mathcal{U}) \leq R^* \times Z\pi(G) \).

We are led to consider units in \( N_\mathcal{U}(G) \) which commute with all unipotent elements of \( \mathcal{U} \), and Blackburn groups come into play. We recall some definitions. A group \( G \) is called a *Dedekind group* if each subgroup is normal in \( G \). Such a group is abelian or the direct product of the quaternion group \( Q_8 \) of order 8, an elementary abelian 2-group \( E_2 \) and an abelian group with all elements of odd order (see [39, (5.3.7)]). A non-abelian Dedekind group is called a *Hamiltonian*. If \( G \) has finite non-normal subgroups, then, following Blackburn [3], we denote by \( R(G) \) the intersection of all finite non-normal subgroups of \( G \) (sometimes saying that \( R(G) \) is ‘defined’) and we call \( G \) a *Blackburn group* if \( R(G) \neq 1 \). We recall the following definition:

**Definition 4.4.** A group \( G \) is called a *\( Q \)-group* if \( G \) has an abelian subgroup \( A \) of index 2 which is not elementarily abelian, and \( G = \langle A, b \rangle \) for some \( b \in G \) of order 4 with \( x^b = x^{-1} \) for all \( x \in A \). If in addition there is \( a \in A \) with \( a^2 = b^2 \), then \( G \) is said to be a *\( Q^* \)-group*.

The condition \( R(G) \neq 1 \) severely restricts the structure of \( G \). The finite groups \( G \) which are not Dedekind groups and satisfy \( R(G) \neq 1 \) were classified by Blackburn [3]: a finite \( p \)-group \( G \) which is not a Dedekind group and satisfies \( R(G) \neq 1 \) is a 2-group, and the 2-groups \( G \) which are not Dedekind groups and satisfy \( R(G) \neq 1 \) are \( Q \)-groups, or of the form \( Q_8 \times C_4 \times E_2 \) or \( Q_8 \times Q_8 \times E_2 \). A finite Blackburn group which is not a \( p \)-group belongs to one of five classes listed in [3, Theorem 2].

For an element \( g \) of \( G \) of finite order, we shall write \( g \) for the sum of the elements of the cyclic group \( \langle g \rangle \), and \( H \), for a finite subgroup \( H \) of \( G \), will denote the sum of the elements of \( H \).

A scrutiny of what is needed for and what is done in the proof of [2, Proposition 2.1] leads to the following proposition.

**Proposition 4.5.** Suppose that \( G \) has a finite non-normal cyclic subgroup, and assume that some \( u \in N_\mathcal{U}(G) \) commutes with all unipotent elements of \( ZG \). Then \( [G, u] \leq R(G) \).

*Proof.* Let \( C = \langle c \rangle \) be a non-normal cyclic subgroup of \( G \), and take any \( g \in G \). We show that \( [g, u] \in C \), by considering the following two possibilities.
Case 1. Suppose that \(c^y \notin C\). Clearly \(y = (1 - c)g \hat{e} = g \hat{e} - cg \hat{e}\) is a nilpotent element. If \(ge^m = cg^e^m\) for some integers \(n\) and \(m\), then \(c^y = e^{n-m}\), a contradiction. Hence \(\text{supp}(g \hat{e}) \cap \text{supp}(cg \hat{e}) = \emptyset\). Therefore \(g^a\) appears with coefficient 1 in \(y^a = y\), so that \(g^a = gc^a\) for some \(n \in \mathbb{N}\), and \([g, u] = e^a \in C\).

Case 2. Suppose that \(c^y \in C\). Choose \(h \in G\) with \(c^h \notin C\), and note that \([h, u] \in C\) by Case 1. Consider the nilpotent element \(y = (1 - c)hg^{-1} \hat{e} = hg^{-1} \hat{e} - chg^{-1} \hat{e}\). If \(hg^{-1} e^m = chg^{-1} e^m\) for some integers \(n\) and \(m\), then \(c^h = g^{-1} e^{n-m}g \in C\), a contradiction. So \(\text{supp}(hg^{-1} \hat{e}) \cap \text{supp}(chg^{-1} \hat{e}) = \emptyset\), and as \((hg^{-1})^u\) appears with coefficient 1 in \(y^u = y\), it follows that \((hg^{-1})^u = hg^{-1} e^n\) for some \(n \in \mathbb{N}\), that is, \([h, u] = g^{-1} e^u g [g, u]\), and therefore \([g, u] \in C\). □

The following simple observation is useful for the determination of \(R(G)\).

Remark 4.6. Suppose that \(G\) has a finite non-normal subgroup. Then \(R(G)\) is the intersection of all finite non-normal cyclic subgroups of prime-power order of \(G\).

We immediately formulate the following corollaries.

Corollary 4.7. Suppose that \(G\) has a finite non-normal cyclic subgroup. Then \([\mathcal{U}, Z_{\infty}(\mathcal{U})] = [G, Z_{\infty}(\mathcal{U})] \leq R(G)\). Thus \(Z_{\infty}(\mathcal{U}) = Z(\mathcal{U})\) if \(R(G) = 1\). Suppose further that \([\mathcal{U}, Z_{\infty}(\mathcal{U})]\) is non-trivial, and let \(p^n\) be its order, for some prime \(p\) and \(n \in \mathbb{N}\) (see Remark 4.6). Then the following hold:

1. \(Z_{\infty}(\mathcal{U}) = Z_{n+1}(\mathcal{U})\);
2. \(Z_{\infty}(\mathcal{U})/Z(\mathcal{U})\) is a \(p\)-group of exponent \(\leq p^n\);
3. for each \(u \in \mathcal{U}\) there is \(x \in \text{supp}(u)\) such that \([ux^{-1}, G \cap Z_{\infty}(\mathcal{U})] = 1\);
4. a torsion-free normal subgroup of \(G\) centralizes \(Z_{\infty}(\mathcal{U})\).

Furthermore, for \(u \in Z_{\infty}(\mathcal{U})\) the group \(<\text{supp}(u)>^G\) is virtually cyclic and, if it is infinite, \(Z(G)\) is not periodic.

Proof. From Proposition 4.1, the elements of \(Z_{\infty}(\mathcal{U})\) normalize \(G\) and commute with all unipotent elements of \(ZG\), so that \(N = [G, Z_{\infty}(\mathcal{U})] \leq R(G)\) by Proposition 4.5. Note that the image of \(Z_{\infty}(\mathcal{U})\) under the natural map \(\pi : RG \to RG/N\) is central in \(RG/N\). Set \(C = [\mathcal{U}, Z_{\infty}(\mathcal{U})]\), a normal subgroup of \(\mathcal{U}\), contained in \(G\) by Proposition 4.1(2). Since \(\pi(C) = 1\), it follows that \(C \leq N\) and therefore \(C = N\).

Now suppose that \(C\) is non-trivial, of order \(p^n\). As \(C \leq Z_{\infty}(\mathcal{U})\), we have \([\mathcal{U}, C] < C\), and so \(\mathcal{U}\) acts as a \(p\)-group on \(C\), by the properties of the automorphism group of a cyclic \(p\)-group. Therefore \(\mathcal{U}\) acts trivially on the composition factors of \(C\), and (1) follows as \(C = [\mathcal{U}, Z_{\infty}(\mathcal{U})]\).

As to (2), we shall prove inductively that \([\mathcal{U}, Z_{\infty}(\mathcal{U})]^{p^n} \leq \mathcal{C}^{p^n}\) for all \(m \geq 0\). The case \(m = 0\) being obvious, suppose that the claim holds for some \(m \geq 0\), and take any \(u \in \mathcal{U}\) and \(v \in Z_{\infty}(\mathcal{U})\). Setting \(w = v^{p^m}\), we have...
\[ [u, v^{p^{n+1}}] = [u, v^p] = [u, w^{w^{u\cdot w^{-1}}}]. \]

Inductively, we have \([u, w] \in C^{p^n}\). Since \(u\) (in particular, \(w\)) acts trivially on the composition factors of \(C\), all factors \([u, w]^{w}, u, \) lie in the same coset of \(C^{p^{n+1}}\), and it follows that \([u, v^{p^{n+1}}] \in C^{p^{n+1}}\). This proves (2).

