1. Introduction

This paper is a substantially extended version of the note [7]. Let $G$ be a finite group, and denote its integral group ring by $\mathbb{Z}G$. There are many interesting questions dealing with the unit group $U(\mathbb{Z}G)$ of $\mathbb{Z}G$. Then, it is obviously sufficient to consider the normalized unit group $V(\mathbb{Z}G)$, which consists of the units of $\mathbb{Z}G$ with augmentation 1. Some of the questions concern the finite subgroups of $V(\mathbb{Z}G)$. For example, is any finite subgroup $H$ of $V(\mathbb{Z}G)$ isomorphic to a subgroup of $G$? If $H$ is a group basis of $\mathbb{Z}G$, which means that $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras, this question has become known as the ‘isomorphism problem for integral group rings’ (a negative answer to this problem has been given recently in [5, 6]). Over the last twenty years, the following conjecture of H. Zassenhaus has been at the centre of research.

(ZC) For any group basis $H$ of $\mathbb{Z}G$ there is a central automorphism $\alpha$ of $\mathbb{Z}G$ with $H = G\alpha$.

A survey about (ZC) and related questions is given in [11]. By definition, a central automorphism fixes the centre element-wise, so with $\alpha$ as above, $G$ and $G\alpha$ are rationally conjugate, that is, conjugate in the units of the rational group ring $\mathbb{Q}G$, by the Noether–Skolem Theorem. In particular, (ZC) does not only imply a positive answer to the isomorphism problem, it also makes a strong statement about the embedding of group bases in $\mathbb{Z}G$. Though it has been known for some time that (ZC) is not true in general, it has given a conceptual point of view to many problems related to the integral group ring. The conjecture is closely connected with the names of K. W. Roggenkamp and L. L. Scott, who gave the first counter-examples [24] and proved in the fundamental paper [22] that for a finite $p$-group $G$, any two group bases of $\mathbb{Z}_pG$ are $p$-adically conjugate. (At this point one should remark that all definitions made above also make sense when the coefficient ring $\mathbb{Z}$ is replaced by an arbitrary commutative ring.)

This paper deals with another theorem due to Roggenkamp and Scott (see [23; 27; 21, VI.1]), henceforth called the ‘F’-Theorem’. In order to state it, let $R$ be a $p$-adic ring, that is, the integral closure of the $p$-adic integers $\mathbb{Z}_p$ in a finite extension field of the $p$-adic field $\mathbb{Q}_p$. (Then $R$ is a complete discrete valuation ring.)

If $N$ is a normal subgroup of $G$, then $I_R(N)G$ denotes the kernel of the natural map $RG \rightarrow RG/N$. 

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Let \( G \) be a finite group with a normal \( p \)-subgroup \( N \) containing its centralizer, \( C_G(N) \subseteq N \). Then for any augmented automorphism \( \alpha \) of \( RG \) which stabilizes \( I_R(N)G \), the groups \( G\alpha \) and \( G \) are conjugate in the units of \( RG \).

It should be remarked that Roggenkamp and Scott have stated this theorem only for unramified extensions \( R \) of \( \mathbb{Z}_p \) (however, see [17, (4.2) Remarks] and [27, p. 266]).

The hypothesis on \( G \) is that \( G \) is \( p \)-constrained with \( O_{p'}(G) \), or, equivalently, that the generalized Fitting subgroup \( F^*(G) \) equals \( O_p(G) \) (whence the name of the theorem).

Though the theorem presents itself in the work of others and appears to be needed (cf. [10, 12, 13, 27, 28]), there is no published account. Some ingredients of the proof are given in [15–19]. There seems to be a flaw in the proof, because a ‘theorem’ appearing in the surveys [16, 17] is false; see § 4, where it is shown how the use of it can be avoided if one is only interested in automorphisms of \( ZG \). Also a complete, yet short, proof of the \( F^* \)-Theorem is given in the case that \( G \) has a normal Sylow \( p \)-subgroup (Theorem 4.6; then, the assumption on the stabilization of \( I_R(N)G \) becomes superfluous).

It will become clear that the strong results of A. Weiss on permutation modules [30, 31] constitute the basis for any proof of the \( F^* \)-Theorem. In § 3, some consequences of these results which fit well in the given context are recorded. Section 2 contains some preparatory results.

It should be remarked that it is not known whether the stabilization of the ideal \( I_R(N)G \) is automatic for \( N = O_p(G) \). This seems to be a fundamental question in representation theory. This problem does not occur in the ‘global’ situation, when a ‘class sum correspondence’ is available (see [26; 21, IV]). More precisely, let \( S \) be a \( G \)-adapted ring, that is, an integral domain of characteristic 0 in which no prime divisor of \( |G| \) is invertible. (A basic example is \( \mathbb{Z}_\pi(G) \), the intersection of the localizations \( \mathbb{Z}_{(p)} \) with \( p \) in \( \pi(G) \), the set of prime divisors of \( |G| \).) Then any augmented automorphism \( \alpha \) of \( SG \) automatically stabilizes \( I_S(O_p(G))G \). Since questions about isomorphisms tend to reduce to questions about automorphisms (see [9, 5.3; 22, p. 597]), the following theorem, which includes a positive answer to the isomorphism problem for \( G \), is a consequence of the \( F^* \)-Theorem.

**Theorem 1.1.** Let \( G \) be a finite group with a normal \( p \)-subgroup \( N \) containing its centralizer, \( C_G(N) \subseteq N \). Then \( (ZC) \) holds for \( \mathbb{Z}_{\pi(G)}G \), that is, any two group bases of \( \mathbb{Z}_{\pi(G)}G \) are rationally conjugate.

The following theorems emphasize the impact of the \( F^* \)-Theorem on the structure of unit groups of integral group rings. As a general reference, we refer to the books [21, 29]. The first theorem has been proved in [13], and depends heavily on Theorem 1.1.

**Theorem 1.2.** Let \( G \) be a finite solvable group and \( H \) a group basis of \( \mathbb{Z}G \). Then for all \( p \), Sylow \( p \)-subgroups of \( H \) and \( G \) are conjugate in the units of \( \mathbb{Q}G \).

Sometimes statements about (the automorphisms of) \( \mathbb{Z}G \) can be deduced from knowledge about the automorphisms of \( \mathbb{Z}G/N_i \), where the non-identity \( N_i \trianglelefteq G \) are such that \( G \) naturally embeds into \( \prod_i G/N_i \). In inductive proofs, the \( F^* \)-Theorem
may guarantee the existence of such normal subgroups. This strategy has been successfully applied in [10] to give the following theorem.