As to (3), let \(u \in \mathcal{U}\), and set \(H = G \cap Z_{\infty}(\mathcal{U})\). By Corollary 4.2, \(u \in N_{\mathcal{U}}(H)\). Let \(H\) act on \(\text{supp}(u)\) as described before Lemma 2.9 (the Ward–Coleman Lemma for infinite groups). By (2), \(H\) acts as a p-group on \(\text{supp}(u)\), and as in the proof of Lemma 2.9, it follows that there is \(x \in \text{supp}(u)\) such that \([ux^{-1}, H] = 1\), as desired.

To prove (4), let \(H\) be a normal torsion-free subgroup of \(G\). As \([H, Z_{\infty}(\mathcal{U})] \leq R(G)\), and \([H, Z_{\infty}(\mathcal{U})]\) maps to the trivial subgroup under the natural map \(RG \to RG/H\), it follows that \([H, Z_{\infty}(\mathcal{U})] \leq R(G) \cap H = 1\).

To prove the final statement, let \(u \in Z_{\infty}(\mathcal{U})\), pick any \(g_0 \in \text{supp}(u)\) and write \(u = g_0w\), \(N = \langle \text{supp}(w) \rangle\). Then \(N\) is a finite normal subgroup of \(G\) by Theorem 2.2 since \(u \in N_{\mathcal{U}}(G)\). Further \([G, g_0] \leq NR(G)\) since \([G, u] \leq R(G)\), and \([G, g_0]\) and \([G, u]\) have the same image under the natural map \(RG \to RG/N\). It follows that \(\text{supp}(u)\) is contained in the virtually cyclic normal subgroup \(<NR(G), g_0>\) of \(G\). Since some power of \(g_0\) is contained in \(Z(G)\), the proof is complete. \(\square\)

Corollary 4.8. Let \(c\) be an element of finite order in \(G\). Then \([c, Z_{\infty}(\mathcal{U})] \leq \langle c \rangle\).

**Proof.** Let \(u \in Z_{\infty}(\mathcal{U})\). By Proposition 4.1(1) and Proposition 2.6, conjugation by \(u\) induces a class-preserving automorphism of \(G\), and there is \(g \in G\) such that \(c^u = c^g\). If \(\langle c \rangle\) is a normal subgroup of \(G\), then \([c, u] = c^{-1}c^g \in \langle c \rangle\), and otherwise \([c, u] \in [G, Z_{\infty}(\mathcal{U})] \leq R(G) \leq \langle c \rangle\). In both cases we have the desired conclusion. \(\square\)

The corollary shows that is likely that the group \(C_{\infty} \rtimes C_4\) is embedded in \(G\) if \(R(G) \neq 1\) and the hypercenter of the unit group does not equal the center. However we shall see that for this particular group the hypercenter of the unit group equals the center.

To give some indication of what can be done so far, we mention two classes of groups for which the hypercenter coincides with the center.

**Example 4.9.** We have \(Z_{\infty}(\mathcal{U}) = Z(\mathcal{U})\) if \(G\) is either (1) a simple group or (2) a Frobenius group. Indeed:

1. We can assume that \(G\) is non-abelian. If \(G\) is finite then \(R(G)\) is defined and trivial, so that Corollary 4.7 applies. If \(G\) is infinite then Proposition 4.1(1) in conjunction with Theorem 2.2 shows that \(Z_{\infty}(\mathcal{U})\) is a normal subgroup of \(G\), and so \(Z_{\infty}(\mathcal{U}) = 1\).

2. Let \(G\) be a Frobenius group, and suppose that \(N_{\mathcal{U}}(G) \neq GZ(\mathcal{U})\). Then by [22, remark before Theorem 2.6], a Frobenius complement of \(G\) is finite, and the intersection of its conjugates is the trivial subgroup. Thus \(R(G)\) is defined and trivial, and the claim follows from Corollary 4.7.
We look briefly at Blackburn groups $G$; thus assume that $R(G)$ is defined and that $R(G) \neq 1$. By Remark 4.6, $R(G)$ is a $p$-group for some prime $p$. This leads to the following simple observation.

**Lemma 4.10.** Let $G$ be a Blackburn group, and let $q$ be a prime which does not divide the order of $R(G)$. Then the set $T_q$ of $q$-elements of $G$ forms a normal $q$-subgroup of $G$, all of whose subgroups are normal in $G$. (In particular, $T_q$ for odd $q$ is an abelian $q$-group.)

**Proof.** Let $x$ and $y$ be $q$-elements in $G$. Then $\langle x \rangle$ is normal in $G$ since otherwise we would have $R(G) = 1$. Thus $xy$ is also a $q$-element, and it follows that $T_q$ is a group. If a subgroup of $T_q$ is non-normal in $G$, then $T_q$ also contains a finite cyclic non-normal subgroup of $G$, which again contradicts $R(G) \neq 1$. That $T_q$ is abelian for odd $q$ follows from the classification of Dedekind groups. \(\square\)

Suppose that $R(G)$ is of odd order; then it follows that each finite normal subgroup of $G$ has a normal Sylow 2-subgroup, so that $G$ has the normalizer property (Example 3.5), and Proposition 4.1(1) gives the first statement below:

**Example 4.11.** Suppose that $G$ is a Blackburn group with $R(G)$ of odd order. Then $Z_{\infty}(U(ZG)) \leq Z(U(ZG))G$. If moreover $G$ is generated by elements of finite order, then $Z_{\infty}(U(ZG)) = Z(U(ZG))$.

**Proof.** To prove the second statement, set $\mathcal{U} = U(ZG)$ and suppose that $Z_{\infty}(\mathcal{U}) \neq Z(\mathcal{U})$. Then there exists $g \in G$ of finite order and $v \in Z_{\infty}(\mathcal{U})$ such that $[g, v] \neq 1$. By Lemma 5.5, $g$ is a 2-element and $g^2 = g^{-1}$, so that by Lemma 5.5, $[g, v] \in \langle g \rangle \cap R(G) = 1$, a contradiction. \(\square\)

Apart from the case of torsion-free $G$, which remains special, one also has to consider the case when all finite cyclic subgroups of $G$ are normal in $G$, which we shall assume for the rest of the section. (This situation was considered in [31] where partial results were obtained.) We write $T$ for the subgroup of $G$ consisting of the elements of finite order. We note (see also Corollary 5.6):

**Corollary 4.12.** Suppose that all finite cyclic subgroups of $G$ are normal in $G$. Then $Z_{\infty}(\mathcal{U}) \subseteq N_{\mathcal{U}}(G) = (U(RT) \cap Z(\mathcal{U}))G$.

**Proof.** By Proposition 4.1(1), $Z_{\infty}(\mathcal{U}) \subseteq N_{\mathcal{U}}(G)$. The group $T$ is a Dedekind group. In particular, finite normal subgroups of $T$ are nilpotent, so that $N_{\mathcal{U}}(G) = Z(\mathcal{U})G$ and $Z(\mathcal{U}) = (U(RT) \cap Z(\mathcal{U}))Z(G)$ by Corollary 3.6. \(\square\)

We shall see in Section 5 that if $R = \mathbb{Z}$ then $Z_{\infty}(\mathcal{U})/Z(\mathcal{U})$ is a 2-group. This gives the following result:

**Corollary 4.13.** Let $G$ be a group and $R = \mathbb{Z}$. If all finite subgroups of $G$ are normal and $[Z_{\infty}(\mathcal{U}), T] \neq 1$, then one of the following holds:
Lemma 5.2. Let $H$ be a finite subgroup of $\mathbb{Z}^T G \cap C_{\mathbb{Z}}(T) G$, or

(1) $T$ is an abelian 2-group and $Z_{\infty}(\mathcal{U}) \leq \mathcal{U}(\mathbb{Z}T) G \cap C_{\mathbb{Z}}(T) G$,

(2) $T$ is a Hamiltonian 2-group and $Z_{\infty}(\mathcal{U}) \leq G$.

5 More specific properties of hypercentral units

We keep the notation of the previous sections.

Lemma 5.1. If $\langle t \rangle$ is a finite cyclic non-normal subgroup of $G$ then

$$[Z_n(\mathcal{U}), \mathcal{U}] = [Z_n(\mathcal{U}), G] \subset \langle t \rangle$$

(with proper inclusion), for all $n \geq 1$. In particular, $Z_{\infty}(\mathcal{U}) \subseteq \Delta(\mathcal{U})$, and if $G$ has a non-central element of order 2 then $Z_{\infty}(\mathcal{U}) = Z(\mathcal{U})$.

Proof. Let $u \in Z_n(\mathcal{U})$. By Proposition 4.1, $u \in N_{\mathcal{U}}(G)$ and $u$ commutes with nilpotent elements. Hence it follows, by Proposition 4.5, that $[Z_n(\mathcal{U}), G] \subseteq \langle t \rangle$. Furthermore, since $\langle t \rangle$ is not normal in $G$, we obtain that $[Z_n(\mathcal{U}), G] = \langle d \rangle \subset \langle t \rangle$. Consequently, the image of $Z_n(\mathcal{U})$ in the group ring $R(G/\langle d \rangle)$ is central. Thus the natural image of $[Z_n(\mathcal{U}), \mathcal{U}]$ in $R(G/\langle d \rangle)$ is trivial. But then, since $[Z_n(\mathcal{U}), \mathcal{U}] \subseteq G$, it follows that $[Z_n(\mathcal{U}), \mathcal{U}] \subseteq \langle d \rangle = [Z_n(\mathcal{U}), G]$. \(\square\)

From now on we let $R = \mathbb{Z}$, so that $\mathcal{U} = U(\mathbb{Z}G)$.

It is well known that if $H$ is a finite subgroup in $\mathcal{U}$ consisting of units of augmentation one, then $H$ is a $\mathbb{Q}$-linear independent subset of $\mathbb{Q}G$ (see [34] and [42, Lemma 37.1]). Hence the $\mathbb{Q}$-subalgebra of $\mathbb{Q}G$ generated by $H$ is the group algebra $\mathbb{Q}H$.

Lemma 5.2. Let $H$ be a finite subgroup of $\mathcal{U}$ consisting of units of augmentation one, and $e \in \mathbb{Z}H$ an idempotent. If $u \in Z_{\infty}(\mathcal{U})$ and $H^u = H$ then $eu = ue$.

Proof. It is sufficient to prove the statement for every primitive idempotent $e$ in the semisimple subalgebra $\mathbb{Q}H$ of $\mathbb{Q}G$. Let $e$ be such an idempotent and suppose that $eu \neq ue$. Since $e^u \in \mathbb{Q}H$ it follows that $e^u e = 0$. Indeed, because $\mathbb{Q}H$ is a semisimple algebra and $\mathbb{Q}He \subseteq M_n(D)$ for some division ring $D$, we may assume that $e = E_{11} \in S = M_n(D)$. Now by a previous lemma $u$ commutes with all elements in $E_{11} M_n(D) (1 - E_{11})$ and with all elements in $(1 - E_{11}) M_n(D) E_{11}$ (as all elements in these sets are nilpotent). If $n \geq 2$ then these sets contain the respective elements $E_{1i}$ and $E_{i1}$ with $i \geq 2$. Thus $u$ also commutes with $E_{1i} E_{i1} = E_{11} = e$, in contradiction to the assumption. So $n = 1$ and thus $M_n(D) = D$. Hence $e$ is a central idempotent of $\mathbb{Q}H$. But as $H^u = H$ it follows that $D^u$ is also a simple component of $\mathbb{Q}H$ with defining idempotent $e^u$. Since by assumption $e^u \neq e$ it follows $D^u$ and $D$ are different simple components and thus $D^u D = 0$, in particular $e^u e = 0$, as desired. So $(eu)^2 = 0$. Hence by Proposition 4.1, $eu = (eu)^u = e^u u$. Therefore $e = e^u$, a contradiction. \(\square\)

Corollary 5.3. (1) If $v$ is a unit in $\mathcal{U}$ of finite order and augmentation one then $[Z_n(\mathcal{U}), \langle v \rangle] \subseteq \langle v \rangle$. 

\(\langle AutoPDF V7 5/2/07 10:28 \) WDG (170x240mm) Tmath \ J-1709 JGT, : HC1: elo 2/2/07 pp. 1–28 1709_02 (p. 16)
(2) If \( Z_{x_k}(\mathcal{U}) \) is not central and \( g, h \in G \) are periodic elements of relatively prime order then \( gh \) has finite order.

(3) If \( Z_{x_k}(\mathcal{U}) \) is not central and if there exists a 2-element \( t_0 \in G \) such that \( \langle t_0 \rangle \) is not normal in \( G \), then the elements of \( G \) of finite odd order form an abelian subgroup.

Proof. (1) Let \( v \in \mathcal{U} \) be an element of finite order. If \( n = 1 \) then the statement is obvious. Assume that the result holds for \( n \) and let \( u \in Z_{n+1}(\mathcal{U}) \). Because of Proposition 4.1, \( g_0 = [u, v] \in G \cap Z_n(\mathcal{U}) \). By the induction hypothesis, \([g_0, v] \in \langle v \rangle \). Since \( v \), and thus also \( v^n \), has finite order and \( g_0 \) normalizes \( \langle v \rangle \), it follows that \( g_0 \in T(Z_n(\mathcal{U})) \), the torsion subgroup of \( Z_n(\mathcal{U}) \). Put \( H = \langle v, g_0^k \mid k \in \mathbb{Z} \rangle \) and notice that \( \langle v \rangle \leq H \) and \( H^u = H \). Because \( G \cap Z_{x_k}(\mathcal{U}) \) is a normal subgroup of \( \mathcal{U} \) consisting of linearly independent elements of \( RG \), by a result of Bovdi [4, Theorem 3], every subgroup of \( G \cap Z_{x_k}(\mathcal{U}) \) is normal in \( \mathcal{U} \). In particular, \( g_0^k \in \langle g_0 \rangle \) for all \( k \in \mathbb{Z} \). Consequently, \( H = \langle v, g_0^k \rangle \) is finite and \( \langle v \rangle \leq H \).

Let \( f \) denote the order of \( u \). Lemma 5.2 applied to the group \( H \) yields that the idempotent \( f^{-1}(\sum_{i=0}^{f-1} v^i) \) commutes with \( u \). Since \( H^u = H \) we get in the group algebra \( \mathbb{Q}H \) the equality

\[
\frac{1}{f} \left( \sum_{i=0}^{f-1} (v^i)^f \right) = \frac{1}{f} \left( \sum_{i=0}^{f-1} v^i \right).
\]

Thus \( v^f \in \langle v \rangle \). Hence \([u, v] \in \langle v \rangle \).

(2) Let \( g \) and \( h \) be periodic elements of \( G \) of relatively prime orders. Assume that \( gh \) has infinite order. Then \( \langle g \rangle \) and \( \langle h \rangle \) are both non-normal in \( G \). Hence by Lemma 5.1, \( \langle g \rangle \cap \langle h \rangle = 1 \). Therefore \( Z_{x_k}(\mathcal{U}) \) is central.