**Theorem 1.3.** The isomorphism problem for integral group rings has a positive solution for finite nilpotent by abelian groups.

In [12], the representation of a (solvable) group $G$ as a subdirect product of various factor groups (as above), together with the corresponding projective limits of group rings, has been systematically studied. However, the resulting theory leads to effective results only if Theorem 1.1 holds.

2. Preparatory results

In this section, some well-known results that are needed for the proof of the $F^*$-Theorem are given, along with some additional information. In the first step of this proof, it is shown that the normal $p$-subgroups $N$ and $Na$ are conjugate in the units of $RG$ (cf. Lemma 4.1); the following lemma explains to some extent why conjugacy of $p$-groups is so powerful. It is an extension of a result of D. B. Coleman (see [21, II, Lemma 2.1]; in its present form, it is due to A. I. Saksonov [26, 1.14]).

Throughout this section, $G$ denotes a finite group.

**Lemma 2.1.** Let $P$ be a $p$-subgroup of $G$, and $R$ a commutative ring with $pR \neq R$. Then $N_V(P) = N_G(P) \cdot C_V(P)$, where $V = V(RG)$.

The following proposition is based on this lemma and an observation of F. Gross [4, Corollary 2.4]. It shows that a ‘conjugating unit’ occurring in the $F^*$-Theorem is unique up to central units and group elements.

**Proposition 2.2.** Let $N$ be a normal $p$-subgroup of $G$ containing its own centralizer, $C_G(N) \subseteq N$. Let $R$ be a commutative ring with $pR \neq R$, and let $V = V(RG)$. Then $N_V(G) = G \cdot C_V(G)$.

**Proof.** Take any $u \in N_V(G)$ and define $\varphi \in \text{Aut}(G)$ by $g \varphi = g^u$ for all $g \in G$. We have to show that $\varphi \in \text{Inn}(G)$. Let $S$ be a Sylow $p$-subgroup of $G$. By Lemma 2.1, we may assume that every element of $S$ is fixed by $\varphi$. In particular, all elements of $N$ are fixed by $\varphi$, so $g^{-1}(g \varphi) \in C_G(N) = Z(N) =: A$ for all $g \in G$. Hence $\varphi$ fixes every element of $A$ and every coset of $A$, so there is a 1-cocycle $\delta \in Z^1(G/A, A)$, defined by $\delta(gA) = g^{-1}(g \varphi)$ for all $g \in G$. It suffices to show that $\delta$ is a 1-coboundary (see [8, I, 17.1]). Since $A$ is a $p$-group, the restriction map $H^1(G/A, A) \rightarrow H^1(S/A, A)$ is injective [8, I, 16.21], and the image of $\delta$ under this map is the zero element since every element of $S$ is fixed by $\varphi$.

The following criteria on indecomposability of permutation modules plays a crucial role in the proof of the $F^*$-Theorem (see Lemma 4.1 below). It is proved in [15, p.231, Claim 1]. Here, a different proof is given, thereby pointing out an important detail.

Let $R$ be either a complete discrete valuation ring with residue class field of characteristic $p > 0$, or a field of characteristic $p > 0$.\"
Lemma 2.3. Let $N$ be a normal $p$-subgroup of $G$ with $C_G(N) \subseteq N$. Let $\alpha$ be either a group automorphism of $N$ or an augmented $R$-algebra automorphism of $RG$. Consider $M = RG$ as an $R(G \times N)$-module, the action being given by $v \cdot (g, n) = g^{-1}v(n\alpha)$ for all $n \in N$, $g \in G$ and $v \in M$. Then $M$ is absolutely indecomposable.

Proof. Let $E = \text{End}_{R(G \times N)}(M)$; we show that $E / \text{rad}(E) \cong R / \text{rad}(R)$ (cf. [3, §30B]). Let $\varphi \in E$. Then $\varphi$ is given by right multiplication with $1\varphi$, and $1\varphi$ is centralized by $N\alpha$. In this way, we may identify $E$ with $C_{RG}(N\alpha)$. If $\alpha$ is a group automorphism of $N$, then it follows from $C_G(N) \subseteq N$ that

$$1\varphi \in RC_G(N) + I_R(N)G + pRG \subseteq RN + I_R(N) + pRG \subseteq R + \text{rad}(RG)$$

(for the last step, see [3, 5.26]), and if $\alpha$ is an augmented $R$-algebra automorphism of $RG$, then we obtain similarly

$$1\varphi \in RC_{Ga}(N\alpha) + I_R(N\alpha)G\alpha + pRG \subseteq R + \text{rad}(RG).$$

If $1\varphi \in \text{rad}(RG)$, then $1\varphi \in \text{rad}(C_{RG}(N\alpha))$ (see [3, 5.6, §5B]). This completes the proof. \hfill $\square$

For later use (Proposition 3.12, Remark 4.4), the observation on the centralizer of $N$ just made is recorded in a separate lemma. Let $E$ be a discrete valuation ring with residue class field of characteristic $p > 0$, choose $\pi \in R$ with $\text{rad}(R) = \pi R$, and let $t \in \mathbb{N}$ be such that $p^t R \subseteq \pi^t R$.

Lemma 2.4. Let $N$ be a normal $p$-subgroup of $G$ with $C_G(N) \subseteq N$, and let $R$ and $\pi^t$ be as above. Let $C_i \subseteq RG$ consist of those $x \in RG$ such that $xn \equiv nx \mod \pi^t RG$ for all $n \in N$. Then $C_i$ is contained in $R + I_R(N)G + \pi^t RG$. In particular, elements of $C_i$ of augmentation 1 are units in $RG$. \hfill $\square$

Finally, let us recall the following useful criteria for when two finite subgroups of $V(RG)$ are conjugate; the arguments work for any commutative ring $R$. Let $H$ be a finite group, and $\Lambda = \varepsilon RG$, for some central idempotent $\varepsilon$ of $RG$. We are interested in group homomorphisms $\alpha: H \to \Lambda^x$. Two such homomorphisms $\alpha$ and $\beta$ are called $R$-equivalent if there is a unit $u \in \Lambda^x$ such that $h\alpha = u^{-1} \cdot h\beta \cdot u$ for all $h \in H$. Given $\alpha$, we turn $\Lambda$ into a right $R(G \times H)$-module by linear extension of the group action $m \cdot (g, h) = g^{-1}m(h\alpha)$ for all $g \in G$, $h \in H$ and $m \in \Lambda$. This module is denoted by $\Lambda \alpha$.