(3) Because of the assumption, Lemma 5.1 implies that \( [Z_{x_k}(\mathcal{U}), \mathcal{U}] \) is a 2-group. Let \( q \in G \) be a periodic element of odd order. Assume \( \langle q \rangle \) is non-normal in \( G \). Then, by Lemma 5.1, \( [Z_{x_k}(\mathcal{U}), \mathcal{U}] \subseteq \langle q \rangle \) and hence it has odd order. Since it is also has even order, \( Z_{x_k}(\mathcal{U}) \) is central, a contradiction. Hence \( \langle q \rangle \leq G \). Therefore, by the classification of Dedekind groups, the elements of odd order generate an abelian subgroup.

The following Lemma might be of independent interest. We write \( T(G) \) for the set of periodic elements of \( G \).

**Lemma 5.4.** Let \( G \) be a group and \( A \) a subring of \( RG \) which is a finitely generated \( R \)-module. Then \( G \cap U(A) \subseteq T(G) \).

**Proof.** Let \( \{a_1, \ldots, a_n\} \) be a set of generators of the \( R \)-module \( A \) and put \( X = \bigcup_{i=1}^{n} \text{supp}(a_i) \), a finite subset of \( G \).

If \( g \in G \cap U(A) \) has infinite order then there exists \( m \in \mathbb{N} \) such that \( g^m \notin X \). Since \( A \) is a subring of \( RG \) we have \( g^m \in A \) and thus \( g^m = \sum r_j a_j \), with \( r_j \in R \). However this implies that \( g^m \in X \), a contradiction.
We shall write $\omega(G)$ for the augmentation ideal of $\mathbb{Z}G$. The next result was proved by Li and Parmenter in case $n = 2$ (see [31]).

**Lemma 5.5.** Let $G$ be a group, $u \in \mathbb{Z}_n(\mathcal{U})$ and $v$ an element of finite order and augmentation one in $\mathcal{U}$. If $c = [u, v] \neq 1$ then

$$v^c = v^{-1} \quad \text{and} \quad v^2 \in \mathcal{U} \cap Z_{n-1}(\mathcal{U}) \subseteq Z_{n-1}(G).$$

Let $f$ denote the order of $v$. Then $f = 2^m$ with $m \leq n$, and $v^{2^n-1}$ is central in $\mathcal{U}$. If $n = 2$ then $m = 2$. In particular, elements of $\mathcal{U}$ that are of finite order and whose order is not a power of 2 commute with $Z_{\infty}(\mathcal{U})$, and $Z_{\infty}(\mathcal{U})^2 \subseteq C_{\mathcal{U}}(T(G)).$

**Proof.** Let $u \in \mathbb{Z}_n(\mathcal{U})$ and $v$ an element of finite order in $\mathcal{U}$. Assume that $[u, v] \neq 1$. By Corollary 5.3 we know that $c \in \langle v \rangle$. It is well known that the trivial units $\langle v \rangle$ in the integral group ring $\mathbb{Z}\langle v \rangle$ have a torsion-free normal complement. In particular $U(\mathbb{Z}\langle v \rangle) = \langle v \rangle \times U(1 + \omega^2 \langle v \rangle)$ (see [42, Theorem 30.1]). So if $x \in U(1 + \omega^2 \langle v \rangle)$ then by Corollary 5.3 we get that $x^{-1}x^u \in U(1 + \omega^2 \langle v \rangle)$. Hence, by Proposition 4.1, we get that $x^{-1}x^u = [x, u] \in G \cap (1 + \omega^2 \langle v \rangle)$.

Thus, by Lemma 5.4, $x^{-1}x^u$ is a periodic unit in the torsion-free group $(1 + \omega^2 \langle v \rangle)$. Thus $x = x^u$.

We have $Q\langle v \rangle = \sum_{d \mid n} Q(\zeta_d^e)$, with $\zeta_d$ a primitive $d$th root of unity. The direct summand $L = Q(\zeta_f) = Q\langle v \rangle e$ contains the subfield $K = Q(1 + \omega^2 \langle v \rangle)e$, where $e$ is the idempotent corresponding to $L$. Let $I_L$, resp. $I_K$, denote the rings of integers of the respective fields. Since their unit groups have the same torsion-free rank (see [42, Theorem 30.1]) we have $|U(I_L) : U(I_K)| < \infty$. By [42, Lemma 21.2], $K$ is a totally real field and $[L : K] = 2$. So $K = Q(\zeta_f + \bar{\zeta_f} - 1)$ and thus, by the above, $(\zeta_f + \bar{\zeta_f} - 1)^u = \zeta_f^u + \bar{\zeta_f}$. This argument is valid in any component $Q(\zeta_d)$ of $Q\langle v \rangle$. Hence $(v + v^{-1})^u = v + v^{-1}$. By Corollary 5.3 we know that $v^u \in \langle v \rangle$. Since $v^u \neq v$ it follows that $v^u = v^{-1}$. Hence $(v^u)^u = v^{-u}$ and thus $[v^u, u] = v^{2k}$, for any positive integer $k$.

Proposition 4.1 also yields that $v^{-2} = [v, u] \in Z_{n-1}(G)$ and thus, by induction, we obtain that $v^{-2^k} = [v, u]^k \in Z_{n-k}(G)$ for $k \leq n-1$. From this it easily follows that $f = 2^m$ for some $m$. Since $v^{2^m}$ is central if and only if $k \in \{m - 1, m\}$ it follows that $m \leq n$, $v^{2^m} \in Z(G)$ and $v^2 = 1$. Therefore $m \leq n$.

If $n = 2$, then $v^2 \in Z(G)$ and $v^2 = (v^2)^u = v^{-2}$. As $v$ is not central and, by assumption, $Z_{\infty}(\mathcal{U}) \neq Z(\mathcal{U})$, we obtain from Lemma 5.1 that $f = 4$.

Finally, $u^2$ commutes with all torsion elements and hence $Z_{\infty}(\mathcal{U})^2 \subseteq C_{\mathcal{U}}(T(G))$. \hfill \Box

A weaker version of this result, with $Z_{\infty}(\mathcal{U})$ replaced by $Z_2(\mathcal{U})$, was proved by Li and Parmenter [31, Lemma 3.1(a)].

We can sharpen the result from Corollary 4.12 as follows:

**Corollary 5.6.** Suppose that all finite cyclic subgroups of $G$ are normal in $G$, and let $T$ be the subgroup of $G$ consisting of the elements of finite order. Then the following hold.
(1) If $T$ is a 2-group then $Z_{\infty}(\mathcal{U}) \subseteq Z_{\infty}(G)$.

(2) If $T$ is not a 2-group, then $Z_{\infty}(\mathcal{U}) \subseteq (U(\mathbb{Z}T) \cap \mathbb{Z}(\mathcal{U})) \cdot (Z_{\infty}(G) \cap C_G(T))$.

**Proof.** If the Dedekind group $T$ is a 2-group, then $U(\mathbb{Z}T) = \pm T$ (see [42, Corollary 2.3]), and so (1) holds by Corollary 4.12.

If $T$ is not a 2-group, then, as $T$ is nilpotent, $Z_{\infty}(G)$ centralizes $T$ by Lemma 5.5, and so (2) holds, again by Corollary 4.12. \qed

Lemma 5.5 admits a short proof (see [31, Theorem 3.5]) of the following result which was already proved in [30, Theorem 2]. (The definition of a Q*-group is given in Definition 4.4.)

**Proposition 5.7.** Suppose that $G$ is a periodic group, and that $Z_{\infty}(G) \neq \mathcal{U}$. Then $G$ is a Q*-group.

**Proof.** Suppose that $G$ is a Dedekind group. Then $G$ is a 2-group by Corollary 5.6(2), and since $G$ is non-abelian, it follows that $G$ is a Q*-group. Hence we can assume that $G$ has non-normal cyclic subgroups, and then $[\mathcal{U}, Z_{\infty}(\mathcal{U})] \leq R(G)$ by Corollary 4.7.