Lemma 2.5. Group homomorphisms $\alpha, \beta: H \to \Lambda^x$ are $R$-equivalent if and only if $\Lambda \alpha \cong \Lambda \beta$ as $R(G \times H)$-modules.

Proof. If there is $u \in \Lambda^x$ with $h\beta = u^{-1} \cdot h\alpha \cdot u$ for all $h \in H$, then right multiplication with $u$ is an isomorphism $\Lambda \alpha \to \Lambda \beta$ of $R(G \times H)$-modules. Conversely, let $\theta: \Lambda \alpha \to \Lambda \beta$ be an $R(G \times H)$-isomorphism. Then $\theta$ is given by multiplication with $u \in \varepsilon\theta$ from the right, $h\alpha \cdot u = u \cdot h\beta$ for all $h \in H$, and $\Lambda u = \Lambda$ implies that $u$ is a unit in $\Lambda$. \hfill $\square$

The next lemma shows how results on permutation lattices come into play.
A homomorphism $\alpha: H \to V(RG)$ is $R$-equivalent to a homomorphism $H \to G$ (followed by the inclusion $G \hookrightarrow V(RG)$) if and only if $iRG_\alpha$ is a permutation lattice for $G \times H$ over $R$.

**Proof.** Suppose that $iRG_\alpha$ is a permutation lattice for $G \times H$ over $R$. Since the restriction of $iRG_\alpha$ to $G \times 1$ is a transitive permutation lattice, it follows that there is $u \in RG$ such that $Gu$ is an $R$-basis of $RG$ whose elements are permuted under the action of $G \times H$. It follows that $u \in U(RG)$ and $H \alpha \leq G^u$. The converse is obvious.

### 3. Some applications of Weiss’ results

The paper [30] of A. Weiss contains results which are of fundamental interest for the representation theory of $p$-group permutation modules. The proof of the F$^*$-Theorem relies heavily on them, or respectively their generalization (to arbitrary $R$), first given by K. W. Roggenkamp [20], and then by A. Weiss in [31].

In this section, some useful conclusions of these results, which stand in direct relationship to the F$^*$-Theorem, are recorded.

Let us recall the following definitions from [31]. Let $G$ be a finite group and $R$ a $p$-adic ring. An $R$-representation of $G$ is an $RG$-module which is free of finite rank as an $R$-module. A monomial lattice for $G$ over $R$ is an $RG$-module which is isomorphic to a direct sum of modules of the form $\phi|_H^G$, for some homomorphism $\phi$ from the subgroup $H$ into the group $R^\times$ of units of $R$. Here $\phi|_H^G$ stands for $1^* RG$ with $H$ acting on $1^* R$ via $\phi: H \to R^\times$, and if $H$ acts trivially on $1^* R$, then $1^* H$ is written instead of $\phi|_H^G$. A permutation lattice for $G$ over $R$ is an $RG$-module which is isomorphic to a direct sum of modules of the form $1^* H$.

Throughout this section, $\zeta$ denotes a primitive $p$th root of unity.

For the convenience of the reader, we state Weiss’ results as given in [31].

A short proof of the first theorem can be found in [14, Appendix 1].

**Theorem 3.1.** Let $M$ be an $R$-representation of the finite $p$-group $G$. Suppose $N$ is a normal subgroup of $G$ so that:

(a) the $N$-module $M_N$ is a free $RN$-module;

(b) the fixed-point module $M^N$ is a permutation lattice for $G/N$ over $R$.

Then $M$ is a permutation lattice for $G$ over $R$.

The discussion of the F$^*$-Theorem in the next section will show that this theorem provides the basic tool for proving conjugacy of Sylow $p$-subgroups.

**Theorem 3.2.** Assume that $\zeta \in R$, and let $M$ be an $R$-representation of the finite $p$-group $G$. Suppose $M/(1-\zeta)M$ is the reduction modulo $(1-\zeta)$ of some monomial lattice of $G$ over $R$. Then $M$ itself is a monomial lattice of $G$ over $R$.

**Lemma 3.3.** Let $G$ be a finite $p$-group.

(a) If $M$ is a permutation lattice for $G$ over $R$ then $H^1(G,M) = 0$.

(b) If $\zeta \in R$ and $M$ is a monomial lattice for $G$ over $R$ then $1-\zeta$ annihilates $H^1(G,M)$.
For the rest of this section, \( G \) always denotes an arbitrary finite group. We fix \( \pi \in \mathbb{C} \) with \( \text{rad}(\mathbb{C}) = \pi \mathbb{C} \) and denote by \( e \) the ramification index of \( K / \mathbb{Q}_p \), where \( K \) is the quotient field of \( \mathbb{C} \). Thus \( \mathbb{C} = \pi^e \mathbb{C} \). We also fix a \( p \)-adic ring \( S \) which contains \( R \) and \( \xi \).

The following corollaries of Theorem 3.2 and Lemma 3.3 extend their scope considerably. We begin with an easy lemma.

**Lemma 3.4.** Let \( t \in \mathbb{N} \). Then \( \pi^t S \subseteq (1 - \xi) S \) if and only if \( \pi^{t(p-1)} R \subseteq \pi R \), and if one inclusion is proper, then so is the other.

**Proof.** Since \( (1 - \xi)^{p-1} / p \in S^\times \), it follows that \( \pi^t S \subseteq (1 - \xi) S \) if and only if \( \pi^{t(p-1)} S \subseteq \pi S \). In that case, \( \pi^{t(p-1)} / p \) is integral, so \( \pi^{t(p-1)} = pr \) for some \( r \in \mathbb{C} \). If one of the inclusions is proper, then \( r \) lies in \( \text{rad}(\mathbb{C}) \). This proves the lemma. \( \square \)

In particular, it follows that if \( t > e/(p-1) \), then the roots of unity of \( K \) map injectively into \( (\mathbb{C}/\pi \mathbb{C})^\times \), which is also proved in [1, Lemma 4.3].

The following corollary is a consequence of Lemma 3.3, but we should mention that a proof which avoids this lemma has been given by R. Boltje and B. Külshammer [1, Theorem 4.5(e)].

**Corollary 3.5.** Let \( M \) and \( N \) be monomial lattices for \( G \) over \( R \), and let \( t \in \mathbb{N} \). If \( t > e/(p-1) \) and \( M/\pi^t M \equiv N/\pi^t N \) as \( R/\pi^t R \)-representations for \( G \), then \( M \equiv N \).

**Proof.** The reductions modulo \( \pi^t \) of the \( S \)-representations \( M' = S \otimes_R M \) and \( N' = S \otimes_R N \) of \( G \) are isomorphic.

By [31, Lemma (53.1)(a)], \( \text{Hom}_S(M', N') \) is a monomial lattice for \( G \) over \( S \), and by Lemma 3.3 and [8, 1,16.18], \( \text{Ext}^1_G(M', N') \), which is isomorphic to \( \text{H}^1(G, \text{Hom}_S(M', N')) \), is annihilated by \( 1 - \xi \).

Since \( \pi^t S \subseteq (1 - \xi) S \) by Lemma 3.4, it follows as in [3, 30.14] (a result due to Maranda), that a monomial lattice for \( G \) over \( S \) is uniquely determined by its reduction modulo \( \pi^t \), and application of [3, 30.25] concludes the proof. \( \square \)

Using Theorem 3.2, this result can be strengthened to give the following corollary.

**Corollary 3.6.** Let \( M \) be an \( R \)-representation of \( G \), and \( N \) a monomial lattice for \( G \) over \( R \). Suppose that \( t \in \mathbb{N} \) is chosen such that \( t > e/(p-1) \). If \( M/\pi^t M \equiv N/\pi^t N \) as \( R/\pi^t R \)-representations for \( G \), then \( M \equiv N \).

**Proof.** The reductions modulo \( \pi^t \) of the \( S \)-representations \( M' = S \otimes_R M \) and \( N' = S \otimes_R N \) of \( G \) are isomorphic. Let \( P \) be a Sylow \( p \)-subgroup of \( G \). Since \( \pi^t S \subseteq (1 - \xi) S \) by Lemma 3.4, it follows from Theorem 3.2 that with \( N'_P \), the restriction of \( N' \) to \( P \), also the restriction \( M'_P \) is a monomial lattice for \( G \) over \( S \). By Lemma 3.3 and [8, 1,16.18], it follows that \( 1 - \xi \) annihilates \( \text{H}^1(G, \text{Hom}_S(N'_P, M'_P)) \). Again, it follows as in [3, 30.14] that \( M' \equiv N'_P \), and therefore \( M \equiv N \) by [3, 30.25]. \( \square \)
The difference between $\pi tS$ and $(1 - \zeta)S$ is crucial. The example given in [20, p. 343; 21, VI.1.4] does not illustrate this, since there $\pi^{(p-1)}R \neq pR$ for all $t$. The following example was communicated to the first author by G. Nebe.

**Example 3.7.** Let $R = \mathbb{Z}_3[\sqrt{3}]$, $i = -1$, and $\zeta = \frac{1}{3}(1 + i\sqrt{3})$, a primitive 3rd root of unity. Then $S = R[\zeta] = \mathbb{Z}_3[\sqrt{3}, i]$ is the integral closure of $R$ in the quotient field of $R[\zeta]$, $\text{rad}(R) = \pi R$ with $\pi = \sqrt{3}$ and $\pi S = (1 - \zeta)S$.

Let

$$A = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & -2 \end{bmatrix}.$$ 

Then $A$ is a unit of order 3 in $\text{Mat}_2(R)$, with 

$$A \equiv 1 \mod \sqrt{3} \quad \text{and} \quad T^{-1}AT = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix},$$

where

$$T = \begin{bmatrix} i & i \\ \zeta^2 & -\zeta \end{bmatrix} \in \text{GL}_2(S).$$

So if $G = \langle A \rangle$ acts naturally on $M := R \oplus R$ by multiplication, then $G$ acts trivially on $M/\pi M$, and $S \otimes_R M$ is a monomial lattice for $G$ over $S$, but $M$ itself is not a monomial lattice.

The following proposition improves Proposition 3.1 of [7]. (Since these notes were written, the work [2] of Cliff and Weiss has come to our attention; in it the same argument – counting fixed point dimensions – is used). Let $K$ be the quotient field of $R$.

**Proposition 3.8.** Suppose that $M$ is a direct summand of a monomial lattice for $G$ over $R$ such that $M/\pi M$ is the reduction modulo $\pi$ of some permutation lattice $N$ for $G$ over $R$, and $K \otimes_R M \equiv K \otimes_R N_\pi$ as KP-modules for some Sylow $p$-subgroup $P$ of $G$. Then $M \equiv N$.

**Proof.** Suppose that $G$ is a $p$-group. By Green’s Indecomposability Theorem, a module of the form $\varphi_{G}^{H}$, for some homomorphism $\varphi: H \to R^{n}$, is indecomposable. Hence $M = \bigoplus_{i=1}^{n} \varphi_{G}^{H}$, is a monomial lattice for $G$ over $R$, by the Krull–Schmidt Theorem. Then $M/\pi M$ is the reduction of the permutation lattice $L = \bigoplus_{i=1}^{n} \varphi_{G}^{H}$, so $L \equiv N$ (see [3, 81.17]). The fixed point dimension $d = \dim_{K}(K \otimes_{R} M)^{G}$ can be computed from the character $\xi$ of $M$: it is the number of times the trivial character occurs in $\xi$, which is by Frobenius reciprocity the number of homomorphisms $\varphi_{i}$ which are trivial. Since $K \otimes_{R} M \equiv K \otimes_{R} L$, the same argument shows that $d = n$, whence $M \equiv L \equiv N$.

Now let $G$ be arbitrary, and let $P$ be a Sylow $p$-subgroup of $G$. We have just proved that $M_{\pi}$ is a permutation lattice for $P$ over $R$. Hence $M$ is a direct summand of a permutation lattice for $G$ over $R$. But $M/\pi M$ lifts uniquely (up to isomorphism) to a direct summand of a permutation lattice for $G$ over $R$ [3, 81.17], and it follows that $M \equiv N$. \qed

For $t \in \mathbb{N}$, let $\kappa_{t}: RG \to (R/\pi^{t}R)G$ denote reduction modulo $\pi^{t}$. The next theorem deals with groups of units in $RG$ which, modulo $\pi^{t}$, are conjugate to subgroups of $G$. 