Let $u \in Z_2(\mathcal{U})$ so that $|G, u| \neq 1$. Take any $b \in G$ with $|b, u| \neq 1$. Then $b^u = b^{-1}$ by Lemma 5.5, and $b^2 = [u, b] \in Z(\mathcal{U})$ implies that $b$ has order 4. Since $[G, u]$ is contained in the cyclic group $R(G)$, it follows that $[G, u]$ is a cyclic group of order 2 contained in the center of $G$. Consequently, $A = C_G(u)$ is a subgroup of index 2 in $G$ (as the kernel of the homomorphism $G \rightarrow [G, u], g \mapsto [g, u]$). If $a \in A$, then $ab^{-1} = (ab)^u = (ab)^{-1}$ by Lemma 5.5, i.e., $a^b = a^{-1}$. Thus $A$ is abelian. From Example 3.1 it now follows that class-preserving automorphisms of $G$ are inner automorphisms, and so we obtain from Proposition 4.1(1) that $u = az$ for some $a \in G$, $z \in Z(\mathcal{U})$. Note that $[a, u] = [u, u] = 1$, i.e., $a \in A$, and that $b^u = b^{az} = b^u = b^{-1}$. Thus $b^2 = b^{-2} = b^{-1}b^u = [b, a] = a^{-b}a = a^2$, and the proof is complete. \qed

The converse also holds. We formulate it in a slightly stronger form as follows (cf. also [31], [32]). Recall from Example 7.1 that for an arbitrary Q*-group $G$, the hypercenter $Z_{\infty}(U(\mathbb{Z}G))$ still has the expected description.

**Proposition 5.8.** Suppose that $G$ is generated by its periodic elements, and let $R = \mathbb{Z}$. Then $Z_{\infty}(\mathcal{U})$ is non-central and centralized by $G_{\infty}$, the set of elements of $G$ of infinite order, if and only if $G$ is a Q*-group.

**Proof.** One direction follows from Example 7.1.

Assume that $Z_{\infty}(\mathcal{U})$ is non-central and centralized by $G_{\infty}$. We shall follow the previous proof to show that $G$ is a Q*-group. Again, we can assume that $G$ has non-normal finite cyclic subgroups, since otherwise $G$ is periodic, the case we have just dealt with. Let $u \in Z_2(\mathcal{U})$ so that $|G, u| \neq 1$. Again, we can take $b \in G$ of finite order with $|b, u| \neq 1$. As above, it follows that $b$ has order 4, and that $A = C_G(u)$ is a sub-
group of index 2 in $G$. If $a \in A$, then $ab \neq ab^{-1} = (ab)^n$, so that $ab$ is of finite order by assumption, and we obtain again $a^b = a^{-1}$. Hence $A$ is abelian, and the proof is completed as before. 

It seems possible that the hypothesis on the elements of infinite order can be removed.

In a nilpotent group, the elements of finite order form a characteristic subgroup (see, for example, [39, (5.2.7)]). Thus the elements of finite order in $Z_{\infty}(\mathcal{U})$ form a characteristic subgroup $T(Z_{\infty}(\mathcal{U}))$ of $\mathcal{U}$.

A normal torsion subgroup of $U(ZG)$ is contained in $\pm G$, by a result of Bovdi [4, Theorem 1]. Further, the structure of a normal torsion subgroup $N$ of $U(ZG)$ is severely restricted: every subgroup of $N$ is normal in $U(ZG)$, and $N$ is either abelian or a Hamiltonian 2-group (see [5, Theorem 2]). The papers of Bovdi also contain other results. For example, if $N$ is a Hamiltonian 2-group, then the set of elements of finite order in $G$ forms a Hamiltonian 2-group in which every subgroup is normal in $G$.

As a consequence, $T(Z_{\infty}(U(ZG))) \leq G$. For an elementary proof of this result, see [31, Lemma 2.1]. We shall give a direct proof along our lines.

**Lemma 5.9.** The set $T(Z_{\infty}(\mathcal{U}))$ of hypercentral torsion units in $\mathcal{U}$ form a normal subgroup of $\mathcal{U}$ contained in $\mathcal{U}$, where $\mathcal{U}$ denotes the group of roots of unity in $R$. Each subgroup of $T(Z_{\infty}(\mathcal{U}))$ is normal in $\mathcal{U}$. In particular, $T(Z_{\infty}(\mathcal{U})) \cap G$ is a Dedekind group.

**Proof.** That $T(Z_{\infty}(\mathcal{U})) \leq \mathcal{U}$ follows from Proposition 4.1(1) and Corollary 2.5.

Set $H = T(Z_{\infty}(\mathcal{U})) \cap G$. Assume that there is $h \in H$ such that $\langle h \rangle$ is not normal in $\mathcal{U}$, and choose $u \in \mathcal{U}$ with $g = hu \notin \langle h \rangle$. Note that $g \in G$ since $g \in T(Z_{\infty}(\mathcal{U}))$. Then $x = (h-1)uh = huh - uh$ is a nilpotent element, so that $x = x^h$ by Proposition 4.1(3), and it follows that

$$uh - h^{-1}uh = h^{-1}x = x = huh - uh.$$ 

Multiplying by $u^{-1}$ on the left, we reach the contradiction $h^{-1}g = g - h$ and $h = gh$, i.e., $g \in \langle h \rangle$. The proof is complete. 

**6 An Engel condition**

Motivated by some results of Section 4, we wish to explore how hypercentral units can act on units of finite order, remembering that the integral group ring of a finite cyclic group usually has non-trivial units (of infinite order). Using these results we present in Corollary 6.2 a different proof of the fact proved in Lemma 5.5 that conjugation by a hypercentral unit centralizes or inverts a torsion element of $\mathcal{U}(RG)$.

For elements $x, y$ of a group, the (left-normed) commutator

$$[x, y, \ldots, y]_n$$

is defined inductively by
\[ [x, y, \ldots, y] = \left[ [x, y, \ldots, y], y \right] \]

for \( n \geq 2 \).

The next proposition is probably well known; its proof, based on elementary properties of cyclotomic units, is technical but simple. For background information on cyclotomic units, we refer the reader to [44]. Knowledge about the automorphism group of a cyclic group is taken for granted.

**Proposition 6.1.** Let \( x \) and \( y \) be elements of a group, with \( x \) of finite order, and \( y \) acting non-trivially on \( X = \langle x \rangle \). Then there exists \( u \in U(\mathbb{Z}X) \) such that \( u \) and \( y \) does not satisfy an Engel condition, i.e., for all \( n \in \mathbb{N} \)

\[ [u, y, \ldots, y] \neq 1, \]

unless \( x \) is a 2-element and \( x^2 = x^{-1} \). (In the latter case, there is for any \( u \in U(\mathbb{Z}X) \) a group element \( x_u \in X \) such that \( [u, y] = [x_u, y] \).)

**Proof.** Let \( \sigma \) denote the automorphism of \( \mathbb{Z}X \) induced by conjugation by \( y \), written exponentially. For all primes \( p \) dividing the order of \( X \), let \( X_p \) be the Sylow \( p \)-subgroup of \( X \).

If \( \sigma \) restricted to \( X_p \) is not of \( p \)-power order, then \( X_p = \langle z, y \rangle \) for any generator \( z \) of \( X_p \), and all commutators \([z, y, \ldots, y]\) are non-trivial. Hence we can assume that \( \sigma \) acts as an automorphism of \( p \)-power order on \( X_p \), for all \( p \).