Theorem 3.9. Let \( H \) be a finite group, and \( \varphi, \sigma : H \rightarrow U(RG) \) group homomorphisms with \( H \sigma \subseteq G \) and \( \varphi \kappa_i = \sigma \kappa_i \) for some \( i \in \mathbb{N} \) such that \( i \geq e/(p - 1) \). If the latter is an equality, assume further that \( \varphi|_p \) and \( \sigma|_p \) are \( K \)-equivalent, for some Sylow \( p \)-subgroup \( P \) of \( H \). Then \( \varphi \) is \( R \)-equivalent to \( \sigma \). More precisely, there is a unit \( u \in 1 + \pi^t RG \) such that \( x \varphi = x \sigma^u \) for all \( x \in H \).

Proof. Let \( M = _1RG_{\varphi} \) and \( N = _1RG_{\sigma} \) as \( R \)-representations of \( G \times H \). By assumption, \( M/\pi^t M \) is the reduction modulo \( \pi^t \) of the permutation lattice \( N \). It is shown that \( M \cong N \). If \( t > e/(p - 1) \), this follows from Corollary 3.6. So let us assume that \( \varphi|_p \) and \( \sigma|_p \) are \( K \)-equivalent, for some Sylow \( p \)-subgroup \( P \) of \( H \). By Theorem 3.2 and Lemma 3.4, \( M^t := S \otimes_R M \) is a direct summand of a monomial lattice for \( G \times H \) over \( S \). Clearly \( M^t/\pi^t M^t \) is the reduction modulo \( \pi^t \) of the permutation lattice \( N^t := S \otimes_R N \) for \( G \times H \) over \( S \), and by Lemma 2.5, \( L \otimes_S M^t_{p\times p} \cong L \otimes_S N^t_{p\times p} \), where \( L \) is the quotient field of \( S \). Hence \( M^t \cong N^t \) by Proposition 3.8, and \( M \cong N \) by the Noether–Deuring Theorem [3, 30.25]. By Lemma 2.5, it follows that \( \varphi \) and \( \sigma \) are \( R \)-equivalent.

Let us make the connection with cohomology explicit. Let \( V = RG \), considered as an \( R \)-representation of \( H \), the action being given by \( v \cdot x = (v \alpha)^{-1}v(x \varphi) \) for all \( x \in H \) and \( v \in V \). A 1-coboundary \( \theta \in B^1(H, V) \) is given by
\[
\theta(x) = 1 \cdot (1 - x) = 1 - (x \alpha)^{-1}(x \varphi)
\]
for all \( x \in H \). Since \( \varphi \kappa_i = \sigma \kappa_i \), there is \( \beta \in Z^1(H, V) \) with \( \beta = \pi^t \cdot \beta \). We already know that \( V \) is a permutation lattice for \( H \) over \( R \). By Lemma 3.3, and [8, I, 16.18], \( \beta \) is a coboundary, so there is \( v \in V \) with
\[
1 - (x \alpha)^{-1}(x \varphi) = \pi^t(v - (x \alpha)^{-1}v(x \varphi)) \quad \text{for all} \quad x \in H.
\]
With \( u = 1 - \pi^t v \), it follows that \( (x \alpha)u = u(x \varphi) \) for all \( x \in H \), and \( u \) is a unit in \( RG \) since \( u \in 1 + \text{rad}(RG) \). The theorem is proven. \( \square \)

Remark 3.10. A similar statement holds for homomorphisms
\[
\varphi, \sigma : H \rightarrow U(eRG)
\]
with \( H \sigma \subseteq eG \), where \( e \) is a central idempotent of \( RG \). Then, it suffices to consider the case where \( G \rightarrow eG \) is an isomorphism, so that via
\[
\begin{array}{ccc}
H & \xrightarrow{\sigma} & eG \\
\end{array}
\]
an action of \( H \) on \( RG \) can be defined. One may consider the \( R \)-representations
\[
M = _1eRG_e \oplus (1 - e)RG_e \quad \text{and} \quad N = _1RG_e
\]
of \( G \times H \), and apply the Krull–Schmidt Theorem after showing that \( M \cong N \).

The following example shows that in the situation of Theorem 3.9, some additional assumptions besides \( t \geq e/(p - 1) \) are necessary.

Example 3.11. Let \( G = \langle a : a^2 \rangle \times \langle b : b^n \rangle \) for some odd number \( n > 1 \); so \( G \) is a cyclic group of order \( 2n \). Then \( e = n^{-1} \sum_{i=1}^{n} b_i \) is a central idempotent in \( \mathbb{Z}_2 G \), and a unit of order 2 is given by \( c = (e - (1 - e))a = (-1 + 2e)a \).

A normalized automorphism \( \varphi \) of \( \mathbb{Z}_2 G \) is defined by \( a \varphi = c \) and \( b \varphi = b \). Though \( \varphi \) induces the identity modulo 2, it is not an inner automorphism.
However, note the following remarkable instance.

**Proposition 3.12.** Assume that $G$ has a normal $p$-subgroup $N$ with $C_G(N) \subseteq N$. Let $H$ be a finite group, $P$ a Sylow $p$-subgroup of $H$, and $\varphi, \sigma: H \rightarrow \text{V}(RG)$ group homomorphisms with $N \subseteq H \sigma \subseteq G$ and $\varphi|_P \cdot \kappa_t = \sigma|_P \cdot \kappa_t$ for some $t \in \mathbb{N}$ with $t \geq e/(p - 1)$. Then $\varphi$ is $R$-equivalent to $\sigma$.

**Proof.** Consider the augmentation ideal $V = I_S(G)$ as an $S$-representation of $H$, the action of $H$ given by $v \cdot x = (x \sigma)^{-1}v(x \varphi)$ for all $x \in H$ and $v \in V$. A 1-cocycle $\theta \in Z^1(H, V)$ is given by $\theta(x) = 1 - (x \sigma)^{-1}(x \varphi)$ for all $x \in H$.

We shall show that $\theta$ is a coboundary. By [8, I, 16.18], it suffices to show that $\theta|_P$ is a coboundary. By assumption, and Theorem 3.2, $V_P$ is a monomial lattice for $P$ over $S$. There is $\beta \in Z^1(P, V)$ with $\theta|_P = \pi^t \cdot \beta$. Since $\pi^t \in (1 - \xi)S$, it follows from Lemma 3.3 that $\theta|_P$ is a coboundary.

Hence there is $v \in V$ with

$$1 - (x \sigma)^{-1}(x \varphi) = v - (x \sigma)^{-1}v(x \varphi) \quad \text{for all } x \in H.$$ 

With $u = 1 - v$, it follows that $(x \sigma)u = u(x \varphi)$ for all $x \in H$, and $u$ is a unit in $SG$ by Lemma 2.4.

By Lemma 2.5, $1SG_{\varphi} \cong 1SG_{\sigma}$, and application of the Noether–Deuring Theorem concludes the proof.

We finish this section with two corollaries concerning automorphisms of group rings.

**Corollary 3.13.** Let $t \in \mathbb{N}$ with $t > e/(p - 1)$. Let $\alpha$ be an automorphism of $RG$ which induces the identity on $(R/\pi^t R)G$. Then $\alpha$ is an inner automorphism of $RG$. In other words, the homomorphism Picent$(RG) \rightarrow$ Picent$((R/\pi^t R)G)$ is injective.

**Proof.** This follows from Theorem 3.9. As to the reformulation, see [3, remark after (55.23)].

**Corollary 3.14.** Let $A = \mathbb{Z}_{\pi(G)}$, and $\alpha$ be an automorphism of $AG$ which induces the identity on $(A/pA)G$, for some prime $p \in \pi(G)$. Then $\alpha$ is given by conjugation with a unit from $\mathbb{Z}_{(p)}G$.

**Proof.** Since $A$ is $G$-adapted, the class sum correspondence is available (see [21, IV]). Since $\alpha$ fixes all class sums of $G$, it is given by conjugation with a rational unit (for this kind of reasoning, see [29, 37.6, 41.4]). It follows from Theorem 3.9 that $\alpha$ is given by conjugation with a unit from $\mathbb{Z}_{(p)}G$, and hence also with a unit from $\mathbb{Z}_{(p)}G$ (see [29, 41.18]).

4. **Towards the proof of the $F^*$-Theorem**

The intention of this section is to give at least an idea of how a proof of the $F^*$-Theorem should work. Emphasis is put on the additional difficulties that arise when $p = 2$. Some ingredients towards a proof are given in [15–17]. Additionally in this section, a short proof for Proposition 4.3 below is presented, which enables
one to avoid [17, (4.8) Lemma]. Also, Lemma 4.2 below is proved, so that a complete, yet short, proof of [15, Theorem 6] (the case of a normal Sylow $p$-subgroup) can be given.

Let $RG$, $N$ and $\alpha$ satisfy the assumptions of the $F^*$-Theorem. Essentially, any proof of it should consist of the following steps.

**Step 1.** (a) Modify $\alpha$ by an inner automorphism such that $N\alpha = N$.

(b) Modify $\alpha$ by an inner automorphism such that $P\alpha = P$.

**Step 2.** Modify $\alpha$ by a group automorphism of $G$ such that $\alpha$ fixes $P$ element-wise.

**Step 3.** Show that $\alpha$ is an inner automorphism.

There might be some overlap between the first two steps, for example if the proof involves an induction on the group order, using calculations in the ‘small group ring’.

4.1. **Towards Step 1(a)**

This step has been sketched in [15, p. 231]. For the convenience of the reader, the missing details are given below.

As before, $M = RG$ is considered as an $R(G \times G)$-module, the action of $G \times G$ being given by $m \cdot (x, y) = x^{-1}m(y\alpha)$ for all $(x, y) \in G \times G$. The proof of the next lemma makes use of the elementary theory of vertices and sources, the Krull–Schmidt Theorem, and the Mackey decomposition. As general reference we give [3].

**Lemma 4.1.** Assume that $\alpha$ stabilizes $I_R(N)G$. Then the groups $N$ and $N\alpha$ are conjugate in the units of $RG$.

**Proof.** The fixed point module $M^{N \times 1}$ is the annihilator in $RG$ of $I := I_R(N)G$, and the fixed point module $M^{1 \times N}$ is the annihilator in $RG$ of $I_R(N\alpha)G\alpha = I\alpha$. Therefore $M^{N \times 1} = M^{1 \times N}_1$, and $M^{1 \times N}$ is a permutation lattice for $P \times 1$ over $R$. By Theorem 3.1,

$$M_{P \times N} \cong \bigoplus_{j=1}^{\frac{|G: P|}{P \times 1}} 1_{U_j}^{P \times N}.$$  \hfill (1)

The number of summands equals the $R$-rank of $M^{P \times N}$. As $M^{P \times N} = M^{P \times 1}$, it follows that $n = |G : P|$. Now

$$M_{P \times 1} \cong \bigoplus_{j=1}^{\frac{|G: P|}{U_j \setminus a \cap (P \times 1)}} 1_{U_j}^{P \times 1} \cap (P \times 1),$$

and similarly for $M^{1 \times N}$. Since both modules are free, it follows that $U_j \cap (P \times 1) = 1$ and $U_j \cap (1 \times N) = 1$ for all $j$. In particular, $|U_j| \leq |N|$ for all $j$, and comparing ranks in (1) gives $|U_j| = |N|$ for all $j$. Restricting to $N \times N$ in (1) gives

$$M_{N \times N} \cong \bigoplus_{j=1}^{\frac{|G: P|}{U_j \cap (N \times N)}} 1_{U_j \cap (N \times N)}^{N \times N}.$$
The number of summands in this decomposition is the $R$-rank of $M^{N \times N}$. Since $M^{N \times N} = M^{N \times 1}$, there are $|G : N|$ summands. It follows that $U_j \leq N \times N$ for all $j$. Hence $U_j = \{(x, x\alpha_j) : x \in N\}$ for some $\alpha_j \in \text{Aut}(N)$, and the $R(G \times N)$-module $1_{U_j}^{G \times N}$ is isomorphic to $RG$, with action $m \cdot (g, x) = g^{-1}m(x\alpha_j^{-1})$ for all $g \in G$, $x \in N$ and $m \in RG$. By Lemma 2.3, the modules $1_{U_j}^{G \times N}$ and $M_{G \times N}$ are indecomposable. Since $M_{G \times N}$ is relatively $(P \times N)$-projective, it follows from (1) that $M_{G \times N} \cong 1_{U_j}^{G \times N}$ for some $j$. Hence $M$ is a permutation lattice for $G \times N$ over $R$, and $N\alpha$ is conjugate to a subgroup of $G$ in the units of $RG$, by Lemma 2.6. More precisely, the given isomorphism tells us that there is $u \in M$ so that $u \cdot RG = RG$ and $u \cdot U_j = u$; so $u$ is a unit in $RG$ and $u^{-1}xu = (x\alpha_j)\alpha$ for $x \in N$, that is, $N^u = N\alpha$.