Set \( C = C_X(\sigma) \), the subgroup of \( X \) of fixed points of \( \sigma \), and let \( \pi : \mathbb{Z}X \rightarrow \mathbb{Z}X/C \) be the natural homomorphism. Suppose that a unit \( u \) of \( \mathbb{Z}X \) satisfying \( \pi(u) = 1 \) is not fixed by \( \sigma \). Then also \( \pi(u^{-1}u^\sigma) = 1 \) and \( u^{-1}u^\sigma \) is not fixed by \( \sigma \). The first statement is obvious since \( \sigma \) induces an automorphism of \( \mathbb{Z}X/C \). To prove the second statement, set \( c = u^{-1}u^\sigma \) and assume by way of contradiction that \( c \) is fixed by \( \sigma \). Let \( n \) be the order of \( \sigma \). We have \( u^\sigma = uc \) and \( c \neq 1 \) as \( u \) is not fixed by \( \sigma \). Also \( c^n = 1 \) since \( u = u^\sigma = (uc)^{\sigma^{-1}} = \cdots = uc^n \). Hence \( c \) is a unit of finite order in \( \mathbb{Z}X \), with \( \varepsilon(c) = 1 \) (since \( \pi(u) = 1 \); here \( \varepsilon \) denotes augmentation). It follows that \( c \) is a group element, so that \( c \in C \). Write \( u = m_1 + m_2 \) with \( m_1 \in \mathbb{Z}C \), \( m_2 \in \mathbb{Z}X \) and \( \supp(m_2) \subseteq X \setminus C \). Then \( \varepsilon(m_1) = 1 \) as \( \pi(u) = 1 \). On the other hand,

\[ m_1c + m_2c = uc = u^\sigma = m_1 + m_2^\sigma \quad \text{and} \quad \supp(m_2c), \supp(m_2^\sigma) \subseteq X \setminus C, \]

so that \( m_1c = m_1 \). Therefore \( m_1 \in (\mathbb{Z}C)c \) and \( \varepsilon(m_1) \) is divisible by the order of \( c \), a contradiction.

We have seen that once we are able to construct a unit \( u \) of \( \mathbb{Z}X \) with \( \pi(u) = 1 \) which is not fixed by \( \sigma \), then all commutators \([u, y, \ldots, y]\) are non-trivial.

Suppose that \( x \) is a 2-element and \( x^2 = x^{-1} \). Then \( X \) has a complement in \( U(\mathbb{Z}X) \) which is element-wise fixed by \( \sigma \) (see [39, (2.10)]), and it follows immediately that for any \( u \in U(\mathbb{Z}X) \), there is a group element \( x_u \in X \) satisfying \([u, y] = [x_u, y] \).
In all other cases, we will construct a unit $u$ of $\mathbb{Z}X$ with $\pi(u) = 1$ and $u \neq u^\sigma$. Let $\tau$ be a power of $\sigma$ such that the restriction of $\tau$ to $X_p$ is of order $p$ if $\sigma$ acts non-trivially on $X_p$, for all $p$. Then if $v \in U(\mathbb{Z}X)$, setting $u = v^{-1}v^\tau$ we have $\pi(u) = 1$. Further, if $u = u^\sigma$ (and therefore $u = u^\tau$), then $v^\tau = vc$ for some $c \in C$ (by the same argument as above). Thus it suffices to find a unit $v$ in $\mathbb{Z}X$ such that $v^{-1}v^\tau \notin C$.

Let $\chi : C\mathbb{X}X \rightarrow C$ be a faithful complex representation. Then $\chi(x) = \xi$ for a primitive $|X|$-th root of unity $\xi$. Set $m = \varphi(|X|)$, where $\varphi$ denotes Euler’s totient function.

Suppose that $X = X_p$ for an odd prime $p$. Then $1 + \xi$ is a cyclic unit, and there exists a Bass cyclic unit $v \in U(\mathbb{Z}X)$ (see, for example, [42, §10]) such that $\chi(v) = (1 + \xi)^m$. Assume that $v^{-1}v^\tau \in C$. Then $(1 + \xi^l\xi)^m \epsilon (1 + \xi^m), \xi > l \epsilon \mathbb{N}$ with $l \notin \langle \xi^l \rangle < \langle \xi \rangle$ (as $\tau$ is of order $p$). Taking absolute values gives $|1 + \xi^l\xi| = |1 + \xi|$, which is only possible if $\xi^l = \xi = \xi^{-1}$, when $\langle \xi^l \rangle = \langle \xi^2 \rangle = \langle \xi \rangle$. This contradiction shows that $v^{-1}v^\tau \notin C$, and we are done in this case.

Suppose that $X$ is a 2-group, and $x^\tau \neq x^{-1}$. Then a real cyclotomic unit is given by $\gamma = \xi^{-1}(1 - \xi)/(1 - \xi)$, and there exists a unit $v \in U(\mathbb{Z}X)$ such that $\chi(v) = \gamma^m$. Assume that $v^{-1}v^\tau \in C$. As $\tau$ is of order 2, we have $\chi(x^\tau) = \chi(\gamma)$ (as $\gamma \epsilon \mathbb{Z}$). Thus if $\gamma = \xi^{-1}(1 - \xi)/(1 + \xi) \epsilon \mathbb{R}$, then $\chi(v^\tau) = \gamma^m$, and $\gamma^{-1} \gamma = \pm 1$ as $\gamma^{-1} \gamma \epsilon \mathbb{R}$ and $\chi(v^{-1}v^\tau) = (\gamma^{-1} \gamma)^m$ is a root of unity. But this implies $\xi^4 = 1$, a contradiction (as $x^\tau \neq x^{-1}$).

Finally, suppose that $X$ is not a $p$-group. By the previous results, we can assume that $X$ contains an element $a$ of odd prime order which is fixed by $\sigma$, and an element $b$ of order 4 which is inverted by $\sigma$. Let us replace $X$ by $\langle a, b \rangle$. Again, we could make use of a Bass cyclic unit, but for simplicity, we choose to work with $\gamma = 1 - \xi$ (which is a unit, by [44, Proposition 2.8]). Then, there exists $v \in U(\mathbb{Z}X)$ such that $\chi(v) = \gamma^k$ for some $k \epsilon \mathbb{N}$. As $\chi(v^\sigma) = (1 + \xi)^k$, and $(1 - \xi)^{-1}(1 + \xi)$ is not a root of unity, it follows that $\chi(v^{-1}v^\sigma)$ is not a root of unity. Hence $v^{-1}v^\sigma \notin C$, which completes the last case. □

What we have really shown is the following: if $u$ is a unit such that $u$ and $y$ do not satisfy an Engel condition, and $\chi$ is a faithful representation of $\langle x \rangle$, then $u$ can be chosen such that $\chi([u, y, \ldots, y]) \neq 1$ for all commutators $[u, y, \ldots, y]$.

Let $R$ be an integral domain of characteristic 0, with quotient field $K$. Let $v$ be a unit of finite order in $RG$. Then $K[v]$, the $K$-span of the elements of $\langle v \rangle$ in $KG$, is a semisimple algebra, and the projection onto some block of $K[v]$ yields a faithful representation of $\langle v \rangle$. Thus we have the following result:

**Corollary 6.2.** Let $u$ and $v$ be units in $\mathbb{Z}G$, with $v$ of finite order and $u$ acting non-trivially on $\langle v \rangle$. Then there exists $w \in U(\mathbb{Z}[v])$ such that all commutators $[u, w, \ldots, w]$ are non-trivial unless $v$ is a 2-element and $v^n = v^{-1}$. In particular, the latter holds if $u \epsilon Z_\infty(\mathbb{Z}[\mathbb{W}])$.

### 7 Some examples

We shall give some examples where $Z_\infty(\mathbb{W})$ does not consist only of central units. For simplicity, we shall restrict to the case when $R = \mathbb{Z}$.
We first deal with Q*-groups $G$, where a description of $\mathbb{Z}_G(\mathcal{U})$ is known for periodic $G$ (see [30, Theorem 2]). This description remains valid for an arbitrary Q*-group $G$. For the reader’s convenience, we include a proof which is based on an observation of Bovdi [5, Theorem 11] and some of the results in this paper.