Thus we assume from now on that $N = N\alpha$.

4.2. Towards Step 1(b)

Aiming for this step, one can, in the same spirit, prove the following: if the fixed-point module $M^{P \times 1}$ is a permutation lattice for $1 \times P$ over $R$, then $P$ and $P\alpha$ are conjugate in the units of $RG$. Also, if $M = k \otimes_R M$ is a $k(P \times P)$-permutation module, where $k = R/\text{rad}(R)$, it follows in the very same way that $\hat{P}$ and $P\alpha$ are conjugate in the units of $kG$. However, the difficulty is to verify the assumption, that is, that $M^{P \times 1}$ is a permutation lattice for $1 \times P$ over $R$, especially if one tries to avoid a complicated induction. (In this context, it should be noted that one might also try to prove the $F^*$-Theorem under the assumption that $\alpha$ only stabilizes $I_1(N)G$, but the difficulties remain.) Note that $M^{P \times 1}$ is a permutation lattice for $1 \times P$ over $R$ if $1_{RG}/N\alpha$ (which is isomorphic to $M^{N \times 1}$) is a permutation lattice for $P \times P$ over $R$. To get a better understanding of the problem, consider the following diagram, where $\mathcal{R} = R/pR$:

\[
\begin{array}{ccc}
1_{RG\alpha} & \longrightarrow & 1_{RG}/N\alpha \\
\downarrow & & \downarrow \\
1_{\mathcal{R}G\alpha} & \longrightarrow & 1_{\mathcal{R}G}/N\alpha
\end{array}
\]

Let $g \in G$. Then $g\alpha \in N_{V(RG)}(N)$ since $N = N\alpha$, so there is $h \in G$ such that $(g\alpha)h^{-1} \in C_{V(RG)}(N)$, by Lemma 2.1. As $(g\alpha)h^{-1}$ has augmentation 1, it follows from Lemma 2.4 that $(g\alpha)h^{-1} \in 1 + I_k(N)G + pRG$, so $g\alpha$ and $h$ map to the same element under the natural map $RG \rightarrow \mathcal{R}G/N\alpha$. We have shown that $\alpha$ induces on $\mathcal{R}G/N\alpha$ a group automorphism of $G/N$.

In particular, $1_{\mathcal{R}G}/N\alpha$ is an $\mathcal{R}(P \times P)$-module (and therefore the reduction modulo $p$ of a permutation lattice for $P \times P$ over $R$). If $p > 2$, then $e > e/(p - 1)$, and it follows from Corollary 3.6 that $1_{RG}/N\alpha$ is a permutation lattice for $P \times P$ over $R$. Hence we are left with the case $p = 2$.

If $1_{kG\alpha}$ were a $k(P \times P)$-permutation module, so that by the above considerations $\hat{P}$ and $P\alpha$ were conjugate in the units of $kG$, then in the $e = 1$ case, it would follow from Proposition 3.12 that $P$ and $P\alpha$ were conjugate in the units of $RG$. This might have been the intention of [16, Theorem 26] and [17, Theorem (4.10)], where the following is claimed to be true ($R$ denotes an unramified extension of $\mathbb{Z}_p$, with residue field $k = R/p\mathcal{R}$, and bars denote reduction modulo $p$).
Suppose that \( P \) and \( Q \) are \( p \)-groups, and let \( M \) be an \( R(P \times Q) \)-module (with the action \( (x, y) \cdot m = x \cdot m \cdot y^{-1}, x \in P, y \in Q, m \in M \), which is free of finite rank as an \( RQ \)-module. If \( M/(M \cdot 1_R(Q)) \) is a permutation module for \( kP \), then \( M \) is a permutation module for \( k(P \times Q) \).

Note that this would really give new information only when \( p = 2 \). The following example shows, however, that for \( p = 2 \) the statement is not true in general.

Let \( G = \langle x, y \rangle \cong C_2 \times C_2 \) be Klein’s four-group, and let \( R = \mathbb{Z}_2 \). An \( R \)-representation of \( G \) is given by

\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

The underlying \( RG \)-module \( M \) is an absolutely indecomposable module with vertex \( G \). Obviously, \( M \) is a free \( R \langle y \rangle \)-module, and

\[
M/M \cdot 1_R(\langle y \rangle) = M/M \cdot (y - 1)
\]

is, as a trivial \( k \langle x \rangle \)-module, of course a permutation module. However, it is easily seen that \( M \) is not a permutation module for \( kG \).

So let us return to the investigation of \( \hat{1}RG/N_\alpha \). The automorphism which \( \alpha \) induces on \( RG/N \) may be written as the product of the group automorphism of \( G/N \) which \( \alpha \) induces on \( RG/N \) and an automorphism \( \gamma \). Then \( \gamma \) induces the trivial automorphism on \( RG/N \). If one could deduce that \( \gamma \) is an inner automorphism, then \( \hat{1}RG/N_\alpha \) would be a permutation lattice as desired. In general, there seems to be no simple argument for that. However, if \( G\alpha \subseteq \mathbb{Z}_{\pi(G)}G \), then \( \gamma \) is an inner automorphism by Corollary 3.14. Thus, if one is only interested in automorphisms of \( \mathbb{Z}_{\pi(G)}G \), Step 1(b) does not cause additional difficulties.

4.3. Towards Step 2

For this step, suppose that \( P\alpha = P \). Then \( \hat{1}RG_\alpha \) is a trivial source module with vertex \( \{(x, x\alpha^{-1}) : x \in P\} \). It would be desirable to have a proof which shows directly that \( \hat{1}RG_\alpha \) is in fact a permutation module (so that Lemma 2.6 can be applied), but this seems to be a difficult task. However, the following lemma is easy to prove.

**Lemma 4.2.** If \( P\alpha = P \), then there is \( \beta \in \text{Aut}(N_G(P)) \) with \( \beta|_P = \alpha|_P \).

**Proof.** By the Schur–Zassenhaus Theorem, there is a complement \( K \) to \( P \) in \( N_G(P) \). By Lemma 2.1, there is a map \( f : K \to N_G(P) \) with \( x^{\alpha} = x^f(k) \) for all \( x \in P \). Let \( x \in P \) and \( h, k \in K \). Then

\[
x^{f(h)f(k)} = x^{h\alpha \cdot k\alpha} = x^{(hk)\alpha} = x^{f(hk)},
\]

so

\[
f(h) f(k) \in \mathbb{Z}(P) \cdot f(hk)
\]

since \( C_G(P) \subseteq \mathbb{Z}(P) \). Let \( Q = \langle Z(P), f(K) \rangle \). Then \( \mathbb{Z}(P) \leq Q \) and \( Q/\mathbb{Z}(P) \equiv K \) by the above. Again by the Schur–Zassenhaus Theorem, there is a complement \( L \) to \( \mathbb{Z}(P) \) in \( Q \). Let \( \overline{f} : K \to L \) be defined by \( \overline{f}(k) = g \) if \( f(k) = zg \) with \( z \in \mathbb{Z}(P) \) and \( g \in L \). By (2), \( \overline{f} \) is a homomorphism, which is clearly bijective. Define
\[ \beta: N_G(P) \rightarrow N_G(P) \] by \( (xk)\beta = x\alpha \cdot \tilde{f}(k) \) for all \( x \in P \) and \( k \in K \). Then \( \beta \in \text{Aut}(N_G(P)) \) as \( (x^k)\alpha = x^{\alpha \cdot k} = x^\alpha \cdot (k) \) for all \( x \in P \) and \( k \in K \).

4.4. Towards Step 3

This step is unproblematic. Roggenkamp and Scott proved it using ‘Green correspondence of automorphisms’ (see [17, Lemma (4.8)]), but in the case of the principal block, one can argue more directly.

**Proposition 4.3.** Let \( G \) be a finite group, \( \alpha \) a normalized automorphism of \( B \), the principal block of \( RG \), and \( P \) a Sylow \( p \)-subgroup of \( G \). Suppose that \( \alpha \) fixes the image of \( P \) in \( B \) elementwise. Then \( \alpha \) is an inner automorphism.

**Proof.** The ‘untwisted’ \( R(G \times G) \)-module \( {}_1B_1 \) is a trivial source module with vertex \( \Delta P = \{(x, x): x \in P\} \). By the assumption on \( \alpha \), this holds for the twisted \( R(G \times G) \)-module \( {}_1B_\alpha \) as well. Put \( L = N_{G \times G}(\Delta P) \), and let \( V \) be the \( R_L \)-lattice corresponding to \( {}_1B_\alpha \) under the Green correspondence. Since Green correspondents have the same vertices and sources, \( V \) is a direct summand of \( 1_{\Delta P} \), that is, \( V \) is an indecomposable projective module of \( RL/\Delta P \). We shall show that \( V \) is the projective cover of the trivial module. By Frobenius reciprocity [3, 10.8],

\[ \text{Hom}_{RL}(V, 1_L) \cong \text{Hom}_{R(G \times G)}(V \cdot L^{G \times G}, 1_{G \times G}). \] (3)

Since \( {}_1B_\alpha \) and \( V \) are Green correspondents, \( {}_1B_\alpha \) is a direct summand of \( V \cdot L^{G \times G} \). Moreover, \( B \cap L(G) \) is a submodule of \( {}_1B_\alpha \), and the quotient is the trivial \( R(G \times G) \)-module \( R \) since \( \alpha \) is a normalized automorphism. Hence the right-hand side of (3) is not zero, and \( V \) has the trivial module as epimorphic image. It follows that \( {}_1B_1 \) and \( {}_1B_\alpha \) have isomorphic Green correspondents, so \( {}_1B_1 \cong {}_1B_\alpha \), and \( \alpha \) is an inner automorphism by Lemma 2.5.

The shortest proof of Step 3 makes use of a Maschke argument.

**Remark 4.4.** If \( G \) has a normal \( p \)-subgroup \( N \) such that \( C_G(N) \subseteq N \), then, in the situation of the above proposition, one can also give a Maschke argument.

Let \( X \) be a set of (right) coset representatives for \( P \) in \( G \), and put

\[ u = \frac{1}{|G:P|} \sum_{x \in X} x^{-1}(x\alpha). \]

Note that the definition of \( u \) does not depend on the particular choice of \( X \), so \( g^{-1}u(g\alpha) = u \) for all \( g \in G \). Moreover \( u \) has augmentation 1 and is contained in \( C_{RG}(N) \), so is a unit in \( RG \) by Lemma 2.4, and \( \alpha \) is conjugation with \( u \).

**Remark 4.5.** Proposition 4.3 is clearly a result for the principal block, as the following example shows. Let \( G = \langle \langle x: x^3 \rangle \times \langle y: y^3 \rangle \rangle \rtimes \langle a: a^2 \rangle \), the operation given by \( x^a = x^{-1} \) and \( y^a = y^{-1} \). An automorphism \( \alpha \) of \( G \) is defined by \( x\alpha = x^{-1}, y\alpha = y \) and \( a\alpha = a \). Since \( \alpha \) does not preserve the conjugacy classes, it induces a non-inner automorphism of \( \mathbb{Z}_3G \). But \( \mathbb{Z}_3G \) decomposes into two blocks, so \( \alpha \) induces a non-inner automorphism of the non-principal block, by the last proposition.

From the above discussion, the last theorem follows.
Theorem 4.6. Let $G$ be a finite group with a normal Sylow $p$-subgroup $P$ containing its own centralizer, $C_G(P) \subseteq P$. Then for any normalized automorphism $\alpha$ of $RG$, the groups $G\alpha$ and $G$ are conjugate in the units of $RG$.

Proof. The automorphism $\alpha$ stabilizes the induced augmentation ideal $I_R(P)G$ since $RG/P$ is the largest homomorphic image of $RG$, as an $R$-algebra, which is torsion free and whose reduction modulo $\pi$ is semi-simple (cf. [16, p. 377]). So $P$ and $P\alpha$ are conjugate in the units of $RG$, by Lemma 4.1, and we may assume that $P\alpha = P$. By Lemma 4.2, there is $\beta \in \text{Aut}(G)$ with $\beta|_P = \alpha|_P$, so we may even assume that $\alpha|_P = \text{id}|_P$. But then $\alpha$ is an inner automorphism by Proposition 4.3. 

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