**Example 7.1.** Let $G$ be a Q*-group, with $G = \langle A, b \rangle$ and $a \in A$ as in Definition 4.4. Then either $\mathbb{Z}_G(\mathcal{U}) = \mathbb{Z}(\mathcal{U}) \langle a \rangle$, or $G$ is a Hamiltonian 2-group and $\mathbb{Z}_G(\mathcal{U}) = \mathbb{Z}(\mathcal{U}) G$.

Here is a proof. Let $H = G \cap \mathbb{Z}_G(\mathcal{U})$, a normal subgroup of $\mathcal{U}$ by Corollary 4.2.

Take any $x \in H \cap A$. Note that $x^2 = x^{-b} x = [b, x]$. We have $[b, x] \leq \langle x \rangle$ as $\langle x \rangle \leq G$, and $[b, x] \leq \langle b \rangle$ by Lemma 5.5. As $b^2 = a^2$, the intersection $\langle b \rangle \cap \langle x \rangle$ is either trivial or equal to $\langle a^2 \rangle$. It follows that either $x^2 = 1$ and $x \in Z(G)$, or $x^2 = a^2$.

On the other hand, let $y \in A$ with $y^2 = a^2$, and take any $u \in \mathcal{U}$. Let $*$ be the usual anti-involution of $\mathbb{Z}G$ (defined by $g^* = g^{-1}$ for $g \in G$ and $\mathbb{Z}$-linear extension).

Write $u = x_1 + x_2 b$ with $x_i \in \mathbb{Z}A$; then $uu^* = (x_1 x_1^* + x_2 x_2^*) + x_1 x_2 (b + b^{-1})$ commutes with $y$ (since $y^2 = a^2$), and it follows that $y^u (y^u)^* = 1$ (this is Bovdi’s observation). Thus $y^u \in G$ by a classical result of Berman. Note that $y^u$ and $y$ have the same image under the natural map $\mathbb{Z}G \rightarrow \mathbb{Z}G / \langle a^2 \rangle$ since $[G, y] = \langle a^2 \rangle$, so that $[u, y] \in \langle a^2 \rangle \leq Z(G)$. It follows that $y \in Z_2(\mathcal{U})$.

So far, we have seen that $H \cap A = Z(G) \langle g \in G \mid g^2 = a^2 \rangle = Z(G) \langle a \rangle$.

Assume that $H \not\subseteq A$, and choose $h \in H \setminus A$. Note that $h^2 = a^2$; in particular, $h$ is of finite order. Thus if $\langle h \rangle$ is not normal in $G$ we immediately obtain by Proposition 4.1(3) and Proposition 4.5 the contradiction that $[G, h] \leq R(G) = \langle a^2 \rangle \leq \langle h \rangle$. Hence $\langle h \rangle \leq G$. It follows that

$$y^2 = y^{-b} y = [h, y] \in \langle h \rangle \cap \langle y \rangle \leq \langle a^2 \rangle$$

for all $y \in A$, so that $G$ is a Hamiltonian 2-group. But then $A$, $b$ and $a$ can be chosen such that $a$ is any given non-central element of $G$, so that $H = G$.

Finally, we note that $\mathbb{Z}_G(\mathcal{U}) = \mathbb{Z}(\mathcal{U}) G$ by Example 3.1 and Proposition 4.1(1), which completes the proof.

The next two examples show that the central height of the unit group $\mathcal{U}$ can be arbitrarily high (as was already noticed in [31], [32]).

**Example 7.2.** Fix $n \geq 2$, and let

$$G = \langle a, x \mid a^{2^n} = 1, a^x = a^{-1} \rangle \cong C_{2^n} \rtimes \mathbb{Z}_n.$$

It is well known that $\mathcal{U} = U(\mathbb{Z}G) = U(\mathbb{Z} \langle a \rangle) \rtimes \langle x \rangle$. Further, by [42, (2.10)], $U(\mathbb{Z} \langle a \rangle) = \langle a \rangle \times K$ with the complement $K$ centralized by $x$. It follows that $[\mathcal{U}, U(\mathbb{Z} \langle a \rangle)] = \langle a^2 \rangle$, and $[\mathcal{U}, \mathcal{U}] = \langle a^2 \rangle$, so that

$$1 \leq \langle a^{2^{n-1}} \rangle \leq \cdots \leq \langle a^4 \rangle \leq \langle a^2 \rangle \leq \mathcal{U}$$
is the descending central series of $\mathcal{U}$. Hence $\mathcal{U}$ is nilpotent of class $n$. We have $Z(\mathcal{U}) = \langle K, a^{2^{n+1}}, x^2 \rangle$, and the upper central series of $\mathcal{U}$ is

$$1 \lhd Z(\mathcal{U}) \lhd Z(\mathcal{U})\langle a^{2^{n+1}} \rangle \lhd \cdots \lhd Z(\mathcal{U})\langle a^2 \rangle \lhd \mathcal{U}. $$

Finally, we note that

$$\mathcal{U} = Z\infty(\mathcal{U}) = N_{\mathfrak{G}}(G) = \Delta(\mathcal{U}).$$

**Example 7.3.** Let $p$ be an odd prime and $n \geq 2$. Let

$$G = \langle a, x \mid a^{p^n} = 1, a^r = a^{p^{r+1}} \rangle \cong C_{p^n} \rtimes C_\infty. $$

As before, we have $\mathcal{U} = U(\mathfrak{Z}G) = U(\mathfrak{Z}\langle a \rangle) \rtimes \langle x \rangle$. By previous results, $G$ has the normalizer property, and $Z\infty(\mathcal{U}) = Z(\mathcal{U})G$. Let $U(\mathfrak{Z}\langle a \rangle)\langle s \rangle$ be the group of units in $\mathfrak{Z}\langle a \rangle$ which commute with $x$. By Corollary 3.6, $Z(\mathcal{U}) = U(\mathfrak{Z}\langle a \rangle)\langle s \rangle\langle x^{p^{r+1}} \rangle$. From the description of $\mathcal{U}$ it follows that $\langle a \rangle \leq \mathcal{U}$ and $\langle a \rangle \leq Z\infty(\mathcal{U})$.

Let $m = \varphi(p^n) = p^{n-1}(p - 1)$, write $2m = 1 + kp^n$, and set $u = (1 + a)^m - ka$. Then $u$ is a Bass cyclic unit in $\mathfrak{Z}\langle a \rangle$ (see [42, (10.3)]). Set $y = x^{p^{r+1}}$, and assume that $y \in Z\infty(\mathcal{U})$. Since $G \cap Z\infty(\mathcal{U}) \lhd \mathcal{U}$ by Corollary 4.2, $\langle y \rangle$ acts according to the Ward–Coleman Lemma (for infinite groups) on $\text{supp}(u)$, and since $y$ acts as an automorphism of order $p$ on $\langle a \rangle$, it follows from Lemma 2.9 that there is $a^s \in \text{supp}(u)$ such that $y$ centralizes $ua^s$. Note that $ua^s = \sum_{i=0}^{m} \binom{m}{i} a^{i-s}$, and that the powers of $a$ appearing in this sum are distinct. It follows that $y$ commutes with $a^{s-x} + a^{m-s}$ (the coefficients of other powers of $a$ being $1$), so that $|a^s, y| = 1$ (as $p$ is odd) and $a^s \in \langle a^p \rangle$. Also, $y$ commutes with $\binom{m}{i} a^{i-s} + \binom{m}{i} a^{m-i-1-s}$, and it follows that $y$ commutes with $a^{1-s}$, and therefore also with $a$, a contradiction. Thus $y \notin Z\infty(\mathcal{U})$. Therefore

$$1 \lhd Z(\mathcal{U}) \lhd Z(\mathcal{U})\langle a^{p^n} \rangle \lhd \cdots \lhd Z(\mathcal{U})\langle a \rangle = Z\infty(\mathcal{U})$$

is the upper central series of $\mathcal{U}$, and $\mathcal{U}$ is of central height $n$.

Finally, we note that $u \in \Delta(\mathcal{U})$ and $u \notin N_{\mathfrak{G}}(G)$.

The next example shows that even if $G$ contains non-normal finite subgroups, $Z_2(\mathcal{U})$ need not be equal to $Z(\mathcal{U})T$, where $T$ denotes the torsion subgroup of $Z_2(\mathcal{U})$ (and so the result of Arora and Passi [2] does not generalize to infinite groups).

**Example 7.4.** Let

$$G = \langle a, b, x, y \mid a^4 = b^2 = y^2 = 1, a^b = a^{-1}, x^{-1} = xa^2, a^x = b^i, y = [a, x] = [b, x] = 1 \rangle$$

$$\cong \langle (a, b) \times \langle x \rangle \times \langle y \rangle \rangle \cong (\mathbb{Q}_8 \rtimes C_\infty) \rtimes C_\infty.$$
Then $G$ is nilpotent of class 3. Note that $\langle a \rangle$ and $\langle b \rangle$ are the non-normal finite cyclic subgroups of $G$, so that $R(G) = \langle a^2 \rangle \cong C_2$. Set $\mathbb{U} = U(ZG)$. By Corollary 3.6 and Proposition 4.1(1), $G$ has the normalizer property, and $Z_\infty(\mathbb{U}) \triangleleft Z(\mathbb{U})G$. Note that $U(ZQ_8) = \pm Q_8$ (cf. [42, Theorem (2.7)]). By Corollary 3.6,

$$Z(\mathbb{U}) = (U(ZQ_8) \cap Z(\mathbb{U}))Z(G),$$

and so $Z(\mathbb{U}) = \langle a^2, x^2, y^2 \rangle$. We shall determine $G \cap Z_\infty(\mathbb{U})$. Note that

$$QG = \bigoplus_{i,j=-\infty} (QQ_8)x^iy^j.$$

Since $\frac{1}{2}(1 - a^2)QQ_8$ is the division ring of rational quaternions, and since $G/Q_8$ is ordered, it follows by a classical argument (see, for example, [42, (45.3)]) that $U(\frac{1}{2}(1 - a^2)QG) = U(\frac{1}{2}(1 - a^2)QQ_8)$. As $U(\frac{1}{2}(1 - a^2)ZQ_8) \cong Q_8$, it follows that each unit in $ZG$, when multiplied by a suitable element of $G$, lies in

$$ZG \cap \left(\frac{1}{2}(1 - a^2)QG + \frac{1}{2}(1 + a^2)QG\right) = 1 + (1 + a^2)ZG.$$ 

Since the images of $ab$ and $x$ in $G/\langle a^2 \rangle$ are central, it follows that their images in $\mathbb{U}/\langle a^2 \rangle$ are central units, and therefore $ab, x \in Z_\infty(\mathbb{U})$.

Set $e_a = \frac{1}{2}a - \frac{1}{2}Q_8$ and $e_b = \frac{1}{2}b - \frac{1}{2}Q_8$, two rational idempotents satisfying $e_a e_b = 0$ and $e_b^2 = e_b$. Thus $(e_a y)^2 = 0$, and $(e_a y)^a = -e_a y$, $(e_b y)^y = e_b y$, $(e_b y)^a y = -e_b y$ are all different from $e_a y$, and so we can apply Proposition 4.1(3) to conclude that $a, y, ay \not\in Z_\infty(\mathbb{U})$.

Summarizing, we have shown that $Z_\infty(\mathbb{U}) = Z_2(\mathbb{U}) = \langle ab, x, y^2 \rangle$.

Finally, we consider briefly the units in the integral group ring of the group which seems latently present within the study of hypercentral units, as remarked after Corollary 4.8.

**Example 7.5.** From Corollary 4.7 it follows that $Z_\infty(\mathbb{U}) = Z(\mathbb{U})$ if $G$ contains non-central involutions, which, for example, holds for the infinite dihedral group. Let

$$G = \langle a, x \mid a^4 = 1, x^a = x^{-1} \rangle \cong C_\infty \rtimes C_4.$$ 

This group has a unique involution which is central. Set $\mathbb{U} = U(ZG)$. Each element of $N_\mathbb{U}(G)$ induces a class-preserving automorphism of $G$. Since class-preserving automorphisms of metacyclic groups are inner automorphisms (see [28, Folgerung 5.15]), it follows that $N_\mathbb{U}(G) = Z(\mathbb{U})G$.

Since $G \cap Z_\infty(\mathbb{U}) \triangleleft Z_\infty(G) = \langle a^2 \rangle$, it follows from $Z_\infty(\mathbb{U}) \triangleleft N_\mathbb{U}(G)$ (Proposition 4.1(1)) that $Z_\infty(\mathbb{U}) = Z(\mathbb{U})$. Since conjugacy classes of elements of the coset $\langle x \rangle a$ have infinite length, $Z(\mathbb{U})$ is contained in $Z(\langle x \rangle \times \langle a^2 \rangle)$. The units of $Z(C_\infty \times C_2)$ being trivial, we have $Z(\mathbb{U}) = \pm \langle a^2 \rangle$. 


Set $D_{\infty} = G/\langle a^2 \rangle$, the infinite dihedral group. Units of $\mathbb{Z}D_{\infty}$ are studied in [42, Section 46]. We have

$$\mathcal{U} = U(\mathbb{Z}G) = U(1 + 2\mathbb{Z}D_{\infty}) \rtimes G,$$

where $G$ acts on the units of $1 + 2\mathbb{Z}D_{\infty}$ via the homomorphism $G \to D_{\infty}$. To justify this claim, note that a homomorphism $\varphi$ from $\mathbb{Z}G$ to the ring of $2 \times 2$ matrices over the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$ is given by

$$x\varphi = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, \quad a\varphi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The kernel of $\varphi$ is the ideal $(1 + a^2)\mathbb{Z}G$. Thus it suffices to show that $\mathcal{U}\varphi = G\varphi$. For this purpose, let $u \in \mathcal{U}$, and write $u = m_1 + m_2a$ with $m_i \in \mathbb{Z}\langle x, a^2 \rangle$. Let $*\ $ denote the classical anti-involution on $\mathbb{Z}G$. Then

$$uu^* = (m_1m_1^* + m_2m_2^*) + m_1m_2(a + a^{-1}),$$

so that $(uu^*)\varphi = (m_1m_1^* + m_2m_2^*)\varphi$ is a diagonal matrix whose diagonal entries are in the units $\pm \langle t^k \mid k \in \mathbb{Z} \rangle$ of $\mathbb{Z}[t, t^{-1}]$. It follows easily that one of the $m_i$ has the form $\pm x^k + (1 + a^2)\mathbb{Z}\langle x \rangle$ for some $k$, and the other lies in $(1 + a^2)\mathbb{Z}\langle x \rangle$. In any case, it follows that $uu^* \in G\varphi$.

We also remark that $U(1 + 2\mathbb{Z}D_{\infty}) \neq \pm 1$. For example,

$$\begin{bmatrix} 1 + 2(t - t^{-1}) & 2(-1 + t^2) \\ 2(-1 + t^2)^* & 1 + 2(t - t^{-1})^* \end{bmatrix}$$

has determinant 1, and therefore corresponds to a non-trivial unit in $1 + 2\mathbb{Z}D_{\infty}$ (see [42, (46.1)]) which can be viewed as a unit in $\mathcal{U}$:

$$u = \frac{1}{2}(1 - a^2) + \frac{1}{2}(1 + a^2)(1 + 2(x - x^{-1}) + 2(-1 + x^2)a) \in \mathcal{U}.$$

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**References**


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